COFINITELY AMPLY WEAKLY SUPPLEMENTED MODULES

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ABSTRACT

We study amply weak supplemented modules and cofinitely amply weakly supplemented modules in this thesis. We prove that every factor module, homomorphic image, supplemented submodule of an amply (cofinitely) weak supplemented module is amply (cofinitely) weak supplemented.

ÖZET

Bu tezde bol zayıf tümleyen ve dual sonlu bol zayıf tümleyen modüller incelenmiştir. (Dual sonlu) Bol zayıf tümleyen modüllerin her faktor modülü, homomorf görüntüsünün ve tümlenen alt modüllerininde (Dual sonlu) bol zayıf tümleyen olduğu kanıtlanmıştır.

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CHAPTER 1

INTRODUCTION

R will be an associative ring with identity and we will consider left unital R-modules. Let M be an R-module. A module M is supplemented, if every submodule K of M has a supplement, ie., a submodule L minimal with respect to $K + L = M$. It is well known that a submodule L of M is a supplement of a submodule K if $K + L = M$ and $K \cap L \ll K$. If replace the last condition by $K \cap L \ll M$ we obtain the definition of weak supplement of a submodule. If every cofinite submodule K of M (that is $K \leq M$ with M/K finitely generated) has a weak supplement M is called a cofinitely weak supplemented module. Weakly supplemented modules are studied in (Lomp 1996), cofinitely weak supplemented modules are studied in (Alizade and Büyükaşık 2003). We say a submodule L of the R-module M has ample supplements in M if every submodule K such that $M = L + K$ contains a supplement of L in M. A module M is called amply supplemented if every submodule has ample supplements in M. Clearly amply supplemented modules are supplemented. Amply supplemented modules are studied in (Wisbauer 1991).

We study amply weak supplemented modules which are defined as follows. Let M be an R-module and $U \leq M$. If for every $V \leq M$ with $U + V = M$ there exists a weak supplement V' of U with $V' \leq V$ then we say that U has ample weak supplements in M. If every submodule of M has ample weak supplements in M then M is called amply weak supplemented module. We show that every factor module, homomorphic image, supplement submodule of an amply weak supplemented module is amply weak supplemented modules are amply weak supplemented and also the direct sum of projective amply weak supplemented modules are amply weak supplemented.

We study cofinitely amply weakly supplemented modules, i.e., modules M whose cofinite submodules have ample weak supplements in M. We show that every factor module, homomorphic image, supplement submodule of an cofinitely amply weakly supplemented module is cofinitely amply weakly supplemented.

CHAPTER 2

WEAKLY SUPPLEMENTED MODULES AND COFINITELY WEAK SUPPLEMENTED MODULES

2.1 Weakly Supplemented Modules and Cofinitely Weak Supplemented Modules

Definition 2.1.1. A submodule N of a module M is called *small or superfluous* if there is no proper submodule K of M such that $N + K = M$. Equivalently $N + K = M$ implies that $K = M$. It is denoted by $N \ll M$.

Proposition 2.1.2. Let M be a module

1. If $K \leq N \leq M$ and K is small in N then K is small in M.

2. Let N be a small submodule of a module M, then any submodule of N is also small in M.

3. If K is a small submodule of module M and K is contained in a direct summand N of M then K is small in N.

4. $K \ll M$ and $N \ll M$ iff $K + N \ll M$.

5. If $K \leq N \leq M$, then $N \ll M$ iff $K \ll M$, $N/K \ll M/K$.

6. Finite sum of small submodules N_i of M is a small submodule of M.

7. Let $f : M \to N$ be a homomorphism of modules M and N, let K be a submodule of M. If K is a small submodule of M, then $f(K)$ is a small submodule of N.

Proof. 1. Let $K + L = M$ for a submodule L of M.
$N = N \cap M = N \cap (K + L) = K + (N \cap L)$. Since K is small in N, $N = N \cap L$ so $N \leq L$. $K \leq N$ and $N \cap L$ so $K \leq L$. Therefore $M = K + L = L$. Thus $K \ll M$.

2. Let K be a submodule of N and $K+L=M$ for a submodule L of M. Since $K \leq N$, $N+L=M$ and also since $N \ll M$, $L=M$. So $K \ll M$.

3. $K \leq N \leq M$, $K \ll M$ and $M=N \oplus L$ for a submodule L of M. Let $K+U=N$ for a submodule U of N. $M=N+L=K+U+L$ since $K \ll M$, $M=U+L$ and $U \cap L \leq N \cap L=0$ implies $U \cap L=0$ so $M=U \oplus L$.

$N=N \cap M=N \cap (U \oplus L)=U \oplus (N \cap L)=U$. So $K \ll N$.

4. $(\Rightarrow)$ Let $(K+N)+L=M$ for some $L \leq M$.

Since $K+(N+L)=(K+N)+L=M$ and $K \ll M$, we have $N+L=M$. Since $N \ll M$, $L=M$.

$(\Leftarrow)$ $K \leq K+N \ll M$ by 2) $K \ll M$. Similarly $N \leq K+N \ll M$ by 2) $N \ll M$.

5. $(\Rightarrow)$ Since $N \ll M$ by 2) $K \ll M$. Suppose that $N/K+X/K=M/K$ where X/K is a submodule of M/K, then $N+X=M$ by assumption $X=M$ i.e. $X/K=M/K$.

$(\Leftarrow)$ Let $N+X=M$ then $(N+X)/K=M/K$ i.e. $N/K+(X+K)/K=M/K$ or $N+X+K=M$. Since $N \ll M$ we have $X+K=M$. Now $K \ll M$ implies $X=M$.

6. Let $N=\sum_{i=1}^{n} N_i$ and let $N_1+...+N_n+X=M$ for some $X \leq M$. Since $N_i \leq M$, $N_1+(N_2+...+N_n+X)=M$ then $N_2+...+N_n+X=M$ continuing in this way we obtain $N_n+X=M$ and since $N_n \ll M$, $X=M$.

7. Suppose that $f(K)+L=f(M)$ for some $L \leq f(M)$.

Then $f^{-1}(f(K)+L)=f^{-1}(f(K))+f^{-1}(L)=f^{-1}(f(M))=M$ and therefore

$$M=K+Kerf+f^{-1}(L)=K+f^{-1}(L)$$

Since $K \ll M$, $f^{-1}(L)=M$, hence $f(f^{-1}(L))=f(M)$ implies that $L \cap f(M)=f(M)$. So $L=f(M)$.

$\square$

Definition 2.1.3. A submodule N of a module M is called a *supplement* of a submodule L of M if $N + L = M$ and N is minimal with respect to this property.

Proposition 2.1.4. N is a supplement of L in M if and only if $N+L = M$ and $N \cap L \ll N$. (Wisbauer 1991).

Proof. ($\Rightarrow$)Let N be a supplement of L in M. Then we know that $M = N + L$ and N is minimal with respect to this property.
For $K \leq N$ let $N = K + (N \cap L)$.
By Modular Law $N = K+(N\cap L) = N\cap(K+L)$ that is $N \leq L+K$. $M = N+L = L+K$.
By minimality of N we have $K = N$.
($\Leftarrow$) Let $M = L + K$ for some submodule K of N.
$N = N \cap M = N \cap (K + L) = K + (N \cap L)$. Since $N \cap L \ll N$, $K = N$. So N is minimal with respect to $N + L = M$.

□

If every submodule of M has a supplement in M then M is called a *supplemented* module.

Definition 2.1.5. Let N and L be submodules of M. We call N a weak supplement of L in M if $N + L = M$ and $N \cap L \ll M$. N is called a *weak supplement* in M if there exists a submodule L such that N is a weak supplement of L in M.

Clearly any supplement is a weak supplement. M is *weakly supplemented* module if every submodule of M has a weak supplement.

Lemma 2.1.6. Let M be an R-module. If M is weakly supplemented, every factor module of M is weakly supplemented.

Proof. Let $K \leq M$ and $N/K \leq M/K$. Then $N + L = M$ and $N \cap L \ll M$ for a submodule $L \leq M$. $N/K + (L + K)/K = M/K$ since epimorphic image of small submodule is small and $N/K \cap (L + K)/K = [(N \cap L) + K]/K \ll M/K$ holds.
$\sigma(N \cap L) \ll M/K$ holds, where $\sigma : M \to M/K$ is the canonical epimorphism.

□

A module N is called a *small cover* of a module M if there exist a small epimorphism $f : N \to M$, i.e. $Kerf \ll N$.

Lemma 2.1.7. Let M be a weakly supplemented R-module. If N is a small cover of M then N is weakly supplemented.

Proof. Let $M \cong N/K$ for some $K \ll N$.

Then for every submodule $L \leq N$, $(L + K)/K$ has a weak supplement X/K in N/K, with $((L + K) \cap X)/K \ll N/K$. By 2.1.2(v) $(L + K) \cap X$ is small in N. Thus $L \cap X \leq (L \cap X) + K = (L + K) \cap X \ll N$ and $L + X = N$. Hence X is a weak supplement of L in N.

□

Lemma 2.1.8. Let M be a weakly supplemented R-module. Every supplement in M and every direct summand of M is weakly supplemented.

Proof. If $N \leq M$ is a supplement of M, then $N + K = M$ for some $K \leq M$ and $K \cap N \ll N$. By Lemma 2.1.6, $M/K \simeq N/N \cap K$ is weakly supplemented and by Lemma 2.1.7, N is weakly supplemented. Direct summands are supplements and hence weakly supplemented.

□

Lemma 2.1.9. Let M be an R-module. Let K and L be submodules of M such that L is weakly supplemented and $L + K$ has a weak supplement in M, then K has a weak supplement in M.

Proof. By assumption $L + K$ has a weak supplement $N \leq M$ such that $L + K + N = M$ and $(L + K) \cap N \ll M$. Since L is weakly supplemented $(K \cap N) \cap L$ has a weak supplement $X \leq L$. So

$M = L + K + N = X + ((K + N) \cap L) + K + N = K + (X + N)$ and

$$K \cap (X + N) \leq ((K + X) \cap N) + ((K + N) \cap X) \leq ((K + L) \cap N) + ((K + N) \cap X) \ll M$$

This means that $N + X$ is a weak supplement of K in M.

□

Theorem 2.1.10. Let M be an R-module.

If $M = M_1 + M_2$ with M_1 and M_2 weakly supplemented, then M is weakly supplemented.

Proof. For every submodule $N \leq M$, $M_1 + (M_2 + N)$ has a trivial weak supplement and by Lemma 2.1.9, $M_2 + N$ has a weak supplement. Applying previous lemma once more we get a weak supplement for N.

$\square$

Let M be an R-module. For $K \leq M$ if M/K is finitely generated then K is called a *cofinite submodule* of M. If every cofinite submodule of M has a supplement in M then M is called a cofinitely supplemented module.

Definition 2.1.11. We call M a *cofinitely weak supplemented* module if every cofinite submodule has a weak supplement.

Clearly cofinitely supplemented modules and weakly supplemented modules are cofinitely weak supplemented.

Proposition 2.1.12. If $f : M \to N$ is a homomorphism and a submodule L containing $Kerf$ is a weak supplement in M, then $f(L)$ is a weak supplement in $f(M)$.

Proof. If L is a weak supplement of K in M then

$f(M) = f(L + K) = f(L) + f(K)$ and

since $L \cap K \ll M$, we have $f(L \cap K) \ll f(M)$.

As $L \geq Kerf$, $f(L) \cap f(K) = f(L \cap K)$. So $f(L)$ is a weak supplement of $f(K)$ in $f(M)$.

$\square$

Proposition 2.1.13. A homomorphic image of cofinitely weak supplemented module is cofinitely weak supplemented module.

Proof. Let $f : M \to N$ be a homomorphism and M be a cofinitely weak supplemented module. Suppose that X is cofinite submodule of $f(M)$, then

$$M/f^{-1}(X) \cong (M/Kerf)/(f^{-1}(X)/Kerf) \cong f(M)/X$$

Therefore $M/f^{-1}(X)$ is finitely generated. Since M is a cofinitely weak supplemented module, $f^{-1}(X)$ is a weak supplement in M and by Proposition 2.1.12, $X = f(f^{-1}(X))$

is a weak supplement in $f(M)$.

□

Since every factor module of M is isomorphic to homomorphic image of M we have the following corollary.

Corollary 2.1.14. Any factor module of a cofinitely weak supplemented module is cofinitely weak supplemented module.

Lemma 2.1.15. If K is a weak supplement of N in a module M and $T \ll M$, then K is a weak supplement of $N + T$ in M.

Proof. Let $f : M \to M/N \oplus M/K$ be defined by $f(m) = (m + N, m + K)$ and

$$g : M/N \oplus M/K \to M/N + T \oplus M/K$$

be defined by $g(m + N, m^{'} + K) = (m + N + T, m^{'} + K)$. Then f is an epimorphism as $M = N + K$ and $Kerf = N \cap K \ll M$ as K is a weak supplement of N in M.

So f is a small epimorphism. Now $Kerg = (N + T)/N \oplus 0$ and
$(N + T)/T = \sigma(T) \ll M/N$ since $T \ll M$; where $\sigma : M \to M/N$ is the canonical epimorphism. Therefore g is a small epimorphism, i.e.
$(N + T) \cap K = Kerf(fg) \ll M$. Clearly $(N + T) + K = M$, so K is a weak supplement of $N + T$ in M.

□

Lemma 2.1.16. If $f : M \to N$ is a small epimorphism, then a submodule L of M is a weak supplement in M if and only if $f(L)$ is a weak supplement in N.

Proof. If L is a weak supplement of K in M then by Lemma 2.1.15, $L + Kerf$ is also a weak supplement of K and by Proposition 2.1.12, $f(L) = f(L+Kerf)$ is a weak supplement in N. Let $f(L)$ be a weak supplement of a submodule T of N, i.e. $N = f(L)+T$ and $f(L) \cap T \ll N$. Then $M = L + f^{-1}(T)$. The inverse image of a small submodule of N is small in M. So $L \cap f^{-1}(T) \leq f^{-1}(f(L) \cap T) \ll N$. Thus $f^{-1}(T)$ is a weak supplement of L.

□

Lemma 2.1.17. A small cover of a cofinitely weak supplemented module is a cofinitely weak supplemented module.

Proof. Let N be a cofinitely weak supplemented module, $f : M \to N$ be a small epimorphism and L be a cofinite submodule of M. Then $N/f(L)$ is an epimorphic image of M/L under the epimorphism

$f : M/L \to N/f(L)$ defined by $f(m + K) = f(m) + f(L)$, therefore $f(L)$ is a weak supplement. By Lemma 2.1.16, L is a weak supplement in M.

□

Lemma 2.1.18. Let N and U be submodules of M with cofinitely weak supplemented N and cofinite U. If $N + U$ has a weak supplement in M, then U also has a weak supplement in M.

Proof. Let X be a weak supplement of $(N + U)$ in M. Then we have

$$N/N \cap (X + U) \cong (N + X + U)/(X + U) = M/X + U \cong (M/U)/(X + U/L)$$

The last module is a finitely generated module, hence $N \cap (X+U)$ has a weak supplement Y in N, i.e.,

$$Y + [N \cap (X + U)] = N \quad , \quad Y \cap N \cap (X + U) = Y \cap (X + U) \ll N \leq M$$

$$M = U + X + N = U + X + Y + [N \cap (X + U)] = U + X + Y \quad and$$

$$U \cap (X + Y) \leq [X \cap (Y + U)] + [Y \cap (X + U)] \leq [X \cap (N + U)] + [Y \cap (X + U)] \ll M$$

Therefore $X + Y$ is a weak supplement of U in M.

□

Proposition 2.1.19. Let K be an R-module and let N_i $(i \in I)$ be any collection of cofinitely weak supplemented submodule of M. Then $\sum_{i \in I} N_i$ is a cofinitely weak supplemented submodule of K.

Proof. Let $M = \sum_{i \in I} N_i$ where each submodule N_i is cofinitely weak supplemented and N be a cofinite submodule of M. Then M/N is generated by some finite set $\{x_1 + N, x_2 + N, ..., x_k + N\}$ and therefore $M = R_{x1} + R_{x2} + .. + R_{xk} + N$. Since each x_i

is contained in the sum $\sum_{j\in F_i} N_j$ for some finite subset F_i of I,

$R_{x1} + R_{x2} + .. + R_{xk} \leq \sum_{j\in F} N_j$, j for some finite subset $F = \{i_1, i_2, .., i_r\}$ of I. Then $M = N + \sum_{t=1}^{r} N_{i_t}$.

Since $M = N_{i_r} + (N + \sum_{t=1}^{r-1} N_{i_t})$ has a trivial weak supplement 0 and N_{i_r} is a cofinitely weak supplemented module, $N + \sum_{t=1}^{r-1} N_{i_t}$ has a weak supplement in M. Similarly $N + \sum_{t=1}^{r-2} N_{i_t}$ has a weak supplement in M and by using Lemma 2.1.18 so on. Continuing in this way we will obtain at last N has a weak supplement in M.

□

CHAPTER 3

AMPLY SUPPLEMENTED MODULES

3.1 Amply Supplemented Modules

Definition 3.1.1. We say a submodule U of the R-module M has *ample supplements* in M if every submodule V such that $M = U + V$ contains a supplement of U in M.

A Module M is called *amply supplemented* if every submodule has ample supplements in M. Clearly amply supplemented modules are supplemented.

3.1.2. Properties of Amply Supplemented Modules

Let M be an amply supplemented R-module. Then

(1) Every supplement of a submodule of M is amply supplemented.

(2) Direct summands of M are amply supplemented.

(3) Factor modules of M are amply supplemented.

Proof. (1) Let V be a supplement of $U < M$ and $V = X + Y$ thus $M = X + Y$. Then there is a supplement $Y^{'}$ of $(U + X)$ in M with $Y^{'} < Y$.

We get $X \cap Y^{'} < (U + X) \cap Y^{'} \ll Y^{'}$ and $M = U + X + Y^{'}$ implies $X + Y^{'} = V$. $Y^{'}$ is a supplement of X in V.

(2) Since direct summands are supplements it follows from (1)

(3) If $M = X + Y$, there is a supplement $Y^{'}$ of X with $Y^{'} < Y$. For $L \leq M$, $M = X/L+Y/L$. Since $Y^{'}$ is a supplement of X and $L < X$ (Wisbauer 1991). $(Y^{'}+L)/L$ is a supplement of X/L in M/L.

□

Lemma 3.1.3. Let M be a R-module and $M = U_1 + U_2$. If the submodules U_1, U_2 have ample supplements in M then $U_1 \cap U_2$ has also ample supplements.

Proof. Let $U_1 \cap U_2 + V = M$ for $V \leq M$.

$$U_1 \cap U_2 + U_2 \cap V = U_2 \to U_1 + U_2 \cap V = M$$

$$U_1 \cap U_2 + U_1 \cap V = U_1 \to U_2 + U_1 \cap V = M$$

Therefore there is a supplement $V_2^{'}$ of U_1 in $U_2 \cap V$. So

$$U_1 + V_2^{'} = M,\ U_1 \cap V_2^{'} \ll V_2^{'},\ V_2^{'} \leq U_2 \cap V$$

There is a supplement $V_1^{'}$ of U_2 in $U_1 \cap V$. So

$$U_2 + V_1^{'} = M,\ U_2 \cap V_1^{'} \ll V_1^{'},\ V_1^{'} \leq U_1 \cap V$$

Since $U_1 \cap U_2 + V_2^{'} = U_2$ and $U_1 \cap U_2 + V_1^{'} = U_1$

$$U_1 \cap U_2 + V_1^{'} + V_2^{'} = M,\ V_1^{'} + V_2^{'} \leq V$$

$$(U_1 \cap U_2) \cap (V_1^{'} + V_2^{'}) = U_1 \cap [U_2 \cap (V_1^{'} + V_2^{'})] = U_2 \cap V_1^{'} + U_1 \cap V_2^{'} \ll V_1^{'} + V_2^{'}$$

$\square$

Theorem 3.1.4. For an R-module M the following properties are equivalent:

(a) M is amply supplemented.

(b) Every submodule $U \leq M$ is of the form $U = X + Y$ with X supplemented and $Y \ll M$.

(c) For every submodule $U \leq M$, there is a supplemented submodule $X \leq U$ with $U/X \ll M/X$.

If M is finitely generated, then $(a) - (c)$ are also equivalent to:

(d) Every maximal submodule has ample supplements in M.

Proof. $(a) \Rightarrow (b)$ Let V be a supplement of U in M. $U + V = M$ and $U \cap V \ll V$ and let $X < U$ be a supplement of V in M. Then we have $X + V = M$

$$U \cap (X + V) = U \cap M \to X + U \cap V = U$$

where X is supplemented.

(b)$\Rightarrow (c)$ If $U = X + Y$ with X supplemented and $Y \ll M$, then $Y/X \cap Y \ll M/X \cap Y$.

Let $\pi : M/X \cap Y \to (M/X \cap Y)/(X/X \cap Y) \cong M/X$

$$\pi(Y/X \cap Y) = (Y/X \cap Y + X/X \cap Y)/(X/X \cap Y) \cong U/X \ll M/X$$

(c)$\Rightarrow (a)$ If $U + V = M$ and X is a supplemented submodule of V with $V/X \ll M/X$, then

$V/X + (U + X)/X = M/X \to U + X/X = M/X \to U + X = M$, for a supplement V' of $U \cap X$ in X.

We have $U \cap X + V' = X \to U + V' = M$ and $U \cap V' = (U \cap X) \cap V' \ll V' \to V' < V$ is a supplement of U.

(a)$\Rightarrow (d)$ It is clear by the definition of amply supplemented modules.

(d)$\Rightarrow (a)$ If M is finitely generated and all maximal submodules have supplements,then M is supplemented and $M/RadM$ is semisimple. Then, for $U \leq M$, the factor module $M/(U+RadM)$ is semisimple and $U+RadM$ is an intersection of finitely many maximal submodules.

□

An R-Module M is called $\pi-projective$ if for every submodules U and V with $U+V = M$ there exists a homomorphism $f : M \to M$ such that $Imf \leq U$ and $Im(1 - f) \leq V$.

An R-module M is $\pi - projective$ if and only if the epimorphism

$g : U \oplus V \to M \quad g(u, v) \to u + v$ splits.

Proof. $(\Rightarrow)$ Define $h : M \to U \oplus V$ by $h(m) = (f(m), (1 - f)(m)), \forall m \in M$

Since g is an epimorphism

$$(g \circ h)(m) = f(m) + (1 - f)(m) = m$$

So $g \circ h = 1_M$ and it splits.

($\Leftarrow$) Let $h : M \to U \oplus V$ be a homomorphism defined by $h(m) = (u, v)$ for $m \in M$. Since the epimorphism g splits, $g \circ h = 1m$. Hence $f(m) = u$ and $Imf \leq U$.

$u + v = g \circ h(m) = m, \quad v = m - u = m - f(m) = (1 - f)(m), \quad Im(1 - f) \leq V$. So M is $\pi - projective$.

Let $U + V = M$ and M is M-projective since $U \leq M$ and $V \leq M$ then M is U-projective and V-projective. So M is $U \oplus V$ projective that is

$$\begin{tikzcd}
U \oplus V \arrow[r, "f"] & M \arrow[r] & o \\
& M \arrow[ul, dashed, "g"] \arrow[u, "1_M"'] &
\end{tikzcd}$$

$f \circ g = 1_M \quad U \oplus V \to M$ splits. So M is $\pi - projective$.

$\square$

CHAPTER 4

AMPLY WEAK SUPPLEMENTED MODULES

4.1 Amply Weak Supplemented Modules

Definition 4.1.1. Let M be an R-module and $U \leq M$. If every $V \leq M$ with $U + V = M$ there exists a weak supplement V' of U with $V' \leq V$, then we call U has *ample weak supplements* in M.

Definition 4.1.2. Let M be an R-module. If every submodule of M has ample weak supplements in M then M is called *amply weak supplemented* module.

Lemma 4.1.3. Every factor module of an amply weak supplemented module is amply weak supplemented.

Proof. Let M be an amply weak supplemented module and M/N be any factor module of M. Let $K/N \leq M/N$. For $V/N \leq M/N$, let $K/N + V/N = M/N$. Then $K + V = M$ and because M is an amply weak supplemented module, there exists a weak supplement V' of K with $V' \leq V$. By Lemma 2.1.7, $(V' + N)/N \leq V/N$, K/N has ample weak supplements in M/N. Thus M/N is amply weak supplemented.

□

Corollary 4.1.4. Every homomorphic image of an amply weak supplemented module is amply weak supplemented.

Proof. Let M be an amply weak supplemented module. Since every homomorphic image of M is isomorphic to a factor module of M, then by Lemma 4.1.3, every homomorphic image of M is amply weak supplemented.

□

Lemma 4.1.5. Every supplement submodule of an amply weak supplemented module is amply weak supplemented.

Proof. Let M be an amply weak supplemented module and V be any supplement submodule of M. Let V be a supplement of U in M. Let $K \leq V$ and $K + T = V$ for $T \leq V$. Then $U + K + T = M$. Since M is amply weak supplemented, $U + K$ has a weak supplement $T^{'}$ in M with $T^{'} \leq T$. This case $U + K + T = M$ and $(U + K) \cap T^{'} \ll M$. Since $K + T \leq V$ and V is a supplement of U in M, $K + T^{'} = V$. We prove $K \cap T^{'} \ll V$. Let $K \cap T^{'} + S = V$ for $S \leq V$. Since $U + K \cap T^{'} + S = M$ and $K \cap T^{'} \leq (U + K) \cap T^{'} \ll M$, $U + S = M$. Since $S \leq V$ and V is a supplement of U, $S = V$. Hence $K \cap T^{'} \ll V$ and $T^{'}$ is a weak supplement of K in V. Since $T^{'} \leq T$, K has ample weak supplements in V. Thus V is amply weak supplemented.

□

Corollary 4.1.6. Every direct summand of an amply weak supplemented module is amply weak supplemented.

Proof. Let M be an amply weak supplemented module. Since every direct summand of M is supplement in M, then by Lemma 4.1.5, every direct summand of M is amply weak supplemented.

□

Theorem 4.1.7. Let M be an R-module, $U \leq M$ and $V \leq M$ with $M = U + V$. If U and V have ample weak supplements in M, then $U \cap V$ also has ample weak supplements in M.

Proof. Let $U \cap V + T = M$ for $T \leq M$. Since $U + V = M$, then we can prove $U + V \cap T = V + U \cap T = M$. Then by hypothesis U has a weak supplement $V^{'}$ with $V^{'} \leq V \cap T$ and V has a weak supplement $U^{'}$ with $U^{'} \leq U \cap T$. This case $U + V^{'} = M$ and $V^{'} \leq V$, by Modular law $U \cap V + V^{'} = V$. Since $V + U^{'} = M$ and $U^{'} \leq U$, also by Modular law $U \cap V + U^{'} = U$

$$M = U + V = U \cap V + U^{'} + U \cap V + V^{'} = U \cap V + U^{'} + V^{'}$$

Since

$$(U \cap V) \cap (U^{'} + V^{'}) \leq (U \cap V + U^{'}) \cap V^{'} + (U \cap V + V^{'}) \cap U^{'} = U \cap V^{'} + V \cap U^{'} \ll M$$

$U^{'} + V^{'}$ is a weak supplement of $U \cap V$ in M. Since $U^{'} \leq T$ and $V^{'} \leq T$, $U^{'} + V^{'} \leq T$. Thus $U \cap V$ has ample weak supplements in M.

□

Lemma 4.1.8. Let M be a weakly supplemented and $\pi - projective$ module. Then M is amply weak supplemented.

Proof. Let $U \leq M$ and $U + V = M$ for $V \leq M$. Since M is weakly supplemented, there exists a weak supplement X of U in M. Since M is $\pi - projective$, there exists an homomorphism $f : M \to M$ such that $Imf \leq V$ and $Im(1-f) \leq U$. Then we can prove $f(U) \leq U$ and $(1-f)(V) \leq V$

$$M = f(M) + (1-f)(M) = U + f(U + X) = U + f(U) + f(X) = U + f(X)$$

Let $u \in U \cap f(X)$. Then there exists $x \in X$ with $u = f(x)$. This case
$x - u = x - f(x) = (1-f)(x) \in U$ and then $x \in U$.
Hence $x \in U \cap X$ and $U \cap f(X) \leq f(U \cap X)$. Since $U \cap X \ll M$, then by Proposition 2.1.2, $f(U \cap X) \ll f(M)$. Then $U \cap f(X) \leq f(U \cap X) \ll M$. Hence $f(X)$ is a weak supplement of U in M. Since $f(X) \leq V$, U has ample weak supplements in M. Thus M is amply weak supplemented.

□

Corollary 4.1.9. Every projective and weakly supplemented module is amply weak supplemented.

Proof. We can prove every projective module is $\pi - projective$. Then by Lemma 4.1.8, every projective and weakly supplemented module is amply weak supplemented.

□

Corollary 4.1.10. Let $M_1, M_2, ..., M_n$ be projective modules. Then $\overset{n}{\underset{i=1}{\oplus}} M_i$ is amply weak supplemented if and only if for every $1 \leq i \leq n$, M_i is amply weak supplemented.

Proof. $(\Rightarrow)$ Clear from Corollary 4.1.6.

$(\Leftarrow)$ Since for every $1 \leq i \leq n$, M_i is amply weak supplemented, M_i is weakly supplemented. Then by Theorem 2.1.10, $\overset{n}{\underset{i=1}{\oplus}} M_i$ is also weakly supplemented. Since for every $1 \leq i \leq n$, M_i is projective, $\overset{n}{\underset{i=1}{\oplus}} M_i$ is also projective. Then by Corollary 4.1.9, $\overset{n}{\underset{i=1}{\oplus}} M_i$ is amply weak supplemented.

□

Corollary 4.1.11. Let R be a ring. Then the following statements are equivalent.

(a) R is semilocal.

(b) ${}_R$R is weakly supplemented.

(c) ${}_R$R is amply weak supplemented.

(d) Every finitely generated R-module is weakly supplemented.

(e) Every finitely generated R-module is amply weak supplemented.

Proof. $(a) \Leftrightarrow (b)$ See (Lomp 1999).

$(b) \Leftrightarrow (d)$ See (Lomp 1996).

$(b) \Leftrightarrow (c)$ Clear from Corollary 4.1.9.

$(b) \Leftrightarrow (e)$ Clear from Corollary 4.1.4 and Corollary 4.1.10.

□

CHAPTER 5

COFINITELY AMPLY WEAKLY SUPPLEMENTED MODULES

5.1 Cofinitely Amply Weakly Supplemented Modules

Definition 5.1.1. Let M be an R-module. If every cofinite submodule of M has ample weak supplements in M then M is called $cofinitely\ amply\ weakly\ supplemented\ module$.

Lemma 5.1.2. Every factor module of an cofinitely amply weakly supplemented module is cofinitely amply weakly supplemented.

Proof. Let M be an cofinitely amply weakly supplemented module and M/N be any factor module of M. Let K/N be a cofinite submodule of M/N.
Then $(M/N)/(K/N)$ finitely generated.
By $M/K \cong (M/N)/(K/N)$, M/K also finitely generated. Hence K is a cofinite submodule of M. For $V/N \leq M/N$, let $K/N+V/N = M/N$. Then $K+V = M$ and because M is cofinitely amply weakly supplemented module and K is a cofinite submodule of M, there exists a weak supplement V' of K with $V' \leq V$. By Lemma 2.1.7, $(V'+N)/N$ is weak supplement of K/N in M/N. Since $(V'+N)/N \leq V/N$, K/N has ample weak supplements in M/N. Thus M/N is cofinitely amply weakly supplemented.

□

Corollary 5.1.3. Every homomorphic image of an cofinitely amply weakly supplemented module is cofinitely amply weakly supplemented.

Proof. Let M be an cofinitely amply weakly supplemented module. Since every homomorphic image of M is isomorphic to a factor module of M, then by Lemma 5.1.2, every homomorphic image of M is cofinitely amply weakly supplemented.

□

Lemma 5.1.4. Every supplement submodule of an cofinitely amply weakly supplemented module is cofinitely amply weakly supplemented.

Proof. Let M be an amply cofinitely supplemented module and V be any supplement submodule of M. Let V be a supplement of U in M. Let K be a cofinite submodule of V. Then we can prove $U \cap V + K$ is also a cofinite submodule of V. By

$$M/(U+K) = (U+V)/(U+K) \cong V/(U \cap V + K)$$

$U+K$ is a cofinite submodule of M. Let $K+T = V$ for any $T \leq V$. Then $U+K+T = M$. Since M is cofinitely amply weakly supplemented and $U+K$ is a cofinite submodule of M, $U+K$ has a weak supplement $T^{'}$ in M with $T^{'} \leq T$. This case $U+K+T^{'} = M$ and $(U+K) \cap T^{'} \ll M$. Since $K + T^{'} \leq V$ and V is a supplement of U in M, $K + T^{'} = V$. We prove $K \cap T^{'} \ll V$. Let $K \cap T^{'} + S = V$ for $S \leq V$. Since $U + K \cap T^{'} + S = M$ and $K \cap T^{'} \leq (U+K) \cap T^{'} \ll M$, $U + S = M$. Since $S \leq V$ and V is a supplement of U, $S = V$. Hence $K \cap T^{'} \ll V$ and T is a weak supplement of K in V. Since $T^{'} \leq T$, K has ample weak supplements in V. Thus V is cofinitely amply weakly supplemented.

□

Corollary 5.1.5. Every direct summand of an cofinitely amply weakly supplemented module is cofinitely amply weakly supplemented.

Proof. Let M be an amply cofinitely supplemented module. Since every direct summand of M is supplement in M, then by Lemma 5.1.4, every direct summand of M is cofinitely amply weakly supplemented.

□

Lemma 5.1.6. Let M be a cofinitely weak supplemented and $\pi - projective$ module. Then M is cofinitely amply weakly supplemented.

Proof. Let U be a cofinite submodule of M and $U + V = M$ for $V \leq M$. Since M is cofinitely weak supplemented and U is a cofinite submodule of M, there exists a weak supplement X of U in M. Since M is $\pi - projective$, there exists an homomorphism $f : M \to M$ such that $lmf \leq V$ and $lm(1-f) \leq U$. Then we can prove $f(U) \leq U$ and

$(1-f)(V) \leq V$. This case

$$M = f(M) + (1-f)(M) = U + f(U+X) = U + f(U) + f(X) = U + f(X)$$

Let $u \in U \cap f(X)$. Then there exists $x \in X$ with $u = f(x)$. This case

$x - u = x - f(x) = (1-f)(x) \in U$ and then $x \in U$.

Hence $x \in U \cap X$ and $U \cap f(X) \ll f(U \cap X)$. Since $U \cap X \ll M$, then by Proposition 2.1.2, $f(U \cap X) \ll f(M)$. Then $U \cap f(X) \leq f(U \cap X) \ll M$. Hence $f(X)$ is a weak supplement of U in M. Since $f(X) \leq V$, U has ample weak supplements in M. Thus M is cofinitely amply weakly supplemented.

□

Corollary 5.1.7. Every projective and cofinitely weak supplemented module is cofinitely amply weakly supplemented.

Proof. We can prove every projective module is $\pi - projective$. Then by Lemma 5.1.6, every projective and weakly supplemented module is amply weak supplemented.

□

Corollary 5.1.8. Let $(M_i)_{i \in I}$ be a family of projective modules. Then $\oplus_{i \in I} M_i$ is amply weak supplemented if and only if for every $i \in I$, M_i is amply weak supplemented.

Proof. ($\Rightarrow$) Clear from Corollary 5.1.5.

($\Leftarrow$) Since for every $i \in I$, M_i is cofinitely amply weakly supplemented, M_i is cofinitely weak supplemented. Then by Lemma 1.7. $\oplus_{i \in I} M_i$ is also cofinitely weak supplemented. Since for every $i \in I$, M_i is projective, by Lemma Proposition 2.1.19 $\oplus_{i \in I} M_i$ is also projective. Then by Corollary 5.1.7, $\oplus_{i \in I} M_i$ is cofinitely amply weakly supplemented.

□

Corollary 5.1.9. Let R be a ring. Then the following statements are equivalent.

(a) R is semilocal.

(b) $_R$R is weakly supplemented.

(c) $_R$R is amply weakly supplemented.

(d) Every R-module is cofinitely weak supplemented.

(e) Every R-module is cofinitely amply weakly supplemented.

Proof. $(a) \Leftrightarrow (b)$ See (Lomp 1999).

$(b) \Leftrightarrow (d)$ See (Alizade and Büyükaşık 2003).

$(b) \Leftrightarrow (c)$ Clear from Corollary 5.1.7.

$(b) \Leftrightarrow (e)$ Clear from Corollary 5.1.3, and Corollary 5.1.8.

$\square$

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