OSCILLATION THEORY FOR SECOND ORDER DIFFERENTIAL EQUATIONS AND DYNAMIC EQUATIONS ON TIME SCALES

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Oscillation Theory for Second Order Differential Equations and Dynamic Equations on Time Scales

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ABSTRACT

This thesis provides the oscillation criteria for second order linear differential equations and dynamic equations on time scales.

We establish the comparison theorems and oscillation criteria for self-adjoint and non-self adjoint equations and systems of first order ordinary differential equations. Then we prove the fundamental results concerning the dynamic equations: existence and uniqueness theorem and disconjugacy criteria.

ÖZET

Bu tez ikinci mertebeden lineer differansiyel denklemlerde ve zaman skalasında dinamik denklemlerde salınım teorisini içermektedir.

İkinci mertebeden öz eşlenik ve öz eşlenik olmayan diferansiyel denklemlerde ve birinci mertebeden diferansiyel denklem sistemlerinde karşılaştırma teoremleri ve salınım kriterlerini kurduk. Ek olarak zaman skalasında dinamik denklemlerde varlık teklik teoremini ve disconjugacy kriterlerini ispatladık.

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Chapter 1

INTRODUCTION

Since the most differential equations cannot be solved in terms of eliminating functions, it is important to be able to compare the unknown solutions of one differential equation with the known solutions of another.

One of the most frequently occurring types of differential equations in mathematics and the physical sciences is the linear second order differential equation of the form

$$p_0(x)u''(x) + p_1(x)u'(x) + p_2(x)u(x) = p_3(x). \qquad (1.1)$$

The coefficient functions $p_i(x)$ $[i = 0, 1, 2, 3]$ are assumed continuous and real valued on an interval I of the real axis, which may be finite or infinite. In this thesis we are concerned with the basic tools of comparison theorems and oscillation theory of second order linear differential and dynamic equations. We deal with the homogenous linear differential equation obtained by dropping the forcing term $p_3(x)$.

In Chapter 2 we give brief information for the theory of differential equations. We first give definitions of adjoint and self-adjoint operators and Lagrange Identity. Next we introduced the Sturm Liouville Problem which is the basis of our study. And then we present the definition of oscillatory and nonoscillatory solutions and disconjugate differential equations. Finally very powerful methods of oscillation theory the Prüfer Substitution and the Riccati differential equation are introduced. For the main notions and the facts from the theory of differential equations we refer to [5, 17, 21, 22, 27].

In Chapter 3 we deal with the comparison theorems for self-adjoint differential equations. We used the differential equation

$$L[u] = -\frac{d}{dx}[p(x)u'(x)] + r(x)u(x) = 0$$

as a basis of our study. We try to interpret the zeros of the solutions of the Sturm Majorant of the above equation by comparing the coefficients. Birkhoff and Rota [5] and Hartman [17] give comparison theorems for second order self-adjoint differential equations by using Prüfer Substitution. We present different point of view for the proofs of these theorems. Next we give oscillation and nonoscillation criteria for this type of differential equations by using the Riccati technique and Prüfer substitution.

In Chapter 4 first we generalized the results obtained in Chapter 3 for the differential equations of the type

$$-\frac{d}{dx}[p(x)\frac{du}{dx}] + q(x)\frac{du}{dx} + r(x)u = 0.$$

Since every second order differential equation can be stated as a system of first order equations we also present the oscillation criteria for the systems of first order equations.

In Chapter 5 first the time scale calculus as developed by Stefan Hilger [18] is introduced. The calculus of time scales is included in many of the recent papers interested in the time scales. We refer to [1, 4, 7, 18, 19, 20] for calculus of time scales. For functions $f : \mathbb{T} \rightarrow \mathbb{R}$ we introduce a derivative which unifies the ordinary derivative of continuous case and difference derivative of discrete case and an integral which unifies the ordinary integral of continuous case and the summation of discrete case. Fundamental results, e.g., the product rule, quotient rule, integration by part formula are presented. Next we present the induction principle on time scales to prove the existence and uniqueness of the solutions of initial value problem including a self-adjoint linear dynamic equation. Then we introduce the Hilger's complex plane. We use the so-called cylinder transformation to introduce the exponential function on time scales. This exponential function is then shown to satisfy an initial value problem involving a first order linear dynamic equation [18]. We present some properties of exponential function. Agarwal, Bohner O'Regan and Peterson [1] and Bohner and Peterson [7] present other properties of the exponential functions on time scales. Then we give the oscillation properties of the second order linear self-adjoint differential equation

$$[p(t)y^{\Delta}(t)]^{\Delta} + q(t)y^{\sigma}(t) = 0.$$

Such equations have been well studied in the continuous case (where they are called Sturm-Liouville equations) and the discrete case (where they are called Sturm-Liouville difference equations). Erbe and Hilger unified these two case [12]. They presented the generalization of Sturm's Seperation and Comparison

Theorems to the time scales. We investigate the disconjugacy of self adjoint equations. Also the theory of Riccati equations is developed in the general setting of time scales, and we present a characterization of disconjugacy in terms of a certain quadritic functional. The extended results are examined by Erbe, Peterson and R̃ehãk [14]. The eigenvalue problem, one of the most interesting problem of Sturm Liouville differential equation, is considered by Agarwal, Bohner, Wong [3]. Doš̃ly and Hilger then give a necesssary and sufficient condition for the oscillation Sturm Liouville dynamic equation on time scales [10]. An analogue of the classical Prüfer transformation, which has proved to be a useful tool in the theory of Sturm-Liouville equations, is given as well.

Chapter 2

PRELIMINARIES

2.1 Notations

First of all we need to describe some notations that we used throughout this thesis.

C := The class of continuous functions defined on $\mathbb{R}$.

C^n := The class of functions defined on $\mathbb{R}$ whose first n th derivative and itself are continuous.

$\mathbb{T}^k$:= The region of differentiability of the functions defined on $\mathbb{T}$.

$\mathbb{T}^{k^2}$:= The region of differentiability of the functions defined on $\mathbb{T}^k$.

C_{rd} := The class of all right dense continuous functions defined on $\mathbb{T}$.

C_{prd} := The class of all piecewise right dense continuous functions.

$\mathcal{R}$:= The set of all regressive and rd-continuous functions.

$\mathbb{C}_h$:= Hilger's Complex plane.

2.2 Adjoint and Self-adjoint Operators and Lagrange Identity

Early studies of differential equations concentrated on formal manipulations yielding solutions in terms of familiar functions. Out of these studies emerged many useful concepts, including those of integrating factors, and exact differentials. Birkhoff and Rota [5] extended these concepts to second order linear differential equations and derived the extremely important notions of adjoint and self-adjoint equations from them.

Definition 2.2.1. A second order homogenous linear differential equation

$$L[u] = p_0(x)u''(x) + p_1(x)u'(x) + p_2(x)u(x) = 0 \tag{2.1}$$

is said to be **exact** if and only if for some $P(x),\ Q(x) \in C^1$

$$p_0u'' + p_1u' + p_2u = \frac{d}{dx}[P(x)u' + Q(x)u]$$

is satisfied for all functions $u \in C^2$.

For simplicity we omit the independent variable x in some of the equations of this chapter.

Lemma 2.2.2. *The operator* $L[u]$ *in* (2.1) *is exact if and only if its coefficients satisfy*

$$p_0''(x) - p_1'(x) + p_2(x) = 0$$

Corollary 2.2.3. *A function* $v \in C^2$ *is an integrating factor for* (2.1) *if and only if it is a solution of the second order linear differential equation*

$$M[v] = (p_0v)'' - (p_1v)' + p_2v = 0 \tag{2.2}$$

Definition 2.2.4. The operator M is called the **adjoint** of the operator L.

The concept of the adjoint of a linear operator, which originated historically in the search of integrating factors, is of major importance because of the role which it plays in the theory of oscillation.

Consider the operators M and L. If L is multiplied by v and M is multiplied by u, and the results subtracted it follows that:

$$vL[u] - uM[v] = \frac{d}{dx}[p_0(u'v - uv') - (p_0' - p_1)uv] \tag{2.3}$$

The relation (2.3) is called **Lagrange Identity**.

In Chapter 3 we apply Lagrange Identity for the proofs of the comparison theorems.

Definition 2.2.5. Homogenous linear differential equations that coincide with their adjoint are called **self-adjoint**.

The condition for (2.1) to be self adjoint is $p_0' - p_1 = 0$. Since this relation implies $p_0'' - p_1' = 0$, it is also sufficient. Moreover in self-adjoint case, since $[p(uv' - u'v)]' = u(pv')' - v(pu')'$ Lagrange Identity becomes:

$$vL[u] - uL[v] = \frac{d}{dx}[p_0(u'v - uv')]. \tag{2.4}$$

Theorem 2.2.6. *The second order linear differential equation* (2.1) *is self-adjoint if and only if*

$$\frac{d}{dx}[p(x)\frac{du}{dx}] + q(x)u = 0. \tag{2.5}$$

All second order homogenous differential equations can be turned into self adjoint form by multiplying through by

$$h(x) = \frac{e^{\int \frac{p_1(x)}{p_0(x)}}dx}{p_0(x)}$$

Example 2.2.7. Consider the Chebyshev differential equation:

$$(1 - x^2)u'' - xu' + \lambda u = 0$$

In this case $p_0(x) = 1 - x^2,\ p_1(x) = x$ and $p_2(x) = \lambda$. Then

$$\begin{aligned} h(x) = \frac{e^{\int \frac{p_1(x)}{p_0(x)}}dx}{p_0(x)} &= \frac{e^{\int \frac{x}{1-x^2}dx}}{1 - x^2} \\ &= \frac{e^{-\frac{1}{2}\ln(1-x^2)}}{1 - x^2} \\ &= \frac{1}{(1 - x^2)^{\frac{3}{2}}}. \end{aligned}$$

Hence the self adjoint form of the Chebyshev differential equation is given by:

$$\frac{d}{dx}(\frac{1}{\sqrt{1 - x^2}}\frac{du}{dx}) + \frac{\lambda}{(1 - x^2)^{\frac{3}{2}}}u = 0.$$

2.3 Sturm-Liouville Problem

The Sturm-Liouville problem is the boundary value problem

$$\frac{d}{dx}[p(x)\frac{du}{dx}] + [q(x) + \lambda r(x)]u(x) = 0 \tag{2.6}$$

together with the boundary conditions

$$\alpha u(a) + \beta u(b) = 0 \tag{2.7a}$$

$$\gamma u'(a) + \delta u'(b) = 0 \tag{2.7b}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha^2+\beta^2 \neq 0$ and $\gamma^2+\delta^2 \neq 0$. Here, λ is a parameter and $r(x)$ is a function that is assumed continuous and positive on the interval $a \leq x \leq b$. As before, the functions $p(x), p'(x)$ and $q(x)$are assumed continuous, and $p(x)$ is positive on this interval. Under these conditions the boundary value problem (2.6) and (2.7) is known as a regular Sturm-Liouville problem. This is distinguished from the case when $p(x)$ or $r(x)$ vanishes at some point in the interval $[a, b]$ or when the interval is of infinite length, in which case the problem is called a singular Sturm-Liouville problem.

Other important boundary conditions that may arise with the differential equation (2.6) are

$$u(a) - u(b) = 0 \tag{2.8a}$$

$$u'(a) - u'(b) = 0. \tag{2.8b}$$

These are periodic boundary conditions, since both $u(x)$ and its derivative are required to have same value at the endpoints of the interval $[a, b]$.

2.4 Prüfer Substitution

One of the most powerful method for the study of the solutions of a self-adjoint second order linear differential equation

$$\frac{d}{dx}[p(x)\frac{du}{dx}] + q(x)u(x) = 0, \quad a < x < b \tag{2.9}$$

is the **Prüfer Substitution** which is defined by;

$$p\ u' = \rho(x)\cos\theta(x), \quad u = \rho(x)\sin\theta(x) \tag{2.10}$$

or

$$\rho^2(x) = u^2 + (p\ u')^2, \quad \theta(x) = \arctan(\frac{u}{p\ u'}) \tag{2.11}$$

ρ is called the amplitude and θ is called the phase variable [5, 23]. This method is based on obtaining a system of first order equations from (2.9). When $\rho \neq 0$ the transformation $(p\ u', u) \leftrightharpoons (\rho, \theta)$ defined by (2.10) or (2.11) are analytic with nonvanishing Jacobian.

$$\theta(x) = \arctan(\frac{u}{p\ u'}) \quad \Rightarrow \quad \sin\theta(x) = \frac{u}{\sqrt{u^2 + (p\ u')^2}} = \frac{u}{\rho(x)}$$
$$\Rightarrow \quad \cos\theta(x) = \frac{p\ u'}{\sqrt{u^2 + (p\ u')^2}} = \frac{p\ u'}{\rho(x)}$$

The system is obtained by differentiation of $\rho(x)$

$$\begin{aligned}
\rho'(x) &= \frac{2u(x)u'(x) + 2p(x)p'(x)(u'(x))^2 + 2u'(x)u''(x)p^2(x)}{2\ \sqrt{u^2(x) + (p(x)u'(x))^2}} \\
&= \frac{u'(x)[u(x) + p(x)p'(x)u'(x) + u''(x)p^2(x)]}{\rho(x)} \\
&= \frac{u'(x)p(x)}{\rho(x)} \cdot \frac{1}{p(x)}[u(x) + p(x)p'(x)u'(x) + u''(x)p^2(x)] \\
&= \cos\theta(x)[\frac{u(x)}{p(x)} + u'(x)p'(x) + u''(x)p(x)] \\
&= \cos\theta(x)[\frac{u(x)}{p(x)} + (p(x)u'(x))'] \\
&= \cos\theta(x)[\frac{u(x)}{p(x)} - q(x)u(x)] \\
&= \cos\theta(x)u(x)[\frac{1}{p(x)} - q(x)] \\
&= \cos\theta(x)\frac{u(x)}{\rho(x)}\rho(x)[\frac{1}{p(x)} - q(x)] \\
&= \cos\theta(x)\sin\theta(x)\rho(x)[\frac{1}{p(x)} - q(x)] \\
&= \frac{1}{2}[\frac{1}{p(x)} - q(x)]\rho(x)\sin 2\theta(x)
\end{aligned}$$

and by the differentiation of $\theta(x)$

$$\begin{aligned}
\theta'(x) &= \frac{[u'(x)]^2p(x) - [u'(x)p(x)]'u(x)}{u^2(x) + (u'(x)p(x))^2} \\
&= \frac{[u'(x)]^2p(x) + q(x)u^2(x)}{\rho^2(x)} \\
&= \frac{u^2(x)q(x)}{\rho^2(x)} + \frac{u^2(x)p(x)}{\rho^2(x)} \\
&= \frac{u^2(x)q(x)}{\rho^2(x)} + \frac{1}{p(x)} \cdot \frac{u^2(x)p^2(x)}{\rho^2(x)} \\
&= q(x)\sin^2\theta(x) + \frac{1}{p(x)}\cos^2\theta(x).
\end{aligned}$$

Hence we can transform (2.9) into the following system of first order nonlinear differential equations:

$$\frac{d\theta}{dx} = q(x)\sin^2\theta(x) + \frac{1}{p(x)}\cos^2\theta(x) = F(x,\theta) \tag{2.12a}$$

$$\frac{d\rho}{dx} = \frac{1}{2}[\frac{1}{p(x)} - q(x)]\rho(x)\sin 2\theta(x) \tag{2.12b}$$

The system (2.12) is equivalent to (2.9) in the sense that every nontrivial solution of the system defines a unique solution of the differential equation by Prüfer substitution (2.10) or (2.11), and conversely. This system is called **Prüfer System** associated with self-adjoint differential equation (2.9).

The first differential equation of Prüfer System (2.12) is a first order differential equation in θ, x alone, not containing other dependent variable ρ, it satisfies Lipschitz condition with Lipschitz constant

$$L = \sup_{a<x<b}\left|\frac{\partial F}{\partial \theta}\right|$$

Hence the existence and uniqueness theorems of first order ordinary differential equations are applicable and show that the first differential equation of (2.12) has a unique solution $\theta(x)$ for any initial value $\theta(a) = \nu$, by assuming $p(x)$ and $q(x)$ are continuous at a [5].

With known $\theta(x)$, $\rho(x)$ is given by

$$\rho = \mathcal{K}\exp\{\frac{1}{2}\int_a^x [\frac{1}{p(t)} - q(t)]\sin 2\theta(t)\}dt \tag{2.13}$$

where $\mathcal{K} = \rho(a)$. Each solution of Prüfer System (2.12) depends on two constants: the initial amplitude $\mathcal{K} = \rho(a)$ and the initial phase $\nu = \theta(a)$.

2.5 Riccati Substitution

The first order differential equation

$$\frac{du}{dx} = q_1(x) + q_2(x)u(x) + q_3(x)u^2(x) \tag{2.14}$$

is known as a **Riccati equation**. The Riccati technique relates the zeros of $u(x)$ to the singularities of a solution of the following Riccati equation and constitutes a basic tool in oscillation theory.

Lemma 2.5.1. *If u is a solution of*

$$L[u] = -\frac{d}{dx}[p(x)\frac{du}{dx}] + r(x)u(x) = 0. \tag{2.15}$$

Then

$$h(x) = -\frac{p(x)u'(x)}{u(x)} \tag{2.16}$$

is a solution of the Riccati equation

$$h'(x) - \frac{1}{p(x)}h^2(x) + r(x) = 0. \tag{2.17}$$

Proof. Let $u(x)$ be a real valued nontrivial solution of (2.15) and let $h(x)$ be defined as in (2.16). Then by using (2.15) we get

$$\begin{aligned} h'(x) &= -\frac{[p(x)u'(x)]'u(x) - p(x)u'(x)u'(x)}{u^2(x)} \\ &= -\frac{[p(x)u'(x)]'}{u(x)} + \frac{1}{p(x)} \quad \frac{p^2(x)[u'(x)]^2}{u^2(x)} \\ &= -\frac{r(x)u(x)}{u(x)} + \frac{1}{p(x)} \quad h^2(x) \\ &= -r(x) + \frac{1}{p(x)} \quad h^2(x). \end{aligned}$$

Therefore the Riccati equation of (2.15) is given by

$$h'(x) - \frac{1}{p(x)}h^2(x) + r(x) = 0.$$

□

Chapter 3

COMPARISON AND OSCILLATION CRITERIONS OF SELF ADJOINT EQUATIONS

In this chapter we are concerned with the comparison and oscillation theorems of second order linear equation

$$L[u] = -\frac{d}{dx}[p(x)u'(x)] + r(x)u(x) = 0$$

on infinite domain. We try to interpret the solutions of a differential equation by comparing the coefficients with the solutions of the differential equations with known zeros. The object of interest is the set of zeros of a solution $u(x)$. For the study of zeros of $u(x)$ the Prüfer transformation is a particularly useful tool since $u(x) = 0$ if and only if $\theta(x) = 0 \pmod{\pi}$ [17].

3.1 Comparison Theorems

Let L be the differential operator defined on an open interval (a, ∞) by

$$L[u] = -\frac{d}{dx}[p(x)\frac{du}{dx}] + r(x)u(x) = 0 \qquad a < x < \infty \tag{3.1}$$

where $p(x), r(x)$ are real valued functions on $(a, +\infty)$, $p(x) \geq 0$, $r(x)$ is continuous and $p(x)$ is continuously differentiable. We study the comparison theorem of Sturm Liouville problem dealing with the second order self-adjoint equations:

$$L_1[u] = -\frac{d}{dx}[p_1(x)\frac{du}{dx}] + r_1(x)u(x) = 0 \tag{3.2}$$

$$L_2[v] = -\frac{d}{dx}[p_2(x)\frac{dv}{dx}] + r_2(x)v(x) = 0 \tag{3.3}$$

Theorem 3.1.1. *If x_1 and x_2 are two consecutive zeros of a nontrivial solution of $u(x)$ of (3.2) and if*

(i) $p_1(x) \equiv p_2(x)$, *for* $x \in [x_1, x_2]$,

(ii) $r_1(x) \geq r_2(x)$ *and* $r_1(x) \not\equiv r_2(x)$ *for* $x \in [x_1, x_2]$, *then every solution* $v(x)$ *of* (3.3) *has a zero in* (x_1, x_2).

Proof. Let $u(x)$ and $v(x)$ be the solutions of (3.2) and (3.3) respectively. Since x_1 and x_2 are consecutive zeros of a nontrivial solutions of $L_1[u] = 0$, without loss of generality we can assume that $u(x) > 0$ for all $x \in (x_1, x_2)$. Suppose to the contrary that $v(x) \neq 0$ for all $x \in (x_1, x_2)$. Again without loss of generality we can suppose that $v(x) > 0$ for all $x \in (x_1, x_2)$.

Multiplying equation (3.2) by v , equation (3.3) by u and subtracting the resulting equations and by using Lagrange Identity (2.4) we get

$$\begin{aligned} vL_1[u] - uL_2[v] &= -[p_1u']'v + r_1uv + [p_2v']'u - r_2uv \\ &= [p_2v']'u - [p_1u']'v + (r_1 - r_2)uv \\ &= [p_1(uv' - vu')]' + (r_1 - r_2)uv. \end{aligned}$$

Integrating the last equation from x_1 to x_2, we get

$$\int_{x_1}^{x_2} \{[vL_1[u] - uL_2[v]\}dx = \int_{x_1}^{x_2} [p_1(uv' - vu')]'dx + \int_{x_1}^{x_2} (r_1 - r_2)uvdx.$$

Since

$$\int_{x_1}^{x_2} \{[vL_1[u] - uL_2[v]\}dx = 0,$$

then it follows that

$$\int_{x_1}^{x_2} (r_1 - r_2)u(x)v(x)dx = -\int_{x_1}^{x_2} \frac{d}{dx}[p_1(uv' - vu')]dx.$$

This relation implies that

$$\int_{x_1}^{x_2} (r_1 - r_2)u(x)v(x)dx = -p_1(uv' - vu')\Big|_{x_1}^{x_2}$$

$$\begin{aligned} &= p_1(x_1)\{u(x_1)v'(x_1) - v(x_1)u'(x_1)\} - p_1(x_2)\{u(x_2)v'(x_2) - v(x_2)u'(x_2)\} \\ &= p_1(x_2)v(x_2)u'(x_2) - p_1(x_1)v(x_1)u'(x_1). \end{aligned}$$

Since x_1 and x_2 are consecutive zeros of $u(x)$ and since $u(x) > 0$ for all $x \in (x_1, x_2)$ then $u(x)$ is increasing at x_1 and decreasing at x_2. It implies that $u'(x_1) > 0$ and $u'(x_2) < 0$. By hypothesis the left hand side of the equation is positive but the right hand side is negative. This result leads a contradiction. So the assumption is wrong. Then $v(x)$ has a zero in $[x_1, x_2]$. Note that if we get the equality $0 = 0$ then $r_1 = r_2$. So the equation (3.2) is identical to equation (3.3) on $[x_1, x_2]$. □

The condition $p_1(x) \equiv p_2(x)$ is not essential. We can remove this restriction and establish the following theorem. If the comparison conditions of the following theorem is satisfied then (3.3) is called **Sturm majorant** for (3.2) and (3.2) is called **Sturm minorant** for (3.3).

Theorem 3.1.2. *If x_1 and x_2 are two consecutive zeros of a nontrivial solution $u(x)$ of* (3.2)*, and if*

(i) $p_1(x) \geq p_2(x)$ *for* $x \in [x_1, x_2]$,

(ii) $r_1(x) \geq r_2(x)$ *for* $x \in [x_1, x_2]$,

then every solution $v(x)$ of (3.3) *has a zero in* $[x_1, x_2]$.

Proof. Let $u(x)$ and $v(x)$ be the solutions of (3.2) and (3.3) respectively. Suppose for the contrary that $v(x) \neq 0$ for all $x \in [x_1, x_2]$.

$$L_1[u] = 0 \Rightarrow \frac{d}{dx}[p_1(x)\frac{du}{dx}] = r_1(x)u(x)$$

and

$$L_2[v] = 0 \Rightarrow \frac{d}{dx}[p_2(x)\frac{dv}{dx}] = r_2(x)v(x)$$

$$\begin{aligned}
\frac{d}{dx}[\frac{u}{v}(p_1u'v - p_2uv')] &= \frac{u'v - uv'}{v^2}(p_1u'v - p_2uv') + \frac{u}{v}(p_1u'v - p_2uv')' \\
&= \frac{p_1(u')^2v^2}{v^2} - \frac{p_1uu'vv'}{v^2} - \frac{p_2uu'vv'}{v^2} + \frac{p_2(v')^2u^2}{v^2} \\
&\quad + \frac{u}{v}\big[(p_1u')'v + p_1u'v' - (p_2v')'u - p_2u'v'\big] \\
&= p_1(u')^2 - p_2\frac{uu'v'}{v} + p_2\frac{(v')^2u^2}{v^2} + r_1u^2 - r_2u^2 - p_2\frac{uu'v'}{v}
\end{aligned}$$

This is valid when u, v, p_1u', p_2v' are differentiable and $v(x) \neq 0$. If we subtract and add the term $p_2(u')^2$ to the last equation,

$$\frac{d}{dx}\big[\frac{u}{v}(p_1u'v - p_2uv')\big] = (r_1 - r_2)u^2 + (p_1 - p_2)(u')^2 + p_2\big[(u')^2 - 2\frac{uu'v'}{v} + \frac{(v')^2u^2}{v^2}\big]$$

then we get

$$\frac{d}{dx}\big[\frac{u}{v}(p_1u'v - p_2uv')\big] = (r_1 - r_2)u^2 + (p_1 - p_2)(u')^2 + p_2\big[u' - \frac{uv'}{v}\big]^2. \qquad (3.4)$$

Integrating (3.4) over (x_1, x_2) we obtain

$$\int_{x_1}^{x_2} \frac{d}{dx}\Big[\frac{u}{v}(p_1 u'v - p_2 uv')\Big]dx = \int_{x_1}^{x_2} \Big\{(r_1 - r_2)u^2 + (p_1 - p_2)(u')^2 + p_2[u' - \frac{uv'}{v}]^2\Big\}dx.$$

Since $u(x_1) = u(x_2) = 0$ then

$$\int_{x_1}^{x_2} \frac{d}{dx}\Big[\frac{u}{v}(p_1 u'v - p_2 uv')\Big]dx = 0. \tag{3.5}$$

It follows from (3.5) that

$$-\int_{x_1}^{x_2} p_2[u' - \frac{uv'}{v}]^2 dx = \int_{x_1}^{x_2} (r_1 - r_2)u^2 dx + \int_{x_1}^{x_2} (p_1 - p_2)(u')^2 dx. \tag{3.6}$$

Here right hand side of (3.6) is bigger than or equal to 0. But the left hand side is less than or equal to 0. This is a contradiction unless $r_1(x) \equiv r_2(x)$ and $p_1(x) \equiv p_2(x)$, i.e. (3.2) and (3.3) are the same. So either $u(x) \equiv v(x)$ or $u'(x) - \frac{u(x)v'(x)}{v(x)} \equiv 0. \Rightarrow \frac{u'(x)}{u(x)} = \frac{v'(x)}{v(x)}$, i.e. $u(x) = cv(x)$ where c is a constant. Then $v(x_1) = v(x_2) = 0$, therefore $v(x)$ has at least one zero in $[x_1, x_2]$. □

For another proof obtained by using Prüfer substitution see [5, 17].

3.2 Oscillation Criterions

Now we can establish a number of oscillation and nonoscillation criteria by using the above theorems.

Definition 3.2.1. A self adjoint differential equation

$$L[u] = -\frac{d}{dx}[p(x)\frac{du}{dx}] + r(x)u(x) = 0 \tag{3.7}$$

is said to be **oscillatory** at ∞ if every nontrivial solution has a zero in every interval of the form $[c, +\infty)$ where $c \geq a$.

It is **nonoscillatory** at ∞ if some nontrivial solution has a finite number of zeros in an interval of the form $[c, +\infty)$.

It is said to be **disconjugate** on an interval $(a, +\infty)$ if no solution has more than one zero in $(a, +\infty)$.

Theorem 3.2.2. *If there exist a point* $c \in (a, \infty)$ *such that*

$$\int_c^\infty \frac{1}{p(x)} dx = \infty$$

and

$$\int_c^\infty r(x) dx = -\infty$$

then (3.1) *is oscillatory at* ∞.

Proof. Suppose to the contrary that (3.1) is not oscillatory at ∞. If $u(x) > 0$ on $[\eta, \infty)$ $(\eta > a)$ then $h(x)$ defined in (2.16) is the solution of the Riccati equation (2.17) on $[\eta, \infty)$. Since

$$\int_c^\infty r(x) dx = -\infty$$

i.e. $\forall M > 0\ \exists \gamma\ (\gamma > c)$ such that $\alpha > \gamma$ we have

$$\int_c^\alpha r(x) dx < -M.$$

So we can find η and α such that $c < \eta < \alpha < \infty$ and

$$\int_\eta^x r(t) dt < h(\eta) \tag{3.8}$$

whenever $x \in [\alpha, \infty]$. Integrating the equation (2.17) over $[\eta, x]$ yields

$$\int_\eta^x h'(t) dt = -\int_\eta^x r(t) dt + \int_\eta^x \frac{1}{p(t)} h^2(t) dt.$$

$$h(x) - h(\eta) + \int_\eta^x r(t) dt = g(x) \tag{3.9}$$

where

$$g(x) = \int_\eta^x \frac{1}{p(t)} h^2(t) dt. \tag{3.10}$$

If we choose $M = |h(\eta)|$, it follows from (3.8) and (3.9) that;

$$h(x) - g(x) > 0$$

on $[\eta, \infty)$. From (3.10) we get

$$g'(x) = \frac{1}{p(x)} h^2(x) > \frac{1}{p(x)} g^2(x).$$

It follows that;

$$\frac{1}{p(x)} < \frac{g'(x)}{g^2(x)} \tag{3.11}$$

on $[\eta, \infty)$. Integrating (3.11) on (η, ∞) we obtain

$$\int_{\eta}^{\infty} \frac{1}{p(x)} dx < \int_{\eta}^{\infty} \frac{g'(x)}{g^2(x)} dx = -\frac{1}{g(x)}\Big|_{\eta}^{\infty} = \frac{1}{g(\eta)} - \frac{1}{g(\infty)} < \frac{1}{g(\eta)} < \infty.$$

So from (3.11) we get

$$\int_{c}^{\infty} \frac{1}{p(x)} dx < \infty.$$

This contradicts with the assumption. Then $u(x)$ is oscillatory at ∞. □

Theorem 3.2.3. *If there exist a point $c \in (a, \infty)$ such that*

$$\int_{c}^{\infty} \frac{1}{p(x)} dx < \infty$$

and

$$\int_{c}^{\infty} \mid r(x) \mid dx < \infty$$

then (3.1) *is nonoscillatory at ∞.*

Proof. Let (3.1) is oscillatory at ∞. Then every nontrivial solution of (3.1) has a zero in the interval of the form $[c, \infty)$ where $c \geq a$. For simplicity let $r(x) = -s(x)$. Then

$$\frac{d}{dx}[p(x)u'(x)] + s(x)u(x) = 0 \tag{3.12}$$

If we apply Prüfer substitution (2.10) to (3.12), then we get the following system of the first order equations:

$$\frac{d\theta}{dx} = s(x)\sin^2\theta(x) + \frac{1}{p(x)}\cos^2\theta(x); \tag{3.13}$$

$$\frac{d\rho}{dx} = \Big[\frac{1}{p(x)} - s(x)\Big]\rho(x)\sin\theta(x)\cos\theta(x). \tag{3.14}$$

With known $\theta(x)$, $\rho(x)$ is given by

$$\frac{d\rho}{dx} = \frac{1}{2}\Big[\frac{1}{p(x)} - s(x)\Big]\rho(x)\sin 2\theta(x)$$

$$\frac{d\rho}{\rho} = \frac{1}{2}\Big[\frac{1}{p(x)} - s(x)\Big]\sin 2\theta(x)dx$$

$$\rho(x) = \mathcal{K} \exp\{\frac{1}{2}\int_a^x \big[\frac{1}{p(t)} - s(t)\big] \sin 2\theta(t)dt\} \tag{3.15}$$

where $\mathcal{K} = s(a)$. It is obvious that for any nontrivial solution of (3.1)

$$u(x) = 0 \Leftrightarrow \sin\theta(x) = 0 \Leftrightarrow \theta(x) = k\pi \quad (k \in \mathbb{Z}).$$

Hence (3.1) is oscillatory at ∞ if and only if $\lim_{x\to\infty} \theta(x) = \infty$. Notice that $\theta'(x) > 0$ at all zeros of $u(x)$ since at these points $p(x) > 0$ and $\sin\theta(x) = 0$. Therefore (3.1) is oscillatory at ∞ if and only if $\lim_{x\to\infty} = \infty$. However

$$\begin{aligned}
\Big|\int_c^\infty (\frac{d\theta}{dx})dx\Big| &= \Big|\int_c^\infty \big[s(x)\sin^2\theta(x) + \frac{1}{p(x)}\cos^2\theta(x)\big]dx\Big| \\
&\leq \Big|\int_c^\infty s(x)\sin^2\theta(x)dx\Big| + \Big|\int_c^\infty \frac{1}{p(x)}\cos^2\theta(x)dx \\
&\leq \int_c^\infty \Big|s(x)\sin^2\theta(x)\Big|dx + \int_c^\infty \Big|\frac{1}{p(x)}\cos^2\theta(x)\Big|dx \\
&\leq \int_c^\infty \frac{1}{p(x)}dx + \int_c^\infty |s(x)|dx \\
&= \int_c^\infty \frac{1}{p(x)}dx + \int_c^\infty |r(x)|dx.
\end{aligned}$$

Then

$$\theta(x)\Big|_c^\infty \leq \int_c^\infty \frac{1}{p(x)}dx + \int_c^\infty |r(x)|dx \leq \infty$$

i.e. all the solutions of (3.13) are bounded on $[c, \infty)$. Therefore the phase of (3.1) attains finite number of $n\pi$ on $[c, \infty)$, that is, $\sin\theta$ has finite number of zeros on $[c, \infty)$. Thus (3.1) is not oscillatory at ∞. $\square$

Chapter 4

COMPARISON AND OSCILLATION CRITERIONS OF NONSELF ADJOINT EQUATIONS

4.1 Oscillation Theorems of Second Order Nonself-Adjoint Equations

In this chapter we used some kind of transformations to establish the criteria of oscillation for nonself-adjoint equations. Consider the following differential equations:

$$-\frac{d}{dx}\left[p_1(x)\frac{du}{dx}\right] + q_1(x)\frac{du}{dx} + r_1(x)u = 0 \tag{4.1}$$

$$-\frac{d}{dx}\left[p_2(x)\frac{dv}{dx}\right] + q_2(x)\frac{dv}{dx} + r_2(x)u = 0 \tag{4.2}$$

where q_1, q_2, r_1, r_2 are continuous on (a, ∞), p_1, p_2 are continuously differentiable and $p_1, p_2 > 0$ on (a, ∞).

The generalization of Theorem 3.1.2 to (4.1) and (4.2) can be stated as follows:

Theorem 4.1.1. *If x_1 and x_2 are two consecutive zeros of a nontrivial solution $u(x)$ of (4.1), and if*

(i) $p_1(x) \geq p_2(x) > 0$

(ii) $p_1(x) > p_2(x) > 0$ *whenever* $q_1(x) \neq q_1(x)$

(iii) $r_2(x) + \dfrac{(q_1(x) - q_2(x))^2}{4(p_1(x) - p_2(x))} + \dfrac{q_2(x)^2}{4p_2(x)} \leq r_1(x)$

for $x \in [x_1, x_2]$ then every solution $v(x)$ of (4.2) has a zero in $[x_1, x_2]$

Proof. Suppose to the contrary that $v(x) \neq 0$ for all $x \in [x_1, x_2]$.

$$\begin{aligned}
\frac{d}{dx}\quad & \left[\frac{u}{v}(p_1u'v - p_2v'u)\right] = \frac{u'v - uv'}{v^2}(p_1u'v - p_2v'u) + \frac{u}{v}(p_1u'v - p_2v'u)' \\
= \quad & \frac{u'v - uv'}{v^2}(p_1u'v - p_2v'u) + \frac{u}{v}[(p_1u')'v + p_1u'v' - (p_2v')'u - p_2u'v'] \\
= \quad & \frac{u'v - uv'}{v^2}(p_1u'v - p_2v'u) + \frac{u}{v}[(q_1u' + r_1u)v - (q_2v' + r_2v)u + (p_1 - p_2)u'v']
\end{aligned}$$

$$
\begin{aligned}
&= p_1(u')^2 - p_2\frac{uu'v'}{v} + p_2\frac{u^2(v')^2}{v} + q_1uu' + r_1u^2 - q_2\frac{u^2v'}{v} - r_2u^2 - p_2\frac{uu'v'}{v} \\
&= p_1(u')^2 - 2p_2\frac{uu'v'}{v} + p_2\frac{u^2(v')^2}{v} + (r_1-r_2)u^2 + q_1uu' - q_2\frac{u^2v'}{v} \\
&= (p_1-p_2)(u')^2 + p_2\big[u' - \frac{uv'}{v}\big]^2 + (r_1-r_2)u^2 + (q_1-q_2)uu' + q_2(uu' - \frac{u^2v'}{v}) \\
&= (r_1-r_2)u^2 + (p_1-p_2)(u')^2 + (q_1-q_2)uu' + p_2\big[u' - \frac{uv'}{v}\big]^2 \\
&\quad + q_2u\big[u' - \frac{u^2v'}{v}\big] - \frac{q_2^2}{4p_2}u^2 + \frac{q_2^2}{4p_2}u^2 \\
&= \big[(r_1-r_2)u^2 - \frac{q_2^2}{4p_2}u^2 - \frac{(q_1-q_2)^2}{4(p_1-p_2)}u^2\big] + \frac{(q_1-q_2)^2}{4(p_1-p_2)}u^2 + (p_1-p_2)(u')^2 \\
&\quad + (q_1-q_2)uu' + p_2\big[u' - \frac{uv'}{v}\big]^2 + q_2u\big[u' - \frac{uv'}{v}\big] + \frac{q_2^2}{4p_2}u^2 \\
&= (r_1 - r_2 - \frac{q_2^2}{4p_2} - \frac{(q_1-q_2)^2}{4(p_1-p_2)})u^2 + (p_1-p_2)\big[u' + \frac{q_1-q_2}{2(p_1-p_2)}u\big]^2 \\
&\quad + p_2\big[(u' - \frac{uv'}{v}) + \frac{q_2}{2p_2}u\big]^2
\end{aligned}
$$

where we used (4.1) and (4.2). Hence we find

$$\frac{d}{dx}\big[\frac{u}{v}(p_1u'v - p_2v'u)\big] \geq 0$$

which implies $\frac{u}{v}[p_1u'v - p_2v'u]$ is a nondecreasing function.

$$\frac{u}{v}[p_1u'v - p_2v'u](x_1) = \frac{u}{v}[p_1u'v - p_2v'u](x_2) = 0$$

implies that

$$\frac{u}{v}[p_1u'v - p_2v'u] \equiv 0, \quad \forall x \in [x_1, x_2].$$

Since x_1, x_2 are consecutive zeros of (4.1) and $v(x) \neq 0$ for all $x \in [x_1, x_2]$ then

$$p_1u'v - p_2v'u \equiv 0, \quad \forall x \in [x_1, x_2].$$

At $x = x_1$, $u(x_1) = 0$, $p(x_1) > 0$ and $u'(x_1) > 0$. Then $v(x_1) = 0$.
Similarly at $x = x_2$, $u(x_2) = 0$, $p(x_2) > 0$ and $u'(x_2) < 0$. Then $v(x_2) = 0$. This is a contradiction. Therefore, $v(x)$ has a zero on $[x_1, x_2]$. □

Theorem 4.1.2. *If there exist a point $b \in (a, \infty)$ such that*

$$\int_b^\infty \frac{1}{p_1(x)}dx = \infty \text{ and } \lim_{s\to\infty}\{-\frac{q_1(s)}{2} + \int_b^s [r_1(x) + \frac{q_1^2(x)}{4p_1(x)}]dx\} = -\infty$$

then (4.1) is oscillatory at ∞.

Proof. Assume that (4.1) is not oscillatory at ∞. Then there exist a nontrivial solution $u(x)$ such that

$$u(x) \neq 0; \qquad \forall x \in [\eta, \infty)$$

Without loss of generality we may assume that $u(x) > 0$. Applying Riccati substitution in (2.16), and using (4.1) we obtain

$$\begin{aligned} h'(x) &= \frac{-[p_1(x)u'(x)]'u(x) + p_1(x)[u'(x)]^2}{u^2(x)} \\ &= \frac{[-q_1(x)u'(x) - r_1(x)u(x)]u(x) + p_1(x)[u'(x)]^2}{u^2(x)} \\ &= -r_1(x) - q_1(x)\frac{u'(x)}{u(x)} + p_1(x)\frac{[u'(x)]^2}{u^2(x)} \\ &= -r_1(x) + \frac{q_1(x)}{p_1(x)}h(x) + \frac{1}{p_1(x)}h^2(x). \end{aligned}$$

Then

$$h'(x) + r_1(x) - \frac{q_1(x)}{p_1(x)}h(x) - \frac{1}{p_1(x)}h^2(x) = 0 \tag{4.3}$$

is the Riccati equation of (4.1). By adding and subtracting the term $\frac{q_1^2(x)}{4p_1(x)}$ on the right hand side of (4.3) we find

$$h'(x) = -r_1(x) + \frac{1}{p_1(x)}\Big[\big(h(x) + \frac{q_1(x)}{2}\big)^2 - \frac{q_1^2(x)}{4}\Big]. \tag{4.4}$$

Let

$$H(x) = h(x) + \frac{q_1(x)}{2}$$

Therefore by using (4.4) we find

$$H'(x) = h'(x) + \frac{q_1'(x)}{2} = -r_1(x) + \frac{1}{p_1(x)}\Big[H^2(x) - \frac{q_1^2(x)}{4}\Big] + \frac{q_1'(x)}{2}. \tag{4.5}$$

Integrating (4.5) over $[\eta, s]$ we obtain

$$\int_\eta^s H'(x)\ dx = \int_\eta^s \Big[-r_1(x) - \frac{q_1^2(x)}{4p_1(x)}\Big]\ dx + \int_\eta^s \frac{1}{p_1(x)}H^2(x)\ dx + \int_\eta^s \frac{q_1'(x)}{2}\ dx$$

$$H(s) - H(\eta) = \int_\eta^s \Big[-r_1(x) - \frac{q_1^2(x)}{4p_1(x)}\Big]\ dx + K(s) + \frac{q_1(s)}{2} - \frac{q_1(\eta)}{2}$$

where

$$K(x) = \int_\eta^x \frac{1}{p_1(t)}H^2(t)\ dt. \tag{4.6}$$

Then

$$
\begin{aligned}
H(s) - K(s) &= H(\eta) - \frac{q_1(\eta)}{2} - \big[- \frac{q_1(s)}{2} + \int_\eta^s \big(- r_1(x) - \frac{q_1^2(x)}{4p_1(x)} \big) \, dx \big] \\
&= h(\eta) - \big[- \frac{q_1(s)}{2} + \int_\eta^s \big(- r_1(x) - \frac{q_1^2(x)}{4p_1(x)} \big) \, dx \big]
\end{aligned}
$$

By hypothesis there exists $N > 0$ such that

$$
s > N \Rightarrow -\frac{q_1(s)}{2} + \int_\eta^s \big(- r_1(x) - \frac{q_1^2(x)}{4p_1(x)} \big) \, dx < -M
$$

for given $M > 0$. Hence $H(s) - K(s) > h(\eta) + M$. If we chose $M = |h(\eta)|$ then $H(s) - K(s) > 0$, i.e. $H(s) > K(s)$. From (4.6)

$$
K'(x) = \frac{1}{p_1(x)} H^2(x) > \frac{1}{p_1(x)} K^2(x)
$$

or

$$
\frac{1}{p_1(x)} < \frac{K'(x)}{K^2(x)}
$$

on $[\eta, \ \infty)$ since $K(x) > 0$ on $[\eta, \ \infty)$. Integrating the last inequality over $[b, \infty)$ and using the hypothesis we get

$$
\infty = \int_b^\infty \frac{1}{p_1(x)} dx < \int_b^\infty \frac{K'(x)}{K^2(x)} dx = \frac{1}{K(b)} - \lim_{s \to \infty} \frac{1}{K(s)} < \frac{1}{K(b)} < \infty.
$$

This result leads a contradiction. So the assumption is wrong. Then (4.1) is oscillatory at ∞. □

4.2 Oscillation Theorems For Systems of First Order Equations

Theorem 4.2.1. *Let u, w, v and z are nontrivial solutions of the following first order systems:*

$$
\begin{aligned}
u'(x) &= a_1(x)u(x) + b_1(x)w(x) && (4.7a) \\
w'(x) &= c_1(x)u(x) + d_1(x)w(x) && (4.7b)
\end{aligned}
$$

$$
\begin{aligned}
v'(x) &= a_2(x)v(x) + b_2(x)z(x) && (4.8a) \\
z'(x) &= c_2(x)v(x) + d_2(x)z(x) && (4.8b)
\end{aligned}
$$

where a_i, c_i, d_i are continuous functions for $i = 1, 2$ and $b_i > 0$ on (a, ∞) where $a \in \mathbb{R}$. If x_1 and x_2 are two consecutive zeros of $u(x)$ and;

(i) $b_2(x) \geq b_1(x) > 0$

(ii) $c_1(x) \geq c_2(x)$

(iii) $[b_2(x) - b_1(x)] \cdot [c_1(x) - c_2(x)] \geq \frac{1}{4} \cdot [a_2(x) + d_1(x) - a_1(x) - d_2(x)]^2$

for $x \in [x_1,\ x_2]$ *then* $v(x)$ *has a zero in* $[x_1,\ x_2]$.

Proof. By using Prüfer substitution (2.9) or (2.10) for the systems (4.7) and (4.8) we get following systems:

$$u(x) = \rho_1(x) \sin\theta_1(x) \tag{4.9a}$$

$$w(x) = \rho_1(x) \cos\theta_1(x) \tag{4.9b}$$

and

$$v(x) = \rho_2(x) \sin\theta_2(x) \tag{4.10a}$$

$$z(x) = \rho_2(x) \cos\theta_2(x) \tag{4.10b}$$

Then further by (4.9) and (4.10)

$$\theta_1(x) = \arctan\frac{u}{w}\ ,\quad \theta_2(x) = \arctan\frac{v}{z}\ .$$

$$\theta_1'(x) = \frac{u'(x)w(x) - u(x)w'(x)}{\rho_1^2}$$

By using the original system (4.7) and Prüfer substitution (4.9)

$$\begin{aligned}
\theta_1'(x) &= \rho_1^{-2}(x)\{(a_1(x)u(x) + b_1(x)w(x))w(x) - (c_1(x)u(x) + d_1(x)w(x))u(x)\} \\
&= \rho_1^{-2}\{(a_1\rho_1 \sin\theta_1 + b_1\rho_1\cos\theta_1)\rho_1\cos\theta_1 - (c_1\rho_1\sin\theta_1 + d_1\rho_1\cos\theta_1)\rho_1\sin\theta_1\} \\
&= a_1\sin\theta_1\cos\theta_1 + b_1\cos^2\theta_1 - c_1\sin^2\theta_1 - d_1\sin\theta_1\cos\theta_1.
\end{aligned}$$

And similarly by using the system (4.8) and Prüfer substitution (4.9) we get the following differential equations for the phases of the systems:

$$\theta_1'(x) = b_1\cos^2\theta_1 + (a_1 - d_1)\sin\theta_1\cos\theta_1 - c_1\sin^2\theta_1 = f_1(x,\theta_1) \tag{4.11}$$

and similarly we get

$$\theta_2'(x) = b_2\cos^2\theta_2 + (a_2 - d_2)\sin\theta_2\cos\theta_2 - c_2\sin^2\theta_2 = f_2(x,\theta_2) \tag{4.12}$$

From (4.11) and (4.12) it follows that

$$f_2(x,\theta_2(x)) - f_1(x,\theta_1(x)) = \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} A \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{4.13}$$

where

$$A = \begin{pmatrix} b_2 - b_1 & \frac{1}{2}[(a_2-d_2)-(a_1-d_1)] \\ [(a_2-d_2)-(a_1-d_1)] & c_2 - c_1 \end{pmatrix}$$

A is symmetric matrix and by assumptions all $a_{ij} > 0$. The eigenvalues of A

$$\lambda_{1,2} = \frac{1}{2}\{(b_2-b_1)+(c_2-c_1)$$
$$\pm\sqrt{[(b_2-b_1)+(c_2-c_1)]^2 - 4[(b_2-b_1)+(c_2-c_1) - \frac{1}{4}(a_2+d_1-a_1-d_2)^2]}\ \}$$

are nonnegative if $b_2 \geq b_1,\ c_1 \geq c_2$ and $(b_2-b_1)(c_1-c_2) \geq \frac{1}{4}(a_2+d_1-a_1-d_2)^2$ [8]. Hence $f_2(x,\theta_2(x)) > f_1(x,\theta_1(x))$ holds for all $x \in [x_1,x_2]$. By Sturm Comparison theorem $\theta_2(x) > \theta_1(x)$. □

We used *Mathematica* to compute the eigenvalues of A in the proof the theorem above.

Theorem 4.2.2. *If there exists a point* $c \in (a,\infty)$ *such that*

$$\int_c^\infty b(x)dx = \infty$$

and

$$\lim_{s\to\infty}\{\frac{d_1(s)-a_1(s)}{2b_2(s)} + \int_c^s \left[c_1(x) + \frac{(d_1(x)-a_1(x))^2}{4b_1(x)}\right]dx\} = -\infty$$

then (4.7) *is oscillatory at* ∞.

Chapter 5

SELF ADJOINT DYNAMIC EQUATIONS ON TIME SCALES

In this chapter we are concerned with second order self-adjoint dynamic equations on time scales

$$L[y(t)] = [py]^{\Delta}(t) + q(t)y^{\sigma}(t) = 0. \tag{5.1}$$

5.1 Time Scale Calculus

By a time scale (measure chain) $\mathbb{T}$, we mean an arbitrary nonempty closed subset of real numbers. The set of the real numbers, the integers, the natural numbers, and the Cantor set are examples of time scales. But the rational numbers, the irrational numbers,the complex numbers, and the open interval between 0 and 1, are not time scales.

The calculus of time scales was introduced by Stefan Hilger [18] in order to create the theory that can unify the discrete and continuous analysis. After discovery of time scales in 1988 almost every result obtained in the theory of differential and difference equations are carried into time scales.
Indeed the delta derivative f^{Δ} for a function f on $\mathbb{T}$ is

1. $f^{\Delta} = f'$ if $\mathbb{T} = \mathbb{R}$
2. $f^{\Delta} = \Delta f$ if $\mathbb{T} = \mathbb{Z}$.

5.1.1 Forward and Backward Jump Operators

In order to define Δ derivative (∇ derivative) we need to define forward jump operator σ, (backward jump operator ρ), graininess function μ, and the the region of differentiability $\mathbb{T}^k$ ($\mathbb{T}_k$) which is derived from $\mathbb{T}$.

Definition 5.1.1. Let $\mathbb{T}$ be a time scale. The **forward jump operator** $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \ , \ \forall t \in \mathbb{T}$$

and the **backward jump operator** $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} \ , \ \forall t \in \mathbb{T}.$$

Also $\sigma(\max(\mathbb{T})) = \max(\mathbb{T})$ and $\rho(\min(\mathbb{T})) = \min(\mathbb{T})$. t is called **right dense** if $\sigma(t) = t$, **left dense** if $\rho(t) = t$ and **right scattered** if $\sigma(t) > t$, **left scattered** if $\rho(t) < t$. t is called **dense** point if t is both left and right dense, and t is **isolated** if it is both left and right scattered. The **graininess function** $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$.

If $\mathbb{T} = \mathbb{R}$, then we have for any $t \in \mathbb{R}$

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf(t, \infty) = t$$

and similarly $\rho(t) = t$. Hence every point $t \in \mathbb{R}$ is dense.The graininess function μ turns out to be $\mu(t) \equiv 0$ for all $t \in \mathbb{R}$.

If $\mathbb{T} = \mathbb{Z}$, then we have for any $t \in \mathbb{Z}$

$$\sigma(t) = \inf\{s \in \mathbb{Z} : s > t\} = \inf\{t+1, t+2, \cdots\} = t + 1$$

and similarly $\rho(t) = t - 1$. Hence every point $t \in \mathbb{Z}$ is isolated.The graininess function μ turns out to be $\mu(t) \equiv 1$ for all $t \in \mathbb{Z}$.

Throughout this section and Chapter 5 we make the blanket $[a, b]$ that a and b are the points in $\mathbb{T}$. We define the interval $[a, b]$ in $\mathbb{T}$ by

$$[a, b] = \{t \in \mathbb{T} : a \le t \le b\}.$$

The following theorem unifies the induction principle of $\mathbb{Z}$ and the completeness axiom of real numbers and is used to prove the mean value, and existence and uniqueness theorems on time scales.

Theorem 5.1.2. (Induction Principle on Time Scale) *Let $t_0 \in \mathbb{T}$ and assume that*

$$\{A(t) : t \in [t_0, \infty)\}$$

is a family of statements satisfying:

(i) *The statement* $A(t_0)$ *is true,*

(ii) *If* $t \in [t_0, \infty)$ *is right scattered and* $A(t)$ *is true, then* $A(\sigma(t))$ *is true,*

(iii) *If* $t \in [t_0, \infty)$ *is right dense and* $A(t)$ *is true, then there is a neighborhood* U *of* t *such that* $A(s)$ *is true for all* $s \in U \bigcap (t, \infty)$*,*

(iv) *If* $t \in (t_0, \infty)$ *is right dense and* $A(s)$ *is true for all* $s \in [t_0, t)$*, then* $A(t)$ *is true.*

Then $A(t)$ *is true for all* $t \in [t_0, \infty)$*.*

Throughout this thesis we only deal with Δ dynamic equations. So we only define Δ derivative and Δ integration. The ∇ calculus can be obtained by replacing the operator ρ instead of σ.

5.1.2 Derivative on Time Scales

In order to define the derivative on time scale we also need the following set $\mathbb{T}^k$ (region of differentiability) which is derived from the time scale $\mathbb{T}$ as follows:

$$\mathbb{T}^k = \begin{cases} \mathbb{T} - \{\max \mathbb{T}\}, & \text{if } \max(\mathbb{T}) < \infty \text{ and } \max \mathbb{T} \text{ is right scattered;} \\ \mathbb{T} \qquad\qquad\quad , & \text{otherwise.} \end{cases}$$

Definition 5.1.3. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. If there exists a neighborhood U_t such that

$$|f(\sigma(t)) - f(s) - a[\sigma(t) - s]| \leq \epsilon |\sigma(t) - s| \tag{5.2}$$

is satisfied for all t, $a \in \mathbb{R}$ and $s \in U_t$. Then f is Δ-differentiable at the point t and a is called **Δ-derivative of f** at the point t.

$$a = f^{\Delta}(t) = \begin{cases} \lim\limits_{s \to t} \dfrac{f(\sigma(t)) - f(s)}{\sigma(t) - s} & , \text{ if } \mu(t) = 0; \\ \dfrac{f(\sigma(t)) - f(t)}{\mu(t)} & , \text{ if } \mu(t) > 0. \end{cases}$$

Δ-derivative is defined on $\mathbb{T}^k = \mathbb{T} - \{\max \mathbb{T}\}$, not on the whole time scale. If $t = \max \mathbb{T}$ then a neighborhood U_t of t contains only t. So the inequality (5.2) can be written only for $s = t$,

$$\begin{aligned} |f(\sigma(t)) - f(t) - a[\sigma(t) - t]| &\leq \epsilon|\sigma(t) - t| \\ |f(t) - f(t) - a[t - t]| &\leq \epsilon|t - t| \\ 0 &\leq 0 \end{aligned}$$

Thus the definition of Δ-derivative is satisfied for every value of a. Then a can not be determined uniquely.

Theorem 5.1.4. *Assume that $f,\ g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then for $\alpha,\ \beta \in \mathbb{R}$*

(i) *The linear sum $\alpha f + \beta g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(\alpha f + \beta g)^{\Delta}(t) = \alpha f^{\Delta}(t) + \beta g^{\Delta}(t)$$

(ii) *The product $(fg) : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

(iii) *If $g(t)g(\sigma(t)) \neq 0$ then $\frac{f}{g}$ is differentiable at t with*

$$(\frac{f}{g})^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Example 5.1.5. We consider the function $f(t) = t^2$ on an arbitrary time scale.

Let $\mathbb{T}$ be an arbitrary time scale and $t \in \mathbb{T}^k$. The Δ derivative of f is given by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s \to t} \frac{(\sigma(t))^2 - s^2}{\sigma(t) - s} = \lim_{s \to t} \sigma(t) + s = \sigma(t) + t.$$

1. If $\mathbb{T} = \mathbb{R}$ then $\sigma(t) = t$. Therefore $f^{\Delta}(t) = f'(t) = 2t$
2. If $\mathbb{T} = \mathbb{Z}$ then $\sigma(t) = t + 1$. Therefore $f^{\Delta}(t) = \Delta f(t) = 2t + 1$.
3. If $\mathbb{T} = \mathbb{N}_0^{\frac{1}{2}} = \{\sqrt{n} : n \in \mathbb{N}_0\}$ then $\sigma(t) = \sqrt{t^2 + 1}$. Therefore $f^{\Delta}(t) = t + \sqrt{t^2 + 1}$.

For computational results for Δ-derivative for given time scales see [25, 26].

5.1.3 Integration on Time Scales

We refer to [1, 7, 16, 18] for the time scale integration. Guseinov gives the most detailed theory of Riemann integral, Lebesque measure and integral and a criterion for Riemann integrability [16].

Definition 5.1.6. A function $f : \mathbb{T} \to \mathbb{R}$ is called **regulated** if it has finite right-sided limits at all right dense points in $\mathbb{T}$ and it has finite left-sided limits at all left dense points in $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called **rd-continuous** if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. The set of rd-continuous functions in $\mathbb{T}$ is denoted by C_{rd}.

Theorem 5.1.7. (Existence of Antiderivatives) *Every rd-continuous function has an antiderivative. In particular if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) := \int_{t_0}^{t} f(\tau)\Delta\tau \quad for \quad t \in \mathbb{T}$$

is an antiderivative of f.

Proof. [7, 18]. □

Theorem 5.1.8. *If $a,\ b \in \mathbb{T}$ and $f,\ g \in C_{rd}$, then*

$$\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t$$

and

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t$$

Proof. Trivial from Theorem 5.1.4. □

The formulas above are known as integration by parts formulas.

Theorem 5.1.9. *If $f \in C_{rd}$ and $t \in \mathbb{T}^k$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

Proof. [7]. □

Theorem 5.1.10. *Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$. If $[a, b]$ consists of only isolated points, then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t) & , \quad a < b \ ; \\ 0 & , \quad a = b; \\ -\sum_{t\in[b,a)} \mu(t)f(t) & , \quad a > b. \end{cases}$$

Proof. Assume that $a < b$ and let $[a, b] = \{t_0, t_1, t_2, ..., t_n\}$ where

$$a = t_0 < t_1 < t_2 < ... < t_n = b.$$

$$\begin{aligned} \int_a^b f(t)\Delta t &= \int_{t_0}^{t_1} f(t)\Delta t + \int_{t_1}^{t_2} f(t)\Delta t + ... + \int_{t_{n-1}}^{t_n} f(t)\Delta t \\ &= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(t)\Delta t \\ &= \sum_{i=0}^{n-1} \int_{t_i}^{\sigma(t_i)} f(t)\Delta t \\ &= \sum_{i=0}^{n-1} (\sigma(t_i) - t_i)f(t_i) \\ &= \sum_{t\in[a,b)} (\sigma(t) - t)f(t) \end{aligned}$$

If $a > b$, by using the fact

$$\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$$

we obtain

$$\int_a^b f(t)\Delta t = -\sum_{t\in[b,a)} (\sigma(t) - t)f(t)$$

which is the desired result. □

Example 5.1.11. For $\mathbb{T} = \mathbb{Z}$ consider the indefinite integral

$$\int a^t \Delta t$$

where $a \neq 0$ is a constant.

Since

$$(\frac{a^t}{a-1})^\Delta = \Delta(\frac{a^t}{a-1}) = \frac{a^{t+1} - a^t}{a-1} = a^t,$$

we get that

$$\int a^t \Delta t = \frac{a^t}{a-1} + C$$

where C is an arbitrary constant.

For the computational results for definite integral see [24, 26].

Theorem 5.1.12. *Let $a \in \mathbb{T}^k$ an $b \in \mathbb{T}$ and assume $f : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ is continuous at (t,t), where $t \in \mathbb{T}^k$ with $t > a$. Also assume that $f^\Delta(t,\cdot)$ is rd-continuous on $[a,\sigma(t)]$. Suppose that for each $\epsilon > 0$ there exist a neighborhood U of t, independent of $\tau \in [a,\sigma(t)]$, such that*

$$|f(\sigma(t),\tau) - f(s,\tau) - f^\Delta(t,\tau)(\sigma(t)-s)| \leq \epsilon|\sigma(t)-s| \quad forall \quad s \in U$$

where f^Δ denotes the derivative of f with respect to the first variable. Then

(i) $g(t) = \int_a^t f(t,\tau)\Delta\tau$ *implies* $g^\Delta(t) = \int_a^t f^\Delta(t,\tau)\Delta\tau + f(\sigma(t),t)$

(ii) $h(t) = \int_t^t f(t,\tau)\Delta\tau$ *implies* $h^\Delta(t) = \int_t^b f^\Delta(t,\tau)\Delta\tau - f(\sigma(t),t)$.

Proof. [1, 7] □

5.2 The Exponential Function

In this section we use the cylindrical transformation to define a generalized exponential function for an arbitrary time scale $\mathbb{T}$. First we make some preliminary definitions.

Definition 5.2.1. For $h > 0$ Hilger's complex numbers is defined as

$$\mathbb{C}_h = \{z \in \mathbb{C} : z \neq -\frac{1}{h}\}.$$

For $h = 0$, $\mathbb{C}_0 = \mathbb{C}$.

Definition 5.2.2. For $h > 0$ the cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ is defined by

$$\xi_h(z) = \frac{1}{h}Log(1+hz)$$

where Log is the principle logarithm function and

$$\mathbb{Z}_h = \{z \in \mathbb{C} : -\frac{\pi}{h} \leq Im(z) \leq \frac{\pi}{h}\}$$

and $\mathbb{Z}_0 = \mathbb{C}$. For $h = 0$, $\xi_0(z) = z$ for all $z \in \mathbb{C}$.

Definition 5.2.3. A function $p : \mathbb{T} \to \mathbb{R}$ is said to be **regressive** provided that

$$1 + \mu(t)p(t) \neq 0$$

for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions is denoted by $\mathcal{R}$. If $p \in \mathcal{R}$ then the generalized exponential function is defined by

$$e_p(s,t) = \exp(\int_s^t \xi_{\mu(t)}(p(\tau))\Delta\tau)$$

for $s, t \in \mathbb{T}$. If $p \in \mathcal{R}$ then the first order linear dynamic equation

$$y^{\Delta}(t) = p(t)y(t)$$

is called regressive.

Theorem 5.2.4. *Suppose that $y^{\Delta}(t) = p(t)y(t)$ is regressive and fix $t_0 \in \mathbb{T}^k$. Then $e_p(\cdot, t_0)$ is the solution of the initial value problem*

$$y^{\Delta}(t) = p(t)y(t), \; y(t_0) = 1$$

on $\mathbb{T}$.

Proof. [7]. □

Now we give some important properties of exponential function.

Theorem 5.2.5. *If $p \in \mathcal{R}$ then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$*;*

(ii) $e_p(\sigma(t), s) = [1 + \mu(t)p(t)]e_p(t,s)$*;*

(iii) $e_p(t,s) = \frac{1}{e_p(s,t)}$*;*

(iv) $e_p(t,s)e_q(s,r) = e_p(t,r)$.

Proof. The proof of these properties are included in [1, 7, 18]. We only show the proof of **(ii)** which we will use in Theorem 5.4.1.
(ii) By the definition of Δ derivative we have

$$\begin{aligned} e_p(\sigma(t),s) &= e_p^{\sigma}(t,s) \\ &= e_p(t,s)+\mu(t)e_p^{\Delta}(t,s) \\ &= e_p(t,s)+\mu(t)p(t)e_p(t,s) \\ &= [1+\mu(t)p(t)]e_p(t,s), \end{aligned}$$

where we used Theorem 5.2.4. □

5.3 Existence and Uniqueness Theorem

Throughout we assume that $p,q \in \mathbb{C}_{rd}$ and $p(t) \neq 0$ for all $t \in \mathbb{T}$. Define the set $\mathbb{D}$ to be the set of all functions $y : \mathbb{T}^{k^2} \to \mathbb{R}$ such that $y^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is continuous and such that $(p\, y^{\Delta})^{\Delta} : \mathbb{T}^{k^2} \to \mathbb{R}$ is rd-continuous. A function $y \in \mathbb{D}$ is then said to be solution of (5.1) provided $L[y] = 0$ holds for all $t \in \mathbb{T}^{k^2}$.

Theorem 5.3.1. *If $t_0 \in \mathbb{T}$, and $y_0,\ y_0^{\Delta}$ are given constants then the initial value problem*

$$L[y(t)] = [py^{\Delta}]^{\Delta}(t) + q(t)y^{\sigma}(t) = 0. \tag{5.3}$$

$$y(t_0) = y_0,\ y^{\Delta}(t_0) = y_0^{\Delta} \tag{5.4}$$

has a unique solution that exists on whole time scale $\mathbb{T}$.

Proof. We will use induction principle on time scale (Theorem 5.1.2) to prove the theorem.
Let the statement $A(r)$ be "The initial value problem $(5.3)-(5.4)$ has a unique solution in $[t_0,r] = \{t \in \mathbb{T} : t = 0 \leq t \leq r\}$."
In other words, let the statement $A(r)$ be "The initial value problem

$$[p(t)y^{\Delta}(t)]^{\Delta} + q(t)y^{\sigma}(t) = 0, \quad t \in [t_0,r] \tag{5.5}$$

$$y(t_0) = c_0, \qquad y^{\Delta}(t_0) = c_1$$

has a unique solution $y_r(t)$."

Note that since the operator in (5.5) has a degree of two, then the solution $y_r(t)$ is defined on $[t_0, \sigma^2(r)]$.

I) We will first show that $A(t_0)$ is true, i.e.

$$[p(t)y^{\Delta}(t)]^{\Delta} + q(t)y^{\sigma}(t) = 0, \qquad t \in \{t_0\} \tag{5.6}$$

$$y(t_0) = c_0, \qquad y^{\Delta}(t_0) = c_1 \tag{5.7}$$

has a unique solution.

Case 1) Let t_0 be right dense. So $y^{\Delta}(t) = y'(t)$. In this case, since t_0 has no neighborhood, we can not calculate $y^{\Delta}(t)$. So the statement is meaningless. Thus there is nothing to prove.

Case 2) Let t_0 be right scattered.

i) If $\sigma(t_0)$ is right dense, then $[t_0, \sigma^2(t_0)] = [t_0, \sigma(t_0)] = \{t_0, \sigma(t_0)\}$. In this set we cannot calculate $y^{\Delta\Delta}(t_0)$. So the statement is meaningless. Thus there is nothing to prove.

ii) If $\sigma(t_0)$ is right scattered, then it follows from (5.6) that

$$[p(t_0)y^{\Delta}(t_0)]^{\Delta} + q(t_0)y^{\sigma}(t_0) = 0$$

and from (5.7) that

$$y(t_0) = c_0, \quad y^{\Delta}(t_0) = \frac{y(\sigma(t_0)) - y(t_0)}{\sigma(t_0) - t_0}.$$

Here $y(t) = [t_0, \sigma^2(t_0)] \to \mathbb{R}$. Since t_0 and $\sigma(t_0)$ are right scattered then the interval $[t_0, \sigma^2(t_0)]$ turns out to the set $\{t_0, \sigma(t_0), \sigma^2(t_0)\}$. Since $\{t_0, \sigma(t_0), \sigma^2(t_0)\}$ is a discrete set then the problem becomes

"Does $y(t)$ attain a value at all the elements of $\{t_0, \sigma(t_0), \sigma^2(t_0)\}$? "

Note that $y(t)$ can not take two values at any point of $\{t_0, \sigma(t_0), \sigma^2(t_0)\}$. If we prove that $y(t)$ takes values at $\{t_0, \sigma(t_0), \sigma^2(t_0)\}$ then we will obtain both existence and uniqueness. From (5.6) and (5.7) we can obtain the followings:

$$\begin{aligned} y(t_0) &= c_0, \\ y^\Delta(t_0) &= \frac{y(\sigma(t_0)) - y(t_0)}{\sigma(t_0) - t_0} = c_1 \\ y(\sigma(t_0)) &= y(t_0) + c_1(\sigma(t_0) - t_0), \end{aligned}$$

$$\begin{aligned} \frac{p(\sigma(t_0))y^\Delta(\sigma(t_0)) - p(t_0)y^\Delta(t_0)}{\sigma(t_0) - t_0} + q(t_0)y^\sigma(t_0) &= 0 \\ \frac{p(\sigma(t_0)\dfrac{y(\sigma^2(t_0)) - y(\sigma(t_0))}{\sigma^2(t_0) - \sigma(t_0)} - p(t_0)y^\Delta(t_0)}{\sigma(t_0) - t_0} + q(t_0)y^\sigma(t_0) &= 0. \end{aligned}$$

Here all of the terms is known. Hence $y(\sigma^2(t_0))$ can be determined uniquely.

II) Let $r > t_0$ be right scattered and $A(r)$ be true, i.e. there exist a unique solution $y_r(t) : [t_0, \sigma^2(r)] :\longrightarrow \mathbb{R}$ of the initial value problem

$$\begin{aligned} [p(t)y^\Delta(t)]^\Delta &+ q(t)y^\sigma(t) = 0, \quad t \in [t_0, r] \\ y(t_0) &= c_0, \qquad y^\Delta(t_0) = c_1 \end{aligned}$$

We must show that there exist a unique function $y_{\sigma(r)}(t) : [t_0, \sigma^3(r)] \longrightarrow \mathbb{R}$ such that following equalities

$$\begin{aligned} &[p(t)y^\Delta_{\sigma(r)}(t)]^\Delta + q(t)y^\sigma_{\sigma(r)}(t) = 0 \\ &y_{\sigma(r)}(t_0) = c_0, \qquad y^\Delta_{\sigma(r)}(t_0) = c_1 \end{aligned}$$

are satisfied for all $t \in [t_0, \sigma^3(r)]$. Since $A(r)$ is true; we have unique solution $y_r(t)$ for $[t_0, \sigma^2(r)]$. Since $[t_0, \sigma^2(r)] \subset [t_0, \sigma^3(r)]$ then

$$y_r(t) = y_{\sigma(r)}(t) \qquad \forall t \in [t_0, \sigma^2(r)] \tag{5.8}$$

To complete the proof of the statement we must show that there exist a unique solution at $\sigma^3(r)$.
For an arbitrary function the equalities

$$f(t) = f(\sigma(t)) + (t - \sigma(t))f^\Delta(t) \tag{5.9}$$

$$f(\sigma^2(t)) = f(\sigma(t)) + (\sigma^2(t) - \sigma(t))f^\Delta(\sigma(t)) \tag{5.10}$$

are satisfied. (5.9) comes from Definition 5.1.3 and (5.10) is obtained by setting $t = \sigma(t)$ in (5.9). Lets take $f(t) = p(t)y^{\Delta}_{\sigma(r)}(t)$ in (5.9).

$$\begin{aligned} p(t)y^{\Delta}_{\sigma(r)}(t) &= p(\sigma(t))y^{\Delta}_{\sigma(r)}(\sigma(t)) + (t-\sigma(t))[p(t)y^{\Delta}_{\sigma(r)}(t)]^{\Delta} \\ &= p(\sigma(t))y^{\Delta}_{\sigma(r)}(\sigma(t)) + (\sigma(t)-t)q(t)y_{\sigma(r)}(\sigma(t)) \end{aligned}$$

So we get

$$y^{\Delta}_{\sigma(r)}(t) = \frac{p(\sigma(t))}{p(t)}y^{\Delta}_{\sigma(r)}(\sigma(t)) + \frac{q(t)}{p(t)}(\sigma(t)-t)y_{\sigma(r)}(\sigma(t)) \tag{5.11}$$

Lets take $f(t) = y_{\sigma(r)}(t)$ in (5.10). Then we get

$$y_{\sigma(r)}(\sigma^2(t)) = y_{\sigma(r)}(\sigma(t)) + (\sigma^2(t)-\sigma(t))y^{\Delta}_{\sigma(r)}(\sigma(t)) \tag{5.12}$$

By using (5.11) for $y^{\Delta}_{\sigma(r)}(\sigma(t))$ in (5.12) we obtain

$$\begin{aligned} y_{\sigma(r)}(\sigma^2(t)) &= y_{\sigma(r)}(\sigma(t)) + (\sigma^2(t)-\sigma(t)) \\ &\quad \{\frac{p(\sigma(t))}{p(t)}y^{\Delta}_{\sigma(r)}(\sigma(t)) + \frac{q(t)}{p(t)}(\sigma(t)-t)y_{\sigma(r)}(\sigma(t))\} \end{aligned} \tag{5.13}$$

Plugging $t = \sigma(r)$ in (5.13) and using (5.11) we get

$$\begin{aligned} y_{\sigma(r)}(\sigma^3(r)) &= y_r(\sigma^2(r)) + (\sigma^3(r)-\sigma^2(r)) \\ &\quad \{\frac{p(\sigma^2(r))}{p(\sigma(r))}y^{\Delta}_r(\sigma^2(r)) + \frac{q(\sigma(r))}{p(\sigma(r))}(\sigma^2(r)-\sigma(r))y_r(\sigma^2(r))\} \end{aligned}$$

We accomplished that $y_r(\sigma^3(r))$ is determined uniquely. So by using (5.8) and uniqueness of $y_r(\sigma^3(r))$ we obtain that $A(\sigma(r))$ is true.

III) Let $r_0 \geq t_0$, r_0 be right dense and $A(r_0)$ be true; i.e. there exist a unique function $y_{r_0} : [t_0, \sigma^2(r_0)] \to \mathbb{R}$ such that the following equalities

$$\begin{aligned} &[p(t)y^{\Delta}_{r_0}(t)]^{\Delta} + q(t)y^{\sigma}_{r_0}(t) = 0, \qquad t \in [t_0, r_0] \\ &y_{r_0}(t_0) = c_0, \qquad y^{\Delta}_{r_0}(t_0) = c_1 \end{aligned}$$

are satisfied. We must show that there exist r_1 in right neighborhood of r_0 such that $A(r)$ is true for all $r \in [r_0, r_1]$; i.e. there exist a unique function $y_r : [t_0, \sigma^2(r)] \to \mathbb{R}$ such that following equalities

$$\begin{aligned} &[p(t)y^{\Delta}_r(t)]^{\Delta} + q(t)y^{\sigma}_r(t) = 0, \qquad \forall t \in [t_0, r] \qquad (5.14) \\ &y_r(t_0) = c_0, \qquad y^{\Delta}_r(t_0) = c_1 \end{aligned}$$

are satisfied. Since r_0 is right dense then

$$r_0 = \sigma(r_0) = \sigma^2(r_0). \tag{5.15}$$

From the truth of $A(r_0)$ and (5.15), we get

$$y_r(t) = y_{r_0}(t), \qquad \forall t \in [t_0, r_0]. \tag{5.16}$$

To complete the proof of the statement we must determine $y_r(t)$ for the interval $r_0 < t \leq \sigma^2(r)$. Let $t \in (r_0, r]$. Integrating the equation (5.14) from r_0 to t we get

$$\int_{r_0}^{t} [p(s)y_r^{\Delta}(s)]^{\Delta}\Delta s = -\int_{r_0}^{t} q(s)y_r^{\sigma}(s)\Delta s$$

$$p(t)y_r^{\Delta}(t) - p(r_0)y_r^{\Delta}(r_0) = -\int_{r_0}^{t} q(s)y_r^{\sigma}(s)\Delta s$$

By using equation (5.16); we obtain

$$y_r^{\Delta}(t) = \frac{1}{p(t)}p(r_0)y_{r_0}^{\Delta}(r_0) - \frac{1}{p(t)}\int_{r_0}^{t} q(s)y_r^{\sigma}(s)\Delta s. \tag{5.17}$$

By integrating the equation (5.17) from r_0 to t and using the equation (5.16) we get

$$y_r(t) = y_{r_0}(r_0) + p(r_0)y_{r_0}^{\Delta}(r_0)\int_{r_0}^{t} \frac{\Delta\tau}{p(\tau)} - \int_{r_0}^{t} \frac{1}{p(\tau)}\Big\{\int_{r_0}^{\tau} q(s)y_r^{\sigma}(s)\Delta s\Big\}\Delta\tau. \tag{5.18}$$

If we show that the integral equation (5.18) has a continuous solution $y_r(t)$ then we will complete the proof of the statement. If we apply sequential approaching method by taking $q_1(s) = -q(s)$

$$y_r^{(0)}(t) = y_{r_0}(t) + p(r_0)y_{r_0}y_{r_0}^{\Delta}(r_0)\int_{r_0}^{t} \frac{\Delta\tau}{p(\tau)} \tag{5.19}$$

$$y_r^{(j)}(t) = \int_{r_0}^{t} \frac{1}{p(\tau)}\Big\{\int_{r_0}^{\tau} q_1(s)y_r^{(j-1)}(\sigma(s))\Delta s\Big\}\Delta\tau \quad j = 1, 2, ... \tag{5.20}$$

we get

$$y_r(t) = \sum_{j=0}^{\infty} y_r^{(j)}(t). \tag{5.21}$$

If $y_r(t)$ defined by (5.21) is uniformly convergent then it is a solution of (5.18).

$$\begin{aligned} (5.19) \Rightarrow |y_r^{(0)}(t)| &\leq |y_{r_0}(t)| + |p(r_0)| \cdot |y_{r_0}^{\Delta}(r_0)| \cdot \Big|\int_{r_0}^{t} \frac{\Delta\tau}{p(\tau)}\Big| \\ &\leq |y_{r_0}(t)| + |p(r_0)| \cdot |y_{r_0}^{\Delta}(r_0)| \cdot \int_{r_0}^{r} \frac{\Delta\tau}{|p(\tau)|} = M_0 \end{aligned}$$

$$\begin{aligned}
|y_r^{(1)}(t)| &\leq \int_{r_0}^{t} \frac{1}{|p(\tau)|}\Big\{\int_{r_0}^{\tau} |q(s)|\cdot|y_r^{(0)}(\sigma(s))|\Delta s\Big\}\Delta\tau \\
&\leq \int_{r_0}^{r} \frac{1}{|p(\tau)|}\Big\{\int_{r_0}^{\tau} |q(s)|\cdot|y_r^{(0)}(\sigma(s))|\Delta s\Big\}\Delta\tau \\
&\leq \int_{r_0}^{r} \Big\{\frac{1}{|p(\tau)|}\Big\{\max_{r_0\leq s\leq\rho(s)} |y_r^{(0)}(s)|\Big\}\cdot\int_{r_0}^{\tau} |q(s)|\Delta s\Big\}\Delta\tau \\
&\leq \Big\{\max_{r_0\leq s\leq\rho^2(s)} |y_r^{(0)}(s)|\Big\}\cdot\int_{r_0}^{r} \frac{1}{|p(\tau)|}\Big\{\int_{r_0}^{\tau} |q(s)|\Delta s\Big\}\Delta\tau \\
&\leq M_0 M_1
\end{aligned}$$

where

$$M_1 = \int_{r_0}^{r} \frac{1}{|p(\tau)|}\Big\{\int_{r_0}^{\tau} |q(s)|\Delta s\Big\}\Delta\tau. \tag{5.22}$$

By induction it can be shown that

$$|y_r^{(j)}(t)| \leq M_0 M_1^j, \qquad j = 1, 2, ...$$

Thus the necessary condition for (5.21) to be uniformly convergent is $M_1 < 1$. Since r_1 is in right neighborhood of r_0 it follows that

$$\int_{r_0}^{r_1} \frac{1}{|p(\tau)|}\Big\{\int_{r_0}^{\tau} |q(s)|\Delta s\Big\}\Delta\tau < 1$$

and therefore $M_1 < 1$ for all $r \in [r_0, r_1]$. Thus the integral equation (5.18) has a solution. If we show the uniqueness of the solution we complete the proof of the statement. Let $y_r(t)$ and $z_r(t)$ be two different solution of (5.18).

$$\begin{aligned}
|y_r(t) - z_r(t)| &\leq \int_{r_0}^{t} \frac{1}{|p(\tau)|}\Big\{\int_{r_0}^{\tau} |q(s)|\cdot|y_r^{\sigma}(s) - z_r^{\sigma}(s)|\Delta s\Big\}\Delta\tau \\
&\leq \int_{r_0}^{r} \frac{1}{|p(\tau)|}\Big\{\int_{r_0}^{\tau} |q(s)|\cdot|y_r^{\sigma}(s) - z_r^{\sigma}(s)|\Delta s\Big\}\Delta\tau \\
&\leq \Big\{\max_{r_0\leq s\leq\rho^2(r)} |y_r(\sigma(s)) - z_r(\sigma(s))|\Big\}\int_{r_0}^{r} \frac{1}{|p(\tau)|}\Big\{\int_{r_0}^{\tau} |q(s)|\Delta s\Big\}\Delta\tau \\
&\leq \Big\{\max_{\sigma(r_0)\leq s\leq\rho(r)} |y_r(s) - z_r(s)|\Big\}\int_{r_0}^{r} \frac{1}{|p(\tau)|}\Big\{\int_{r_0}^{\tau} |q(s)|\Delta s\Big\}\Delta\tau
\end{aligned}$$

$$N := \Big\{\max_{\sigma(r_0)\leq s\leq\rho(r)} |y_r(s) - z_r(s)|\Big\}$$

So we get

$$|\, y_r(t) - z_r(t)| \leq N\int_{r_0}^{r} \frac{1}{|p(\tau)|}\Big\{\int_{r_0}^{\tau} |q(s)|\Delta s\Big\}\Delta\tau$$

If $N \neq 0$, then

$$\int_{r_0}^{r} \frac{1}{|p(\tau)|} \{ \int_{r_0}^{\tau} | q(s)|\Delta s \} \Delta\tau > 1$$

It is a contradiction. So $N = 0$. Thus the integral equation (5.18) has a unique solution.

IV) Let $r_0 > t_0$ be left dense point and $A(r)$ is true for all $[t_0, r_0]$, i.e. there exist a unique function $y_r(t) : [t_0, \sigma^2(r)] \to \mathbb{R}$ such that

$$\begin{aligned} &[p(t)y_r^{\Delta}(t)]^{\Delta} + q(t)y_r^{\sigma}(t) = 0 \quad \forall t \in [t_0, r] \\ &y_r(t_0) = c_0, \qquad y_r^{\Delta}(t_0) = c_1 \end{aligned}$$

are satisfied. We must show that $A(r_0)$ is true, i.e. there exist a unique function $y_{r_0}(t) : [t_0, \sigma^2(r_0)] \to \mathbb{R}$ such that

$$\begin{aligned} &[p(t)y_{r_0}^{\Delta}(t)]^{\Delta} + q(t)y_{r_0}^{\sigma}(t) = 0 \quad \forall t \in [t_0, r_0] \\ &y_{r_0}(t_0) = c_0, \qquad y_{r_0}^{\Delta}(t_0) = c_1 \end{aligned}$$

are satisfied.
Lets choose $r_1 \in (t_0, r_0]$ such that

$$\int_{r_1}^{r_0} \frac{1}{|p(\tau)|} \{ \int_{r_1}^{\tau} |q(s)|\Delta s \} \Delta\tau < 1 \tag{5.23}$$

is satisfied. Since r_0 is left dense, there exist a point in left neighborhood of r_0 that satisfies (5.23). Thus by our assumption there exist a unique solution $y_{r_1}(t)$ on $[t_0, r_1]$. Let $t \in [r_1, r_0]$.

$$\begin{aligned} \int_{r_1}^{t} [p(s)y^{\Delta}(s)]^{\Delta}\Delta s &= -\int_{r_1}^{t} q(s)y^{\sigma}(s)\Delta s \\ p(t)y^{\Delta}(t) &= p(r_1)y^{\Delta}(r_1) - \int_{r_1}^{t} q(s)y^{\sigma}(s)\Delta s \\ y^{\Delta}(t) &= \frac{1}{p(t)}p(r_1)y^{\Delta}(r_1) - \frac{1}{p(t)}\int_{r_1}^{t} q(s)y^{\sigma}(s)\Delta s \\ \int_{r_1}^{t} y^{\Delta}(\tau)\Delta\tau &= p(r_1)y^{\Delta}(r_1)\int_{r_1}^{t} \frac{1}{p(\tau)}\Delta\tau - \int_{r_1}^{t} \frac{1}{p(\tau)}\{ \int_{r_1}^{\tau} q(s)y^{\sigma}(s)\Delta s \}\Delta\tau \end{aligned}$$

$$y(t) = y(r_1) + p(r_1)y^{\Delta}(r_1)\int_{r_1}^{t} \frac{1}{p(\tau)}\Delta\tau - \int_{r_1}^{t} \frac{1}{p(\tau)}\{ \int_{r_1}^{\tau} q(s)y^{\sigma}(s)\Delta s \}\Delta\tau$$

So by using the assumption we get

$$y(t) = y(r_1) + p(r_1)y^{\Delta}(r_1)\int_{r_1}^{t}\frac{1}{p(\tau)}\Delta\tau - \int_{r_1}^{t}\frac{1}{p(\tau)}\{\int_{r_1}^{\tau}q(s)y^{\sigma}(s)\Delta s\}\Delta\tau \quad (5.24)$$

By using (5.23) and sequential approaching method , we can show that (5.24) has a unique solution $y(t)$. Since r_1 is in left neighborhood of r_0, then r_1 is right dense, i.e. $\sigma(r_1) = r_1$. From (5.24) the following results are obtained:

$$y(r_1) = y_{r_1}(r_1), \quad y(\sigma(r_1)) = y_{r_1}(\sigma(r_1)) \quad (5.25)$$

$$y^{\Delta}(r_1) = y^{\Delta}_{r_1}(r_1), \quad y^{\Delta}(\sigma(r_1)) = y^{\Delta}_{r_1}(\sigma(r_1)) \quad (5.26)$$

Claim:

$$z(t) = \begin{cases} y_{r_1}(t), & t \in [t_0, \sigma^2(r_1)]; \\ y(t) \quad , & t \in [\sigma(r_1), \sigma^2(r_0)]. \end{cases}$$

is the unique solution of the initial value problem on $[t_0, r_0]$. To prove this claim we must show that $y(\sigma(r_1)) = y_{r_1}(\sigma(r_1))$ and $y(\sigma^2(r_1)) = y_{r_1}(\sigma^2(r_1))$. First statement is trivial from (5.25).

If we plug $f(t) = y_{r_1}(t)$ and $f(t) = y(t)$ respectively into (5.9) and plugging $t = r_1$ we get;

$$y_{r_1}(\sigma^2(r_1)) = y_{r_1}(\sigma(r_1)) + [\sigma^2(r_1) - \sigma(r_1)]y^{\Delta}_{r_1}(\sigma(r_1)) \quad (5.27)$$

$$y(\sigma^2(r_1)) = y(\sigma(r_1)) + [\sigma^2(r_1) - \sigma(r_1)]y^{\Delta}(\sigma(r_1)). \quad (5.28)$$

If we subtract (5.27) from (5.28) and use the equalities (5.25) and (5.26) we obtain the desired result. □

5.4 Basic Tools of Second Order Dynamic Equations

This section brings together the basic tools of second order dynamic equations on time scales. For the main notions, proofs and the facts from the theory of dynamic equations on time scales we refer to [1, 7, 10, 12, 15].

Theorem 5.4.1. *If $a, b \in C_{rd}$ and*

$$1 - a(t)\mu(t) + b(t)\mu^2(t) \neq 0 \quad (5.29)$$

for all $t \in \mathbb{T}^{k^2}$ *then the second order dynamic equation*

$$y^{\Delta\Delta} + a(t)y^{\Delta} + b(t)y = 0 \tag{5.30}$$

can be written in self-adjoint form (5.1)*, where*

$$p(t) = e_{\alpha}(t, t_0) \tag{5.31}$$

with $t_0 \in \mathbb{T}^k$*,*

$$\alpha(t) = \frac{a(t) - \mu(t)b(t)}{1 - a(t)\mu(t) + b(t)\mu^2(t)} \tag{5.32}$$

and

$$q(t) = e^{\sigma}_{\alpha}(t, t_0)b(t) = [1 + \mu(t)\alpha(t)]p(t)b(t). \tag{5.33}$$

Proof. Assume that (5.29) holds and α is given by (5.32). Then

$$1 + \alpha(t)\mu(t) = \frac{1}{1 - a(t)\mu(t) + b(t)\mu^2(t)} \neq 0,$$

and hence $\alpha \in \mathcal{R}$ so that p defined as (5.30) exists. Replacing $y = y^{\sigma} - \mu y^{\Delta}$ in the third term on the left side of equation (5.30), we obtain

$$y^{\Delta\Delta}(t) + [a(t) - \mu(t)b(t)]y^{\Delta} + b(t)y^{\sigma}(t) = 0$$

Multiplying both sides by $e^{\sigma}_{\alpha}(t, t_0)$, we get the equation

$$e^{\sigma}_{\alpha}(t, t_0)y^{\Delta\Delta}(t) + e^{\sigma}_{\alpha}(t, t_0)[a(t) - \mu(t)b(t)]y^{\Delta} + e^{\sigma}_{\alpha}(t, t_0)b(t)y^{\sigma}(t) = 0$$

The coefficient of $y^{\Delta}(t)$ is

$$\begin{aligned} e^{\sigma}_{\alpha}(t, t_0)[a(t) - \mu(t)b(t)] &= [1 + \mu(t)\alpha(t)]e_{\alpha}(t, t_0)[a(t) - \mu(t)b(t)] \\ &= \frac{a(t) - \mu(t)b(t)}{1 - a(t)\mu(t) + b(t)\mu^2(t)} e_{\alpha}(t, t_0) \\ &= \alpha(t)e_{\alpha}(t, t_0) \\ &= e^{\Delta}_{\alpha}(t, t_0). \end{aligned}$$

See the proof of Theorem 5.2.4 and Theorem 5.2.5 (ii) for the last equation. Hence the equation (5.30) is equivalent to

$$\begin{aligned} 0 &= e^{\sigma}_{\alpha}(t, t_0)y^{\Delta\Delta}(t) + e^{\Delta}_{\alpha}(t, t_0)y^{\Delta} + e^{\sigma}_{\alpha}(t, t_0)b(t)y^{\sigma}(t) \\ &= [e_{\alpha}(\cdot, t_0)y^{\Delta}]^{\Delta}(t) + e^{\sigma}_{\alpha}(t, t_0)b(t)y^{\sigma}(t). \end{aligned}$$

This equation is in self adjoint form with p given by (5.31) and q given by (5.32).

□

Example 5.4.2. Here we use Theorem 5.4.1 to write the dynamic equation

$$y^{\Delta\Delta} + 4y = 0$$

in self adjoint form on an arbitrary time scale $\mathbb{T}$.

In this case we have $a(t) \equiv 0$ and $b(t) \equiv 4$. This implies

$$1 - a(t)\mu(t) + b(t)\mu^2(t) = 1 + 4\mu^2(t) > 0.$$

So Theorem 5.4.1 can be applied. From (5.32), $\alpha(t) = \dfrac{-4\mu(t)}{1 + 4\mu^2(t)}$. Therefore, with the arbitrary $t_0 \in \mathbb{T}$, we get from (5.31) that

$$p(t) = e_\alpha(t, t_0) = e_{-4\mu/(1+4\mu^2)}(t, t_0)$$

and from (5.33) that

$$q(t) = [1 + \mu(t)\alpha(t)]p(t)b(t) = \frac{4}{1 + 4\mu^2(t)} e_{-4\mu/(1+4\mu^2)}(t, t_0)$$

Hence the self adjoint form of the above dynamic equation is

$$(e_{-4\mu/(1+4\mu^2)}(t, t_0)y^\Delta)^\Delta + \frac{1}{1 + 4\mu^2(t)} e_{-4\mu/(1+4\mu^2)}(t, t_0)y^\sigma = 0$$

Theorem 5.4.3. *If $a \in \mathcal{R}$, then the second order dynamic equation*

$$y^{\Delta\Delta} + a(t)y^{\Delta^\sigma} + b(t)y^\sigma = 0$$

can be written in self-adjoint form (5.1), *where*

$$p = e_a(., t_0), \qquad q = bp. \tag{5.34}$$

Proof. [7] □

The Prüfer Substitution has proved to be a useful tool in the qualitative theory of Sturm-Liouville differential equations (5.1) (with $\mathbb{T} = \mathbb{R}$). Bohner and Peterson [7] develop the extension of Prüfer Substitution to the time scales case. Let y be a nontrivial solution of (5.1). Then for all $t \in \mathbb{T}$

$$y^2(t) + (p(t)y^\Delta(t))^2 \neq 0.$$

and we can find real numbers $\rho(t)$ and $\theta(t)$ with $0 \leq \theta(t) \leq 2\pi$ such that the equations

$$y(t) = \rho(t)\sin\theta(t) \tag{5.35}$$

$$p(t)y^\Delta(t) = \rho(t)\cos\theta(t) \tag{5.36}$$

are satisfied for all $t \in \mathbb{T}$.

Definition 5.4.4. The equations (5.35) and (5.36) are said to be the **Prüfer Substitution**.

The Prüfer Substitution on time scales unifies the continuous Prüfer substitution and the discrete Prüfer substitution examined by Bohner and Dosly in 2001 [6].

Theorem 5.4.5. *If y is a nontrivial solution of* (5.1) *and if ρ and θ are defined by* (5.35) *and* (5.36)*, then the following equations hold:*

$$\begin{aligned}\rho^{\Delta} = \rho\Big\{\frac{1}{p}\cos\theta(\sin\theta)^{\sigma} &- q\sin\theta(\cos\theta)^{\sigma} - \frac{\mu q}{p}\cos\theta(\cos\theta)^{\sigma} \\ &- (\sin\theta)^{\Delta}(\sin\theta)^{\sigma} - (\cos\theta)^{\Delta}(\cos\theta)^{\sigma}\Big\} \qquad (5.37)\end{aligned}$$

$$\begin{aligned}(\sin\theta)^{\Delta}(\cos\theta)^{\sigma} - (\sin\theta)^{\sigma}(\cos\theta)^{\Delta} &= \frac{1}{p}\cos\theta(\cos\theta)^{\sigma} + q\sin\theta(\sin\theta)^{\sigma} \\ &\quad + \frac{\mu q}{p}\cos\theta(\sin\theta)^{\sigma}. \qquad (5.38)\end{aligned}$$

Proof. [7]. □

Remark 5.4.6. Theorem 5.4.5 suggests a method to construct solutions of self adjoint dynamic equation (5.1): Observe that the dynamic equation (5.38) for θ is independent from ρ. It is a nonlinear equation and might be difficult to solve but once (5.38) is solved, the linear dynamic equation (5.37) for ρ is readily solved.

Definition 5.4.7. We say that $y : \mathbb{T} \times \mathbb{T}^{k^2} \to \mathbb{R}$ is the **Cauchy function** for (5.1) provided for each $s \in \mathbb{T}^{k^2}$, $y(\cdot, s)$ is the solution of the initial value problem

$$L[y(\cdot,s)] = 0, \qquad y(\sigma(s),s) = 0, \qquad y^{\Delta}(\sigma(s),s) = \frac{1}{p(\sigma(s))}$$

Theorem 5.4.8. *(Variation of Constants Formula) Assume $f \in C_{rd}$ and let $y(t,s)$ be the Cauchy function for* (5.1)*. Then*

Proof. [7]. Also examine the partial differentiation on time scale [2] for the proof. □

Example 5.4.9. We use the variation of constants formula in Theorem 5.4.8 to solve the initial value problem

$$y^{\Delta\Delta}(t) = 1,\ y(0) = y^{\Delta}(0) = 0$$

for $\mathbb{T} = h\,\mathbb{Z}$.

Here $f(t) = 1,\ p(t) = 1,\ q(t) = 0$. Then $y(t) = \int_0^t y(t,s)\Delta s$ is the solution of the initial value problem where $y(t,s)$ is the Cauchy function for (5.1). Then by definition 5.4.7 $y(t,s)$ is the solution of the initial value problem

$$L[y(\cdot,s)] = 0,\ y(\sigma(s),s) = 0,\ y^{\Delta}(\sigma(s),s) = \frac{1}{p(\sigma(s))}.$$

It can be verified that if $q(t) = 0$ then the Cauchy function for 5.1 is given by $y(t,s) = \int_{\sigma(s)}^t \frac{1}{p(\tau)}\Delta\tau$. Hence $y(t,s) = \int_{\sigma(s)}^t \Delta\tau = t - \sigma(s)$. Then

$$y(t) = \int_0^t (t - \sigma(s))\Delta s.$$

If $s \in \mathbb{T} = h\,\mathbb{Z}$ then $s = h\,n$ for some $n \in \mathbb{Z}$. $\Rightarrow \sigma(s) = h\,(n+1) = h\,n+h = s+h$. And if $t \in \mathbb{T} = h\,\mathbb{Z}$ then $s = h\ m$ for some $m \in \mathbb{Z}$. Then

$$\begin{aligned} y(t) &= \int_0^{hm} (t - h - s)\Delta s \\ &= (t-h)\ t - \int_0^{hm} s\Delta s \\ &= (t-h)\ t - \{\int_0^h s\Delta s + \int_h^{2h} s\Delta s + \cdots \int_{(m-1)h}^{mh} s\Delta s\} \end{aligned}$$

by Theorem 5.1.9

$$\begin{aligned} &= (t-h)\ t - (h\cdot h + h\cdot 2h + \cdots h\cdot(m-1)h) \\ &= (t-h)\ t - h^2(1 + 2 + \cdots + (m-1)) \\ &= (t-h)\ t - \frac{hm\cdot h(m-1)}{2} \\ &= (t-h)\ t - \frac{t\cdot\rho(t)}{2} = \frac{1}{2}t(t-h). \end{aligned}$$

Theorem 5.4.10. *(Comparison Theorem for IVPs) Assume the Cauchy function y for (5.1) satisfies $y(t,s) \geq 0$ for $t \geq \sigma(s)$. If $u, v \in \mathbb{D}$ are functions satisfying*

$$L[u(t)] \geq L[v(t)], \quad \forall t \in [a,b], \qquad u(a) = v(a), \qquad u^{\Delta}(a) = v^{\Delta}(a)$$

then

$$u(t) \geq v(t), \qquad \forall t \in [a, \sigma^2(b)].$$

Proof. Let u and v satisfy the conditions of the theorem and define

$$w(t) = u(t) - v(t), \quad \forall t \in [a, \sigma^2(b)].$$

Then

$$h(t) = L[u(t)] - L[v(t)] \geq 0, \quad \forall t \in [a, \sigma^2(b)].$$

Hence w solves the initial value problem

$$L[w(t)] = h(t), \;\; w(a) = w^{\Delta}(a) = 0.$$

So by variation of constants formula (Theorem 5.4.8)

$$\begin{aligned} w(t) &= \int_a^t y(t,s)h(s)\Delta(s) \\ &= \int_a^{\rho(t)} y(t,s)h(s)\Delta(s) + \int_{\rho(t)}^t y(t,s)h(s)\Delta(s) \end{aligned}$$

by Theorem 5.1.9;

$$\begin{aligned} &= \int_a^{\rho(t)} y(t,s)h(s)\Delta(s) + \mu(\rho(t))y(t,\rho(t))h(\rho(t)) \\ &= \int_a^{\rho(t)} y(t,s)h(s)\Delta(s) \\ &\geq 0. \end{aligned}$$

Since $y(t,s)$ is the Cauchy function for (5.1), $y(t,\rho(t)) = y(\sigma(\rho(t)),\rho(t)) = 0$. ☐

Definition 5.4.11. If $x, y : \mathbb{T} \to \mathbb{R}$ are differentiable on $\mathbb{T}^k$, then we define Wronskian of x and y by

$$W(x,y)(t) = det \begin{pmatrix} x(t) & y(t) \\ x^{\Delta}(t) & y^{\Delta}(t) \end{pmatrix}.$$

Lemma 5.4.12. *If $x, y : \mathbb{T} \to \mathbb{R}$ are differentiable on $\mathbb{T}^k$, then*

$$W(x,y)(t) = det \begin{pmatrix} x^{\sigma}(t) & y^{\sigma}(t) \\ x^{\Delta}(t) & y^{\Delta}(t) \end{pmatrix} \qquad \forall t \in \mathbb{T}^k.$$

Proof. For $t \in \mathbb{T}^k$, by Definition 5.1.3 we have

$$\begin{aligned} det \begin{pmatrix} x^{\sigma}(t) & y^{\sigma}(t) \\ x^{\Delta}(t) & y^{\Delta}(t) \end{pmatrix} &= det \begin{pmatrix} x(t) + \mu(t)x^{\Delta}(t) & y(t) + \mu(t)y^{\Delta}(t) \\ x^{\Delta}(t) & y^{\Delta}(t) \end{pmatrix} \\ &= det \begin{pmatrix} x(t) & y(t) \\ x^{\Delta}(t) & y^{\Delta}(t) \end{pmatrix} \\ &= W(x,y)(t) \end{aligned}$$

which gives us the desired result. $\square$

Definition 5.4.13. If $x, y : \mathbb{T} \to \mathbb{R}$ are differentiable on $\mathbb{T}^k$, then the Lagrange bracket of x and y is defined by

$$\{x; y\}(t) = p(t)W(x, y)(t) \qquad \forall t \in \mathbb{T}^k.$$

Theorem 5.4.14. *(Lagrange Identity). If $x, y \in \mathbb{D}$, then*

$$x^\sigma(t)L[y(t)] - y^\sigma(t)L[x(t)] = \{x; y\}^\Delta(t), \qquad \forall t \in \mathbb{T}^k.$$

Proof.

$$\begin{aligned}
\{x; y\}^\Delta &= (p\, W(x, y))^\Delta \\
&= (x\, p\, y^\Delta - p\, x^\Delta y)^\Delta \\
&= x^\sigma (p\, y^\Delta)^\Delta + x^\Delta p\, y^\Delta - y^\sigma (p\, x^\Delta)^\Delta - y^\Delta p\, x^\Delta \\
&= x^\sigma (p\, y^\Delta)^\Delta - y^\sigma (p\, x^\Delta)^\Delta \\
&= x^\sigma \{(p\, y^\Delta)^\Delta + q\, y^\sigma\} - y^\sigma \{(p\, x^\Delta)^\Delta + q\, x^\sigma\} \\
&= x^\sigma\ L[y] - y^\sigma\ L[x]
\end{aligned}$$

on $\mathbb{T}^{k^2}$. $\square$

Corollary 5.4.15. *(Abel's Formula). If x and y both solve* (5.1)*, then*

$$W(x, y)(t) = \frac{C}{p(t)}, \qquad \forall t \in \mathbb{T}^k$$

where C is a constant.

Proof. From Theorem 5.4.14

$$\{x; y\}^\Delta = (p\, W(x, y))^\Delta = x^\sigma\ L[y] - y^\sigma\ L[x] = 0.$$

Hence $p(t)W(x, y)(t) = C$ for all $t \in \mathbb{T}^k$. $\square$

Corollary 5.4.16. *If x and y both solve* (5.1)*, then*

$$W(x, y)(t) \equiv 0, \qquad \forall t \in \mathbb{T}^k$$

or

$$W(x, y)(t) \neq 0, \qquad \forall t \in \mathbb{T}^k.$$

Proof. Trivial from Corollary 5.4.16. $\square$

5.5 The Riccati Equation

In this section the Riccati technique, very powerful method of studying oscillation theory is considered and the oscillation and disconjugacy criterions of second order dynamic equations are given. Erbe, Peterson and R̃ehãk give the extended results for the comparison of linear dynamic equations on time scales in [14]. Erbe and Peterson studied detailed the Riccati equation on measure chain in [13]. The basic results in Sturmian theory, oscillation, nonoscillation, and Riccati techniques are extended to dynamic equations are presented by Erbe and Hilger [12].

Definition 5.5.1. We say that a solution y of (5.1) has a **generalized zero** at t, in case $y(t) = 0$. A solution y has a generalized zero in $(t,\ \sigma(t))$, in case $y(\sigma(t))y(t) < 0$ and $\mu(t) > 0$. We say that (5.1) is **disconjugate** on an interval $[a,\ b]$, if there is no nontrivial solution of (5.1) with two (or more) generalized zeros in $[a,\ b]$. (5.1) is said to be **nonoscillatory** on $[\tau,\ \infty)$ if there exist $a \in [\tau,\ \infty)$ such that (5.1) is disconjugate on $[a,\ b]$ for $b > a$. (5.1) is **oscillatory** on $[\tau,\ \infty)$ if it has infinitely many generalized zeros in $[\tau,\ \infty)$.

Assume y is a solution of (5.1) with no generalized zeros. If we apply The Riccati substitution

$$z = \frac{p\ y^{\Delta}}{y} \tag{5.39}$$

on $\mathbb{T}^k$, then

$$p + \mu\ z = p + \frac{p\ y^{\Delta}}{x} = \frac{p\ (y + \mu\ y^{\Delta})}{y} = \frac{p\ y^{\sigma}}{y} > 0 \tag{5.40}$$

on $\mathbb{T}^k$.

$$\begin{aligned} z^{\Delta} &= \frac{y\ (p\ y^{\Delta})^{\Delta} - p\ (y^{\Delta})^2}{y\ y^{\sigma}} \\ &= \frac{-q\ y\ y^{\sigma} - p\ (y^{\Delta})^2}{y\ y^{\sigma}} \\ &= -q - \frac{y}{p\ y^{\sigma}}(\frac{p\ y^{\Delta}}{y})^2 \\ &= -q - \frac{y}{p\ y^{\sigma}} z^2 \\ &= -q - \frac{z^2}{p + \mu\ z} \end{aligned}$$

on $\mathbb{T}^k$, here we used (5.40) for the last equation. Hence z is the solution of the Riccati Equation

$$R[z](t) = z^{\Delta}(t) + q(t) + \frac{z^2(t)}{p(t) + \mu(t)z(t)} = 0 \tag{5.41}$$

on $\mathbb{T}^k$ satisfying

$$p(t) + \mu(t)z(t) > 0 \tag{5.42}$$

for all $t \in \mathbb{T}^k$. Hence we establish the proof of following result.

Theorem 5.5.2. *Let y be the solution of* (5.1)*. If y has no generalized zeros on $\mathbb{T}$ then z defined as in* (5.39) *is a solution of the Riccati equation* (5.41) *on $\mathbb{T}^k$, and* (5.42) *holds for all $t \in \mathbb{T}^k$.*

Example 5.5.3. Consider the Riccati equation

$$z^{\Delta} + 4(\frac{1}{9})^{t+1} + \frac{z^2}{(\frac{1}{9})^t + \mu(t)z} = 0$$

on $\mathbb{T} = \mathbb{Z}$.

Since $\mathbb{T} = \mathbb{Z}$, the self adjoint equation corresponding to the equation above is

$$\Delta\big[(\frac{1}{9})^t \Delta y(t)\big] + 4(\frac{1}{9})^{t+1} y(t+1) = 0.$$

Expanding this equation and simplifying we get the difference equation

$$y(t+2) - 6y(t+1) + 9y(t) = 0.$$

A general solution of this equation is

$$y(t) = c_1 3^t + c_2 t 3^t.$$

Hence

$$\begin{aligned} z(t) &= \frac{p(t)\Delta y(t)}{y(t)} \\ &= (\frac{1}{9})^t \cdot \frac{3^t(2c_1 + 3c_2 + 2c_2 t)}{3^t(c_1 + c_2 t)} \end{aligned}$$

For $c_1 \neq 0$ we get that $z(t) = (\frac{1}{9})^t \cdot \dfrac{2 + 3c + 2ct}{1 + ct}$.

Theorem 5.5.4. *Assume $p > 0$. Then* (5.1) *has a positive solution on $\mathbb{T}$ if and only if the Riccati equation* (5.41) *has a solution z on $\mathbb{T}^k$ satisfying* (5.42) *on $\mathbb{T}^k$.*

Proof. [7] □

Theorem 5.5.5. *Assume $y \in \mathbb{D}$ has no generalized zeros in $\mathbb{T}$ and z is defined by the Riccati substitution* (5.39) *for $t \in \mathbb{T}^k$. Then* (5.42) *holds for $t \in \mathbb{T}^k$ and*

$$L[y](t) = y^\sigma(t)R[z](t) \qquad \forall t \in \mathbb{T}^{k^2}.$$

Definition 5.5.6. Let $w = \sup \mathbb{T}$ and if $w < \infty$ assume $\rho(w) = w$. Further assume $a \in \mathbb{T}$ and $q \in C_{rd}$. We say (5.1) is **oscillatory** on $[a, w)$ provided every nontrivial real-valued solution has infinitely many generalized zeros in $[a, w)$.

The following theorem is an oscillation criteria for time scales which include only isolated points. More general types of oscillation criteria can be found in Section 5.6.

Theorem 5.5.7. *(Wintner's Theorem). Assume* $\sup \mathbb{T} = \infty, \ a \in \mathbb{T}$, $\mu(t) \geq K > 0$ *and* $0 < p(t) \leq M$ *for all* $t \in [a, \infty)$*, and*

$$\int_a^\infty q(t)\Delta t = \infty.$$

Then (5.1) *is oscillatory on* $[a, \infty)$.

Proof. Assume (5.1) is nonoscillatory on $[a, \ \infty)$. Then there exist $t_0 \geq a$ such that (5.1) has a positive solution y on $[t_0, \ \infty)$. Then we define z by the Riccati substitution (5.39) on $[t_0, \ \infty)$. By Theorem 5.5.4 z is the solution of the Riccati equation (5.41) on $[t_0, \ \infty)$ and (5.42) is satisfied on $[t_0, \ \infty)$. Integrating both sides of Riccati equation from t_0 to t we get that

$$\begin{aligned} z(t) &= z(t_0) - \int_{t_0}^t q(\tau)\Delta\tau - \int_{t_0}^t \frac{z^2(\tau)}{p(\tau) + \mu(\tau)z(\tau)}\Delta\tau \\ &\leq z(t_0) - \int_{t_0}^t q(\tau)\Delta\tau. \end{aligned}$$

Letting $t \to \infty$ we get that

$$\lim_{t\to\infty} z(t) = -\infty. \tag{5.43}$$

But from (5.42) we have that

$$z(t) \geq -\frac{p(t)}{\mu(t)} \geq -\frac{M}{K} \quad for \ \ t \geq t_0,$$

and this contradicts (5.43) so the proof is complete. □

Example 5.5.8. Consider the dynamic equation

$$x^{\Delta\Delta}(t) + \frac{1}{t}x^{\sigma}(t) = 0$$

on $\mathbb{T} = h\mathbb{N}$ where $h > 1$ is a constant.

Here $\sup \mathbb{T} = \infty$, $\mu(t) = h > 1 > 0$ and $p(t) \equiv 1$ for all $t \in \mathbb{T}$. Hence Theorem 5.5.7 is applicable. If $a \in \mathbb{T}$ then $a = hk$ for some $k \in \mathbb{N}$.

$$\begin{aligned}
\int_a^{\infty} q(t)\Delta t &= \sum_{k=\frac{a}{h}}^{\infty} q(kh)h = h\sum_{k=\frac{a}{h}}^{\infty} q(kh) = h\sum_{k=0}^{\infty} q((k+\frac{a}{h})h) = h\sum_{k=0}^{\infty} q(a+kh) \\
&= \{\frac{1}{a+h} + \frac{1}{a+2h} + \frac{1}{a+3h} + \cdots\} \\
&\geq h\{\frac{1}{h} + \frac{1}{2h} + \frac{1}{3h} + \cdots\} \\
&= h\frac{1}{h}\sum_{k=0}^{\infty}\frac{1}{k} = \sum_{k=0}^{\infty}\frac{1}{k}.
\end{aligned}$$

Since $\sum_{k=0}^{\infty}\frac{1}{k}$ is divergent then by comparison test $\int_a^{\infty} q(t)\Delta t = \infty$. Therefore $x^{\Delta\Delta}(t) + \frac{1}{t}x^{\sigma}(t) = 0$ is oscillatory on $[a, \infty)$.

5.6 Disconjugacy and Oscillatory Criterions

The following theorem summarizes and unifies the basic oscillatory and disconjugacy properties of (5.1). This theorem is due to [14].

Theorem 5.6.1. *(Reid-Roundabout theorem) The following statements are equivalent:*

(i) (5.1) *is disconjugate on* $[a, b]$.

(ii) (5.1) *has a solution without generalized zeros on* $[a, b]$.

(iii) *The Riccati dynamic equation* (5.41) *has a solution* z *with* $p(t)+\mu(t)z(t) > 0$ *for* $t \in [a, b]^k$.

(iv) *The quadratic functional*

$$\mathcal{F}(\xi; a, b) = \int_a^b \{p(t)(\xi^{\Delta}(t))^2 - q(t)(\xi^{\sigma}(t))^2\}\Delta t \tag{5.44}$$

is positive definite for $\xi \in U(a,b)$, where

$$U(a,b) = \{\xi \in C^1_{prd}[a,b] : \xi(a) = \xi(b) = 0\}.$$

Here C^1_{prd} denotes the set of all continuous functions whose derivatives are piecewise rd-continuous.

Proof. **(i)⇒(ii)** Assume that (5.1) is disconjugate on $[a,b]$. Let u and v be two solutions of (5.1) satisfying the initial conditions

$$u(a) = 0\ ,\ u^{\Delta}(a) = 1$$

$$v(b) = 0\ ,\ v^{\Delta}(b) = -1$$

By the definition of disconjugacy on $[a,b]$,

$$u(t) > 0\ ,\ \forall t \in (a,b] \tag{5.45a}$$

$$v(t) > 0\ ,\ \forall t \in [a,b) \tag{5.45b}$$

Let $y(t) = u(t) + v(t)$. From (5.45) it follows that $y(t) > 0$ for $t \in (a,b)$. Also

$$y(a) = u(a) + v(a) = v(a) > 0$$

and

$$y(b) = u(b) + v(b) = u(b) > 0$$

Then (5.1) has a solution without generalized zeros.

(ii)⇒(i) Let $u(t)$ be the solution of (5.1) without generalized zeros. Without loss of generality assume that $u(t) > 0$ on $[a,b]$. If we suppose for the contrary that (5.1) is not disconjugate on $[a,b]$ then (5.1) has a nontrivial solution v with at least two generalized zeros. Without loss of generality there are points $t_1 < t_2$ in $[a,b]$ such that $v(t_1) \le 0$, $v(t_2) \le 0$ and $v(t) > 0$ on (t_1,t_2) where $(t_1,t_2) \neq \emptyset$.

$$(\frac{v}{u})^{\Delta} = \frac{v^{\Delta}u - u^{\Delta}v}{uu^{\sigma}} = \frac{W(u,v)}{uu^{\sigma}} = \frac{C}{puu^{\sigma}}.$$

So $(\frac{v}{u})^{\Delta}$ is of one sign. Hence $\frac{v}{u}$ is strictly monotone on $[a,b]$. But

$$\frac{v}{u}(t_1) \le 0\ ,\ \frac{v}{u}(t) > 0\ ,\ \frac{v}{u}(t_2) \le 0$$

where $t \in (t_1, t_2)$. It is a contradiction. So (5.1) is disconjugate on $[a, b]$.

(ii)⇒(iii) Trivial from Theorem 5.5.4.

(iii)⇒(iv) Let z be the solution of (5.41) satisfying (5.42) for $t \in [a, b]^k$. If $\xi \in U(a, b)$, then

$$\begin{aligned}
(z\xi^2)^\Delta(t) &= \xi^2(\sigma(t))z^\Delta(t) + (\xi^2(t))^\Delta z(t) \\
&= \xi^2(\sigma(t))z^\Delta(t) + \xi(\sigma(t))\xi^\Delta(t)z(t) + \xi^\Delta(t)\xi(t)z(t) \\
&= \xi^2(\sigma(t))(-q(t) - \frac{z^2(t)}{p(t) + \mu(t)u(t)}) + \xi(\sigma(t))\xi^\Delta(t)z(t) \\
&\quad +\xi^\Delta(t)[\xi(\sigma(t)) - \mu(t)\xi^\Delta(t)]z(t) \\
&= p(t)(\xi^\Delta(t))^2 - q(t)\xi^2(\sigma(t)) - \frac{z^2(t)\xi^2(\sigma(t))}{p(t) + \mu(t)z(t)} + 2\xi(\sigma(t))\xi^\Delta(t)z(t) \\
&\quad -[p(t) + \mu(t)z(t)](\xi^\Delta)^2.
\end{aligned}$$

So

$$(z\xi^2)^\Delta(t) = p(t)(\xi^\Delta(t))^2 - q(t)\xi^2(\sigma(t)) - \Big\{\frac{z(t)\xi(\sigma(t))}{\sqrt{p(t) + \mu(t)z(t)}} - \sqrt{p(t) + \mu(t)z(t)}\xi^\Delta(t)\Big\}^2$$

is satisfied for all $[a, b]^k$. By integrating the last equation over $[a, b]$ and using $\xi(a) = \xi(b) = 0$ we get

$$\begin{aligned}
\mathcal{F}(\xi; a, b) &= \int_a^b \big\{p(t)(\xi^\Delta(t))^2 - q(t)\xi^2(\sigma(t))\big\}\Delta t \\
&= \int_a^b \Big\{\frac{z(t)\xi(\sigma(t))}{\sqrt{p(t) + \mu(t)z(t)}} - \sqrt{p(t) + \mu(t)z(t)}\xi^\Delta(t)\Big\}^2 \Delta t.
\end{aligned}$$

It follows that $\mathcal{F}(\xi; a, b) \geq 0$ for all $\xi \in U(a, b)$. Assume that $\mathcal{F}(\xi; a, b) = 0$. Then

$$\frac{z(t)\xi(\sigma(t))}{\sqrt{p(t) + \mu(t)z(t)}} - \sqrt{p(t) + \mu(t)z(t)}\xi^\Delta(t) = 0, \quad \forall t \in [a, b]^k.$$

Hence ξ solves the initial value problem

$$\xi^\Delta(t) = \frac{z(t)}{p(t) + \mu(t)u(t)}\xi(\sigma(t)), \quad \xi(a) = 0.$$

Existence and uniqueness theorem (Theorem 5.2.4) this implies that $\xi(t) = 0$ for all $t \in [a, b]^k$. This is a contradiction. Hence $\mathcal{F}$ is positive definite for $\xi \in U(a, b)$.

(iv)⇒(i) Let $\mathcal{F}(\xi; a, b)$ is positive definite on $U(a, b)$. Assume that (5.1) is not disconjugate on $[a, b]$. Then there exist at least two zeros of (5.1). Let t_0 and t_1 be the consecutive zeros of (5.1) in $[a, b]$ such that $a \leq t_0 \leq \sigma(t_0) < t_1 \leq b$ and y of be a solution of (5.1) satisfying the following conditions:

$$y(t_0) = 0 \;\; if \;\; \mu(t_0) = 0$$

$$y(t_0)x(\sigma(t_0)) < 0 \;\; if \;\; \mu(t_0) > 0,$$

and

$$y(t_1) = 0 \;\; if \;\; \mu(t_1) = 0$$

$$y(t_1)x(\sigma(t_1)) < 0 \;\; if \;\; \mu(t_1) > 0,$$

and

$$y(t) \neq 0 \;\; in \;\; (t_0, t_1).$$

Let

$$\xi(t) = \begin{cases} 0, & a \leq t \leq t_0; \\ y(t), & t_0 < t \leq t_1; \\ 0, & t_1 \leq t \leq b. \end{cases}$$

$$\begin{aligned}
\mathcal{F}(\xi; a, b) &= \int_a^b \{p(t)(\xi^\Delta(t))^2 - q(t)\xi^2(\sigma(t))\}\Delta t \\
&= \int_{t_0}^{t_1} \{p(t)(\xi^\Delta(t))^2 - q(t)\xi^2(\sigma(t))\}\Delta t \\
&= \int_{t_0}^{\sigma(t_1)} \{p(t)(\xi^\Delta(t))^2 - q(t)\xi^2(\sigma(t))\}\Delta t - \int_{t_1}^{\sigma(t_1)} \{p(t)(\xi^\Delta(t))^2 - q(t)\xi^2(\sigma(t))\}\Delta t \\
&= \int_{t_0}^{\sigma(t_1)} p(t)(\xi^\Delta(t))^2\Delta t - \int_{t_0}^{\sigma(t_1)} q(t)\xi^2(\sigma(t))\Delta t - \int_{t_1}^{\sigma(t_1)} p(t)(\xi^\Delta(t))^2\Delta t \\
&\quad + \int_{t_1}^{\sigma(t_1)} q(t)\xi^2(\sigma(t))\Delta t \\
&= \int_{t_0}^{\sigma(t_1)} p(t)(\xi^\Delta(t))^2\Delta t - \int_{t_0}^{t_1} q(t)\xi^2(\sigma(t))\Delta t - \int_{t_1}^{\sigma(t_1)} p(t)(\xi^\Delta(t))^2\Delta t \\
&= \int_{t_0}^{\sigma(t_0)} p(t)(\xi^\Delta(t))^2\Delta t + \int_{\sigma(t_0)}^{t_1} p(t)(\xi^\Delta(t))^2\Delta t + \int_{t_1}^{\sigma(t_1)} p(t)(\xi^\Delta(t))^2\Delta t \\
&\quad - \int_{t_0}^{\sigma(t_0)} q(t)\xi^2(\sigma(t))\Delta t - \int_{\sigma(t_0)}^{t_1} q(t)\xi^2(\sigma(t))\Delta t \\
&= \int_{t_0}^{\sigma(t_0)} p(t)(\xi^\Delta(t))^2\Delta t + \int_{t_1}^{\sigma(t_1)} p(t)(\xi^\Delta(t))^2\Delta t - \int_{t_0}^{\sigma(t_0)} q(t)\xi^2(\sigma(t))\Delta t \\
&\quad + \int_{\sigma(t_0)}^{t_1} \{p(t)(y^\Delta(t))^2 - q(t)y^2(\sigma(t))\}\Delta t
\end{aligned}$$

$$\begin{aligned}
&= \mu(t_0)p(t_0)(\xi^\Delta(t_0))^2 + \mu(t_1)p(t_1)(\xi^\Delta(t_1))^2 - \mu(t_0)q(t_0)(\xi^2(\sigma(t_0)) \\
&\quad +[p(t)y^\Delta(t)y(t)]_{\sigma(t_0)}^{t_1} - \int_{\sigma(t_0)}^{t_1} L[y(t)]y(\sigma(t))\Delta t \\
&= \mu(t_0)p(t_0)(\xi^\Delta(t_0))^2 + \mu(t_1)p(t_1)(\xi^\Delta(t_1))^2 - \mu(t_0)q(t_0)(\xi^2(\sigma(t_0)) \\
&\quad +[p(t)y^\Delta(t)y(t)]_{\sigma(t_0)}^{t_1} \\
&= C + D
\end{aligned}$$

(we used integration by parts formula Theorem 5.1.8 and $L[y(t)] = 0$) where

$$C = \mu(t_0)p(t_0)(\xi^\Delta(t_0))^2 - \mu(t_0)q(t_0)\xi^2(\sigma(t_0)) - p(\sigma(t_0))y^\Delta(\sigma(t_0))y(\sigma(t_0)) \tag{5.46}$$

and

$$D = \mu(t_1)p(t_1)(\xi^\Delta(t_1))^2 + p(t_1)y^\Delta(t_1)y(t_1). \tag{5.47}$$

Further from (5.46)

$$C = \begin{cases} -p(t_0)y(t_0)y^\Delta(t_0), & \mu(t_0) = 0; \\ \dfrac{p(t_0)y(t_0)y(\sigma(t_0))}{\mu(t_0)}, & \mu(t_0) > 0. \end{cases} \tag{5.48}$$

and

$$D = \begin{cases} p(t_1)y(t_1)y^\Delta(t_1), & \mu(t_1) = 0; \\ \dfrac{p(t_1)x(t_1)x(\sigma(t_1))}{\mu(t_1)}, & \mu(t_0) > 0. \end{cases} \tag{5.49}$$

It follows from (5.48) and (5.6) that $\mathcal{F}(\xi; a, b) \leq 0$. This contradiction completes the proof. □

Theorem 5.6.1 makes it clear that there are at least two methods of investigation of (non)oscillation of (5.1). The first one is based on the equivalence of **(i)** and **(iv)** and this method can be formulated as follows:

Lemma 5.6.2. *If for any $T \in (\tau, \infty)$ there exists $0 \neq \xi$ in $U(T)$, where*

$$\begin{aligned}
U(T) = \{\xi \in C_p^1[T,\infty) \quad &: \quad \xi(t) = 0, \quad \forall t \in [\tau, T] \ \ \textit{and} \ \ \exists \hat{T}, \ \hat{T} > \sigma(T) \ \ \textit{such} \\
\textit{that} \ \ \xi(t) \quad &= \quad 0, \quad \forall t \in [\sigma(\hat{T}), \infty)\}
\end{aligned}$$

such that $\mathcal{F}(\xi; T, \infty) = \mathcal{F}(\xi; T, \sigma(\hat{T})) \leq 0$ then (5.1) *is oscillatory.*

Another method of investigation for the oscillation theory of (5.1) based on the equivalence of **(i)** and **(iv)** of Theorem 5.6.1.

Lemma 5.6.3. *Consider the equation*

$$[p_1(t)x^{\Delta}(t)]^{\Delta} + q_1(t)x^{\sigma}(t) = 0 \tag{5.50}$$

where $p_1(t)$, $q_1(t)$ *satisfy the same conditions as* $p(t)$ *and* $q(t)$ *of* (5.1). *Suppose that* $p_1(t) \leq p(t)$ *and* $q(t) \leq q_1(t)$ *on* $[T, \infty)$ *for all large* T. *Then* (5.50) *is nonoscillatory on* $[T, \infty)$ *implies* (5.1) *is nonoscillatory on* $[T, \infty)$.

Proof. Let (5.50) is nonoscillatory on $[T, \infty)$. Then there exists $a \in [T, \infty)$ such that (5.50) is disconjugate on $[a, b]$ for all $b > a$. Then the equivalence of **(i)** and **(iv)** of Theorem 5.6.1 implies

$$\mathcal{F}_1(\xi; a, b) = \int_a^b \{p_1(\xi^{\Delta}(t))^2 - q_1(t)(\xi^{\sigma}(t))^2\}\Delta t$$

is positive definite for all $\xi \in U(a, b)$.

$$\begin{aligned} 0 < \mathcal{F}_1(\xi; a, b) &= \int_a^b \{p_1(t)(\xi^{\Delta}(t))^2 - q_1(t)(\xi^{\sigma}(t))^2\}\Delta t \\ &\leq \int_a^b \{p(\xi^{\Delta}(t))^2 - q(t)(\xi^{\sigma}(t))^2\}\Delta t \\ &= \mathcal{F}(\xi; a, b) \end{aligned}$$

Hence $\mathcal{F}(\xi; a, b)$ is positive definite. Then (5.1) is disconjugate on $[a, b]$. Then (5.1) is nonoscillatory $[T, \infty)$. □

REFERENCES

[1] P. Agarwal, M. Bohner, D. O'Regan, A. Peterson, "Dynamic Equations on Time Scales: A Survey", *J. Comp. Appl. Math., Special Issue on Dynamic Equations on Time Scales*, edited by R. P. Agarwal, M. Bohner, and D. ORegan, (Preprint in Ulmer Seminare 5) **141(1-2)**, (2002), 1-26.

[2] C.D. Ahlbrandt, C. Morian, "Partial Diffrential Equations on Time Scales", *Journal of Computational and Applied Mathemetics* **141**, (2002) 35-55

[3] P. Agarwal, M. Bohner,P.J.Y. Wong, "Sturm Liouville Eigenvalue Problems on Time Scales", *Applied Mathematics and Computation* **99**, (1999) 153-166.

[4] P. Agarwal, M. Bohner. "Basic Calculus on Time Scales and Some of its Applications", *Resultate der Mathematik*, **35 (1-2)**, (1999) 3-22.

[5] G. Birkhoff, G.C. Rota, Ordinary Differential Equations, Cambridge-Massachusets, (1962).

[6] M. Bohner, O. Došly, "The Discrete Prüfer Substitution", *Prooceedings of American Math Society* **129**, (2001) 2715-2726.

[7] M. Bohner, A. Peterson, Dynamic Equations on Time Scales An Introduction with Applications, Birkhäuser Boston, (2001).

[8] R. Bronson, Matrix Methods An Introduction, Academic Press New York, (1969).

[9] S. Chen, F. Mong, Z. Zheng, "Oscillation of Linear Hamiltonian Systems", *Dynamic Systems and Applications* **8**, (1999) 353-360.

[10] O. Došly, S. Hilger, "A Necessary and Sufficient Condition for The Oscillation of Sturm-Liouville Dynamic Equations on Time Scales", *Journal of Computational and Applied Mathematics* **141**, (2002) 147-158.

[11] L. Erbe, "Oscillation Theorems for Second Order Linear Differential Equations", *Pasific Journal of Mathematics* **35**, (1970) 337-343.

[12] L. Erbe, S. Hilger, "Sturmian Theory on Measure Chains", *Differential Equations and Dynamical Systems*, **1**, (1999) 121-127.

[13] L. Erbe and A. Peterson, "Riccati equations on a measure chain", *In G. S. Ladde, N. G. Medhin, and M. Sambandham, editors, Proceedings of Dynamic Systems and Applications* **3**, (2001)193-199, Dynamic Publishers, Atlanta, GA.

[14] L. Erbe, A. Peterson, R̃ehãk P., "Comparison Theorems For Linear Dynamic Equations on Time Scales", *Journal of Mathematical Analysis and Applications* **275**, (2002) 418-438.

[15] G. Sh. Guseinov, B. Kaymakcalan, "On Disconjugacy Criterion for Seceond Order Dynamic Equations", *Journal of Computational and Applied Mathematics* **141**, (2002) 187-196.

[16] G. Sh. Guseinov, "Integration on Time Scales", *Journal of Mathematical Analysis and Applications* **285**, (2003) 107-127.

[17] P. Hartman, Ordinary Differential Equations (second edition), Birkhäuser Boston, (1982).

[18] S. Hilger, "Analysis on Measure Chains-a Unified Approach to Continuous and Discrete Calculus", *Results in Mathematics*, **18**, (1990) 18-56.

[19] S. Hilger, "Differential and Difference Calculus-Unified!", *Nonlinear Analysis, Theory, Methods & Applications* **30**, (1997) 2683-2694

[20] V. Lakshmikantham, S. Sivasundaran, B. Kaymakcalan, Dynamic Systems on Measure Chains, Kluwe Academic Publishers (1996).

[21] W.T. Reid, Ordinary Differential Equations, John Wiley & Sons INc., New York (1971).

[22] W.T. Reid, Sturmian Theory for Ordinary Differential Equations, Springer-Verlag, New York (1980).

[23] G. Teschl, Lecture Notes: Ordinary Differential Equations and Dynamical Systems , Institut für Mathematik Universität Wien

[24] A. Yantir, "Integration on Time Scale with *Mathematica*", *Challanging the Boundaries of Symbolic Computation: The Proceeding of 5th Mathematica Symposium*, Imperial College Press, (2003) 325-331.

[25] A. Yantir, Ü. Ufuktepe, "Basic Calculus on Time Scale with *Mathematica*", *Lectute Notes in Computer Science*, **2567**, (2003), 821-827.

[26] A. Yantir, Ü. Ufuktepe, "Mathematica Applications on Time Scales for Calculus", *Future Generations Computer Science* (to appear)

[27] C. Young Eutiquio, Partial Differential Equations An introduction, Allyn and Bacon Inc. (1972).