# VERTEX COLORING OF A GRAPH

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in Mathematics

by
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# ABSTRACT

Vertex coloring is the following optimization problem; given a graph, how many colors are required to color its vertices in such a way that no two adjacent vertices receive the same color? The required number of colors is called the chromatic number of $G$ and is denoted by $\chi(G)$. In this thesis, we reviewed the vertex coloring concepts and theorems. The package ColorG which we have improved has many functions for dealing with graph coloring. This package uses a heuristic method due to Brelaz to color the graph so that adjacent vertices have distinct colors.

# ÖZET

Tepe boyama, verilen bir çizgenin komşu tepelerinin farklı renklerle boyanması koşuluyla gereken en az renk sayısının bulunmasını konu alan bir optimizasyon problemidir. Gereken en az renk sayısı çizgenin kromatik sayısıdır ve $\chi(G)$ ile gösterilir. Geliştirdiğimiz ColorG isimli *Mathematica* paketin çizgelerin boyanmasıyla ilgili birçok fonksiyonu vardır. Bu paket çizgelerin boyanması için Brelaz algoritmasını kullanmaktadır.

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# CHAPTER 1

# INTRODUCTION

Recent years have seen an increased demand for the application of mathematics. Graph theory has proven to be particularly useful to a large number of rather diverse fields. The exciting and rapidly growing area of graph theory is rich in theoretical results as well as applications to real-world problems. With the increasing importance of the computer, there has been a significant movement away from the traditional calculus courses and toward courses on discrete mathematics, including graph theory.

The origins of graph theory can be traced back to puzzles that were designed to amuse mathematicians and test their ingenuity. The classic puzzle concerned the bridges of Königsberg, a town in Prussia, which surrounded an island in the River Pregel (the town is now known as Kliningrad on the Pregolya river). Graph theory is considered to have begun in 1736 with the publication of Leonard Euler's solution to the Königsberg bridge problem.

Many real-life situation can be described by means of a diagram consisting of a set of points with lines joining certain pairs of points. Loosely speaking, such a diagram is what we mean by a graph. Graphs lend themselves naturally as models for a variety of situations. Instances of graphs abound: for example, the points might represent cities with lines representing direct flights between certain pairs of these cities in some airline system, or perhaps the lines represent pipelines between certain pairs of these cities in an oil network. On the other hand, the points might represent factories with lines representing communication links between them. Electrical networks, multiprocessor computers or switching circuits may clearly be represented by graphs.

Chromatic theory goes back to a problem, posed some 140 years ago, relating to the coloring of maps, either real or imaginary. The condition postulated was that countries with a common border line (and not just a border point) should receive different colors.

The question was, "How many colors are needed to cover all the different maps imaginable?" The practical answer turned out to be four at most, but this was only proved theoretically by K.Appel and W.Hakken some 28 years ago. The first proof was published in 1976 as a 140 page document with microfiches of some 1482 cases, after many hundreds of hours of computer work.

Apart from being an exercise in abstract thinking, what practical application does this have? The coloring theory brings one immediate application to mind. If you want to make a timetable for an exam, one common condition is that you cannot have two papers written by students at the same time if one or more of the students has to write both papers. If you rephrase the problem correctly it turns out to be a simple coloring matter. The idea of using the minimum number of colors then translates to, "What is the minimum number of sessions you need to set up the timetable?"

We begin our study in Chapter 2, by introducing many of the basic concepts that we shall encounter throughout this thesis. In Chapter 3, we introduce the vertex coloring problem. In Section 3.1, we define vertex coloring and chromatic number of a graph, then in Section 3.2 basic principles for calculating chromatic numbers. We present the proof of Brooks' Theorem in Section 3.3. Chromatic numbers for generated graphs and common graph families are given in Section 3.4 and Section 3.5. Then, in Section 3.6 we give definition of critical graphs and some important theorems about the critical graphs. Obstructions to $k$-chromaticity and chromatic polynomials are given in Section 3.7 and 3.8. Finally, in Chapter 4, we describe some web-based interactive examples on vertex coloring with web*Mathematica* by using our package ColorG.

# CHAPTER 2

# PRELIMINARIES

How can we lay cable at minimum cost to make every telephone reachable from every other? What is the fastest route from the national capital to each state capital? How can $n$ jobs be filled by $n$ people with maximum total utility? What is the maximum flow per unit time from source to sink in a network of pipes? How many layers does a computer chip need so that wires in the same layer don't cross? How can the season of a sports league be scheduled into the minimum number of weeks? In what order should a travelling salesman visit cities to minimize travel time? Can we color the regions of every map using four colors so that neighboring regions receive different colors? These and many other practical problems involve graph theory. In this chapter we will give some definitions and properties of graphs.

The problem that is often said to have been the birth of graph theory will suggest our basic definitions of a graph.

**The Königsberg Bridge Problem**

The city of Königsberg was located on the Pregel river in Prussia. The city occupied two islands plus areas on both blanks. These regions were linked by seven bridges as shown in the Figure 2.1. The citizens wondered whether they could leave home, cross every bridge exactly once, and return home. The problem reduces to traversing the figure on the right, with heavy dots representing land masses and curves representing bridges.

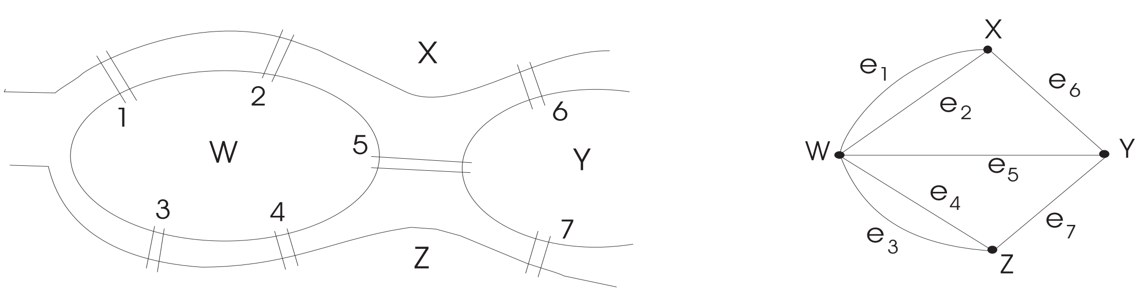


Figure 2.1: Königsberg Bridge Problem

The model on the right makes it easy to argue that the desired traversal does not exist. Each time we enter and leave a land mass, we use two bridges ending at it. We can also pair the first bridge with the last bridge on the land mass where we begin and end. Thus existence of the desired traversal requires that each land mass be involved in an even number of bridges. This necessary condition did not hold in Königsberg.

## 2.1 Graphs

A **graph** is a mathematical structure consisting of two sets $V$ and $E$. The elements of $V$ are called **vertices** and the elements of $E$ are called **edges**. Each edge is identified with a pair of vertices. If the edges of a graph $G$ are identified with ordered pairs of vertices, then $G$ is called a **directed** graph. Otherwise $G$ is called an **undirected** graph. Our discussions in this thesis are concerned with undirected graphs.

We use the symbols $v_1, v_2, v_3, ...$ to represent the vertices and the symbols $e_1, e_2, e_3, ...$ to represent the edges of a graph. The vertices $v_i$ and $v_j$ associated with an edge $e_l$ are called the **end vertices** of $e_l$. The edge $e_l$ is then denoted as $e_l = v_iv_j$. Note that while the elements of $E$ are distinct, more than one edge in $E$ may have the same pair of end vertices. All edges having the same pair of end vertices are called **parallel** or **multiple edges**. Further, the end vertices of an edge need not be distinct. If $e_l = v_iv_i$, then the edge $e_l$ is called a **self-loop** at vertex $v_i$. A graph is called a **simple graph** if it has no parallel edges or self-loops. In this thesis we will work with the simple graphs. A graph $G$ is **planar** if there exists a drawing of $G$ in the plane in which no two edges intersect in a point other than a vertex of $G$.

The cardinality of the vertex set of a graph $G$ is called the **order** of $G$ and is commonly denoted by $n(G)$, or more simply by $n$ when the graph under consideration is clear; while the cardinality of its edge set is the **size** of $G$ and is often denoted by $m(G)$ or $m$. An $(n, m)$ graph has order $n$ and size $m$. A graph with no edges is called an empty graph. A graph with no vertices (and hence no edges) is called a null graph.

An edge is said to be **incident** on its end vertices. Two vertices are **adjacent** if they are the end vertices of an edge. The neighborhood of $v$, $N(v)$, is the set of vertices adjacent to $v$. If two edges have a common end vertex, then these edges are said to be **adjacent**.

For example, in the Figure 2.2 edge $e_1$ is incident on vertices $v_1$ and $v_2$; $v_1$ and $v_3$ are two adjacent vertices, while $e_1$ and $e_3$ are two adjacent edges.

An **independent set** of vertices in a graph is a set of mutually non-adjacent vertices. The **independence number** of a graph $G$ is the maximum cardinality of an independent set of vertices. It is denoted by $\alpha(G)$.

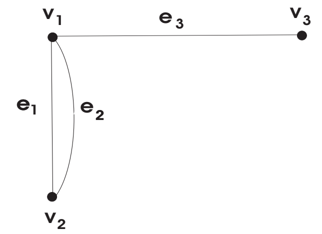


Figure 2.2: Graph $G = (V, E)$. $V = \{v_1, v_2, v_3\}$; $E = \{e_1, e_2, e_3\}$.

The number of edges incident on a vertex $v_i$ is called the **degree** of the vertex, and it is denoted by $deg(v_i)$. Sometimes the degree of a vertex is also referred to as its valency. By definition, a self-loop at a vertex $v_i$ contributes 2 to the degree of $v_i$. A vertex is called **even or odd** according to whether its degree is even or odd. A vertex of degree 0 in $G$ is called **isolated** vertex and a vertex of degree 1 is an end-vertex of $G$. The **minimum degree** of $G$ is the minimum degree among the vertices of $G$ and is denoted by $\delta(G)$. The **maximum degree** is defined similarly and is denoted by $\Delta(G)$. The **degree sequence** of a graph is the sequence formed by arranging the vertex degrees in non-decreasing order.

**Proposition 1.** *A non-trivial simple graph $G$ must have at least one pair of vertices whose degrees are equal.*

**Theorem 2.** *(Euler) The sum of the degrees of a graph is twice the number of edges.*

**Corollary 3.** *In a graph, there is an even number of vertices having odd degree.*

*Proof.* Consider separately, the sum of the degrees that are odd and the sum of those that are even. The combined sum is even by the previous theorem, and since the sum of the even degrees is even, the sum of the odd degrees must also be even. Hence, there must be even number of vertices of odd degree. □

**Corollary 4.** *The degree sequence of a graph is a finite, non-decreasing sequence of non-negative integers whose sum is even.*

Conversely, any non-decreasing, nonnegative sequence of integers whose sum is even is the degree sequence of some graph. For example to construct a graph whose degree sequence is $< 0, 1, 2, 3, 4, 5 >$, start with seven isolated vertices $v_1, v_2, ..., v_7$. For the even-valued terms of the sequence, draw the appropriate number of self-loops on the corresponding vertices. Thus, $v_1$ remains isolated, $v_3$ gets one self-loop, and $v_6$ gets two self-loops. For the four remaining odd-valued terms, group the corresponding vertices into any two pairs, for instance, $v_2, v_4$ and $v_5, v_7$. Then join each pair by a single edge and add to each vertex the appropriate number of self-loops. The resulting graph is shown in Figure 2.3 but a degree sequence doesn't represent a unique graph.

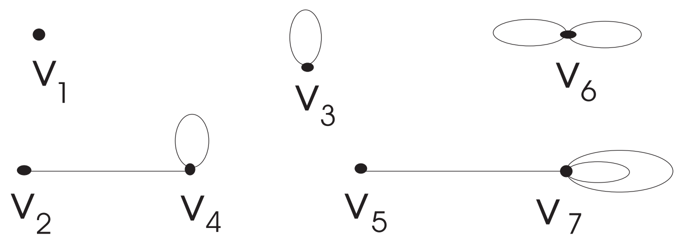


Figure 2.3: Constructing a graph with degree sequence $< 0, 1, 2, 3, 4, 5 >$ .

## 2.2 Subgraphs

A graph $H$ is called a **subgraph** of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph $H$ of a graph $G$ is a **proper subgraph** of $G$ if either $V(H) \neq V(G)$ or $E(H) \neq E(G)$. A subgraph $H$ of $G$ is said to be an **induced subgraph** of $G$ if each edge of $G$ having its ends in $V(H)$ is also an edge of $H$. A subgraph $H$ of $G$ is a **spanning subgraph** of $G$, if $V(H) = V(G)$. The induced subgraph of $G$ with vertex set $S \subseteq V(G)$ is called the subgraph of $G$ induced by $S$ and is denoted by $G[S]$.

**Definition 5.** *A maximal subset of $V(G)$ of mutually adjacent vertices is called a clique in a graph $G$. The clique number $\omega(G)$ of a graph $G$ is the number of the vertices in a largest clique in $G$.*

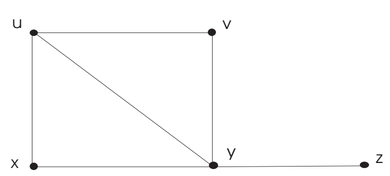


Figure 2.4: A graph with 3-cliques.

## 2.3 Graph Operations

The **union** of two graphs is formed by taking the union of the vertices and edges of the graphs. Thus the union of two graphs is always disconnected.

The **join** $G \vee H$ of the graph $G$ and $H$ is obtained from the graph union $G \cup H$ by adding an edge between each vertex of $G$ and each vertex of $H$.

The **cartesian product** $G = G_1 \times G_2$ has $V(G_1) \times V(G_2)$, and two vertices $(u_1, u_2)$ and $(v_1, v_2)$ of $G$ are adjacent if and only if either $u_1 = u_2$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. A convenient way of drawing $G_1 \times G_2$ is first to place a copy of $G_2$ at each vertex of $G_1$ and then to join corresponding vertices of $G_2$ in those copies of $G_2$ placed at adjacent vertices of $G_1$.

The **complement** $\overline{G}$ of a simple graph $G$ is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(\overline{G})$.

**Vertex Removal:** If $v_i$ is a vertex of a graph $G = (V, E)$, then $G - v_i$ is the induced subgraph of $G$ on the vertex set $V - v_i$; that is, $G - v_i$ is the graph obtained after removing from $G$ the vertex $v_i$ and all the edges incident on $v_i$.

**Edge Removal:** If $e_i$ is an edge of a graph $G = (V, E)$, then $G - e_i$ is the subgraph of $G$ that results after removing from $G$ the edge $e_i$. Note that the end vertices of $e_i$ are not removed from $G$.

**Definition 6.** *Let $G$ be a graph, $e = uv \in E_G$, and let $x = x(uv)$ be a new contracted vertex. The graph $G * e$ on $V_{G*e} = (V_G - \{u, v\}) \cup \{x\}$ is obtained from $G$ by contracting the edge $e$, when*
$E_{G*e} = \{f \mid f \in E_G, f \text{ has no end } u \text{ or } v\} \cup \{wx \mid wu \in E_G \text{ or } wv \in E_G\}$.
*Hence $G * e$ is obtained by introducing a new vertex $x$, and by replacing all edges $wu$ and $wv$ by $wx$, and the vertices $u$ and $v$ are deleted. (See Figure 2.5)*

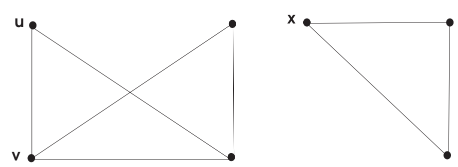


Figure 2.5: Contraction of edge $uv$.

## 2.4 Common Families of Graphs

### Complete Graphs

A **complete graph** is a simple graph such that every pair of vertices is joined by an edge. Any complete graph on $n$ vertices is denoted by $K_n$. It may be seen that $K_n$ has $n(n-1)/2$ edges.

**Bipartite Graphs**

A **bipartite graph** $G$ is a graph whose vertex-set $V$ can be partitioned into two subsets $U$ and $W$, such that each edge of $G$ has one endpoint in $U$ and one endpoint in $W$. The pair $U$, $W$ is called a vertex bipartition of $G$, and $U$ and $W$ are called the bipartition subsets.

A **complete bipartite** graph is a simple bipartite graph such that every vertex in one of the bipartition subsets is joined to every vertex in the other bipartition subset. Any complete bipartite graph that has $m$ vertices in one of its bipartition subsets and $n$ vertices in the other is denoted by $K_{m,n}$.

**Regular Graphs**

A **regular graph** is a graph whose vertices all have equal degree. A $k$-regular graph is a regular graph whose common degree is $k$.

The **Petersen graph** is the 3-regular graph represented by the line drawing in the Figure 2.6. Because it possesses a number of interesting graph-theoretic properties, the Petersen graph is frequently used both to illustrate established theorems and to test conjectures.

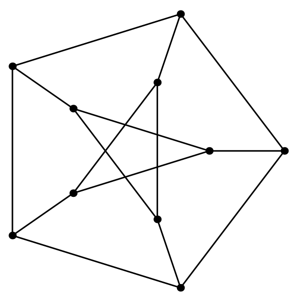

Figure 2.6: Petersen Graph

**Path Graphs and Cycle Graphs**

A **path** graph $P$ is a simple connected graph with $|V_P| = |E_P| + 1$ that can be drawn so that all of its vertices and edges lie on a single straight line. An $n$-vertex path graph is denoted $P_n$.

A **cycle** graph is a simple vertex with a self-loop or a simple connected graph $C$ with $|V(C)| = |E(C)|$ that can be drawn so that all of its vertices and edges lie on a circle. An $n$-vertex cycle graph is denoted $C_n$.

A graph with no cycle is **acyclic**. A **forest** is an acyclic graph. A **tree** is a connected acyclic graph.

## 2.5 Walk and Distance

In a graph, a **walk** from vertex $v_0$ to vertex $v_n$ is an alternating sequence

$$W =< v_0, e_1, v_1, e_2, ..., v_{n-1}, e_n, v_n >$$

of vertices and edges, such that $endpts(e_i) = \{v_{i-1}, v_i\}$, for $i = 1, ..., n$.
Thus, in a physical representation of a graph, a walk models a continuous traversal along some edges and vertices. The **length** of a walk is the number of edge-steps in the walk sequence. An $x - y$ walk is said to be closed if $x$ and $y$ are the same vertex and open if not.

The **concatenation** of two walks, denoted by $W_1 o W_2$,
$W_1 =< v_0, e_1, v_1, e_2, ..., v_{k-1}, e_k, v_k >$ and $W_2 =< v_k, e_{k+1}, v_{k+1}, e_{k+2}, ..., v_{n-1}, e_n, v_n >$ such that walk $W_2$ begins where walk $W_1$ ends, is the walk

$$W_1 o W_2 =< v_0, e_1, ..., v_{k-1}, e_k, v_k, e_{k+1}, ..., v_{n-1}, e_n, v_n >.$$

A **subwalk** of the walk $W =< v_0, e_1, v_1, e_2, ..., v_{n-1}, e_n, v_n >$ is a subsequence of consecutive entries $S =< v_j, e_{j+1}, v_{j+1}, ..., e_k, v_k >$ such that
$0 \leq j \leq k \leq n$, that begins and ends at a vertex of $W$. Thus, the subwalk is itself a walk.

In a graph, the **distance** from vertex $s$ to vertex $t$ is the length of a shortest walk from $s$ to $t$, or $\infty$ if there is no walk from $s$ to $t$.

## 2.6 Connectedness

A graph is **connected** if for every pair of vertices $u$ and $v$, there is a walk from $u$ to $v$. The disconnected graph is made up of connected pieces called **components**.

A **cut-edge** or **cut-vertex** of a graph is an edge or a vertex whose deletion increases the number of components.

A **separating set** or **vertex cut** of a graph $G$ is a set $S \subseteq V(G)$ such that $G - S$ has more than one component.

A **disconnecting set** of edges is a set $F \subseteq E(G)$ such that $G - F$ has more than one component. A graph is **$k$-edge connected** if every disconnecting set has at least $k$ edges. An **edge cut** is an edge set of the form $[S, \overline{S}]$, where $S$ is a nonempty proper subset of $V(G)$ and $\overline{S}$ denotes $V(G) - S$.

A **nonseparable** graph is a connected graph with no cut-vertices. All other graphs are separable. A block of a separable graph $G$ is a maximal nonseparable subgraph of $G$.

# CHAPTER 3

# VERTEX COLORINGS

The first known mention of coloring problems was in 1852, when August De Morgan, Professor of Mathematics at University College, London, wrote Sir William Rowan Hamilton in Dublin about a problem posed to him by a former student, named Francis Guthrie. Guthrie noticed that it was possible to color the countries of England using four colors so that no two adjacent countries were assigned the same color. The question raised thereby was whether four colors would be sufficient for all possible decompositions of the plane into regions.

The Poincaré duality construction transforms this question into the problem of deciding whether it is possible to color the vertices of every planar graph with four colors so that no two vertices assigned the same color. Wolfgang Haken and Kenneth Appel provided an affirmative solution in 1976 (Appel and Haken 1976).

## 3.1 The Minimization Problem for Vertex Coloring

In the most common kind of graph coloring, colors are assigned to the vertices. From a standard mathematical prospective, the subset comprising all the vertices of a given color would be regarded a cell of a partition of the vertex-set. Drawing the graph with colors on the vertices is simply an intuitive way to represent such a partition.

We begin with a practical application of graph coloring known as the storage problem. Suppose the Department of Chemistry of a college wants to store its chemicals. It is a quite probable that some chemicals cause violent reactions when brought together. Such chemicals are incompatible chemicals. For safe storage, incompatible chemicals should be kept in distinct rooms. The easiest way to accomplish this is, of course, to store one chemical in each room. But this is certainly not the best way of doing it since we will be using more rooms than are really needed (unless, of course, all the chemicals are mutually incompatible!). So we ask: What is the minimum number of rooms required to store all the chemicals so that in each room only compatible chemicals are stored?

We convert the above storage problem into a problem in graphs. Form a graph $G = G(V, E)$ by making $V$ correspond bijectively to the set of available chemicals and making u adjacent to v if and only if the chemicals corresponding to u and v are incompatible. Then, any set of compatible chemical corresponds to a set of independent vertices of $G$. Thus a safe storing of chemicals corresponds to a partition of V into independent subsets of $G$. The cardinality of such a minimum partition of $V$ is then the required number of rooms. This minimum cardinality is called the chromatic number of the graph $G$.

The chromatic number, $\chi(G)$, of a graph $G$ is the minimum number of independent subsets that partition the vertex set of $G$. Any such minimum partition is called a chromatic partition of $V(G)$ (Balakrishnan and Ranganathan 2000).

The storage problem just described is actually a vertex coloring problem of $G$. A vertex coloring of $G$ is a map $f : V(G) \rightarrow C$, where $C$ is a set of distinct colors; it is proper if adjacent vertices of $G$ receive distinct colors of $C$; that is, if $uv \in E(G)$ then $f(u) \neq f(v)$. Thus $\chi(G)$ is the minimum cardinality of $C$ for which there exists a proper vertex coloring of $G$ by colors of $C$. Clearly, in any proper vertex coloring of $G$, the vertices that receive the same color are independent. The vertices that receive a particular color make up a color class. Thus, in any chromatic partition of $V(G)$, the parts of the partition constitute the color classes. This allows an equivalent way of defining the chromatic number.

**Definition 7.** *The chromatic number of a graph $G$ is the minimum number of colors needed for a proper vertex coloring of $G$. If $\chi(G) = k$, $G$ is said to be $k$-chromatic.*

For example, the chromatic number of the graph in Figure 3.1 is 3.

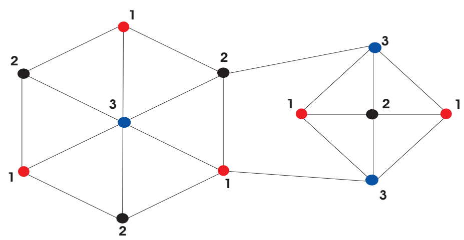


Figure 3.1: A 3-chromatic graph

**Definition 8.** *A $k$-coloring of a graph $G$ is a vertex coloring of $G$ that uses $k$ colors.*

**Definition 9.** *A graph $G$ is said to be $k$-colorable, if $G$ admits a proper vertex coloring using $k$ colors.*

*Thus, $\chi(G) = k$ if graph $G$ is $k$-colorable but not $(k-1)$-colorable.*

In considering the chromatic number of a graph only the adjacency of vertices is taken into account. A graph with a self-loop is regarded as uncolorable, since the endpoint of the self-loop is adjacent to itself. Moreover, a multiple adjacency has no more effect on the colors of its endpoints than a single adjacency. As a consequence, we may restrict ourselves to simple graphs when dealing with chromatic numbers.

It is clear that $\chi(G) = 1$ if and only if $G$ has no edges and $\chi(G) = 2$ if and only if $G$ is bipartite.

## 3.2 Basic Principles for Calculating Chromatic Numbers

Although the chromatic number is one of the most studied parameters in graph theory, no formula exists for the chromatic number of an arbitrary graph. Thus, for the most part, one must be content with supplying bounds for the chromatic number of graphs.

A few basic principles recur in many chromatic-number calculations. Now, we will try to find upper and lower bound to provide a direct approach to the chromatic number of a given graph.

**Upper bound:** Show $\chi(G) \leq k$ by exhibiting a proper $k$-coloring of $G$.

**Lower bound:** Show $\chi(G) \geq k$ by using properties of graph $G$, most especially, by finding a subgraph that requires $k$-colors.

**Proposition 10.** *Let $G$ be a graph with $k$-mutually adjacent vertices. Then $\chi(G) \geq k$.*

*Proof.* Using fewer than $k$ colors on graph $G$ would result in a pair from the mutually adjacent set of $k$ vertices being assigned the same color. □

**Proposition 11.** *Let $H$ be a subgraph of $G$. Then $\chi(G) \geq \chi(H)$.*

*Proof.* Whatever colors are used on the vertices of subgraph $H$ in a minimum coloring of $G$ can also be used in coloring of $H$ by itself. □

**Corollary 12.** *Let $G$ be a graph. Then $\chi(G) \geq \omega(G)$.*

*Proof.* Since clique is a subgraph of $G$, we get this inequality. □

The 4-coloring of the graph $G$ shown in Figure 3.2 establishes that $\chi(G) \leq 4$, and the $K_4$-subgraph (drawn in bold) shows that $\chi(G) \geq 4$. Hence, $\chi(G) = 4$.

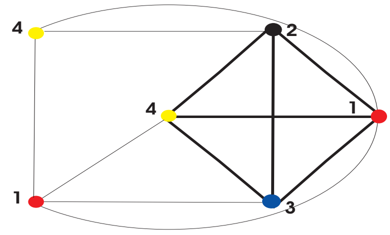


Figure 3.2: A 4-colorable graph

**Proposition 13.** *Let $G$ be any graph. Then*

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}.$$

*Proof.* Given a $k$-coloring of $G$, the vertices being colored with the same color form an independent set. Let $G$ be a graph with $n$ vertices and $c$ a $k$-coloring of $G$. We define

$$V_i = \{v \mid c(v) = i\} \ for \ i = 0, 1, ..., k.$$

Each $V_i$ is an independent set. Let $\alpha(G)$ be the independence number of $G$, we have $V_i \leq \alpha(G)$. Since

$$n = |V(G)| = |V_1| + |V_2| + ... + |V_k| \leq k\ \alpha(G) = \chi(G)\ \alpha(G)$$

we have

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}.$$

□

Most upper bounds on the chromatic number come from algorithms that produce colorings. For example, assigning distinct colors to the vertices yields $\chi(G) \leq n(G)$. This bound is best possible, since $\chi(K_n) = n$, but it holds with equality only for complete graphs. We can improve a "best possible" bound by obtaining another bound that is always at least as good. For example $\chi(G) \leq n(G)$ uses nothing about the structure of $G$; we can do better by coloring the vertices in some order and always using the "least available" color.

**Definition 14.** *The greedy coloring relative to a vertex ordering $v_1, v_2, ..., v_n$ of $V(G)$ is obtained by coloring vertices in order $v_1, v_2, ..., v_n$, assigning to $v_i$ the smallest-indexed color not already used on its lower-indexed neighbors.*

**Theorem 15.** *For any graph $G$,*

$$\chi(G) \leq \Delta(G) + 1.$$

*Proof.* In a vertex ordering, each vertex has at most $\Delta(G)$ earlier neighbors, so the greedy coloring cannot be forced to use more than $\Delta(G) + 1$ colors. This proves constructively that $\chi(G) \leq \Delta(G) + 1$. □

The bound $\Delta(G)+1$ is the worst upper bound that greedy coloring could produce. Choosing the vertex ordering carefully yields improvements. We can avoid the trouble caused by vertices of high degree by putting them at the beginning, where they won't have many earlier neighbors.

**Proposition 16.** *If a graph $G$ has the nonincreasing degree sequence $d_1 \geq d_2 \geq ... \geq d_n$, then $\chi(G) \leq 1 + \max_i\{\min\{d_i, i-1\}\}$* (Welsh and Powel 1967)*.*

*Proof.* We apply the greedy coloring to the vertices in nonincresing order of degree. When we color the $i^{th}$ vertex $v_i$, it has at most $\min\{d_i, i-1\}$ earlier neighbors, so at most this many colors appear on its earlier neighbors. Hence the color we assign to $v_i$ is at most $1 + \min\{d_i, i-1\}$. This holds for each vertex, so we maximize over $i$ to obtain the upper bound on the maximum color used. □

**Theorem 17.** *For any graph $G$, $\chi(G) \leq 1 + \max(\delta(G'))$, where the maximum is taken over all induced subgraphs $G'$ of $G$* (Szekeres and Wilf 1968)*.*

*Proof.* Let $\chi(G) = k$, and let $H$ be the minimal induced subgraph such that $\chi(H) = k$. So for any vertex $v$ in $H$, the graph $(H - v)$ is $(k-1)$-colorable. Fix vertex $v$ in $H$, and consider any $(k-1)$ coloring of $(H - v)$. In that case, if the degree of $v$ in $H$ is less than $(k-1)$, it is possible to color the vertices of $H$ using at most $(k-1)$ colors. Hence, the degree of any vertex $v$ in $H$ is at least $(k-1)$. Thus, $(k-1) \leq \delta(H) \leq \max(\delta(G'))$. □

## 3.3 Brooks' Theorem

We are now prepared to present bounds for the chromatic number of a graph. We give here several upper bounds, beginning with the best known and most applicable.

The bound $\chi(G) \leq 1 + \Delta(G)$ holds with equality for complete graphs and odd cycles. By choosing the vertex ordering more carefully, we can show that these are essentially the only such graphs. This implies, for example, that the Petersen graph is 3-colorable, without finding an explicit coloring. To avoid unimportant complications, we phrase the statement only for connected graphs. It extends to all graphs because the chromatic number of a graph is maximum chromatic number of its components. Many proofs are known; we present a modification of the proof by (Melnikov and Vizing 1969). For an alternative proof see (Lovász 1975).

**Theorem 18.** *(Brooks' Theorem) If a connected graph $G$ is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta(G)$.*

*Proof.* If $\Delta(G) \leq 2$, then $G$ is either a path or a cycle. For a path $G$ (other than $K_1$ and $K_2$), and for an even cycle $G$, $\chi(G) = 2 = \Delta(G)$. According to our assumption $G$ is not an odd cycle. So let $\Delta(G) \geq 3$.

The proof is by contradiction. Suppose the result is not true. Then there exists a minimal graph $G$ of maximum degree $\Delta(G) \geq 3$ such that $G$ is not $\Delta$-colorable, but for any vertex $v$ of $G$, $G - v$ is $\Delta(G - v)$-colorable and therefore $\Delta$-colorable.

**Claim 1.** Let $v$ be any vertex of $G$. Then in any proper $\Delta$-coloring of $G - v$, all the $\Delta$-colors must be used for coloring the neighbors of $v$ in $G$. Otherwise, if some color $i$ is not represented in $N_G(v)$, then $v$ could be colored using $i$, and this would give a $\Delta$-coloring of $G$, a contradiction to the choice of $G$. Thus, $G$ is a regular graph satisfying claim 1.

For $v \in V(G)$, let $N(v) = v_1, v_2, ..., v_n$. In a proper $\Delta$-coloring of $G - v = H$, let $v_i$ receive color $i$, $1 \leq i \leq \Delta$. For $i \neq j$, let $H_{ij}$ be the subgraph of $H$ induced by the vertices receiving the $i$th and $j$th colors.

**Claim 2.** $v_i$ and $v_j$ belong to the same component of $H_{ij}$. Otherwise, the colors $i$ and $j$ can be interchanged in the component of $H_{ij}$ that contains the vertex $v_j$. Such an interchange of colors once again yields a proper $\Delta$-coloring of $H$. In this new coloring, both $v_i$ and $v_j$ receive the same color, namely, $i$, a contradiction to claim 1. This proves claim 2.

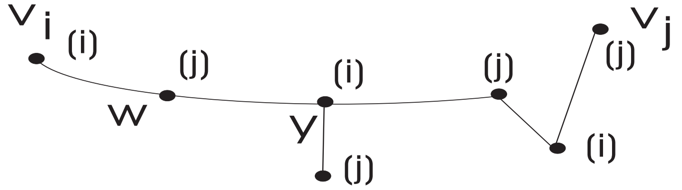


Figure 3.3: Brooks' Theorem (a)

**Claim 3.** If $C_{ij}$ is the component of $H_{ij}$ containing $v_i$ and $v_j$, then $C_{ij}$ is a path in $H_{ij}$. As before, $N_H(v_i)$ contains exactly one vertex of color $j$. Further, $C_{ij}$ cannot contain a vertex, say $y$, of degree at least 3; for, if $y$ is the first such vertex on a $v_i - v_j$ path in $C_{ij}$

that has been colored, say, with $i$, then at least three neighbors of $y$ in $C_{ij}$ have the color $j$. Hence, we can recolor $y$ in $H$ with a color different from both $i$ and $j$, and in this new coloring of $H$, $v_i$ and $v_j$ would belong to distinct components of $H_{ij}$. (See Figure 3.3.) Note that by our choice of $y$, any $v_i - v_j$ path in $H_{ij}$ must contain $y$. But this contradicts claim 2.

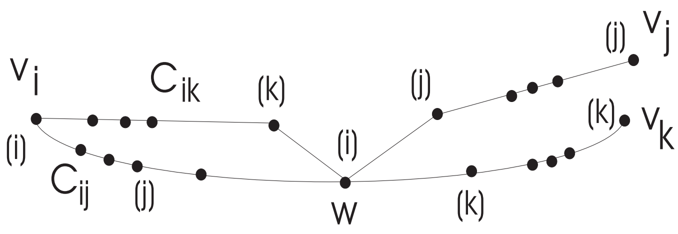


Figure 3.4: Brooks' Theorem (b)

**Claim 4.** $C_{ij} \cap C_{ik} = \{v_i\}$ for $j \neq k$. Indeed, if $w \in C_{ij} \cap C_{ik}, w \neq v_i$, then $w$ is adjacent to two vertices of color $j$ on $C_{ij}$ and two vertices of color $k$ on $C_{ik}$. (See Figure 3.4.) Again, we can recolor $w$ in $H$ by giving a color different from the colors of neighbors of $w$ in $H$. In this new coloring of $H$, $v_i$ and $v_j$ belong to distinct components of $H_{ij}$, a contradiction to claim 1. This completes the proof of claim 4.

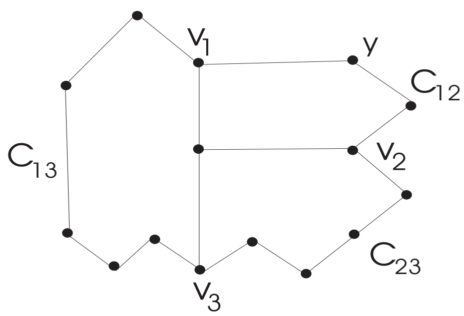


Figure 3.5: Brooks' Theorem (c)

We are now in a position to complete the proof of the theorem. By hypothesis, $G$ is not complete. Hence, $G$ has a vertex $v$, and a pair of nonadjacent vertices $v_1$ and $v_2$ in $N_G(v)$. Then the $v_1 - v_2$ path $C_{12}$ in $H_{12}$ of $H = G - v$ contains a vertex $y(\neq v_2)$ adjacent to $v_1$. Naturally, $y$ would receive color 2. Since $\Delta \geq 3$, there exists a vertex $v_3 \in N_G(v)$. Now interchange colors 1 and 3 in the path $C_{13}$ of $H_{13}$. This would result in a new coloring of $H = G - v$. Denote the $v_i - v_j$ path in $H$ under this new coloring by $C'_{ij}$. (See Figure 3.5). Then $y \in C'_{23}$, since $v_1$ receives color 3 in the new coloring (whereas $y$ retains color 2). Also, $y \in C_{12} - v_1 \subset C'_{12}$. Thus, $y \in C'_{23} \cap C'_{12}$. This contradicts claim 4 (since $y \neq v_2$) and the proof is complete. $\square$

## 3.4 Chromatic Numbers of Generated Graphs

**Proposition 19.** *For the disjoint union,*

$$\chi(G \cup H) = \max\{\chi(G), \chi(H)\}.$$

**Proposition 20.** *The join of the graphs $G$ and $H$ has chromatic number*

$$\chi(G \vee H) = \chi(G) + \chi(H).$$

*Proof.* Lower Bound: In the join $G \vee H$, no color used on the subgraph $G$ can be the same as a color used on the subgraph $H$, since every vertices of $G$ is adjacent to every vertices of $H$. Since $\chi(G)$ colors are required for the subgraph $G$ and $\chi(H)$ colors are required for the subgraph $H$, it follows that $\chi(G \vee H) \geq \chi(G) + \chi(H)$.
Upper Bound: Just use any $\chi(G)$ colors to properly color the subgraph $G$ of $G \vee H$ and use $\chi(H)$ different colors to color the subgraph $H$. □

**Proposition 21.** (Vizing 1963, Albert 1964)

$$\chi(G \times H) = \max\{\chi(G), \chi(H)\}.$$

*Proof.* The cartesian product $\chi(G \times H)$ contains copies of $G$ and $H$ as subgraphs so $\chi(G \times H) \geq \max\{\chi(G), \chi(H)\}$.
Let $k = \max\{\chi(G), \chi(H)\}$. To prove the upper bound, we produce a proper $k$-coloring of $G \times H$ using optimal colorings of $G$ and $H$. Let $g$ be a proper $\chi(G)$- coloring of $G$, and let $h$ be a proper $\chi(H)$- coloring of $H$. Define a coloring $f$ of $G \times H$ by letting $f(u, v)$ be the congruence class of $g(u) + h(v)$ modulo $k$. Thus $f$ assigns color to $V(G \times H)$ from a set of size $k$.

We claim that $f$ properly colors $G \times H$. If $(u, v)$ and $(u', v')$ are adjacent in $G \times H$, then $g(u) + h(v)$ and $g(u') + h(v')$ agree in one summand and differ by between 1 and $k$ in the other. Since the difference of the two sums is between 1 and $k$, they lie in different congruence classes modulo $k$. □

## 3.5 Chromatic Numbers for Common Graph Families

It is straightforward to establish the chromatic number of graphs in some of the most common graph families, by using the basic principles given above.

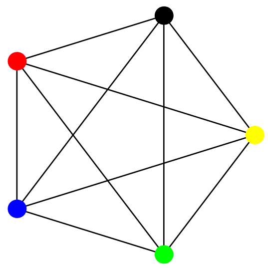

Figure 3.6: A Complete Graph

**Proposition 22.** *For complete graphs,* $\chi(K_n) = n$.

*Proof.* Since $n$ complete graphs have $n$ mutually adjacent vertices using fewer than $n$ colors result in a pair of mutually adjacent vertices being assigned the same color. □

**Proposition 23.** *For bipartite graphs,* $\chi(G) = 2$.

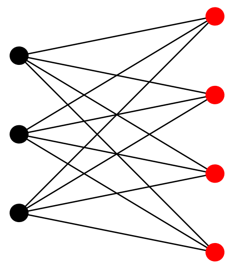

Figure 3.7: A Complete Bipartite Graph

*Proof.* A 2-coloring is obtained by assigning one color to every vertex in one of the bipartition parts and another color to every vertex in the other part. □

**Corollary 24.** *For path graphs, we have* $\chi(P_n) = 2$ *since they are bipartite.*

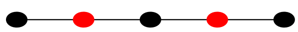

Figure 3.8: A Path Graph

**Corollary 25.** *Since trees are bipartite,* $\chi(T) = 2$*.*

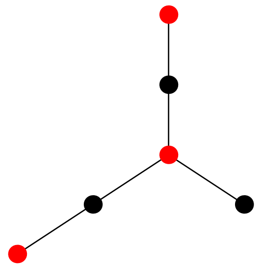

Figure 3.9: A Tree

**Corollary 26.** *Cube graphs are bipartite graphs so we have* $\chi(Q_n) = 2$*.*

**Corollary 27.** *Even cycles have the chromatic number* $\chi(C_{2n}) = 2$ *since they are bipartite.*

**Proposition 28.** *For odd-cycle graphs,* $\chi(C_{2n+1}) = 3$*.*

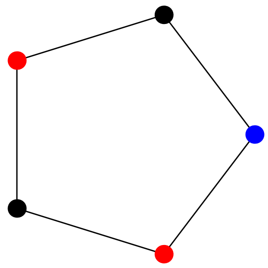

Figure 3.10: An Odd-Cycle

*Proof.* Let $v_1, v_2, ..., v_{2n+1}$ be the vertices of cycle graph $C_{2n+1}$. If two colors were to suffice, then they would have to alternate around the cycle. Thus, the odd-subscripted vertices would have to be one color and the even-subscripted ones the other. But vertex $v_{2n+1}$ is adjacent to $v_1$, which means that the odd cycle graph $C_{2n+1}$ is not 2-colorable. □

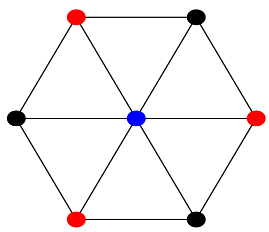

Figure 3.11: An Odd-Order Wheel

**Proposition 29.** *For odd-order wheel graphs, $\chi(W_{2m+1}) = 3$.*

*Proof.* Using the fact that the wheel graph $\chi(W_{2m+1})$ is the join of an even cycle $C_{2m}$ and complete graph $K_1$, by the proposition we have
$\chi(W_{2m+1}) = \chi(C_{2m} + K_1) = \chi(C_{2m}) + \chi(K_1) = 2 + 1 = 3.$ □

**Proposition 30.** *For even-order wheel graphs, $\chi(W_{2m}) = 4$.*

*Proof.* The wheel graph $\chi(W_{2m})$ is the join of an odd-cycle $C_{2m+1}$ and the complete graph $K_1$. Then we have
$\chi(W_{2m}) = \chi(C_{2m+1} + K_1) = \chi(C_{2m+1}) + \chi(K_1) = 3 + 1 = 4.$ □

## 3.6 Critical Graphs

**Definition 31.** *A $k$-chromatic graph $G$ is said to be a critically $k$-chromatic (or a $k$-critical) graph if the vertex-deletion subgraph $G-v$ is (k-1)-colorable and $\chi(G-v) = k-1$ for every vertex of $G$.*

**Definition 32.** *A $k$-chromatic graph $G$ is said to be a minimally $k$-chromatic graph if $\chi(G-e) = k-1$ for every edge of $G$.*

**Proposition 33.** *Let $G$ be a $k$-critical graph.*

**(a)** *For $v \in V(G)$, there is a proper $k$-coloring of $G$ in which the color on $v$ appears nowhere else, and the other $(k-1)$ colors appear on $N(v)$.*

**(b)** *For $e \in E(G)$, every proper $(k-1)$ coloring of $G-e$ gives the same color to the two endpoints of $e$.*

*Proof.*

**(a)** Given a proper $(k-1)$-coloring $f$ of $G-v$, adding color $k$ on $v$ alone completes a proper $k$-coloring of $G$. The other colors must all appear on $N(V)$, since otherwise assigning a missing color to $v$ would complete a proper $(k-1)$-coloring of $G$.

**(b)** If some proper $(k-1)$-coloring of $G-e$ gave distinct colors to the endpoints of $e$, then adding $e$ would yield a proper $(k-1)$-coloring of $G$.

□

**Theorem 34.** *Every $k$-chromatic graph contains a critically (minimally) $k$-chromatic graph.*

*Proof.* Suppose the $k$-chromatic graph $G$ is not critically $k$-chromatic. Then there exists a vertex $v$ such that $G - v$ is $k$-chromatic. If $G - v$ is critically $k$-chromatic, we are done. Otherwise, there is a vertex $w$ in $G - v$ such that $(G - v) - w$ is $k$-chromatic. Continue this process. We eventually arrive at a critically $k$-chromatic graph. The proof is similar for the case of minimally $k$-chromatic graphs. □

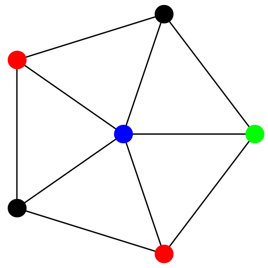

Figure 3.12: $W_6$ is both critically and minimally $k$-chromatic.

**Proposition 35.** *Every minimally $k$-chromatic graph (without isolated vertices) is critically $k$-chromatic.*

The converse is not true in general. For $k = 2$ and $k = 3$ it is true. In fact, $K_2$ is the only critically 2-chromatic graph as well as the only minimally 2-chromatic graph without isolated vertices; while the odd cycles are the only critically 3-chromatic graphs and the only minimally 3-chromatic graphs having no isolated vertices. For $k \geq 4$, neither the critically $k$-chromatic graphs nor the minimally $k$-chromatic graphs have been characterized.

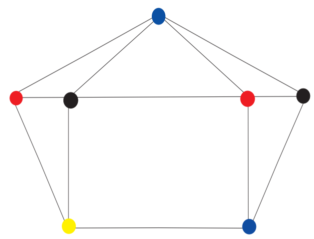

Figure 3.13: A critically 4-chromatic graph that is not minimally 4-chromatic

A $k$-chromatic subgraph of $G$ of minimum order is critically $k$-chromatic, while a $k$-chromatic subgraph of $G$ of minimum size is minimally $k$-chromatic. Every critically $k$-chromatic graph is nonseperable and every minimally $k$-chromatic graph without isolated vertices is nonseperable.

**Theorem 36.** *If $G$ is a critically $k$-chromatic graph then $G$ is $(k-1)$-edge connected. (Equivalently, any connected minimally $k$-chromatic graph is $(k-1)$-edge connected.)*

*Proof.* If $k = 2$, the graph is $K_2$, which is 1-edge-connected. If $k = 3$, the graph is an odd cycle, which is 2-edge connected. Let $k \geq 4$. Suppose $G$ is not $(k-1)$-edge connected. Then there is a partition of the vertex set of $G$ into two sets $X$ and $Y$ such that the cardinality of cut $[X, Y]$ is less than $(k-1)$. So the subgraphs induced by $X$ and $Y$ are $(k-1)$-colorable. Since the chromatic number of $G$ is $k$, cut $[X, Y]$ should have at least $(k-1)$ edges. This contradiction establishes that $G$ is $(k-1)$-edge connected. □

**Theorem 37.** *If $G$ is critically $k$-chromatic (or connected and minimally $k$-chromatic), then no vertex of $G$ has degree less than $k-1$, i.e. $\delta(G) \geq k-1$.*

*Proof.* Suppose that $v$ were a vertex of degree less than $k-1$. Since $G$ is $k$-critical, the vertex deletion subgraph $G - v$ is $(k-1)$-colorable. The colors assigned to the neighbors of $v$ would not include all the colors of $(k-1)$-coloring because vertex $v$ has fewer than $(k-1)$ neighbors. Thus, if $v$ were restored to the graph, it could be colored with any one of the $k-1$ colors that was not used on any of its neighbors. This would achieve a $(k-1)$-coloring of $G$, which is a contradiction. Thus $\delta(G) \geq k-1$. □

**Corollary 38.** *Every $k$-chromatic graph has at least $k$ vertices of degree at least $k-1$.*

*Proof.* Let $G$ be a $k$-chromatic graph and let $H$ be a $k$-critical subgraph of $G$. By the Theorem 37, each vertex of $H$ has degree at least $k-1$ in $H$, and hence also in $G$. Since $H$ is $k$-chromatic, clearly has at least $k$ vertices. □

**Proposition 39.** *A critically $k$-chromatic graph $G$ with exactly one vertex whose degree exceeds $(k-1)$ is minimally $k$-chromatic.*

*Proof.* Since $G$ is critically $k$-chromatic, the degree of each vertex is at least $(k-1)$. In this case, the degree of each vertex is $(k-1)$ except for one vertex. If $e$ is any edge of $G$, $\delta(G-e) = k-2$. Then $\chi(G-e) \leq 1 + \max\ \delta(G')$, where $G'$ is any induced subgraph of $(G-e)$. Thus $\chi(G-e) \leq 1 + (k-2) = k-1$. Since the minimum degree in $(G-e)$ is $k-2$, $\chi(G-e) = k-1$. □

## 3.7 Obstructions to $k$-Chromaticity

**Definition 40.** *An obstruction to $k$-chromaticity (or $k$-obstruction) is a subgraph that forces every graph that contains it to have chromatic number greater than $k$.*
*The complete graph $K_{k+1}$ is an obstruction to $k$-chromaticity.*

**Remark 41.** *If any edge is deleted from a $(k+1)$-critical graph, then by the definition, the resulting graph is not an obstruction to $k$-chromaticity. Thus, a $(k+1)$-critical graph is an edge-minimal obstruction to $k$-chromaticity.*

**Definition 42.** *A set $\{G_j\}$ of chromatically $(k+1)$-critical graphs is a complete set of obstructions if every $(k+1)$-chromatic graph contains at least one member of $\{G_j\}$ as a subgraph.*

The singleton set $\{K_2\}$ is a complete set of obstructions to 1-chromaticity.
The set $\{C_{2j+1} \mid j = 0, 1, ...\}$ of odd cycles is a complete set of 2-obstructions since a graph is bipartite if and only if it contains no odd cycles. Although the even order wheel graphs $W_{2m}$ with $m \geq 2$ are 4-critical, they do not form a complete set of 3-obstructions, since there are 4-chromatic graphs that contain no such wheel. The graph in Figure 3.14 does not contain an even-order wheel graph, but case-by-case analysis can be used to show that the graph is 4-chromatic.

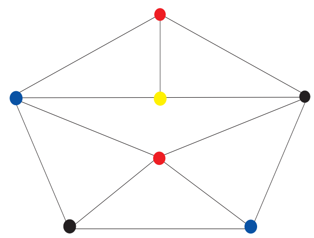

Figure 3.14: A 4-chromatic graph that contains no $W_{2m}$

**Remark 43.** *A more elegant approach to proving that the graph of Figure 3.14 requires at least four colors is based on consideration of the maximum number of vertices that can be colored with a single color.*

## 3.8 Chromatic Polynomials

In the study of colorings, some insight can be gained by considering not only the existence of colorings but also the number of such colorings. This approach was developed by (Birkhoff 1912) as a possible means of attacking the four-color conjecture. We shall denote the number of distinct $k$-colorings of $G$ by $\pi_k(G)$; thus $\pi_k(G) > 0$ if and only if $G$ is $k$-colorable. Two colorings are to be regarded as distinct if some vertex is assigned different colors in the two colorings; in other words, if $(V_1, V_2, ..., V_k)$ and $(V_1', V_2', ..., V_k')$ are two colorings, then $(V_1, V_2, ..., V_k) = (V_1', V_2', ..., V_k')$ if and only if $V_i = V_i'$ for $1 \leq i \leq k$. For example, a triangle has the six distinct 3-colorings shown in Figure 3.15. Note that even though there is exactly one vertex of each color in each coloring, we still regard these six colorings as distinct. When coloring the vertices of $\overline{K}_n$, complement of the complete

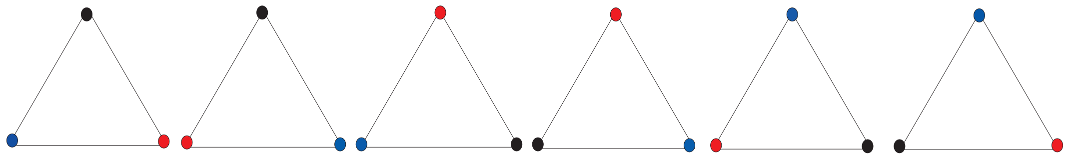

Figure 3.15: Six distinct 3-colorings of $K_3$

graph with $n$ vertices, we can use any of the $k$-colors at each vertex no matter what colors we have used at other vertices. Therefore

$\pi_k(\overline{K}_n) = k^n$. On the other hand, when we color the vertices of $K_n$, there are $k$ choices of color for the first vertex, $(k-1)$ choices for the second, $(k-2)$ for the third, and so on. Thus, in this case,

$$\pi_k(K_n) = k(k-1)...(k-n+1).$$

**Theorem 44.** *If $G$ is a simple graph and $e \in E(G)$, then*

$$\pi_k(G) = \pi_k(G-e) - \pi_k(G*e).$$

*Proof.* Let $u$ and $v$ be the ends of $e$. To each $k$-coloring of $G-e$ that assigns the same color to $u$ and $v$, there corresponds a $k$-coloring of $G*e$ in which the vertex of $G*e$ formed by identifying $u$ and $v$ is assigned the common color of $u$ and $v$. This correspondence is clearly a bijection. Therefore $\pi_k(G*e)$ is precisely the number of $k$-colorings of $G-e$ in which $u$ and $v$ are assigned the same color.

Also, since each $k$-coloring of $G-e$ that assigns different colors to $u$ and $v$ is a $k$-coloring of $G$, and conversely, $\pi_k(G)$ is the number of $k$-colorings $G-e$ in which $u$ and $v$ are assigned different colors. It follows that,

$$\pi_k(G-e) = \pi_k(G) + \pi_k(G*e).$$

□

**Theorem 45.** *For a simple graph $G$ of order $n$ and size $m$, $\pi_k(G)$ is a monic polynomial of degree $n$ in $k$ with integer coefficients and constant term zero. In addition, its coefficients alternate in sign and the coefficient of $k^{n-1}$ is $-m$.*

*Proof.* Proof is induction on $m$. If $m = 0$, $G$ is $\overline{K}_n$ and $\pi_k(\overline{K}_n) = k^n$, and the statement of the theorem is trivially true in this case. Suppose, now, that the theorem holds for all graphs with fewer than $m$ edges, where $m \geq 1$. Let $G$ be any simple graph of order $n$ and size $m$, and let $e$ be any edge of $G$. Both $G - e$ and $G * e$ (after removal of multiple edges, if necessary) are simple graphs with at most $(m-1)$ edges, and hence, by induction hypothesis,

$$\pi_k(G - e) = k^n - a_0 k^{n-1} + a_1 k^{n-2} + ... + (-1)^{n-1} a_{n-2} k,$$

and

$$\pi_k(G * e) = k^{n-1} - b_1 k^{n-2} + ... + (-1)^{n-2} b_{n-2} k,$$

where $a_0, ..., a_{n-1}; b_1, ..., b_{n-2}$ are nonnegative integers (so that the coefficients alternate in sign), and $a_0$ is the number of edges in $G - e$, which is $m - 1$. By Theorem 44, $\pi_k(G) = \pi_k(G - e) - \pi_k(G * e)$, and hence
$\pi_k(G) = k^n - (a_0 + 1)k^{n-1} + (a_1 + b_1)k^{n-2} - ... + (-1)^{n-1}(a_{n-2} + b_{n-2})k$.
Since $a_0 + 1 = m$, $\pi_k(G)$ has all the stated properties. □

**Proposition 46.** *A simple graph $G$ on $n$ vertices is a tree if and only if*

$$\pi_k(T) = k(k-1)^{n-1}.$$

*Proof.* Let $G$ be a tree. We prove that $\pi_k(T) = k(k-1)^{n-1}$ by induction on $n$. If $n = 1$, the result is trivial. So assume the result for trees with at most $(n-1)$ vertices, $n \geq 2$. Let $G$ be a tree with $n$ vertices, and $e$ be a pendant edge of $G$. By Theorem 44, $\pi_k(G) = \pi_k(G-e) - \pi_k(G*e)$. Now, $G-e$ is a forest with two component trees of orders $(n-1)$ and 1, and hence $\pi_k(G-e) = (k(k-1)^{n-2})k$. Since $G*e$ is a tree with $(n-1)$ vertices, $\pi_k(G*e) = k(k-1)^{n-2}$. Thus, $\pi_k(G*e) = (k(k-1)^{n-2})k - k(k-1)^{n-2} = k(k-1)^{n-1}$.

Conversely, assume that $G$ is a simple graph with
$\pi_k(G) = k(k-1)^{n-1} = k^n - (n-1)k^{n-1} + ... + (-1)^{n-1}k$. Hence, by Theorem 45, $G$ has $n$ vertices and $(n-1)$ edges. Further, the last term, $(-1)^{n-1}k$, ensures that $G$ is connected. Hence $G$ is a tree. □

Since the number of proper $k$-colorings is unaffected by multiple edges, we discard multiple copies of edges that arise from the contraction, keeping only one copy of each to form a simple graph.

By virtue of corollary, we can now refer to the function $\pi_k(G)$ as the chromatic polynomial of $G$. Theorem 44 provides a means of calculating the chromatic polynomial of a graph recursively. It can be used in either of two ways:

**i)**by repeatedly applying the recursion

$\pi_k(G) = \pi_k(G-e) - \pi_k(G*e)$, and thereby expressing $\pi_k(G)$ as a linear combination of chromatic polynomials of empty graphs, or

**ii)**by repeatedly applying the recursion

$\pi_k(G-e) = \pi_k(G) + \pi_k(G*e)$ and thereby expressing $\pi_k(G)$ as a linear combination of chromatic polynomials of complete graphs.

**(i)**

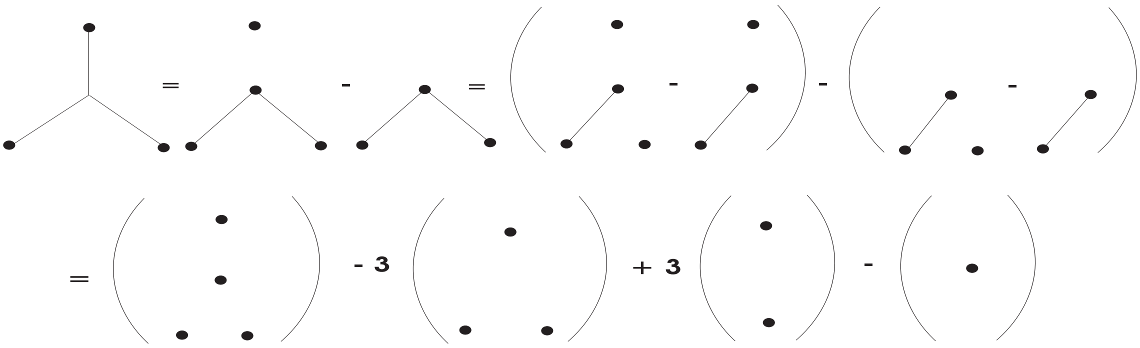


Figure 3.16: Recursive calculation of $\pi_k(T_3)$.

$\pi_k(T_3) = k^4 - 3k^3 + 3k^2 - k = k(k-1)^3$

**(ii)**

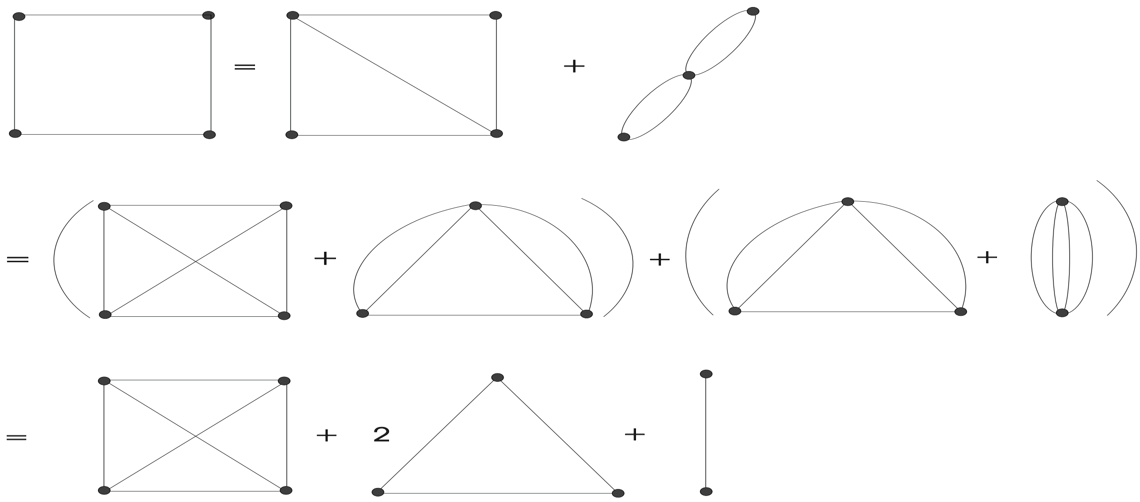


Figure 3.17: Recursive calculation of $\pi_k(C_4)$.

$$\pi_k(C_4) = k(k-1)(k-2)(k-3) + 2k(k-1)(k-2) + k(k-1) = k(k-1)(k^2-3k+3)$$

Method (i) is more suited to graphs with few edges, whereas (ii) can be applied more efficiently to graphs with many edges. These two methods are illustrated in Figure 3.16 and Figure 3.17 where the chromatic polynomial of a graph is represented symbolically by the graph itself.

The calculation of chromatic polynomials can sometimes be facilitated by the use of a number formula relating the chromatic polynomial of $G$ to the chromatic polynomials of various subgraphs of $G$. However, no good algorithm is known for finding the chromatic polynomial of a graph.

Although many properties of chromatic polynomials are known, no one has yet discovered which polynomials are chromatic. It has been conjectured by (Read 1968) that the sequence of coefficients of any chromatic polynomial must first rise in absolute value and then fall-in other words, that no coefficient may be flanked by two coefficients having greater absolute value. However, even if true, this condition, together with the conditions of corollary, would not be enough. Consider the polynomial $k^5 - 11k^4 + 14k^3 - 6k^2 + 2k$. If this is the chromatic polynomial of a connected graph $G$, $G$ should have five vertices and eleven edges. But the number of edges in a connected simple graph of order 5 is at least four and at most ten. So there is no graph for which this given polynomial is the chromatic polynomial.

Chromatic polynomials have been used with some success in the study of planar graphs, where their roots exhibit an unexpected regularity (Tutte 1970). Further results on chromatic polynomials can be found in the lucid survey article by (Read 1968).

## 3.9 Game Chromatic Number of Graphs

Games can be used to model situations where different parties have conflicting interests, or to model the "worst type of erroneous behavior of a system". In the latter case, we assume that an erroneous system is malicious and uses an intelligent strategy to try to prevent us from reaching our goal. If we are able to deal with this type of behavior, we are also able to deal with all weaker types of errors.

With notion of "game", we usually mean a class of actual games (instances), where each game (instance) is distinguished from others in the class only by its starting position and playing area, but uses the same type of moves.

In this section we consider the following game: Let $G$ be a finite graph and let $C$ be the set of colors. We consider a modified graph coloring problem posed as a two-person game, with one person (Alice) trying to color a graph, an the other (Bob) trying to prevent this from happening. Alice and Bob alternate turns, with Alice having the first move. A move consist of selecting a previously uncolored vertex $x$ and assigning to it a color from the color set $C$ distinct from the colors assigned previously (by either player) to neighbors of $x$. If after $n = |V(G)|$ moves, the graph $G$ is colored, Alice is the winner. Bob wins if an impass is reached before all nodes in the graph are colored, i.e., for every uncolored vertex $x$ and every color $\alpha$ from $C$, $x$ is adjacent to a vertex having color $\alpha$. The game chromatic number of a graph $G = (V, E)$, denoted by $\chi_g(G)$, is the least cardinality of a color set $C$ for which Alice has a winning strategy. This parameter is well-defined, since Alice always wins if $|C| = |V|$.

The notion of a graph coloring game was first introduced by (Bodlaender 1991). It seems to be very difficult to determine or estimate the game chromatic number of even small graphs. However, for special classes of graphs, non-trivial upper and lower bounds for the game chromatic numbers are obtained. The easiest case is the class of forests. It was proved by Faigle, Kern, Kierstead, and Trotter that the game chromatic number of a forest is at most 4, and that there are forests of game chromatic number 4 (Faigle et al. 1993). The game chromatic number of outerplanar graphs was studied by (Zue 1999, Kierstead 1994). It was shown in (Kierstead 1994) that outerplanar graphs have game chromatic number at most 8, and this upper bound is reduced to 7 (Zue 1999). On the other hand, there are outerplanar graphs with game chromatic number 6. It is unknown if there are outerplanar graphs with game chromatic number 7.

# CHAPTER 4

# VERTEX COLORING WITH web*MATHEMATICA*

## 4.1 Mathematica and Combinatorica Package

Mathematica, created by Stephen Wolfram, is a software system in which you can investigate mathematics, perform calculations, create graphics, and write programs. Mathematica commands are typed on a graphical user interface containing menu options.

Combinatorica, an extension to the popular computer algebra system Mathematica, is the most comprehensive software available for educational and research applications of discrete mathematics, particularly combinatorics and graph theory. It has been perhaps the most widely used software for teaching and research in discrete mathematics since its initial release in (Skiena 1990).

The goal of Combinatorica is to advance the study of combinatorics and graph theory by making a wide variety of functions available for active experimentation. We developed this package and called ColorG. We consider many classes of graphs, how to color and construct them by using our package ColorG.

## 4.2 The Concept of web*Mathematica*

The growing popularity of the internet and the increasing number of computers connected to it, make it an ideal framework for remote education. Many disciplines are rethinking their traditional philosophies and techniques to adapt to the new technologies. Web-based education is an effective framework for such learning, which simplifies theory understanding, encourages learning by discovery and experimentation and undoubtedly makes the learning process more pleasant. There is a need for adequate tools to help in the elaboration of courses that might make it possible to express all the possibilities offered by online teaching. web*Mathematica* is a web-based technology developed by Wolfram Research that allows the generation of dynamic web content with Mathematica. With this technology, distance education students should be able to explore and experiment with mathematical concepts.

web*Mathematica* is a web version of Mathematica that delivers interactive calculations and visualizations over the web. It allows a web site to return results that are marked up with Mathematica computations. When a request is made for one of these pages, the Mathematica commands are evaluated and any computed results inserted into the page. This is done with the standard Java templating mechanism, JavaServer Pages.

web*Mathematica* technology uses the request/response standard followed by web servers. Input can come from HTML forms, applets, javascript, and web-enabled applications. It is also possible to send data files to a web*Mathematica* server for processing. Output can use many different formats, such as HTML, images, Mathematica notebooks, MathML, SVG, PostScript and PDF.

## 4.3 Coloring Common Graph Families

Finding a minimum coloring can be done using brute-force search but to effectively color large graphs heuristics and usually quite effective. ColorG considers a heuristic approach to graph coloring. We use some commands in the ColorG package to color the graphs and to give web-based examples with web*Mathematica*.
We use mainly the following ColorVertices module.

```
ColorVertices[g_] := Module[{c, p, s},
c = VertexColoring[g];
p = Table[Flatten[Position[c, i]], {i, 1, Max[c]}];
s = ShowLabeledGraph[ Highlight[g, p]]]
```

The module ColorVertices colors vertices of the given graph $g$ properly.
web*Mathematica* allows the generation of dynamic web content with Mathematica. The following example draws the given graph and colors the vertices.

```
<FORM ACTION="vertex.jsp" METHOD="POST">
<p>Please select one of the following graphs and
input into the box:(1,2,3,or 4) </p>
<p>1-CompleteGraph,</p> <p>2-RandomTree, </p>
<p>3-Wheel,</p> <p>4-Cycle)</p>
<msp:allocateKernel>
<INPUT type="text" name="m" ALIGN="LEFT" size="6"
value="<msp:evaluate> MSPValue[$$m,"1"]</msp:evaluate>" />
Input the number of the vertices for the selected graph:
```

```
<INPUT type="text" name="n" ALIGN="LEFT" size="6"
value="<msp:evaluate> MSPValue[$$n,"5"]</msp:evaluate>" />
<msp:evaluate> <<DiscreteMath`ColorG`
</msp:evaluate>
<h3>Vertex coloring of the graph is </h3>
<msp:evaluate> MSPBlock[{$$m,$$n},
Which[$$m==1,MSPShow[ColorVertices[CompleteGraph[$$n]]],
$$m==2, MSPShow[ColorVertices[RandomTree[$$n]]],
$$m==3, MSPShow[ColorVertices[Wheel[$$n]]],
$$m==4, MSPShow[ColorVertices[Cycle[$$n]]]]]
</msp:evaluate>
The Chromatic Number of the selected graph is <msp:evaluate>
MSPBlock[{$$m,$$n},
Which[$$m==1, ChromaticNumber[CompleteGraph[$$n]],
$$m==2, ChromaticNumber[RandomTree[$$n]]],
$$m==3, ChromaticNumber[Wheel[$$n]],
$$m==4, ChromaticNumber[Cycle[$$n]]]]
</msp:evaluate>
<input type="submit" name="button"
value="Color the Graph"> </msp:allocateKernel>
</form>
```

In the vertex.jsp, there are two <INPUT> tags: the first one allows the user of the page to enter the number of the vertices in the graph, and the second specifies a button that, when pressed, will submit the FORM. When the FORM is submitted, it will send information from INPUT elements to the URL specified by the ACTION attribute; in this case, the URL is the same MSP. Information entered by the user is sent to a Mathematica session and assigned to a Mathematica symbol (see Figure 4.4). Additionally, the Mathlets refer to Mathematica functions that are not in standard usage. In this example some Mathematica commands; If, Table, Flatten, Position, VertexColoring, CompleteGraph, Cycle, Wheel, ShowGraph, Highlight, ChromaticNumber, True, False and some mathematical operations are used by the Mathlets. The name of the symbol is given by prepending $$ to the value of the NAME attribute. MSPValue, MSPBlock, MSPShow, MSPToExpression are webMathematica commands. MSPValue returns the value of variable or a default if no value. This example also demonstrates the use of page scoped variables with MSPToExpression. MSPToExpression interprets values and returns the result. MSPShow saves an image on the server and returns the necessary HTML to refer to this image. The image uses a GIF format; it is possible to save images in other formats (Wickham 2002). For coloring the vertices of complete bipartite graphs (CompleteGraph[n,m]), we need one more INPUT tag.

## 4.4 Vertex Coloring of Any Graph with web*Mathematica*

The module DrawG draws the simple graph without isolated points. DrawG takes as input the list of edges of a graph. The vertices of the graph of order n must be labeled consecutively $1, 2, 3, \ldots, n$.

```
DrawG[elist_]:=Module[{edgelist=elist,size,vertlist,vnum},
size=Length[edgelist];
vertlist=Union[Flatten[edgelist]];
vnum=Length[vertlist];
Do[edgelist[[i]]={edgelist[[i]]},{i,size}];
vertlist=CompleteGraph[vnum][[2]];
Graph[edgelist,vertlist]]
```

```
<FORM ACTION="anygraph.jsp" METHOD="POST">
Input the list of the edges in order as follows:
<msp:allocateKernel>
<INPUT type="text" name="v" ALIGN="LEFT" size="50"
Value ="<msp:evaluate> MSPValue[ $$v, "{{1,4},{2,3},
{2,4},{3,4}}"]</msp:evaluate>" />
<msp:evaluate> <<DiscreteMath`ColorG`
</msp:evaluate>
<h3>Vertex Coloring is </h3><p>
<msp:evaluate> input=True;
If[MSPValueQ[$$v],
n=MSPToExpression[$$v]; g=DrawG[n];
MSPShow[ColorVertices[g]],input=False;]
</msp:evaluate>
```

```
The Chromatic Number of the given graph is
<msp:evaluate>
kn=ChromaticNumber[g]
</msp:evaluate></p>
<INPUT TYPE="Hidden" NAME="formNo" VALUE="1">
<INPUT TYPE="Submit"NAME="taskValue"
VALUE="Color the graph's vertices" >
</msp:allocateKernel>
</form>
```

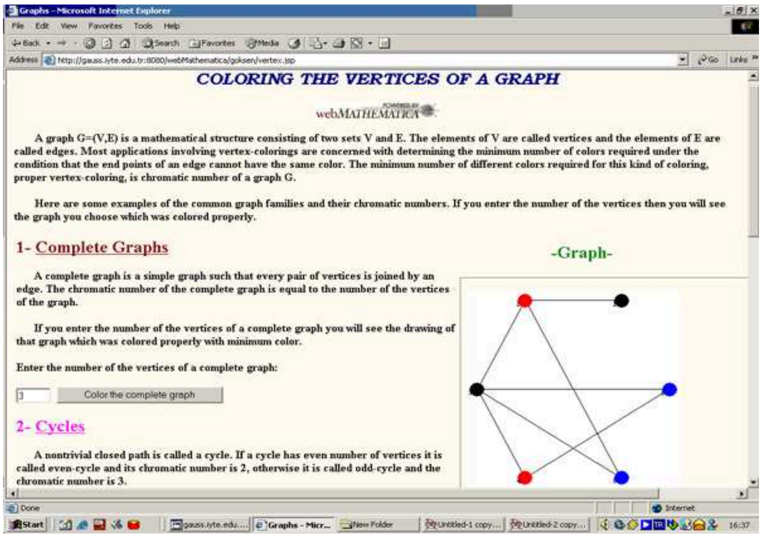


Figure 4.1: Vertex Coloring of a Graph with webMathematica

## 4.5 Generating Graphs with web*Mathematica*

This section presents operations that build graphs from other graphs. The most important operations on graphs are sum, union, join, and product of graphs. We will use these operations with the common graphs; complete graph, random tree, wheel, and cycle.

The join of two graphs is their union with the addition of edges between all pairs of vertices from different graphs. To take the join of two graphs the user should enter the number of the graphs and their vertex numbers into the boxes then he/she sees the join of those graphs and also its proper vertex coloring. The user should enter “1” for the complete graph, for random tree “2”, for wheel “3”, for cycle “4”.

We use some commands in the ColorG package to color the vertices of the generated graphs and to give web-based examples with webMathematica as follows:

```
<FORM ACTION="generate.jsp" METHOD="POST">
<p>Please select one of the following graphs and
input into the box:(1,2,3,or 4) </p> <p>1-CompleteGraph,</p>
<p>2-RandomTree, </p> <p>3-Wheel,</p> <p>4-Cycle</p>
<msp:allocateKernel>
<FORM ACTION="generate.jsp" METHOD="POST">
<font color="#800000" size="6">1- <u>Union of Graphs</u></font>
<INPUT type="text" name="t1" ALIGN="LEFT" size="6" value="
<msp:evaluate> MSPValue[$$t1,"1"]</msp:evaluate>" />
Input the number of the vertices for the selected graph:
<INPUT type="text" name="v1" ALIGN="LEFT" size="6"
value="<msp:evaluate> MSPValue[$$v1,"5"]</msp:evaluate>" />
Enter the number of your second graph:
<INPUT type="text" name="t2" ALIGN="LEFT" size="6"
value="<msp:evaluate> MSPValue[$$t2,"2"]</msp:evaluate>" />
Input the number of the vertices for the selected graph:
<INPUT type="text" name="v2" ALIGN="LEFT" size="6"
value="<msp:evaluate> MSPValue[$$v2,"4"]</msp:evaluate>" />
<msp:evaluate> MSPBlock[{$$v1,$$t1},
Which[$$t1==1, g1=CompleteGraph[$$v1];,
$$t1==2, g1=RandomTree[$$v1];,
$$t1==3, g1=Wheel[$$v1];,
$$t1==4, g1=Cycle[$$v1];]]
</msp:evaluate>
<msp:evaluate> MSPBlock[{$$v2,$$t2},
Which[$$t2 == 1, g2=CompleteGraph[$$v2];,
$$t2 == 2, g2=RandomTree[$$v2];,
$$t2 == 3, g2=Wheel[$$v2];,
$$t2 == 4, g2=Cycle[$$v2];]]
</msp:evaluate>
</FORM>
```

```
<msp:evaluate> input=True; If[MSPValueQ[$$formNo],
Switch[{$$formNo,$$taskValue},
{"A1","Draw and color the graph union"},
MSPBlock[ $$v1,$$t1,$$v2,$$t2,
MSPShow[ShowGraph[GraphUnion[g1,g2]]]]], input=False;]
</msp:evaluate>
<msp:evaluate> input=True; If[MSPValueQ[$$formNo],
Switch[{$$formNo,$$taskValue},
{"A1","Draw and color the graph union"},
MSPBlock[ {$$v1,$$t1,$$v2,$$t2},
MSPShow[ColorVertices[GraphUnion[g1,g2]]]]], input=False;]
</msp:evaluate>
</msp:allocateKernel>
```

This operation above can be extended for the other graph operations: sum, join, and product, also.

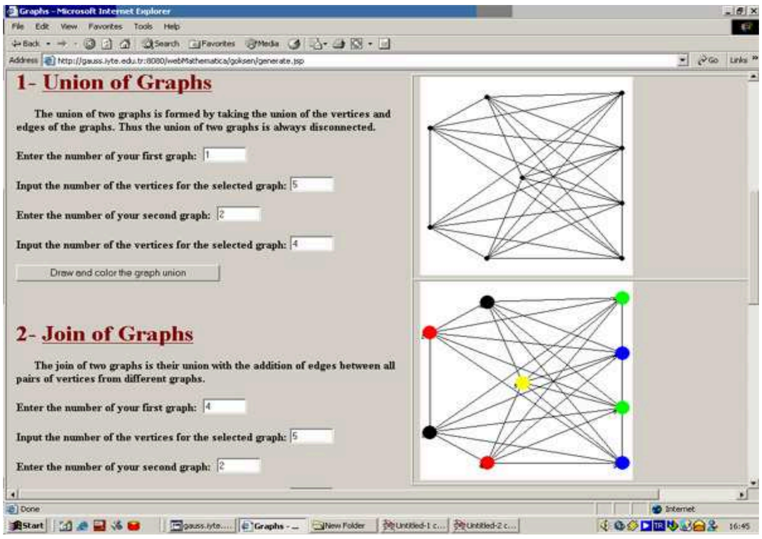


Figure 4.2: A view of join graph with webMathematica

## 4.6 An Application of Vertex Coloring

Some scheduling problems induce a graph coloring, i.e., an assignment of positive integers (colors) to vertices of a graph. We discuss a simple example for coloring the vertices of a graph with a small number $k$ of colors and present computational results for calculating the chromatic number, i.e., the minimal possible value of such a $k$.

Draw up an examination schedule involving the minimum number of days for the following problem:

Set of students: $S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9$

Examination subjects for each group: {algebra, real analysis, and topology}, {algebra, operations research, and complex analysis }, {real analysis, functional analysis, and topology }, {algebra, graph theory, and combinatorics }, {combinatorics, topology, and functional analysis }, {operations research, graph theory, and coding theory }, {operations research, graph theory, and number theory }, {algebra, number theory, and coding theory }, {algebra, operations research, and real analysis}.

Let $S$ be a set of students, $P = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be the set of examinations respectively algebra, real analysis, topology, operational research, complex analysis, functional analysis, graph theory, combinatorics, coding theory, and number theory. $S(p)$ be the set of students who will take the examination $p \in P$. Form a graph $G = G(P, E)$, where $a, b \in P$ are adjacent if and only if $S(a) \cap S(b) \neq \emptyset$. Then each proper vertex coloring of $G$ yields an examination schedule with the vertices in any color class representing the schedule on a particular day. Thus $\chi(G)$ gives the minimum number of days required for the examination schedule. The Mathematica commands for this solution are as follows:

```
k = Input["Input the number of the students"];
S = Table[Input["Input number of the lessons which the student will
choose"],  k];
b = Union[Flatten[Table[KSubsets[S[[i]], 2], i, k], 1]];
ColorVertices[t = DrawG[b]];
h = VertexColoring[t]; d=ChromaticNumber[t];
Print[d"days are requared and you can see below the lessons in the same
paranthesis which are on the same day"]
Table[Flatten[Position[h, i], 2], i, Max[h]]
```

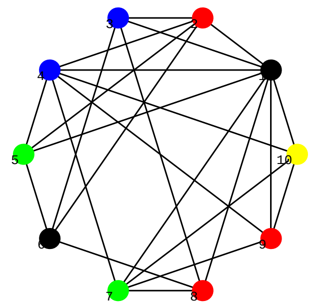

Figure 4.3: The output of the application

5 days are required and you can see below the lessons in the same parenthesis which are on the same day

$$\{\{1, 6\}, \{2, 8, 9\}, \{3, 4\}, \{5, 7\}, \{10\}\}$$

It was very exciting to take 100-year old ideas, simple as they are, and implement them in *Mathematica* and web*Mathematica* for anybody. But, as is often the case, there is more work to be done, both of a theoretical and practical nature. If we consider the game coloring problem mentioned in Section 3.9 as a two-person game, with one person (Alice) trying to color the graph, and the other (Bob) trying to prevent this from happening. We believe that it is possible to play this game online with web*Mathematica*. Of course, this leads to the rich area of game coloring and the many difficult and intriguing questions there.

The following URL address shows our works on graph coloring:
http://gauss.iyte.edu.tr:8080/webMathematica/goksen/

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