INVESTIGATION OF STRESS INTENSITY FACTORS IN AN ELASTIC CYLINDER UNDER AXIAL TENSION WITH A CRACK OF RING-SHAPE

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ABSTRACT

This study is concerned with the fracture of an axisymmetric thick-walled cylinder. The cylinder is under the action of axisymmetric tensile loads at infinity. A ring-shaped crack with surface free tractions is located at the symmetry plane. Material of the cylinder is assumed to be linearly elastic and isotropic. Solution for this problem can be obtained by superposing the solutions for (i) an infinite cylinder subjected to uniformly distributed tensile load at infinity, and (ii) an infinite cylinder having a crack (the perturbation problem). The Hankel and Fourier transform techniques are used for the solution of the field equations. Applying the boundary conditions, the singular integral equation in terms of crack surface displacement derivative is derived. By using an appropriate quadrature formula the integral equation is reduced to a linear algebraic equation system. Numerical solution is used to develop results for the stress intensity factors at the tips of the crack. Results are presented in graphical and tabular forms.

ÖZET

Bu çalışmada eksenel simetrik ve sonsuzda eksenel çekmeye maruz bir silindirik tüpün kırılması problemi ele alınmıştır. Halka şeklindeki çatlak z=0 simetri düzleminde bulunmaktadır. Silindir malzemesinin lineer elastik ve izotropik olduğu varsayılmıştır. Problemin çözümü (i) sonsuzda düzgün yayılı çekmeye maruz tüp ve (ii) çatlak içeren tüp (pertürbasyon problemi), çözümlerinin süperpozisyonu ile elde edilebilir. İlk problemin çözümü ikinciye nazaran daha temel ve çalışmanın asıl amacı olan gerilme şiddeti katsayılarının hesaplanmasıyla ilgili olmadığından yapılmamış, ikinci problem (pertürbasyon problemi) ise detaylı olarak incelenip çözümü yapılmıştır. Elastisite denklemlerinin çözümünde Fourier ve Hankel integral dönüşümleri kullanılmış, sınır şartları uygulanarak çatlak yüzeyi yer değiştirmesinin türevi cinsinden bir tekil integral denklem elde edilmiştir. Bu tekil integral denklem de uygun bir integrasyon formülü kullanılarak bir cebirsel denklem sistemine dönüştürülmüştür. Çatlak uçlarındaki gerilme şiddeti katsayıları farklı çatlak durumları için nümerik olarak hesaplanmış, sonuçlar grafik ve tablolar halinde sunularak değerlendirilmiştir.

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CHAPTER 1

INTRODUCTION

1.1. Theoretical Background

Fracture mechanics was introduced in the 1950s under the leadership of G.R. Irwin. The concept of fracture mechanics were further developed and refined throughout the 1960s by a collaboration of researchers in some universities and research centers (Sanford 2002).

Consider a structure in which a crack develops. This crack will grow with time because of the application of repeated loads or a combination of loads and environmental attack. The longer the crack, the higher the stress concentration induced by it. Many structures are designed to carry service loads that are high enough to initiate cracks, particularly when pre-existing flaws or stress concentrations are present. The designer must anticipate this possibility of cracking and consequently he has to accept a certain risk that the structure will fail. In order to ensure this safety it must be predicted how fast cracks will grow and how fast residual strength will decrease. Making these predictions and developing prediction methods are the objects of fracture mechanics.
Because of that, the process region do not treat as a continuum, crack and fracture problems can not be solved simply by calculating stresses and strains in the body. On the other hand knowledge of stresses and strains in the continuum outside the process region is essential for understanding the process of fracture. In fracture mechanics problems both analytical and numerical methods are widely used scientific tools. Analytical methods usually are based on partial differential or integral equations while the finite element methods dominate in numerical solution of the problems

Fracture mechanics should be able to answer the following five questions

1)What is the residual strength as a function of crack size ?

2)What size of crack can be tolerated at the expected service load ?

3)How long does it take for a crack to grow from a certain initial size to the critical size?

4)What size of pre–existing flaw can be permitted at the moment the structure starts its service life?

5)How often should be structure be inspected for cracks?

Fracture mechanics provide satisfactory answers to some of above questions and useful answers to the others. In order to make a successful use of fracture mechanics in engineering application it is essential to have some knowledge about the broad field of fracture mechanics. This is schematically explained in (Figure 1.1.).

Fracture	Fracture processes and criteria	Plasticity	Testing	Applications

Material Science ⟷ Engineering ⟷

Applied Mechanics ⟷

Fracture Mechanics ⟷

Figure 1.1. The broad field of fracture mechanics.

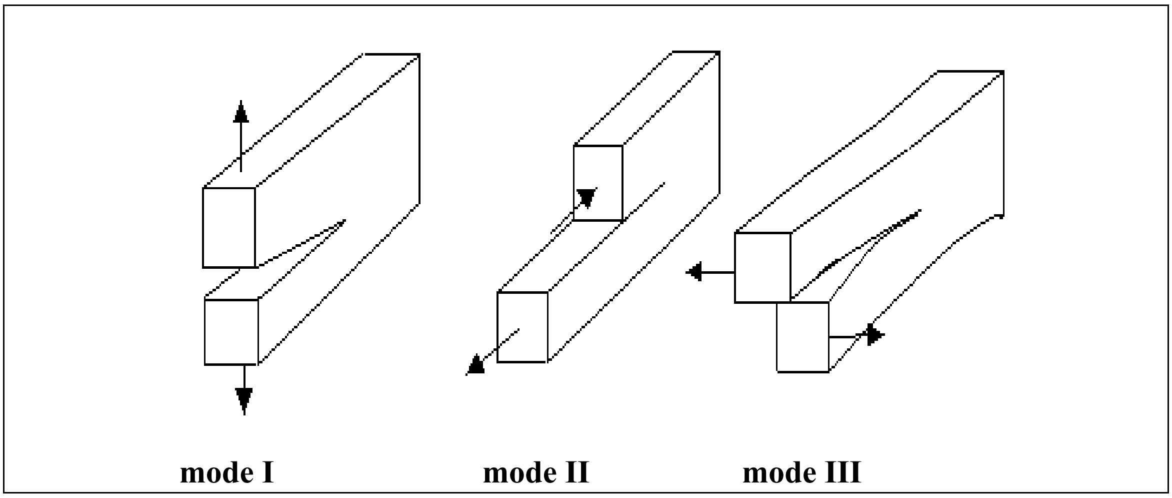

Figure 1.2. The three modes of loading

A crack in a solid can be stressed in three different modes as illustrated in (Figure 1.2.). These modes were introduced by Irwin (1960). Based on the Figure 1.3, the problem subjected to one of the three modes of loadings can be explained as

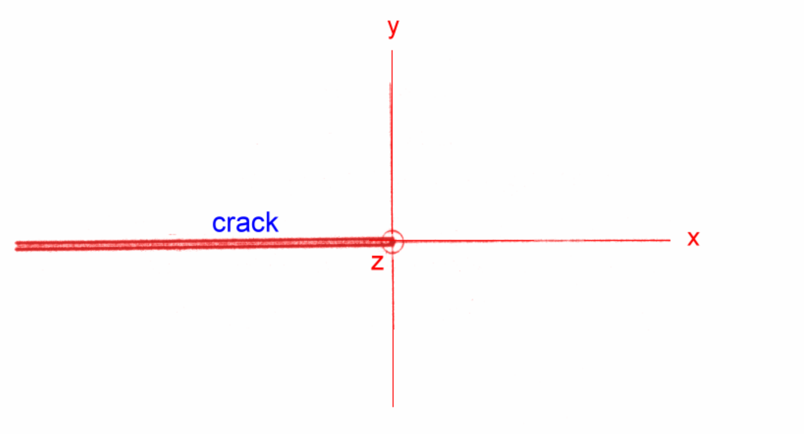

Figure1.3. Crack and coordinate system.(the z- axis toward the reader).

1. Opening Mode (Mode I): A body, under this type of loading, is exhibited two different displacements and stresses behaviors. (i)Horizontal displacements u(*x,y,z*) are symmetric, (ii) vertical displacements *v*(*x,y,z*) are anti-symmetric. Also if the material is isotropic then , (i) normal stresses $\sigma_x(x,y,z), \sigma_y(x,y,z), \sigma_z(x,y,z)$ are symetric, (ii) shear stresses $\tau_{xy}(x,y,z), \tau_{xz}(x,y,z), \tau_{yz}(x,y,z)$ are anti-symetric. These important symmety relations can be described by the following mathematical notation:

$$u(x,-y,z) = u(x,y,z) \qquad w = 0$$
$$v(x,-y,z) = -v(x,y,z) \qquad \frac{\partial^2 w}{\partial z^2} = 0$$

If the material of the body is isotropic, then the stresses given below are valid

$$\begin{aligned}
&\sigma_x(x,-y,z) = \sigma_x(x,y,z)\\
&\sigma_y(x,-y,z) = \sigma_y(x,y,z)\\
&\sigma_z(x,-y,z) = \sigma_z(x,y,z)\\
&\tau_{xy}(x,-y,z) = -\tau_{xy}(x,y,z)\\
&\tau_{xz} = \tau_{yz} = 0
\end{aligned}$$

2. The in plane shearing (or sliding) mode(Mode II): Horizontal displacements are anti - symmetric, vertical displacements are symmetric.

$$u(x,-y,z) = -u(x,y,z) \qquad w = 0$$
$$v(x,-y,z) = v(x,y,z) \qquad \frac{\partial^2 w}{\partial z^2} = 0$$

For in-plane problems there is no dependence on the antiplane coordinate z except for w in mode II. Thus,

$$\begin{aligned} &u = u(x,y) \\ &\sigma_y = \sigma_y(x,y) \\ &etc. \end{aligned}$$

Then, normal stresses are anti-symmetric, shear stresses are symmetric

$$\begin{aligned} &\sigma_x(x,-y) = -\sigma_x(x,y) \\ &\sigma_y(x,-y) = -\sigma_y(x,y) \\ &\sigma_z(x,-y) = -\sigma_z(x,y) \\ &\tau_{xy}(x,-y) = \tau_{xy}(x,y) \\ &\tau_{xz} = \tau_{yz} = 0 \end{aligned}$$

3. The anti-plane shearing mode(Mode III):The only non-vanishing displacement *w* is anti-symetric. The only non-vanishing stresses are τ_{xz} anti-symetric and τ_{yz} symmetric.

$$\begin{aligned} &u = v = 0 \\ &w(x,-y,z) = -w(x,y,z) \\ &\frac{\partial w}{\partial z} = 0 \\ &\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0 \\ &\tau_{xz}(x,-y,z) = -\tau_{xz}(x,y,z) \\ &\tau_{yz}(x,-y,z) = \tau_{yz}(x,y,z) \\ &\frac{\partial \tau_{xz}}{\partial z} = \frac{\partial \tau_{yz}}{\partial z} = 0 \end{aligned}$$

The fracture mechanics approach has three important variables: applied stress, flaw size and fracture toughness. There are two alternative approaches to fracture analysis: the energy criterion and the stress intensity approach. In the present study stress intensity approach is used to analyze the problem. Stress intensity factor is a quantity determined analytically and varies as a function of the crack configuration and the manner in which the external loads are applied. Hence, the analytical expression of k changes from one system to another. It is known that the maximum stress becomes unbounded when the notch root radius tends to zero in the elasticity solutions of problems involving stress concentrations. In this case, it is said that the stress state at the notch root is singular and the asymptotic examination would show that the magnitude of the stresses are of the form $\sigma_{ij} \mapsto k / r^{\alpha}$, $0 < \alpha < 1$ where r is the distance from notch root and k is a constant. Here α and k describe the nature of the stress singularity at the notch root .When the notch becomes a crack, the strength of the stress singularity k is known as the Stress Intensity Factor (SIF). This concept provides a universal description of the fracture process. In other words, no matter what the history or the external conditions in a given system, if the stress intensity factor in any two systems has the same value, the crack tip that they describe will behave in the same way. The universal form of the stress intensity factor allows a complete description of the behavior of the tip of a crack where one need only carry out the analysis of a given problem within the universal elastic region.

1.2. Mathematical Background

1.2.1. Boundary Conditions in Crack Problems:

Conditions on boundaries of the continua may be divided into the following three categories:

1. Conditions on the outer boundaries of the body, including the crack faces: These conditions usually consist of specification of traction or displacements.

2. Continuity conditions on the interfaces between different regions in the continuum such as the elastic region, plastic region and its some regions occupied by different materials.

3. Conditions on the boundary to the process region: These conditions depend on the response of the process region model to loads or displacement.

1.2.2. Methods of Solution

Experiences show that analytical methods often lead to better understanding general properties of the phenomena and also numerical methods can be properly exploited only with a thorough knowledge of analytical methods and results. Analytical techniques are frequently used for controlling the accuracy of numerical methods, therefore this part focus on analytical methods only .

Complex potential, integral transform and singular integral equations are three important methods for the solution of crack problems leading to the calculation of the stress intensity factors. Now, some significant points of these techniques and comparison of them will be given.

1.2.2.1. Complex Potential Method

This method is applicable to only two dimensional problems and provides the simplest analyzing the singular behavior of the solution. Complex potential method can be investigated under the following three important solution techniques (Erdogan 1983):

1. The Method of Conformal Mapping
2. Laurent Series Expansion
3. Boundary Collocation Method

1.2.2.2. Integral Transforms

If the problem is a mixed boundary value problem, then integral transform is the most widely used method(Erdogan 1983). Especially, the crack problems for an elastic plane or an infinite strip containing a line crack, elastic cylinder with an infinite or finite radius containing an axisymetric crack can easily be reduced to dual integral equations by using Fourier, Mellin and Hankel transforms depending on the geometry of the problem.

1.2.2.3. Singular Integral Equations

Dual integral equations arising from the formulation of the crack problems may be reduced to a singular integral equation. The crack problems can also be formulated in terms of a system of singular integral equations by using the related Green's functions (e.g., dislocation and concentrated load solutions). This method has clear advantages in problems involving unusual stress singularities (Erdogan 1983).

1.3. Literature Overview

Hollow cylinders have extensively practical application in engineering. The fracture problem in pressure vessels, pipes and other cylindrical containers has developed rapidly because of various technical applications.

Here, some important examples of previous analytical studies related with the solution of the crack problems leading to the calculation of the stress intensity factors (SIFs) will be given .

(Gupta 1973) analyzed a semi-infinite strip held rigidly on its short end. Stress singularity at the strip corner is obtained from the singular integral equation. Stress along the rigid end is determined and the effect of the material properties on the stress intensity factor is presented. (Sneddon and Welch 1963) considered a long circular cylinder of elastic material containing a penny-shaped crack at the center of the cylinder and analyzed the distribution of stress in the problem. (Erdogan and Erdol 1978) studied an elastostatic axisymmetric problem for a long thick walled cylinder containing a ring shaped internal or edge crack .The problem is formulated in terms of a singular integral equation which has a simple Cauchy kernel for the internal crack and a generalized Cauchy kernel for the edge crack as the dominant part. In the paper by (Gupta 1974) the axisymmetric semi-infinite cylinder with fixed short end is considered. In the study applied loads are far away from the fixed end of the cylinder . In order to formulate the problem, integral transform method is used and a singular integral equation is obtained. (Delale and Erdogan 1982) analyzed a hollow cylinder problem . The cylinder contains an arbitrarily oriented radial crack and it is subjected to arbitrary normal tractions on the crack surfaces. The problem is formulated in terms of a singular integral equation by using the basic dislocation solutions as the Green's functions. A different solution technique has been given by (Benthem and Minderhood 1972) .The problem of a finite strip compressed between two rigid stamps is solved by using eigenfunction technique. There exist also recent papers related with hollow cylinder and stress intensity factor calculation with different boundary conditions. (Uyaner 2004) is considered a problem of ring shaped crack contained in an infinitely thick walled cylinder. The material of the cylinder is assumed to be transversely isotropic (transtropic) and the cylinder is under the action of uniform loading. The stress function is expressed in terms of governing equations. Hankel and Fourier Transform is used and the problem is reduced to a singular integral equation. The singular integral equation is solved by using the Gaussian Quadrature and the stress intensity factors are calculated.(Birinci 2002) analyzed the elastostatic axisymmetric problem for a long thick-walled cylinder containing an axisymmetric circumferential internal or edge crack with cladding at the inner surface of the cylinder. Integral transform techniques are used and the problem is formulated in terms of a singular integral equation. The integral equation is solved numerically by using the quadrature formulas. The stress intensity factors are calculated and influence of the geometrical configuration and the cladding on the SIFs is discussed. (Artem and Gecit 2002) considered the fracture of an axisymmetric hollow cylindrical bar containing rigid inclusions. The cylinder contains a ring shaped crack located at the $z = 0$ plane whose surfaces are free of tractions . The material of the hollow cylinder is to be linearly elastic and isotropic and the cylinder is under the action of uniform loading. Fourier and Hankel transform techniques are used and because of the mixed boundary condition of the problem, a system of three singular integral equation is analyzed and solved numerically. Finally, the normalized stress intensity factors are calculated for crack and two rigid inclusions

The main purpose of the present study is to investigate the stress intensity factors at the tips of the crack. The infinite hollow cylinder containing a ring-shaped crack at $z = 0$ plane (symmetry plane) is considered. The axisymmetric cylinder is under the action of tensile load at infinity and material of the cylinder is assumed to be linearly elastic and isotropic. Solution for the problem is obtained by means of the superposition of two subproblems: (i) an infinite hollow cylinder subjected to uniformly distributed tensile load at infinity, and (ii) an infinite hollow cylinder having a ring shaped crack (perturbation problem). The only load in problem (ii) is the negative of the stresses obtained in the problem (i) at location of the crack. Solution of the problem (i) is relatively simple and straightforward and also not related with the calculation of stress intensity factors. Therefore, in the present study only the perturbation problem will be solved. By using the Fourier and Hankel transform techniques, the general expressions for the displacement and stress components for the perturbation problem are obtained. Applying the boundary conditions on the rigid outer surface and stress free inner surface of the cylinder, a singular integral equation in terms of crack surface displacement derivative is derived. The singular integral equation is reduced to a system of linear algebraic equation by using Gauss-Lobatto quadrature formula. After that the linear algebraic equation system including improper integral is solved numerically by using Gauss-Laguerre integration formula**.** Finally, variations of normalized mode I stress intensity factors at the tips of the crack are calculated. Validation of the problem, comparison of the results are presented in tabular and graphical forms.

CHAPTER 2

PROBLEM DEFINITION AND FORMULATION

2.1. The Infinite Hollow Cylinder Problem

An Infinite hollow cylinder containing a ring-shaped crack of width (*b*-*a*) at the symmetry plane z = 0, is considered. The crack surfaces are free of tractions. The hollow cylinder is subjected to an axial tensile loads of uniform intensity p_0 at infinity. The outer wall of the cylinder is rigid while the inner wall is free of traction. Material of the cylinder is assumed to be linearly elastic and isotropic (See Figure 2.1).

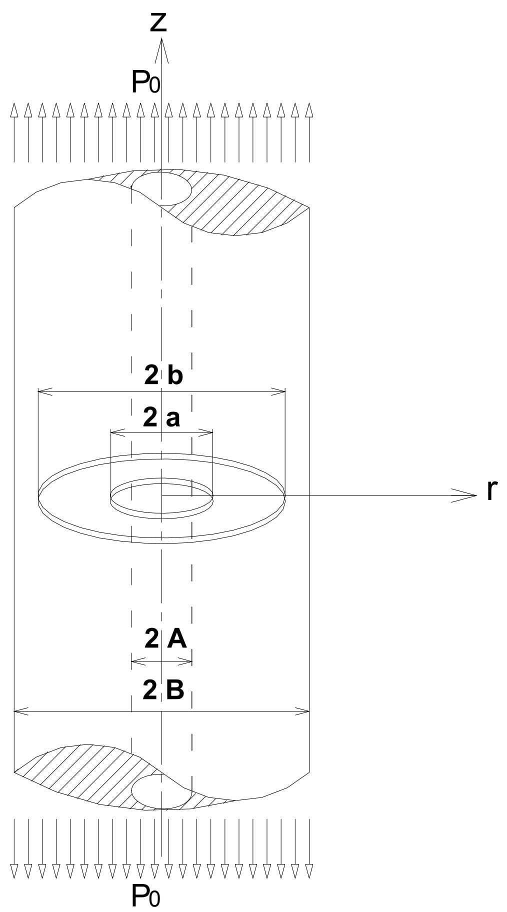

Figure 2.1. Geometry of the problem

Solution for the infinite cylinder having a crack of traction free surfaces and loaded at infinity can be obtained by superposition of the following two sub-problems as illustrated in Figure 2.2: (1) an infinite cylinder subjected to uniformly distributed tensile load of intensity p_0 at infinity with no crack and (2) an infinite cylinder having a ring-shaped crack. The only load in problem (2) is the negative of the stresses in problem (1) at locations of the crack (the perturbation problem). From the viewpoint of fracture mechanics, the relevant problem is the latter. Therefore, the perturbation problem in which the crack surface is subjected to prescribed tractions only is solved.

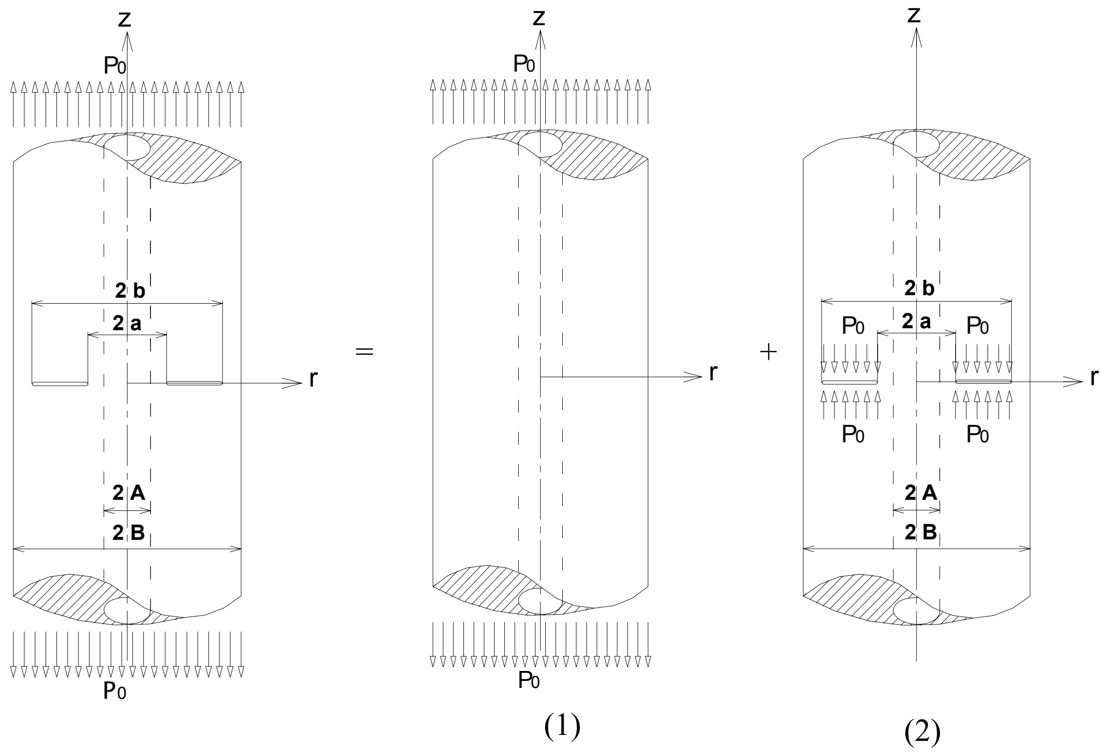

Figure 2.2. Superposition of uniform and crack solutions

For linearly elastic, isotropic and axisymmetric problems, the equilibrium equations can be written in the following form:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \tag{2.1}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\tau_{rz} = 0, \tag{2.2}$$

in which σ and τ denote the normal and the shearing stresses, respectively. The relation between stress and strain (*Generalized Hooke's Law*) in the body in tensor notation is

$$\sigma_{ij} = 2\mu \in_{ij} + \lambda \in_{kk} \delta_{ij} \tag{2.3}$$

Equation (2.3) can be written for the normal and shearing stress components as

$$\sigma_r \equiv \sigma_{rr} = 2\,\mu \in_{rr} + \lambda\,(\in_{rr} + \in_{\theta\theta} + \in_{zz})\delta_{rr}\,, \tag{2.4a}$$
$$\tau_{rz} \equiv \sigma_{rz} = 2\,\mu \in_{rz} + \lambda\,(\in_{rr} + \in_{\theta\theta} + \in_{zz})\delta_{rz}\,, \tag{2.4b}$$
$$\sigma_z \equiv \sigma_{zz} = 2\,\mu \in_{zz} + \lambda\,(\in_{rr} + \in_{\theta\theta} + \in_{zz})\delta_{zz}\,, \tag{2.4c}$$
$$\sigma_\theta \equiv \sigma_{\theta\theta} = 2\,\mu \in_{\theta\theta} + \lambda\,(\in_{rr} + \in_{\theta\theta} + \in_{zz})\delta_{\theta\theta}\,. \tag{2.4d}$$

where $\delta_{ij} = \begin{cases} 1 & for\ i = j \\ 0 & for\ i \neq j \end{cases}$, $\in_{kk} = \in_{rr} + \in_{\theta\theta} + \in_{zz}$, $\lambda = \mu \dfrac{2\nu}{1-2\nu}$ and μ is the shear modulus, ν being the Poisson's ratio.

Rearranging the equation (2.4), the following stress – strain relations can be obtained

$$\sigma_r \equiv \sigma_{rr} = 2\,\mu \in_{rr} + \lambda\,(\in_{rr} + \in_{\theta\theta} + \in_{zz})\,, \tag{2.5a}$$
$$\tau_{rz} \equiv \sigma_{rz} = 2\mu \in_{rz}\,, \tag{2.5b}$$
$$\sigma_z \equiv \sigma_{zz} = 2\,\mu \in_{zz} + \lambda\,(\in_{rr} + \in_{\theta\theta} + \in_{zz})\,, \tag{2.5c}$$
$$\sigma_\theta \equiv \sigma_{\theta\theta} = 2\,\mu \in_{\theta\theta} + \lambda\,(\in_{rr} + \in_{\theta\theta} + \in_{zz})\,. \tag{2.5d}$$

For the axisymmetric (θ independent) problem, the strain–displacement relations are as follows :

$$\in_{rr} = \frac{\partial u}{\partial r}\,, \qquad \in_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{2.6a,b}$$

$$\in_{zz} = \frac{\partial w}{\partial z}\,, \qquad \in_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\,, \tag{2.6c,d}$$

where u and w are displacements in r and z directions in cylindrical coordinate system, respectively.

Substituting expressions given in equations (2.6) into equations (2.5) stress-displacement relations for the axisymmetric cylindrical problem is obtained as

$$\sigma_r = \frac{\mu}{\kappa - 1}\left[(\kappa + 1)\frac{\partial u}{\partial r} + (3 - \kappa)\left(\frac{u}{r} + \frac{\partial w}{\partial z}\right)\right], \tag{2.7a}$$

$$\sigma_\theta = \frac{\mu}{\kappa - 1}\left[(\kappa + 1)\frac{u}{r} + (3 - \kappa)\left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}\right)\right], \tag{2.7b}$$

$$\sigma_z = \frac{\mu}{\kappa - 1}\left[(\kappa + 1)\frac{\partial w}{\partial z} + (3 - \kappa)\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right)\right], \tag{2.7c}$$

$$\tau_{rz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right), \tag{2.7d}$$

where, $\kappa = 3 - 4\nu$ for plain strain.

Substituting equations (2.7) into equations (2.1), the following second order partial differential equation system, named *the Navier Equations* can be obtained

$$(\kappa + 1)\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) + (\kappa - 1)\frac{\partial^2 u}{\partial z^2} + 2\frac{\partial^2 w}{\partial r \partial z} = 0, \tag{2.8a}$$

$$2\left(\frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r}\frac{\partial u}{\partial z}\right) + (\kappa - 1)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + (\kappa + 1)\left(\frac{\partial^2 w}{\partial z^2}\right) = 0. \tag{2.8b}$$

These equations must be solved subjected to the following boundary conditions

$$\sigma_z(r, \pm\infty) = p_0, \qquad (A \langle r \langle B), \tag{2.9a,b}$$

$$w(r, 0) = 0, \qquad (A \langle r \langle a, \quad b \langle r \langle B), \tag{2.9c}$$

$$\sigma_z(r, 0) = 0, \qquad (a \langle r \langle b), \tag{2.9d}$$

$$\sigma_r(A, z) = 0, \qquad (-\infty \langle z \langle \infty), \tag{2.9e}$$

$$\tau_{rz}(A, z) = 0, \qquad (-\infty \langle z \langle \infty), \tag{2.9f}$$

$$u\left(B,z\right)=0\,, \qquad \left(-\infty \langle\; z \;\langle\; \infty\right), \tag{2.9g}$$

$$w\left(B,z\right)=0\,, \qquad \left(-\infty \langle\; z \;\langle\; \infty\right). \tag{2.9h}$$

2.1.1. The Perturbation Problem

Solution for the infinite cylinder containing a ring-shaped crack may be obtained conveniently by the superposition of the solutions for the following two sub problems: (i) problem of an infinite elastic medium containing a ring shaped crack of width (*b-a*) at the symmetry plane and (ii) problem of an infinite medium without crack subjected to arbitrary symmetric loads. This superposition scheme is illustrated in Figure 2.3.

2.1.1.1. An Infinite Elastic Medium Having a Crack

Consider an infinite medium having a crack at $z = 0$ plane, z being the axis of the medium, whose surfaces are subjected to the opposite of the stresses at the locations of the crack obtained from the first problem. Clearly, it is sufficient to consider one half $(z \geq 0)$ of the medium only.

Using the Hankel transform definition

$$H_n\{q(ax);\zeta\} = \int_0^\infty x\, q(ax)\, J_n(x\zeta)dx, \qquad (a > 0) \tag{2.10}$$

where n = 0 for even and n = 1 for odd functions, and now considering the properties of odd and even functions, noticing that $u(r,z)$ is an odd and $w(r,z)$ is an even functions in r, Hankel transform of the displacement components can be written in the following form

$$H_1\{u(r,z);\alpha\} = \int_0^\infty u(r\,,z)\, r\, J_1(r\alpha)dr \;=U(\alpha,z) \tag{2.11a}$$

$$H_0\{w(r,z);\alpha\} = \int_0^\infty w(r\,,z)\, r\, J_0(r\alpha)dr \;=W(\alpha,z) \tag{2.11b}$$

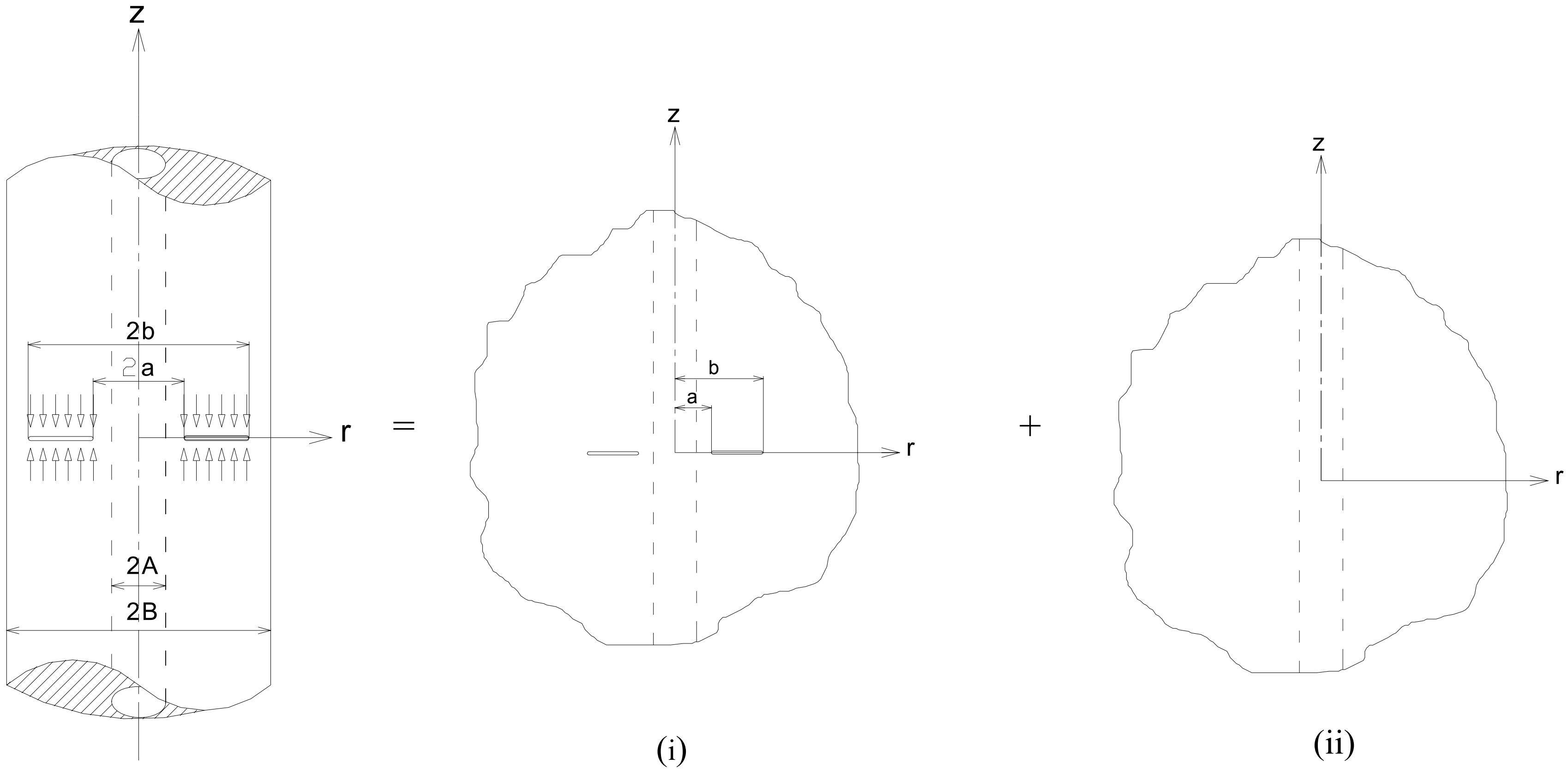

Figure 2.3. The informal superposition scheme (Perturbation problem)

where J_0 and J_1 are the Bessel functions of the first kind of order zero and one, respectively. Applying Hankel transform to equations (2.8) in r-direction

$$(\kappa+1)\mathrm{H}_1\{\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\}+(\kappa-1)\mathrm{H}_1\{\frac{\partial^2 u}{\partial z^2}\}+2\mathrm{H}_1\{\frac{\partial^2 w}{\partial r\partial z}\}=0, \qquad (2.12a)$$

$$2\mathrm{H}_0\{\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\frac{\partial u}{\partial z}\}+(\kappa-1)\mathrm{H}_0\{\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\}+(\kappa+1)\mathrm{H}_0\{\frac{\partial^2 w}{\partial z^2}\}=0, \qquad (2.12b)$$

and now by using the partial derivative properties of Hankel transform, one obtains

$$2\alpha\frac{dU}{dz}-\alpha^2(\kappa-1)W+(\kappa+1)\frac{d^2W}{dz^2}=0, \qquad (2.13a)$$

$$(\kappa+1)(-\alpha^2U)+(\kappa-1)\frac{d^2U}{dz^2}-2\alpha\frac{dW}{dz}=0. \qquad (2.13b)$$

After some routine manipulations, the equation system (2.13) can be reduced to the following fourth order ordinary differential equation

$$\frac{d^4U}{dz^4}-2\alpha^2\frac{d^2U}{dz^2}+\alpha^4U=0. \qquad (2.14)$$

The general solution of equation (2.14) is obtained as

$$U(\alpha,z)=(c_1+c_2z)e^{-\alpha z}+(c_3+c_4z)e^{\alpha z} \qquad (2.15)$$

where c_1, c_2, c_3 and c_4 are arbitrary unknown constants and $\alpha>0$.

To have the finite displacements at infinity, constants c_3 and c_4 must be equal to zero for the upper half - space , moreover constants c_1 and c_2 must be equal to zero for the lower half space. Therefore, considering the subscript 1 and 2, indicate the upper and the lower half-spaces, respectively, equation (2.15) can be written as in the following forms

$$U_1(\alpha,z) = (c_1 + c_2 z)e^{-\alpha z}, \qquad (z \ge 0), \qquad (2.16a)$$

$$U_2(\alpha,z) = (c_3 + c_4 z)e^{\alpha z}, \qquad (z \le 0). \qquad (2.16b)$$

Using the similar procedure for *W(α,z),* the solutions are obtained as follows

$$W_1(\alpha,z) = \left[(c_1 + c_2 z) + \frac{\kappa}{z}c_2\right]e^{-\alpha z}, \qquad (z \ge 0), \qquad (2.17a)$$

$$W_2(\alpha,z) = \left[-(c_3 + c_4 z) + \frac{\kappa}{z}c_4\right]e^{\alpha z}, \qquad (z \le 0). \qquad (2.17b)$$

Taking the inverse transforms of (2.16) and (2.17), displacement components are found to be

$$u_1(r,z) = \int_0^\infty (c_1 + c_2 z)e^{-\alpha z}\ \alpha J_1(\alpha r)d\alpha, \qquad (z \ge 0), \qquad (2.18a)$$

$$u_2(r,z) = \int_0^\infty (c_3 + c_4 z)e^{\alpha z}\alpha\ J_1(\alpha r)d\alpha, \qquad (z \le 0), \qquad (2.18b)$$

$$w_1(r,z) = \int_0^\infty \left[(c_1 + c_2 z) + \frac{1}{\alpha}\ \kappa c_2\right]e^{-\alpha z}\ \alpha\ J_0(\alpha r)d\alpha, \quad (z \ge 0), \qquad (2.18c)$$

$$w_2(r,z) = \int_0^\infty \left[-(c_3 + c_4 z) + \frac{\kappa}{\alpha}\ c_4\right]e^{\alpha z}\alpha\ J_0(\alpha r)d\alpha, \quad (z \le 0). \qquad (2.18d)$$

Substituting equations (2.18) into the expressions given in equations (2.7), one obtains the following expressions for the stress components

$$\sigma_{r1}(r,z)=\mu\int_0^\infty -2(c_1+c_2 z)e^{-\alpha z}\frac{\alpha}{r}\ J_1(\alpha r)d\alpha \qquad (z\le 0), \quad (2.19a)$$

$$+\mu\int_0^\infty [2\alpha(c_1+c_2 z)-(3-\kappa)c_2]e^{-\alpha z}\alpha J_0(\alpha r)d\alpha,$$

$$\sigma_{r2}(r,z)=\mu\int_0^\infty -2(c_3+c_4 z)e^{\alpha z}\frac{\alpha}{r}\ J_1(\alpha r)d\alpha \qquad (z\le 0), \quad (2.19b)$$

$$+\mu\int_0^\infty [2\alpha(c_3+c_4)+(3-\kappa)c_4]e^{az}\alpha\ J_0(\alpha r)d\alpha,$$

$$\sigma_{z1}(r,z)=\mu\int_0^\infty [-2\alpha(c_1+c_2 z)-(\kappa+1)c_2]e^{-\alpha z}\alpha\ J_0(\alpha r)d\alpha, \quad (z\ge 0), \quad (2.19c)$$

$$\sigma_{z2}(r,z)=\mu\int_0^\infty [-2\alpha(c_3+c_4 z)+(\kappa+1)c_4]e^{\alpha z}\alpha\ J_0(\alpha r)d\alpha, \quad (z\le 0), \quad (2.19d)$$

$$\tau_{rz1}(r,z)=\mu\int_0^\infty [-2\alpha(c_1+c_2 z)-(\kappa-1)c_2]e^{-\alpha z}\alpha\ J_1(\alpha r)d\alpha, \quad (z\ge 0), \quad (2.19e)$$

$$\tau_{rz2}(r,z)=\mu\int_0^\infty [2\alpha(c_3+c_4 z)-(\kappa-1)c_4]e^{\alpha z}\alpha\ J_1(\alpha r)d\alpha. \quad (z\le 0), \quad (2.19f)$$

These expressions can be matched on the z=0 plane by the following continuity and symmetry conditions

$$\sigma_{z_1}(r,0)=\sigma_{z_2}(r,0), \qquad (0\le r\le\infty), \quad (2.20a)$$
$$\tau_{rz_1}(r,0)=\tau_{rz_2}(r,0), \qquad (0\le r\le\infty), \quad (2.20b)$$
$$u_1(r,0)=u_2(r,0), \qquad (0\le r\le\infty), \quad (2.20c)$$
$$w_1(r,0)=w_2(r,0), \qquad (0\le r<a,\ \ b\le r<\infty). \quad (2.20d)$$

Conditions (2.20c) and (2.20d) may be replaced by

$$\frac{\partial}{\partial r}[u_1(r,0)-u_2(r,0)]=0, \qquad (0\le r\ \langle\ \infty), \quad (2.21a)$$
$$\frac{\partial}{\partial r}[w_1(r,0)-w_2(r,0)]=2f(r), \qquad (0\le r\ \langle\ \infty), \quad (2.21b)$$

in order to have the conditions of the same type (stress type). Here $f(r)$ is the unknown crack surface displacement derivative such that $f(r)=0$ when $(0 \le r < a,\ \ b \le r < \infty)$. Now using the conditions (2.20a, 2.20b) and (2.21), the unknown constants c_1-c_4 expressed in terms of $F(\alpha)$ become

$$c_1 = c_3 = \frac{(\kappa-1)\,F(\alpha)}{(\kappa+1)\ \alpha}, \tag{2.22a}$$

$$c_2 = -c_4 = -\frac{2\,F(\alpha)}{(\kappa+1)}, \tag{2.22b}$$

where $F(\alpha) = \int_a^b f(r)\,r\,J_1(\alpha\,r)dr$.

Because of that, the hollow cylinder having a ring shaped crack only, is symmetric about z axis, it is sufficient to consider the solution of the axisymmetric problem in the upper or lower half of the space. Therefore, in this study , the general expressions for the displacements and stress components in the upper half space as in terms of *F(α)* are considered in the form

$$u_{Hankel}(r,z) = \frac{1}{\kappa+1}\int_0^\infty \left[-2\alpha\ \ z + (\kappa-1)\right] F(\alpha) e^{-\alpha z} J_1(\alpha\, r) d\alpha, \tag{2.23a}$$

$$w_{Hankel}(r,z) = \frac{1}{\kappa+1}\int_0^\infty \left[-2\alpha\ \ z - (\kappa+1)\right] F(\alpha) e^{-\alpha z} J_0(\alpha\, r) d\alpha, \tag{2.23b}$$

$$\sigma_{rHankel}(r,z) = \frac{2\mu}{\kappa+1}\int_0^\infty \left[2\alpha\, z(\kappa-1)\right] F(\alpha) e^{-\alpha z} \frac{1}{r} J_1(\alpha\, r) d\alpha \tag{2.23c}$$

$$+ \frac{2\mu}{\kappa+1}\int_0^\infty 2(1-\alpha\ \ z) F(\alpha) e^{-\alpha z} \alpha\ \ J_0(\alpha r) d\alpha,$$

$$\sigma_{zHankel}(r,z) = \frac{4\mu}{\kappa+1}\int_0^\infty (\alpha\, z + 1)\, F(\alpha) e^{-\alpha z} \alpha\, J_0(\alpha\, r) d\alpha, \tag{2.23d}$$

. $$\tau_{rz\,Hankel}(r,z) = \frac{4\mu}{\kappa+1}\int_0^\infty \alpha\, z\, F(\alpha) e^{-\alpha z} \alpha\, J_1(\alpha\, r) d\alpha \tag{2.23e}$$

2.1.1.2. An Infinite Elastic Medium with no Crack

It is considered an infinite medium without crack which is loaded symmetrically. The infinite medium is symmetric about both z-axis and z = 0 plane. Using the Fourier sine and cosine transform definitions Fourier transforms of the displacement components can be written in the following form

$$F_c\{u(r,z);\lambda\} = \frac{2}{\pi}\int_0^{\infty} u(r,z)\cos(\lambda r)dr = U_c(r,\lambda), \tag{2.24a}$$

$$F_s\{w(r,z);\lambda\} = \frac{2}{\pi}\int_0^{\infty} w(r,z)\sin(\lambda r)dr = W_s(r,\lambda), \tag{2.24b}$$

where U_c and W_s are Fourier cosine and sine transform functions of u and w, the subscript s and c implies the sine and cosine transform respectively, λ is the Fourier transform variable. Then applying Fourier sine and cosine transforms in z direction to equation (2.8), noticing that $u(r,z)$ is even and $w(r,z)$ is an odd function in z, the following system of second order ordinary differential equation is obtained as :

$$(\kappa+1)\left[\frac{d^2U_c}{dr^2}+\frac{1}{r}\frac{dU_c}{dr}-\frac{U_c}{r^2}\right]-(\kappa-1)\lambda^2U_c+2\lambda\frac{dW_s}{dr}=0, \tag{2.25a}$$

$$-2\lambda\frac{dU_c}{dr}-\frac{2}{r}\lambda U_c+(\kappa-1)\left[\frac{d^2W_s}{dr^2}+\frac{1}{r}\frac{dW_s}{dr}\right]-(\kappa+1)\lambda^2W_s=0. \tag{2.25b}$$

In order to obtain relatively easier problem, after some algebraic manipulations, equation (2.25) can be reduced to a single equation as

$$r^4\frac{d^4U_c}{dr^4}+2r^3\frac{d^3U_c}{dr^3}-(2\lambda^2r^4+3r^2)\frac{d^2U_c}{dr^2}-(2\lambda^2r^3-3r)\frac{dU_c}{dr} \\ +(\lambda^4r^4+2\lambda^2r^2-3)U_c=0 \tag{2.26}$$

Let $\lambda r = x$, then the equation (2.26) becomes

$$x^4 \frac{d^4 U_c}{dx^4} + 2x^3 \frac{d^3 U_c}{dx^3} - (2x^4 + 3x^2)\frac{d^2 U_c}{dx^2} - (2x^3 - 3x)\frac{dU_c}{dx} + (x^4 + 2x^2 - 3)U_c = 0 \tag{2.27}$$

Solution of the equation (2.27) is explained below:
equation (2.27) firstly is considered as the product of two general second order ordinary differential operators in the following form

$$\left(x^2 \frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x) \right)\left(p_3(x)\frac{d^2}{dx^2} + p_4(x)\frac{d}{dx} + p_5(x) \right) U_c = 0 \tag{2.28}$$

When equation (2.28) is written in fourth-order form and compared to the equation (2.27), the equation system given below can be obtained

$$x^2 p_3(x) = x^4 , \tag{2.29a}$$

$$2x^2 \frac{dp_3}{dx} + x^2 p_4 + p_1 p_3 = 2x^3 , \tag{2.29b}$$

$$x^2 \frac{d^2 p_3}{dx^2} + 2x^2 \frac{dp_4}{dx} + x^2 p_5 + \frac{dp_3}{dx} p_1 + p_1 p_4 + p_2 p_3 = -2x^4 - 3x^2 , \tag{2.29c}$$

$$x^2 \frac{d^2 p_4}{dx^2} + 2x^4 \frac{dp_5}{dx} + \frac{dp_4}{dx} p_1 + p_5 p_1 + p_2 p_4 = -2x^3 + 3x \; , \tag{2.29d}$$

$$x^2 \frac{d^2 p_5}{dx^2} + \frac{dp_5}{dx} p_1 + p_2 p_5 = x^4 + 2x^2 - 3 , \tag{2.29e}$$

then solving the equation system (2.29), the functions $p_1(x) - p_5(x)$ can be found in simple polynomial forms. Therefore, equation (2.28) with these polynomials becomes

$$(x^2 \frac{d^2}{dx^2} - 3x\frac{d}{dx} - x^2 + 3)(x^2 \frac{d^2}{dx^2} + x\frac{d}{dx} - x^2 - 1)U_c = 0 \tag{2.30}$$

and hence the general solution of equation (2.30) is found to be (Mclachlan 1955)

$$U_C(r,\lambda) = -\frac{1}{2}c_1 I_1(\lambda\, r) + \frac{1}{2}c_2 K_1(\lambda\, r) + c_3\lambda\, r\ I_0(\lambda\, r) + c_4\lambda\, rK_0(\lambda\, r) \tag{2.31}$$

where I_0, K_0, I_1 and K_1 are the modified Bessel functions of the first and the second kinds of order zero and one, respectively and c_1, c_2, c_3 and c_4 are arbitrary constants. In this case, it is possible to get $W_s(r,\lambda\ )$, substituting equation (2.31) into the equation (2.25), in the following form

$$\begin{aligned} W_S(r,\ \lambda) = \frac{1}{2}c_1 I_0(\lambda\, r) + \frac{1}{2}c_2 K_0(\lambda\, r) - c_3\left[(\kappa+1)I_0(\lambda\, r) + \lambda\, rI_1(\lambda\, r)\right] \\ - c_4\left[(\kappa+1)K_0(\lambda\, r) - \lambda\, rK_1\,(\lambda\, r)\right] \end{aligned} \tag{2.32}$$

Taking the inverse Fourier sine and cosine transforms of equations (2.31) and (2.32), expressions for the displacement components are found to be

$$u(r,z)_{Fourier} = \frac{2}{\pi}\int_0^\infty \left[-\frac{1}{2}c_1 I_1(\lambda\, r) + \frac{1}{2}c_2 K_1(\lambda\, r) + c_3\lambda\, rI_0(\lambda\, r) + c_4\lambda\, rK_0(\lambda\, r)\right]\cos\lambda\, z d\lambda, \tag{2.33a}$$

$$\begin{aligned} w(r,z)_{Fourier} = \frac{2}{\pi}\int_0^\infty \Big\{\frac{1}{2}c_1 I_0(\lambda\, r) + \frac{1}{2}c_2 K_0(\lambda\, r) - c_3\left[(\kappa+1)I_0(\lambda\, r) + \lambda\, rI_1(\lambda\, r)\right] \\ + c_4\left[(\kappa+1)K_0(\lambda r) - \lambda r K_1(\lambda r)\right]\Big\}\sin\lambda z d\lambda\,. \end{aligned} \tag{2.33b}$$

Therefore , in order to obtain the stress components one can use stress-displacement relations given by equation (2.7). These are as follows:

$$\sigma_r(r,z)_{Fourier} = \frac{2\mu}{\pi}\int_0^{\infty}\left\{c_1\left[-\lambda I_0(\lambda r)+\frac{1}{r}I_1(\lambda r)\right]+c_2\left[-\lambda K_0(\lambda r)-\frac{1}{r}K_1(\lambda r)\right]\right. \tag{2.34a}$$
$$+c_3\left[(\kappa-1)\lambda I_0(\lambda r)+2\lambda^2 r I_1(\lambda r)\right]$$
$$\left.+c_4\left[(\kappa-1)\lambda K_0(\lambda r)-2\lambda^2 r K_1(\lambda r)\right]\right\}\cos\lambda z d\lambda,$$

$$\sigma_z(r,z)_{Fourier} = \frac{2\mu}{\pi}\int_0^{\infty}\left\{c_1\lambda I_0(\lambda r)+c_2\lambda K_0(\lambda r)-c_3\left[(\kappa+5)\lambda I_0(\lambda r)+2\lambda^2 r I_1(\lambda r)\right]\right. \tag{2.34b}$$
$$\left.-c_4\left[(\kappa+5)\lambda K_0(\lambda r)-2\lambda^2 r K_1(\lambda r)\right]\right\}\cos\lambda z d\lambda,$$

$$\tau_{rz}(r,z)_{Fourier} = \frac{2\mu}{\pi}\int_0^{\infty}\left\{c_1\lambda I_1(\lambda r)-c_2\lambda K_1(\lambda r)-c_3\left[2\lambda^2 r I_0(\lambda r)+(\kappa+1)\lambda I_1(\lambda r)\right]\right. \tag{2.34c}$$
$$\left.-c_4\left[2\lambda^2 r K_0(\lambda r)-(\kappa+1)\lambda K_1(\lambda r)\right]\right\}\sin\lambda z d\lambda.$$

2.1.2. General Solution

The expressions for displacement and stress components obtained in Hankel and Fourier solutions of the problem (Sections 2.1.1.1. and 2.1.1.2.), will be added together for the solution of the perturbation problem. Therefore the general solutions become

$$u(r,z) = u_{Hankel} + u_{Fourier}, \tag{2.35a}$$
$$w(r,z) = w_{Hankel} + w_{Fourier}, \tag{2.35b}$$
$$\sigma_z(r,z) = \sigma_{zHankel} + \sigma_{zFourier}, \tag{2.35c}$$
$$\sigma_r(r,z) = \sigma_{rHankel} + \sigma_{rFourier}, \tag{2.35d}$$
$$\tau_{rz}(r,z) = \tau_{rzHankel} + \tau_{rzFourier}. \tag{2.35e}$$

Now the arbitrary unknown constants c_1-c_4 appearing in Fourier solution of the problem can be written in terms of unknown function $F(\alpha)$ using the conditions given below at inner and outer lateral surfaces of the cylinder,

$$u(B,z) = 0, \tag{2.36a}$$
$$w(B,z) = 0, \tag{2.36b}$$
$$\sigma_r(A,z) = 0, \tag{2.36c}$$

$$\tau_{rz}(A,z)=0\,. \tag{2.36d}$$

.

Therefore, substituting equations (2.23), (2.33) and (2.34) into the boundary conditions of the problem at inner and outer lateral surfaces, one can obtain the following system of equation

$$-\frac{1}{2}c_1 I_1(\lambda B)+\frac{1}{2}c_2 K_1(\lambda B)+c_3\,\lambda B I_0(\lambda B)+c_4\,\lambda B K_0(\lambda B) \tag{2.37a}$$
$$=-\frac{1}{\kappa+1}\int_0^\infty\left\{\int_0^\infty\left[-2\alpha z+(\kappa-1)\right]F(\alpha)e^{-\alpha z}J_1(\alpha B)d\alpha\right\}\cos\lambda z dz\,,$$

$$\begin{aligned}&-\frac{1}{2}c_1 I_0(\lambda B)+\frac{1}{2}c_2 K_0(\lambda B)+c_3\left[(\kappa+1)I_0(\lambda B)+\lambda B I_1(\lambda B)\right]\\&-c_4\left[(\kappa+1)K_0(\lambda B)-\lambda B K_1(\lambda B)\right]\end{aligned} \tag{2.37b}$$
$$=-\frac{1}{\kappa+1}\int_0^\infty\left\{\int_0^\infty\left[-2\alpha z-(\kappa-1)\right]F(\alpha)e^{-\alpha z}J_0(\alpha B)d\alpha\right\}\sin\lambda z dz\,,$$

$$\begin{aligned}&c_1\lambda I_1(\lambda A)-c_2\lambda K_1(\lambda A)-c_3\left[(\kappa+1)\lambda I_1(\lambda A)+2A\lambda^2 I_0(\lambda A)\right]\\&\qquad-c_4\left[-(\kappa+1)\lambda K_1(\lambda A)+2A\lambda^2 K_0(\lambda A)\right]\end{aligned} \tag{2.37c}$$
$$=-\frac{4}{\kappa+1}\int_0^\infty\left[\int_0^\infty F(\alpha)\alpha^2 J_1(A\alpha)d\alpha\right]ze^{-\alpha z}\sin\lambda z dz\,,$$

$$c_1\left[-\lambda I_0(\lambda A)+\frac{1}{A}I_1(\lambda A)\right]+c_2\left[-\lambda K_0(\lambda A)-\frac{1}{A}K_1(\lambda A)\right] \tag{2.37d}$$
$$+c_3\left[(\kappa-1)\lambda I_0(\lambda A)+2\lambda^2 A I_1(\lambda A)\right]+c_4\left[(\kappa-1)\lambda K_0(\lambda A)-2\lambda^2 A K_1(\lambda A)\right]$$
$$=-\frac{2}{\kappa+1}\int_0^\infty\left[\int_0^\infty\left\{\left[2\alpha z-(\kappa-1)\right]\frac{1}{A}J_1(A\alpha)+2\alpha(1-\alpha z)J_0(A\alpha)\right\}e^{-\alpha z}F(\alpha)d\alpha\right]\cos\lambda z dz$$

After some manupulations, double integral form in system of equation (2.37) can be reduced to a single integration by using the integral formulas given in Appendix A, as

$$-\frac{1}{2}c_1 I_1(\lambda B)+\frac{1}{2}c_2 K_1(\lambda B)+c_3\,\lambda B I_0(\lambda B)+c_4\,\lambda B K_0(\lambda B) \tag{2.38a}$$

$$
= -\frac{1}{\kappa+1}\left\langle \int_0^\infty \left[\frac{-2\alpha(\alpha^2-\lambda^2)+(\kappa-1)\alpha(\lambda^2+\alpha^2)}{(\lambda^2+\alpha^2)^2} \right] F(\alpha) J_1(\alpha B)\, d\alpha \right\rangle ,
$$

$$
-\frac{1}{2}c_1 I_0(\lambda B) + \frac{1}{2}c_2 K_0(\lambda B) + c_3\left[(\kappa+1)I_0(\lambda B)+\lambda B I_1(\lambda B)\right] \tag{2.38b}
$$

$$
-c_4\left[(\kappa+1)K_0(\lambda B) - \lambda B K_1(\lambda B)\right]
$$

$$
= -\frac{1}{\kappa+1}\left\langle \int_0^\infty \left[\frac{-4\alpha^2\lambda-(\kappa+1)\lambda(\lambda^2+\alpha^2)}{(\lambda^2+\alpha^2)^2} \right] F(\alpha) J_0(\alpha B)\, d\alpha \right\rangle ,
$$

$$
c_1\lambda I_1(\lambda A) - c_2\lambda K_1(\lambda A) - c_3\left[(\kappa+1)\lambda I_1(\lambda A) + 2A\lambda^2 I_0(\lambda A)\right]
$$

$$
-c_4\left[-(\kappa+1)\lambda K_1(\lambda A) + 2A\lambda^2 K_0(\lambda A)\right] \tag{2.38c}
$$

$$
= -\frac{2}{\kappa+1}\left(\frac{1}{A}\left\langle \int_0^\infty \left[\frac{2\alpha(\alpha^2-\lambda^2)-(\kappa-1)\alpha(\lambda^2+\alpha^2)}{(\lambda^2+\alpha^2)^2} \right] F(\alpha) J_1(\alpha A)\, d\alpha \right\rangle \right.
$$

$$
\left. +\, 2\int_0^\infty \left[\frac{\alpha^2(\lambda^2+\alpha^2)-\alpha^2(\alpha^2-\lambda^2)}{(\lambda^2+\alpha^2)^2} \right] F(\alpha) J_0(A\alpha) d\alpha \right),
$$

$$
c_1\left[-\lambda I_0(\lambda A) + \frac{1}{A} I_1(\lambda A)\right] + c_2\left[-\lambda K_0(\lambda A) - \frac{1}{A}K_1(\lambda A)\right] \tag{2.38d}
$$

$$
+c_3\left[(\kappa-1)\lambda I_0(\lambda A) + 2\lambda^2 A I_1(\lambda A)\right] + c_4\left[(\kappa-1)\lambda K_0(\lambda A) - 2\lambda^2 A K_1(\lambda A)\right]
$$

$$
= -\frac{8}{\kappa+1}\int_0^\infty \frac{\lambda\alpha^3}{\left(\lambda^2+\alpha^2\right)^2} F(\alpha) J_1(A\alpha) d\alpha .
$$

Equation (2.38) can now be rewritten in the form

$$
-\frac{1}{2}c_1 I_1(\lambda B) + \frac{1}{2}c_2 K_1(\lambda B) + c_3\,\lambda B I_0(\lambda B) + c_4\,\lambda B K_0(\lambda B) = S_1 , \tag{2.39a}
$$

$$
-\frac{1}{2}c_1 I_0(\lambda B) + \frac{1}{2}c_2 K_0(\lambda B) + c_3\left[(\kappa+1)I_0(\lambda B) + \lambda B I_1(\lambda B)\right]
$$

$$
-c_4\left[(\kappa+1)K_0(\lambda B) - \lambda B K_1(\lambda B)\right] = S_2 , \tag{2.39b}
$$

$$
c_1\lambda I_1(\lambda A) - c_2\lambda K_1(\lambda A) - c_3\left[(\kappa+1)\lambda I_1(\lambda A) + 2A\lambda^2 I_0(\lambda A)\right]
$$

$$
-c_4\left[-(\kappa+1)\lambda K_1(\lambda A) + 2A\lambda^2 K_0(\lambda A)\right] = S_3 , \tag{2.39c}
$$

$$
c_1\left[-\lambda I_0(\lambda A) + \frac{1}{A} I_1(\lambda A)\right] + c_2\left[-\lambda K_0(\lambda A) - \frac{1}{A}K_1(\lambda A)\right] \tag{2.39d}
$$

$$
+c_3\left[(\kappa-1)\lambda I_0(\lambda A) + 2\lambda^2 A I_1(\lambda A)\right] + c_4\left[(\kappa-1)\lambda K_0(\lambda A) - 2\lambda^2 A K_1(\lambda A)\right] = S_4
$$

where S_1-S_4 are given in Appendix B, $c_1 - c_4$ are unknown constants. These linear algebraic system of equation are solvable and c_1-c_4 can be obtained in terms of S_1-S_4 in the following forms

$$c_1 = \lfloor c_{11}\, S_3 + c_{12}\, S_4 + c_{13}\, S_2 + c_{14}\, S_1 \rfloor / D, \tag{2.40a}$$

$$c_2 = \lfloor c_{21}\, S_3 + c_{22}\, S_4 + c_{23}\, S_2 + c_{24}\, S_1 \rfloor / D, \tag{2.40b}$$

$$c_3 = \lfloor c_{31}\, S_3 + c_{32}\, S_4 + c_{33}\, S_2 + c_{34}\, S_1 \rfloor / D, \tag{2.40c}$$

$$c_4 = \lfloor c_{41}\, S_3 + c_{42}\, S_4 + c_{43}\, S_2 + c_{44}\, S_1 \rfloor / D, \tag{2.40d}$$

where c_{11}-c_{44} and D are given in Appendix C.

The boundary conditions on the lateral surfaces of the cylinder have already been used in finding expressions for c_1-c_4. Now, the unknown function F(α) can be determined by using the remaining boundary condition, $\sigma_z(r,0) = -p_0$, on the crack.

CHAPTER 3

INTEGRAL EQUATIONS

3.1. Derivation of Integral Equation

Substituting equations (2.34b) and (2.23d) into the equation (2.35c) one obtains

$$\begin{aligned}\sigma_z(r,z) = {} & \frac{4\mu}{\kappa+1}\int_0^\infty (\alpha z+1)\, F(\alpha) e^{-\alpha z} \alpha\, J_0(\alpha r) d\alpha + \frac{2\mu}{\pi}\int_0^\infty \big[c_1 \lambda\, I_0(\lambda r) + c_2 \lambda\, K_0(\lambda r) \\ & - c_3((\kappa+5)\lambda\, I_0(\lambda r) + 2\lambda^2 r I_1(\lambda r)) - c_4((\kappa+5)\lambda\, K_0(\lambda r) - 2\lambda^2 r K_1(\lambda r))\ \big] \\ & \times \cos\lambda\, z d\lambda \end{aligned} \tag{3.1}$$

By using the remaining boundary condition $\sigma_z(r,0) = -p_0$ to the equation (3.1), the following integral form for the normal stress σ_z at $z = 0$ can be obtained

$$\begin{aligned}\sigma_z(r,0) = {} & \frac{4\mu}{\kappa+1}\int_0^\infty F(\alpha)\alpha\, J_0(\alpha r) d\alpha + \frac{2\mu}{\pi}\int_0^\infty \big[c_1 \lambda\, I_0(\lambda r) + c_2 \lambda\, K_0(\lambda r) \\ & - c_3((\kappa+5)\lambda\, I_0(\lambda r) + 2\lambda^2 r I_1(\lambda r)) - c_4((\kappa+5)\lambda\, K_0(\lambda r) \\ & - 2\lambda^2 r K_1(\lambda r))\ \big] d\lambda = -p_0 \end{aligned} \tag{3.2}$$

Now substituting equation (2.40) in equation (3.2), the equation can be written in the form

$$\begin{aligned}\sigma_z(r,0) = {} & \frac{4\mu}{\kappa+1}\int_0^\infty F(\alpha)\alpha\, J_0(\alpha r) d\alpha + \frac{2\mu}{\pi}\int_0^\infty \Big(\frac{1}{D}\big[(c_{11} S_3 + c_{12} S_4 + c_{13} S_2 + c_{14} S_1) \\ & \times \lambda\, I_0(\lambda r) + (c_{21} S_3 + c_{22} S_4 + c_{23} S_2 + c_{24} S_1)\, \lambda\, K_0(\lambda r) \\ & - (c_{31} S_3 + c_{32} S_4 + c_{33} S_2 + c_{34} S_1) \times ((\kappa+5)\lambda\, I_0(\lambda r) + 2\lambda^2 r I_1(\lambda r)) \\ & - (c_{41} S_3 + c_{42} S_4 + c_{43} S_2 + c_{44} S_1) \times ((\kappa+5)\lambda\, K_0(\lambda r) - 2\lambda^2 r K_1(\lambda r))\ \big]\Big) d\lambda = -p_0 \end{aligned} \tag{3.3}$$

Changing the order of integration in Fourier part of the equation (3.3) and rearranging the terms, equation (3.3) becomes:

$$
\begin{aligned}
\sigma_z(r,0) = {} & \frac{4\mu}{\kappa+1}\int_0^\infty F(\alpha)\alpha J_0(\alpha r)d\alpha + \frac{2\mu}{\pi(\kappa+1)}\Big\{\int_a^b f(t)t\Big(\int_0^\infty \frac{1}{D}[a_1(\beta_{11}\lambda I_0(\lambda r) \\
& +\beta_{21}\lambda K_0(\lambda r) - \beta_{31}2\lambda^2 rI_1(\lambda r) + \beta_{41}2\lambda^2 rK_1(\lambda r)) + d_1(-8\beta_{12}\lambda I_0(\lambda r) \\
& -8\beta_{22}\lambda K_0(\lambda r) + 8\beta_{32}2\lambda^2 r I_1(\lambda r) - 8\beta_{42}2\lambda^2 rK_1(\lambda r)) + b_1(-\beta_{13}\lambda I_0(\lambda r) \\
& -\beta_{23}\lambda K_0(\lambda r) + \beta_{33}2\lambda^2 rI_1(\lambda r) - \beta_{43}2\lambda^2 rK_1(\lambda r)) + g_1(-\beta_{14}\lambda I_0(\lambda r) \\
& -\beta_{24}\lambda K_0(\lambda r) + \beta_{34}2\lambda^2 rI_1(\lambda r) - \beta_{44}2\lambda^2 rK_1(\lambda r))]d\lambda\Big)dt\Big\}
\end{aligned}
\tag{3.4}
$$

where $\beta_{11} - \beta_{44}$ are given in Appendix C.

After some lengthy but straightforward algebraic manipulations, equation (3.4) gives the following singular integral equation with kernel having Cauchy-type singularity (Muskhelishvili 1953).

$$
\frac{2\mu}{\pi(\kappa+1)}\int_a^b f(t)\left[\frac{2}{t-r} + 2M_1(r,t) + tN_{11}(r,t)\right]dt = -p_0\,, \quad (a < r < b)\,, \tag{3.5}
$$

where

$$
M_1(r,t) = \frac{M_1^*(r,t)-1}{t-r} \tag{3.6}
$$

$$
M_1^*(r,t) = \begin{cases} \dfrac{2(t-r)}{r}K\left(\dfrac{t}{r}\right) + \dfrac{2r}{t+r}E\left(\dfrac{t}{r}\right) & (r\rangle t) \\[2ex] \dfrac{2t}{t+r}E\left(\dfrac{r}{t}\right) & (r\langle t) \end{cases} \tag{3.7}
$$

in which K and E are the complete elliptic integrals of the first and the second kinds, respectively. Equation (3.5) must be solved under single-valuedness condition for the displacement around the crack given below :

$$\int_a^b f(t)dt = 0 \tag{3.8}$$

The integral equation (equation (3.5)) has three types of singularities :

1) A simple Cauchy-type singularity at $t = r$,
2) Logarithmic singularity in the kernel M_1,
3) N_{11} has singular terms when $t=A$, B and $r = \pm A$, $\pm B$ due to the behavior of the integrand of the integral N_{11} as $\lambda \to \infty$.

In this case, $N_{11}(r,t)$ can be written in the following form

$$N_{11}(r,t) = \int_0^\infty L_{11}(r,t,\lambda)d\lambda \tag{3.9}$$

Then the singular part of the kernel may be separated as

$$N_{11s}(r,t) = \int_0^\infty L_{11\infty}(r,t,\lambda)d\lambda \tag{3.10}$$

where

$$L_{11\infty}(r,t,\lambda) = \lim_{\lambda\to\infty} L_{11}(r,t,\lambda) \tag{3.11}$$

Integrand of integral given by Equation (3.5) contains modified Bessel functions I_0, K_0, I_1 and K_1. By using asymptotic expansions for modified Bessel functions, given in Appendix D, and after some manipulations, $L_{11\infty}(r,t,\lambda)$ can be obtained in the following form

$$\begin{aligned} L_{11\infty}(r,t,\lambda) = \frac{1}{\sqrt{rt}} \Big\{ & e^{-\lambda(2B-r-t)}\left[\frac{1}{\kappa}(-4(B-r)(B-t)\lambda^2 - 2(B-r)\lambda + 6(B-t)\lambda + (\kappa^2+3))\right] \\ & + e^{\lambda(2A-r-t)}\left[4(A-r)(A-t)\lambda^2 + 2(A-r)\lambda + 6(A-t)\lambda + 4\right]\Big\} \end{aligned} \tag{3.12}$$

The singular part of the kernel can be obtained by integrating $L_{11\infty}(r,t,\lambda)$ with the formulae given in Appendix E as

$$N_{11s}(r,t)=\frac{1}{\sqrt{rt}}\left\{\left[\frac{1}{\kappa}(-4(B-r)^2\frac{d^2}{dr^2}-12(B-r)\frac{d}{dr}+(3-\kappa^2)\right]\frac{1}{(t+r-2B)}\right.$$
$$\left.+\left[-4(A-r)\frac{d^2}{dr^2}+12(A-r)\frac{d}{dr}-2\right]\frac{1}{(t+r-2A)}\right\} \qquad (3.13)$$

therefore, the bounded part of the kernel will be

$$N_{11b}(r,t)=\int_0^\infty [L_{11}(r,t,\lambda)-L_{11\infty}(r,t,\lambda)]d\lambda \qquad (3.14)$$

Then, the kernel $N_{11}(r,t)$ may be written in the following form

$$N_{11}(r,t)=N_{11s}(r,t)+N_{11b}(r,t)\,. \qquad (3.15)$$

Now, Equation (3.5) can be written as

$$\frac{2\mu}{\pi(\kappa+1)}\int_a^b f(t)\left[\frac{2}{t-r}+tN_{11s}(r,t)\right]dt=B_1(r), \qquad (a\langle r\langle b), \quad (3.16)$$

where $B_1(r)$ contains all the bounded terms in Equation (3.5).

Singular behavior of the unknown function $f(t)$ may be determined by writing

$$f(t)=G(t)[(t-a)(b-t)]^{-\gamma} \qquad (0\langle \mathrm{Re}(\gamma)\langle 1) \qquad (3.17)$$

where $G(t)$ is Hölder-continuous function in the interval $[a, b]$ and γ is an unknown constant. $G(t)$ has an integrable singularity at the edges of the crack. Evaluating the integral containing singular term $\frac{1}{t-r}$ using the technique given in (Muskhelishvili 1953)

$$\frac{1}{\pi}\int_a^b \frac{f(t)}{t-r}dt = \frac{G(a)\cot\pi\gamma}{[(b-a)(r-a)]^{\gamma}} - \frac{G(b)\cot\pi\gamma}{[(b-a)\ (b-r)]^{\gamma}} + G^*(r) \qquad (3.18)$$

where $G^*(r)$ is bounded everywhere except at the end points a,b and substituting equation (3.18) in equation (3.16) following complex function technique outlined in (Muskhelishvili 1953) and using the procedure described in (Cook and Erdogan 1972), one may obtain the following characteristic equation for γ

$$\cot\pi\gamma = 0 \qquad (3.19)$$

Therefore $\gamma = 1/2$ is obtained as the power of stress singularity at the tips of the crack $(r \to a,b)$ and also satisfy the equation (3.18).

3.2. Solution of Integral Equation

Having determined the singular behavior of the unknown function, the integral appearing in equation (3.5) may be non-dimensionalized by introducing the following dimensionless variables τ,ξ for the crack

$$t = \frac{b-a}{2}\tau + \frac{b+a}{2}\ , \qquad (a\langle t\langle b,\quad -1\langle\tau\langle 1), \qquad (3.20a)$$

$$r = \frac{b-a}{2}\xi + \frac{b+a}{2}, \qquad (a\langle r\langle b,\quad -1\langle\xi\langle 1), \qquad (3.20b)$$

and the singular integral equation (Equation (3.5)) becomes

$$\frac{1}{\pi}\int_{-1}^{1} f\left(\frac{b-a}{2}\tau+\frac{b+a}{2}\right)\left[\frac{2}{\tau-\xi}+(b-a)M_1(\xi,\tau)+\left(\frac{b-a}{2}\right)\left(\frac{b-a}{2}\tau+\frac{b+a}{2}\right)N_{11}(\xi,\tau)\right]d\tau$$
$$=-\frac{p_0(\kappa+1)}{2\mu} \tag{3.21}$$

After some manipulations the singular integral equation given by (3.21), may be obtained in the form:

$$\frac{1}{\pi}\int_{-1}^{1}\overline{f}(\tau)\left[\frac{2}{\tau-\xi}+\hat{M}_1(\xi,\tau)+\hat{N}_{11}(\xi,\tau)\right]d\tau=-\frac{p_0(\kappa+1)}{2\mu} \tag{3.22}$$

where

$$\overline{f}(\tau)=f\left(\frac{b-a}{2}\tau+\frac{b+a}{2}\right) \tag{3.23}$$

$$\hat{M}_1(\xi,\tau)=(b-a)M_1(\xi,\tau) \tag{3.24}$$

$$\hat{N}_{11}(\xi,\tau)=\left(\frac{b-a}{2}\right)\left(\frac{b-a}{2}\tau+\frac{b-a}{2}\right)N_{11}(\xi,\tau) \tag{3.25}$$

Substituting singular behavior of the dimensionless unknown function

$$\overline{f}(\tau)=\overline{G}(\tau)\left(1-\tau^2\right)^{-1/2} \tag{3.26}$$

where

$$\overline{G}(\tau)=G\left(\frac{b-a}{2}\tau+\frac{b+a}{2}\right)\left(\frac{b-a}{2}\right)^{-1} \tag{3.27}$$

in equation (3.22), one can obtain the following integral equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\overline{G}(\tau)}{\sqrt{1-\tau^2}}\left[\frac{2}{\tau-\xi}+\hat{M}_1(\xi,\tau)+\hat{N}_{11}(\xi,\tau)\right]d\tau=-\frac{p_0(\kappa+1)}{2\mu} \tag{3.28}$$

By using the Gauss-Lobatto integration formula given in Appendix F, equation (3.28) can be reduced to an algebraic system given below:

$$\sum_{i=1}^{n} C_i\overline{G}(\tau_i)\left[\frac{2}{\tau_i-\xi_j}+\hat{M}_1(\xi_j,\tau_i)+\hat{N}_{11}(\xi_j,\tau_i)\right]d\tau=-\frac{p_0(\kappa+1)}{2\mu} \tag{3.29}$$

where

$$\tau_i=\cos[(i-1)\pi/(n-1)], \qquad (\mathrm{i}=1,2,\ldots\ldots\ldots,\mathrm{n}), \tag{3.30a}$$

$$\xi_j=\cos[(2j-1)\pi/2(n-1)], \quad (j=1,2,\ldots\ldots\ldots\ldots,n-1). \tag{3.30b}$$

are the roots and the weighting constants of related Lobatto polynomials are

$$C_1=C_n=\frac{1}{2(n-1)}, \qquad C_i=\frac{1}{n-1}, \ (\mathrm{i}=2,3,\ldots.,\mathrm{n}-1). \tag{3.31a-c}$$

Now, as it can be seen easily, the algebraic system given in equation (3.29) has *n* unknowns, $\overline{G}(\tau_i)$ and (*n-1*) equation. Since the number of unknowns is larger than the number of equations, the single valuedness condition, equation (3.8), must be taken into consideration to have *n*-equations for *n*-unknowns. Hence, equation (3.8) become

$$\sum_{i=1}^{n} C_i \overline{G}(\tau_i) = 0, \qquad \left(-1 \langle \tau \langle 1 \right). \tag{3.32}$$

Infinite integral appearing in equation (3.29) can be calculated numerically by using Laguerre (see Appendix F) integration method for each τ_i, ξ_j value. After determining unknowns $\overline{G}(\tau_i)$ at discrete collocation points the field quantities can be computed numerically. Behavior of the unknown function at the tips of the crack, $\tau = \pm 1$, is characterized by the so-called "stress intensity factor" which is particularly important from the viewpoint of fracture mechanics.

3.3. Stress Intensity Factors

In crack problems stresses become infinite at the tips of the crack. Therefore, the stress state at close vicinity of these points wil be presented by means of the stress intensity factor.

3.3.1. Stress Intensity Factors at the Tips of the Crack

Because of the nature of the problem, it is only focused on Mode I stress intensity factor calculations and investigation in this study. Mode I stress intensity factor at the tips of the crack has been defined in the following form (Erdol and Erdogan 1978)

$$k_1(a) = \lim_{r \to a} \sqrt{2(a-r)}\sigma_z(r,0), \tag{3.33a}$$

$$k_1(b) = \lim_{r \to b} \sqrt{2(r-b)}\sigma_z(r,0), \tag{3.33b}$$

in which $\sigma_z(r,0)$ can be expressed by means of equation (3.5) in the form

$$\sigma_z(r,0) = \frac{4\mu}{\pi(\kappa+1)} \int_a^b \frac{f(t)}{t-r} dt + \sigma_{zb}(r,0), \tag{3.34}$$

where σ_{zb} is the bounded part of the cleavage stress

$$\sigma_{zb}(r,0) = \frac{2\mu}{\pi(\kappa+1)} \int_a^b f(t)\left[2M_1(r,t) + tN_{11}(r,t)\right] dt. \tag{3.35}$$

Now, considering

$$f(t) = \frac{f^*(t)}{\sqrt{(t-a)(b-t)}} = \begin{cases} \dfrac{f^*(t)/\sqrt{b-t}}{\sqrt{t-a}} & near \quad t = a, \\ \dfrac{e^{\pi i/2} f^*(t)/\sqrt{t-a}}{\sqrt{b-t}} & near \quad t = b, \end{cases} \tag{3.36}$$

the integral of the sectionally holomorphic function in equation (3.34) can be evaluated by the method given in (Muskhelishvili 1953)

$$\frac{1}{\pi} \int_a^b \frac{f(t)}{t-r} dt = \frac{e^{\pi i/2}}{\sin \pi/2} \frac{f^*(a)}{\sqrt{b-a}} \frac{1}{\sqrt{r-a}} - \frac{1}{\sin \pi/2} \frac{f^*(b)}{\sqrt{b-a}} \frac{1}{\sqrt{r-b}} + G^*(r) \tag{3.37}$$

where $G^*(r)$ is bounded function for $a \langle r \langle b$. When r approaches a, second part of equation (3.37) will be bounded and therefore equation (3.37) becomes

$$\frac{1}{\pi} \int_a^b \frac{f(t)}{t-r} dt = \frac{f^*(a)}{\sqrt{b-a}\sqrt{a-r}} + G^{**}(r) \tag{3.38}$$

where $G^{**}(r)$ contains all the bounded terms. Now, the stress intensity factor given by (3.33) can be expressed in terms of the unknown function $f^*(r)$ with equations (3.34) and (3.38) in the following form.

$$k_1(a) = \frac{4\mu}{(\kappa+1)} \lim_{r\to a} \sqrt{2(a-r)} \frac{f^*(a)}{\sqrt{b-a}\sqrt{a-r}} = \frac{4\mu}{(\kappa+1)} \frac{f^*(a)}{\sqrt{\frac{b-a}{2}}}, \quad (3.39a)$$

$$k_1(b) = -\frac{4\mu}{(\kappa+1)} \frac{f^*(b)}{\sqrt{\frac{b-a}{2}}}. \quad (3.39b)$$

Comparing (3.26) and (3.36) it can be related $f^*(t)$ and $\overline{G}(\tau)$ by

$$f^*(\tau) = \left(\frac{b-a}{2}\right)\overline{G}(\tau), \qquad (-1\langle\tau\langle 1) \quad (3.40)$$

Now substituting (3.40) into (3.39), the normalized stress intensity factors $\overline{k}_1(a)$ and $\overline{k}_1(b)$ becomes

$$\overline{k}_1(a) = \frac{k_1(a)}{p_0\sqrt{\frac{b-a}{2}}} = \frac{2}{s(\kappa+1)}\overline{G}(-1), \quad (3.41a)$$

$$\overline{k}_1(b) = \frac{k_1(b)}{p_0\sqrt{\frac{b-a}{2}}} = -\frac{2}{s(\kappa+1)}\overline{G}(1), \quad (3.41b)$$

where $s \equiv \frac{p_0}{2\mu}$.

CHAPTER 4

NUMERICAL RESULTS AND DISCUSSION

The axisymmetric crack problem is defined by the dimensionless parameters a/B, b/B, (b-a)/B and Poisson's ratio *v*. Distances are normalized with B, the outer radius of the cylinder.

The system of algebraic equation is solved numerically for unknowns $\overline{G}(\tau_i)$, (i = 1,2,…..,n) at discrete collocation points. n = 30 points is used in calculations. In computing the kernels, because of exponentially decaying behavior of the integrand, the improper integral is evaluated by using Laguerre quadrature formula. Some lengthy algebraic manipulations, hard integrations which can not be found in integration tables, asymptotic analyses of the some expressions including Bessel functions in analytic solution parts, numerical difficulty in the problem, computation of weight and discrete function values, are achieved by using Mathematica 4.2 software program and programming language.

Normalized stress intensity factors at the tips of the crack are calculated for various geometric configurations. Numerical results are given in tabular and graphical forms in Tables 4.1-4.10 and Figures 4.1- 4.10, respectively.

The first result which is important for validation of the problem is that, as the crack size becomes very small in comparison with the other dimensions of the cylinder, it is observed that the normalized stress intensity factors at the tips of the crack, $\bar{k}_1(a)$ and $\bar{k}_1(b)$, approach unity. This is an expected result since the problem turns out to be a finite crack in an infinite medium. First case is verified by substituting (b-a)/B = 10^{-5}. The second validation of the problem is realized with the following case: If the outer radius of the cylinder "*B"* goes to infinity, inner radius of the crack "*a*" and the inner radius of the cylinder "*A*" are very small, the problem turns into a penny-shaped crack in a uniformly loaded infinite medium. In this case the results obtained from the present study and literature (Sneddon and Welch 1963) are in good agreement as they are shown in Table 4.1.

Table 4.1. Comparison of the results obtained in the present study with (Sneddon and Welch 1963)

Present Study	Sneddon and Welch
$k_1(b) = \bar{k}_1(b) p_0 \sqrt{\frac{b}{2}}$	$k_1(b) = \frac{2}{\pi} p_0 \sqrt{b}$
$0.638882 p_0 \sqrt{b}$	$0.6366197 p_0 \sqrt{b}$

Table 4.2 shows the mode I normalized stress intensity factors at the tips of the central crack (case 3) which means the thickness of the net ligaments (a-A and B-b) are equal. As the thickness of the net ligaments are decreased, in other words, as the crack size is increased, the normalized stress intensity factor at the inner tip of the crack $\bar{k}_1(a)$ increases while $\bar{k}_1(b)$ decreases.

Table 4.2. Variation of normalized SIFs, $\bar{k}_1(a)$ and $\bar{k}_1(b)$, for the central crack in the thick walled cylinder ($\bar{k}_1(a) = k_1(a) / p_0 \sqrt{(b-a)/2}$, $\bar{k}_1(b) = k_1(b) / p_0 \sqrt{(b-a)/2}$, *A/B*=0.25, $\nu = 0.3$)

$\frac{b-a}{B}$	$\bar{k}_1(a)$	$\bar{k}_1(b)$
$\rightarrow 0$	$\rightarrow 1.0$	$\rightarrow 1.0$
0.05	1.00999	0.990367
0.10	1.02038	0.980565
0.15	1.03145	0.970255
0.20	1.04367	0.959180
0.25	1.05775	0.947087
0.30	1.07464	0.933708
0.35	1.09547	0.918781
0.40	1.12141	0.902146
0.45	1.15331	0.883965
0.50	1.19078	0.865168

When the inner tip of the crack approaches the free lateral surface (case 4), in other words, when the inner radius of the crack gets smaller, it is observed that $\bar{k}_1(a)$ increases as it is expected. This is shown in Table 4.3. Similarly, this increase is observed in Table 4.4 as well. Comparing Table 4.3 and Table 4.4, when the other parameters are kept fixed, A/B is changed from 0.25 to 0.2, which means that the inner radius of the crack is taken away from the free lateral surface of the cylinder, it is seen that $\bar{k}_1(a)$ in Table 4.3 is always greater than the $\bar{k}_1(a)$ in Table 4.4. And this is achieved as it can be seen from the Figure 4.2.

Table 4.3. Variation of normalized SIFs, $\bar{k}_1(a)$ and $\bar{k}_1(b)$, for an internal crack in the thick walled cylinder ($\bar{k}_1(a) = k_1(a) / p_0\sqrt{(b-a)/2}$, $\bar{k}_1(b) = k_1(b) / p_0\sqrt{(b-a)/2}$, A/B=0.25,b/B=0.8 $\nu = 0.3$)

$\frac{a}{B}$	$\bar{k}_1(a)$	$\bar{k}_1(b)$
0.75	1.00717	0.991426
0.7	1.01376	0.980922
0.65	1.02091	0.968933
0.6	1.02991	0.956022
0.55	1.04272	0.942872
0.5	1.06278	0.930148
0.45	1.09547	0.918781
0.4	1.14621	0.909830
0.35	1.20728	0.903917
0.3	1.20466	0.898660

In Table 4.3 with decreasing a/B, $\bar{k}_1(a)$ increases. Only when a/B decreased from 0.35 to 0.30, one can see that there is a slight decrease in $\bar{k}_1(a)$. In this case, it can be seen that, the inner radius of the crack is very close to the inner wall of the cylinder. This results in another problem with the edge crack. In this special case, the kernel used in this study is no longer bounded in the corresponding closed interval and, of course, the single- valuedness condition is no longer valid (Erdol and Erdogan 1978). Therefore, the solution of the problem with edge crack is the subject of another study. However, from Table 4.4, the same result is not observed because the crack does not show edge-crack behavior, that is, it is not close enough to the inner surface of the cylinder.

Table 4.4. Variation of normalized SIFs, $\bar{k}_1(a)$ and $\bar{k}_1(b)$, for an internal crack in the thick walled cylinder ($\bar{k}_1(a) = k_1(a) / p_0\sqrt{(b-a)/2}$, $\bar{k}_1(b) = k_1(b) / p_0\sqrt{(b-a)/2}$, *A/B*=0.20,*b/B*=0.8 $\nu = 0.3$)

$\frac{a}{B}$	$\bar{k}_1(a)$	$\bar{k}_1(b)$
0.75	1.00706	0.991245
0.7	1.01355	0.980538
0.65	1.02007	0.968332
0.6	1.02689	0.954928
0.55	1.03468	0.940581
0.5	1.04513	0.925610
0.45	1.06219	0.910506
0.4	1.09415	0.896074
0.35	1.15603	0.883593
0.3	1.26285	0.874804

The results for normalized stress intensity factors for a different geometric configuration are tabulated in Tables 4.5 and Table 4.6. The related data is obtained when the outer

Table 4.5. Variation of normalized SIFs, $\bar{k}_1(a)$ and $\bar{k}_1(b)$, for an internal crack in the thick walled cylinder ($\bar{k}_1(a) = k_1(a) / p_0\sqrt{(b-a)/2}$, $\bar{k}_1(b) = k_1(b) / p_0\sqrt{(b-a)/2}$,*A/B*=0.25,*a/B*=0.4, $\nu = 0.3$)

$\frac{b}{a}$	$\bar{k}_1(a)$	$\bar{k}_1(b)$
1.000	$\to 1.0$	$\to 1.0$
1.125	1.01544	0.986885
1.250	1.03306	0.978172
1.375	1.05320	0.971415
1.500	1.07471	0.964286
1.625	1.09613	0.955522
1.750	1.11606	0.944342
1.875	1.13322	0.929678
2.000	1.14621	0.909830
2.125	1.15331	0.883965

radius of the crack is increased towards the rigid lateral surface while the inner radius is kept in the same location (case 5).

Table 4.6. Variation of normalized SIFs, $\bar{k}_1(a)$ and $\bar{k}_1(b)$, for an internal crack in the thick walled cylinder ($\bar{k}_1(a) = k_1(a) / p_0\sqrt{(b-a)/2}$, $\bar{k}_1(b) = k_1(b) / p_0\sqrt{(b-a)/2}$, *A/B*=0.20,*a/B*=0.32, $\nu = 0.3$)

$\frac{b}{a}$	$\bar{k}_1(a)$	$\bar{k}_1(b)$
1.000	$\rightarrow 1.0$	$\rightarrow 1.0$
1.125	1.01544	0.986953
1.250	1.03352	0.978876
1.375	1.05493	0.973350
1.500	1.07854	0.967873
1.625	1.10284	0.961168
1.750	1.12644	0.952896
1.875	1.14827	0.943210
2.000	1.16763	0.932427
2.125	1.18412	0.920813

Here, what is expected in both situations is that $\bar{k}_1(b)$ decreases as b/a increases. It can be observed from the tables that this expectation is realized. It is mentioned in literature (Erdol and Erdogan 1978) that the stress intensity factors always increase at the tips of the crack on the traction free surfaces of the cylinder. However, in this thesis study, the stress intensity factor on the outer surface of the cylinder decreases since the outer surface is rigid.

Another case is studied and the results are shown in Tables 4.7 and 4.8. The crack size is kept fixed and a/B is increased, that is, the crack approached the rigid surface (case 6). Here, as it is expected, both $\bar{k}_1(a)$ and $\bar{k}_1(b)$ decreases. Since the inner wall free of tractions, the crack can not withstand opening while approaching to the free end.

Table 4.7. Variation of normalized SIFs, $\bar{k}_1(a)$ and $\bar{k}_1(b)$, for an internal crack in the thick walled cylinder ($\bar{k}_1(a) = k_1(a)/p_0\sqrt{(b-a)/2}$, $\bar{k}_1(b) = k_1(b)/p_0\sqrt{(b-a)/2}$, A/B=0.25,(b-a)/B=0.3, $\nu = 0.3$)

$\frac{a}{B}$	$\bar{k}_1(a)$	$\bar{k}_1(b)$
0.350	1.12730	0.950203
0.375	1.12667	0.947743
0.400	1.11606	0.944332
0.425	1.10206	0.940768
0.450	1.08783	0.937234
0.475	1.07464	0.933708
0.500	1.06278	0.930148
0.525	1.05208	0.926749
0.550	1.04226	0.924292

Table 4.8. Variation of normalized SIFs, $\bar{k}_1(a)$ and $\bar{k}_1(b)$, for an internal crack in the thick walled cylinder ($\bar{k}_1(a) = k_1(a)/p_0\sqrt{(b-a)/2}$, $\bar{k}_1(b) = k_1(b)/p_0\sqrt{(b-a)/2}$, A/B=0.25,(b-a)/B=0.2, $\nu = 0.3$)

$\frac{a}{B}$	$\bar{k}_1(a)$	$\bar{k}_1(b)$
0.350	1.06774	0.961947
0.375	1.07598	0.964289
0.400	1.07471	0.964286
0.425	1.06926	0.963357
0.450	1.06243	0.962202
0.475	1.05558	0.961030
0.500	1.04928	0.960115
0.525	1.04367	0.959180
0.550	1.03868	0.958205

The variation of normalized stress intensity factors, $\bar{k}_1(a)$ and $\bar{k}_1(b)$, are calculated for different values of Poisson's ratio for case 6. Poisson's ratio is taken as 0.25 and 0.35 in addition to 0.3. These results are given in Tables 4.9, 4.10 and in Figures 4.9, 4.10.

Table 4.9. Variation of normalized SIFs, $\bar{k}_1(a)$ and $\bar{k}_1(b)$, for an internal crack in the thick walled cylinder ($\bar{k}_1(a) = k_1(a) / p_0\sqrt{(b-a)/2}$, $\bar{k}_1(b) = k_1(b) / p_0\sqrt{(b-a)/2}$, *A/B*=0.25, (b-a)/*B*=0.2, $\nu = 0.35$)

$\frac{a}{B}$	$\bar{k}_1(a)$	$\bar{k}_1(b)$
0.350	1.18417	0.938140
0.375	1.13562	0.926330
0.400	1.09927	0.917394
0.425	1.07199	0.909989
0.450	1.05113	0.902946
0.475	1.03453	0.895165
0.500	1.02044	0.885513
0.525	1.00740	0.872753
0.550	0.99411	0.855534

Table 4.10. Variation of normalized SIFs, $\bar{k}_1(a)$ and $\bar{k}_1(b)$, for an internal crack in the thick walled cylinder ($\bar{k}_1(a) = k_1(a) / p_0\sqrt{(b-a)/2}$, $\bar{k}_1(b) = k_1(b) / p_0\sqrt{(b-a)/2}$, *A/B*=0.25,(b-a)/*B*=0.2, $\nu = 0.25$)

$\frac{a}{B}$	$\bar{k}_1(a)$	$\bar{k}_1(b)$
0.350	1.213340	0.930970
0.375	1.132510	0.916000
0.400	1.083380	0.906173
0.425	1.052380	0.898591
0.450	1.031680	0.891445
0.475	1.016560	0.883364
0.500	1.004070	0.873016
0.525	0.992241	0.858807
0.550	0.979615	0.838616

Some of the results given in Tables 4.2-4.10 are also shown graphically in Figures 4.1-4.10 in order to observe and/or understand the behavior of the normalized stress intensity factors at the tips of the crack.

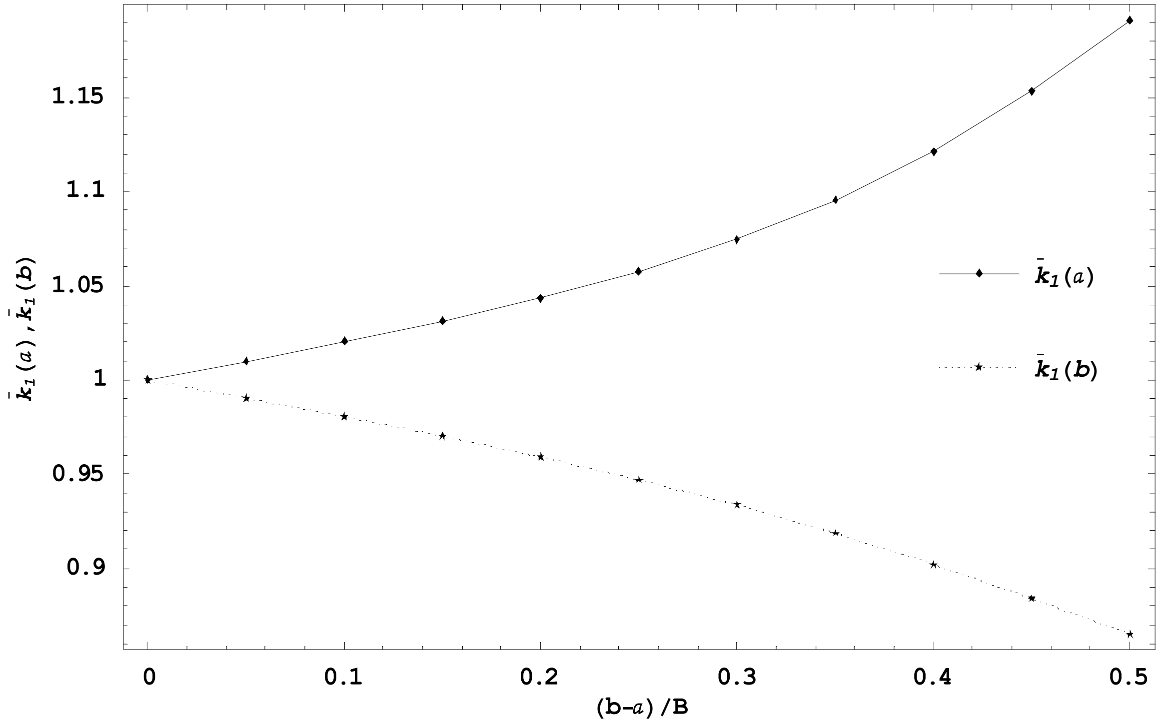

Figure 4.1. Variation of normalized SIFs, $\bar{k}_1(a)$ and $\bar{k}_1(b)$, for the central crack in the thick walled cylinder

($\bar{k}_1(a) = k_1(a) / p_0\sqrt{(b-a)/2}$, $\bar{k}_1(b) = k_1(b) / p_0\sqrt{(b-a)/2}$, A/B=0.25, $\nu = 0.3$)

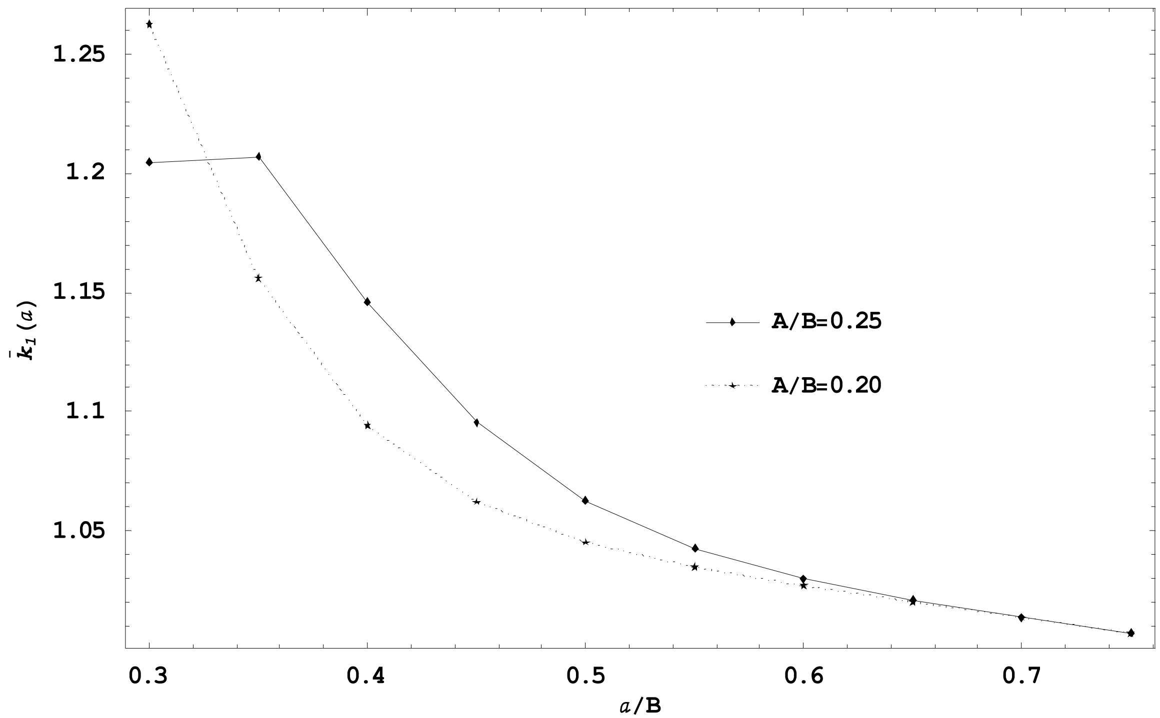

Figure 4.2. Comparison of the results for $\bar{k}_1(a)$ given in Tables 4.3 and 4.4

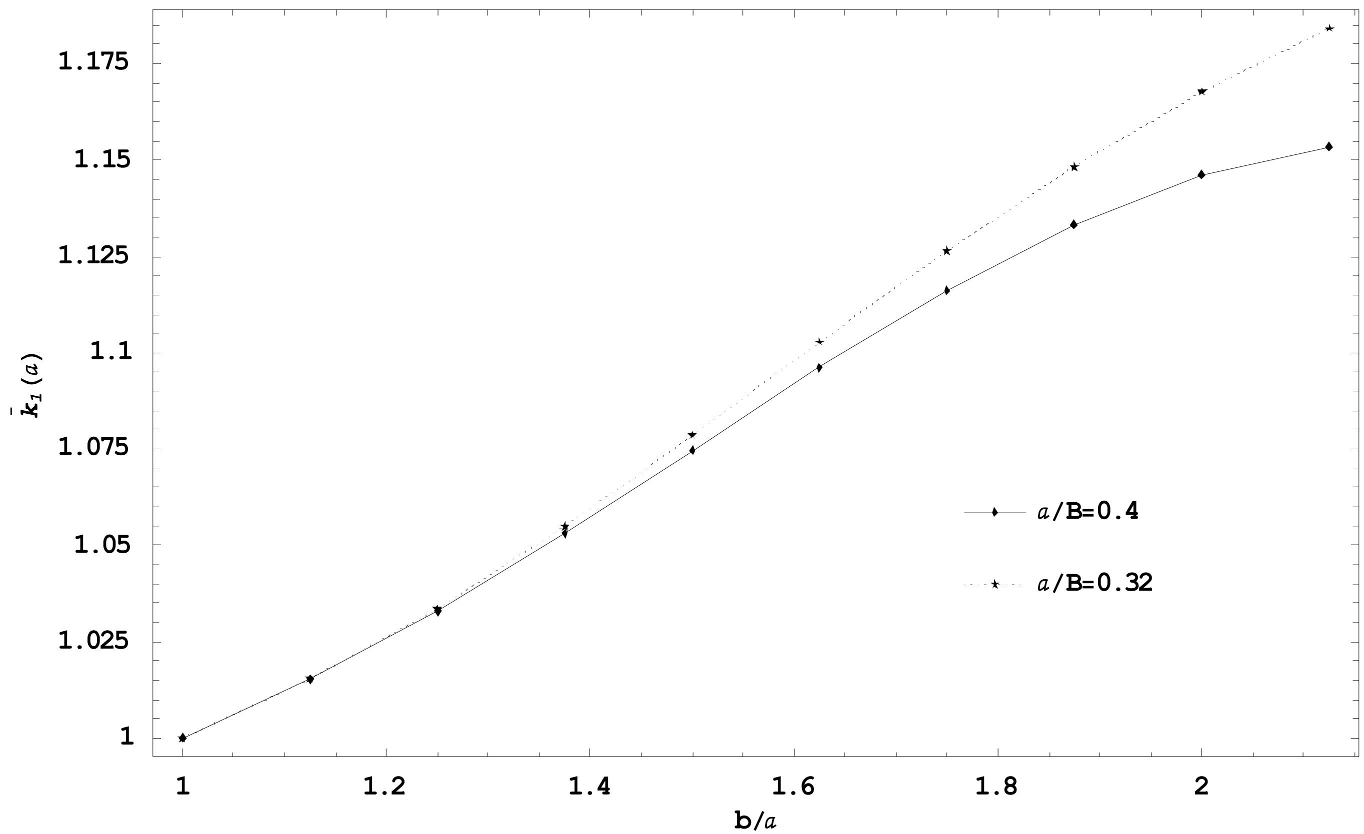

Figure 4.3. Comparison of the results for $\overline{k}_1(a)$ given in Tables 4.5 and 4.6

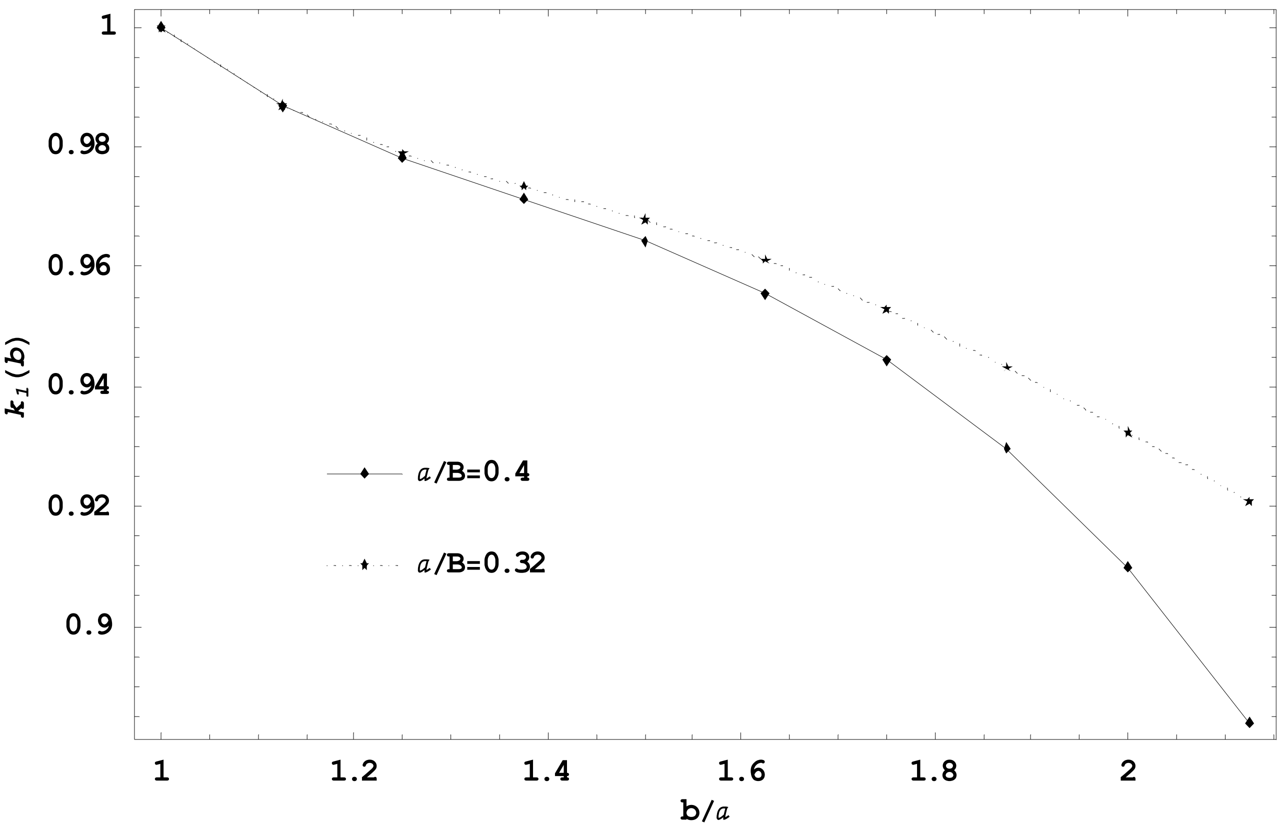

Figure 4.4. Comparison of the results for $\bar{k}_1(b)$ given in Tables 4.5 and 4.6

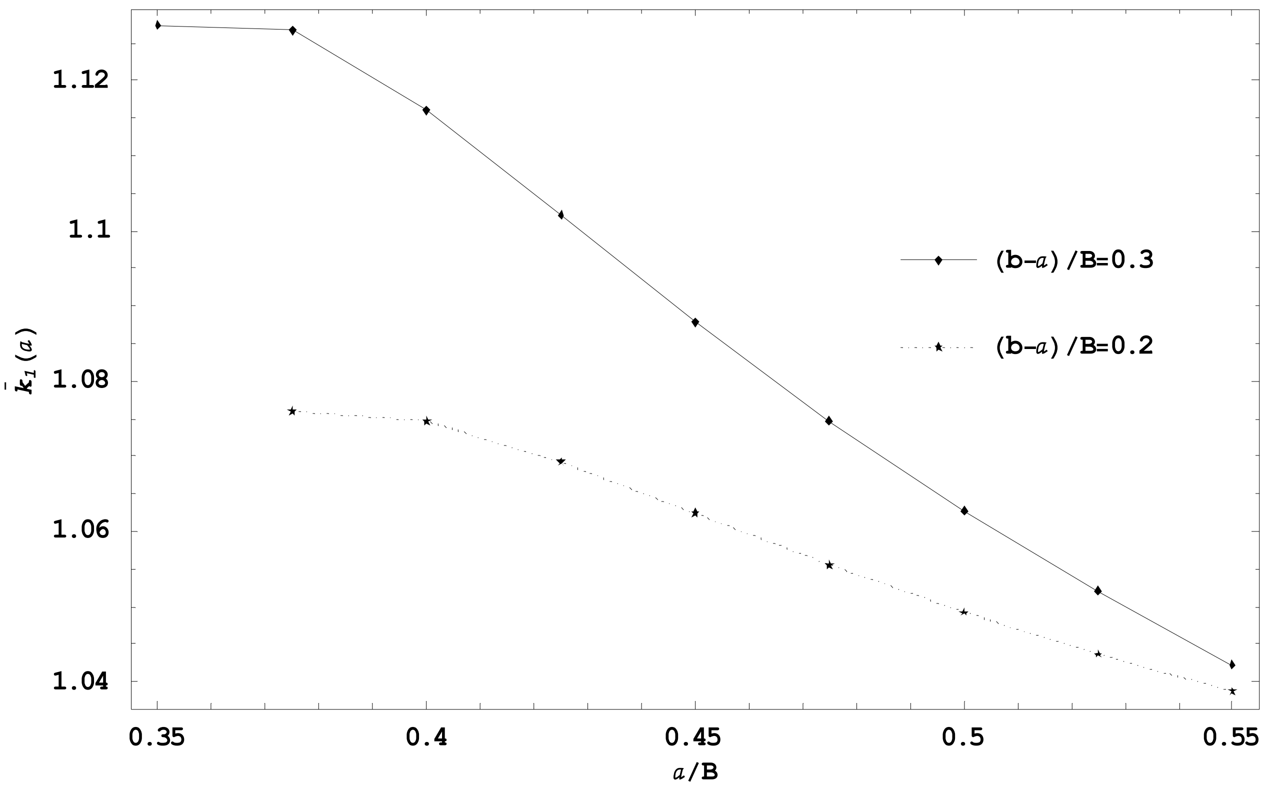

Figure 4.5. Comparison of the results for $\bar{k}_1(a)$ given in Tables 4.7 and 4.8

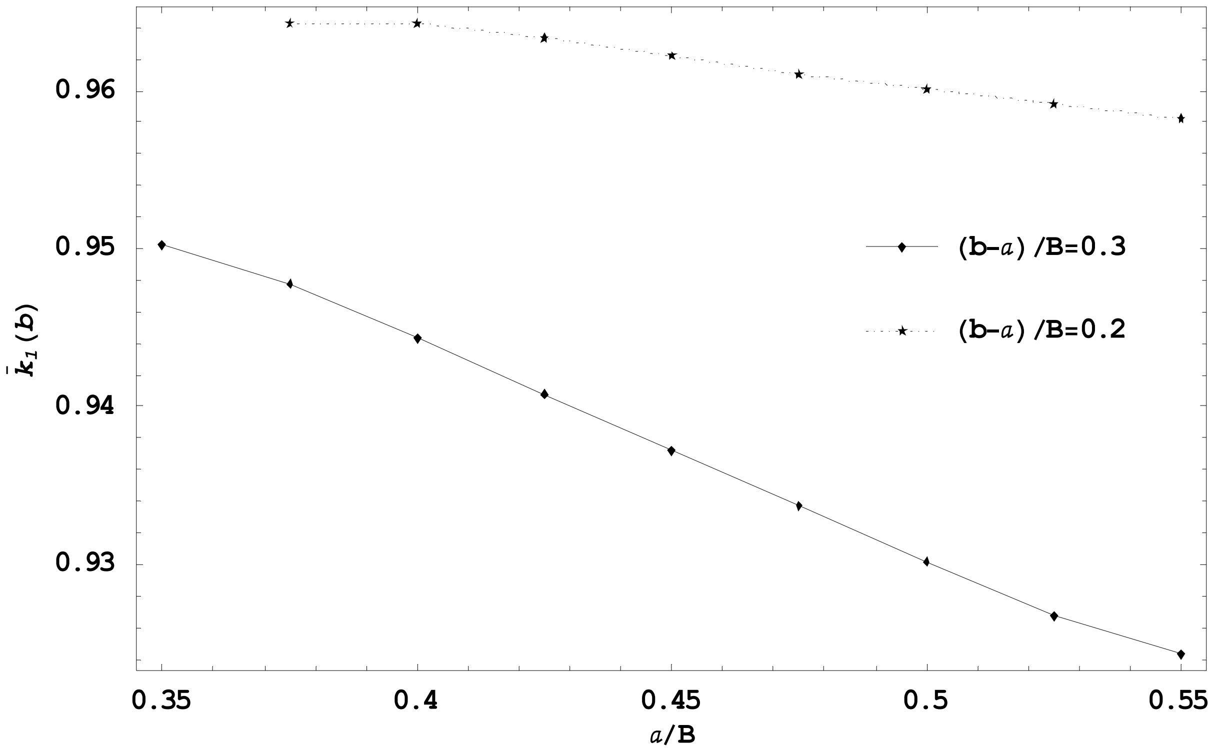

Figure 4.6. Comparison of the results for $\overline{k}_1(b)$ given in Tables 4.7 and 4.8

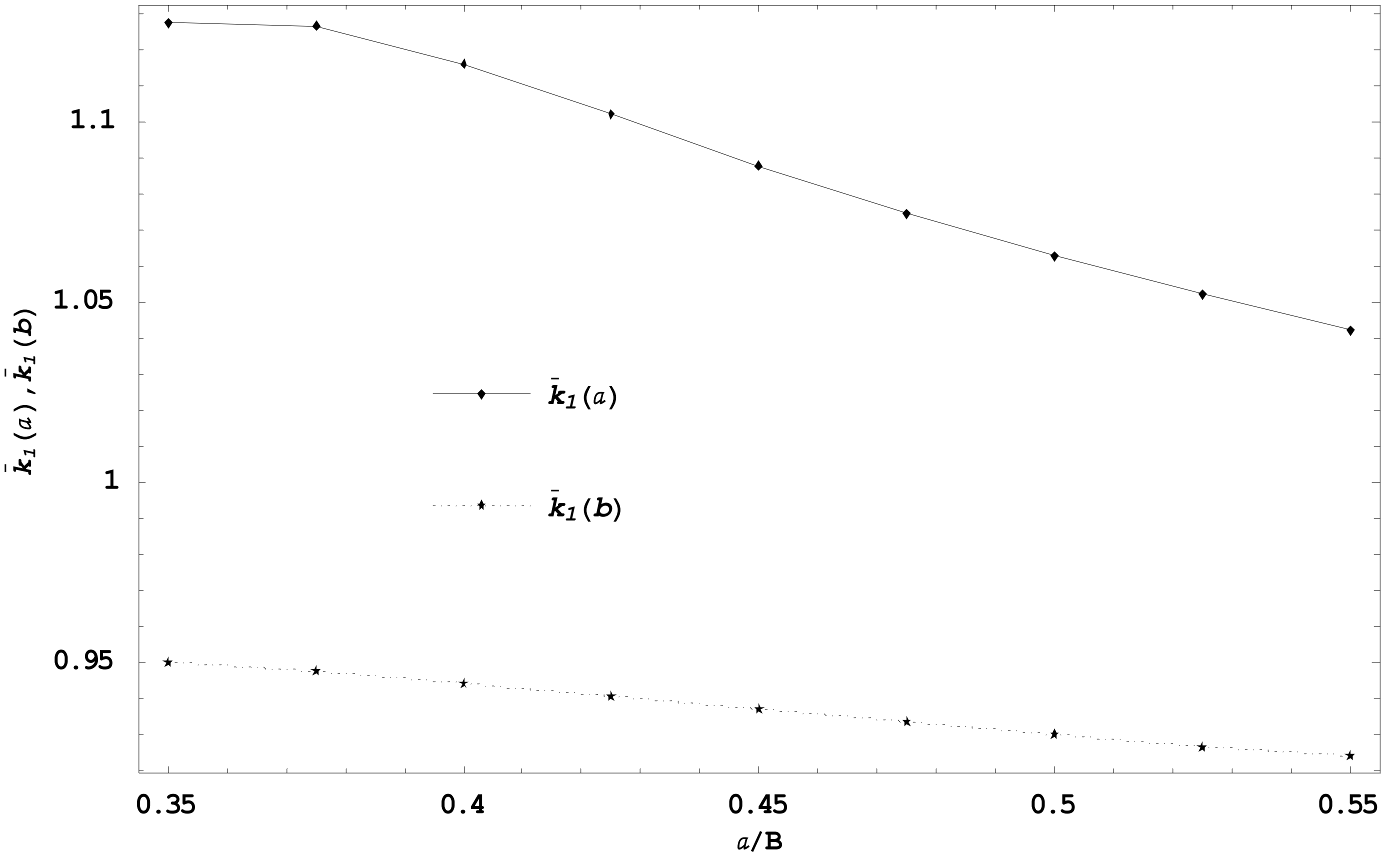

Figure 4.7. Variation of normalized SIFs, $\bar{k}_1(a)$ and $\bar{k}_1(b)$, for an internal crack in the thick walled cylinder ($\bar{k}_1(a) = k_1(a) / p_0\sqrt{(b-a)/2}$, $\bar{k}_1(b) = k_1(b) / p_0\sqrt{(b-a)/2}$, A/B=0.25,(b-a)/B=0.3, $\nu = 0.3$)

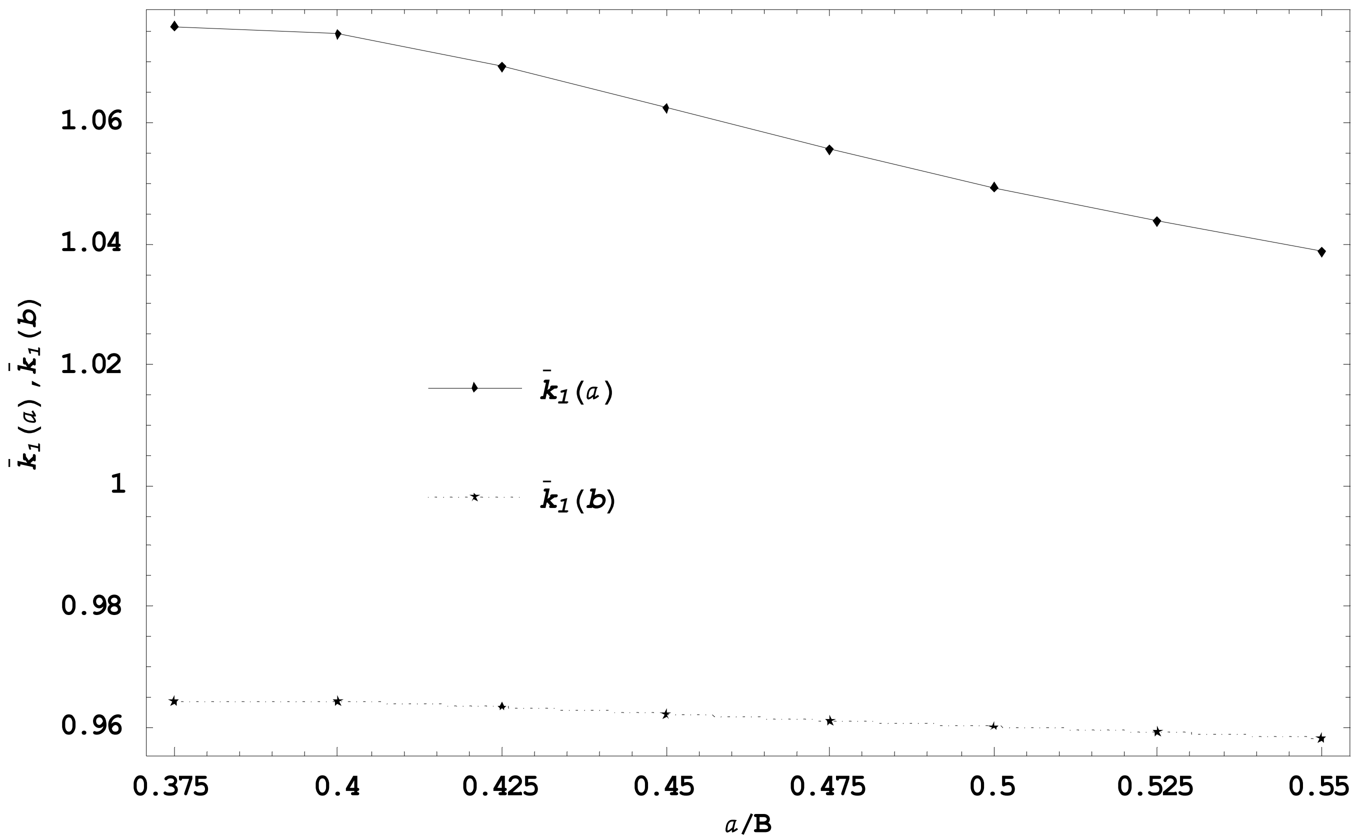

Figure 4.8. Variation of normalized SIFs, $\bar{k}_1(a)$ and $\bar{k}_1(b)$, for an internal crack in the thick walled cylinder ($\bar{k}_1(a) = k_1(a)\,/\,p_0\sqrt{(b-a)/2}$, $\bar{k}_1(b) = k_1(b)\,/\,p_0\sqrt{(b-a)/2}$, A/B=0.25,(b-a)/B=0.2, $\nu = 0.3$)

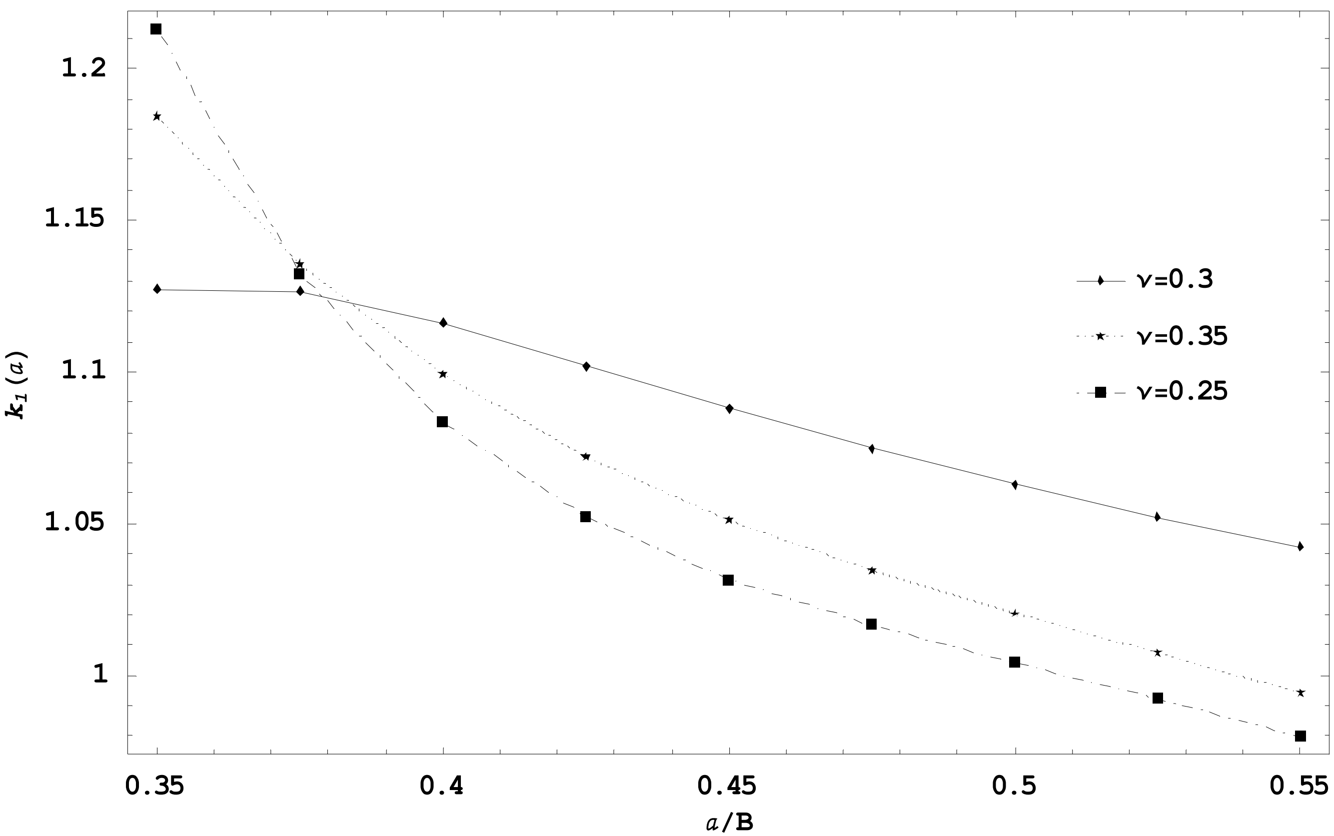

Figure 4.9.Variation of normalized SIFs, $\bar{k}_1(a)$ and $\bar{k}_1(b)$, for an internal crack in the thick walled cylinder

($\bar{k}_1(a) = k_1(a) / p_0\sqrt{(b-a)/2}$, $\bar{k}_1(b) = k_1(b) / p_0\sqrt{(b-a)/2}$, *A/B*=0.25, (b-a)/*B*=0.2)

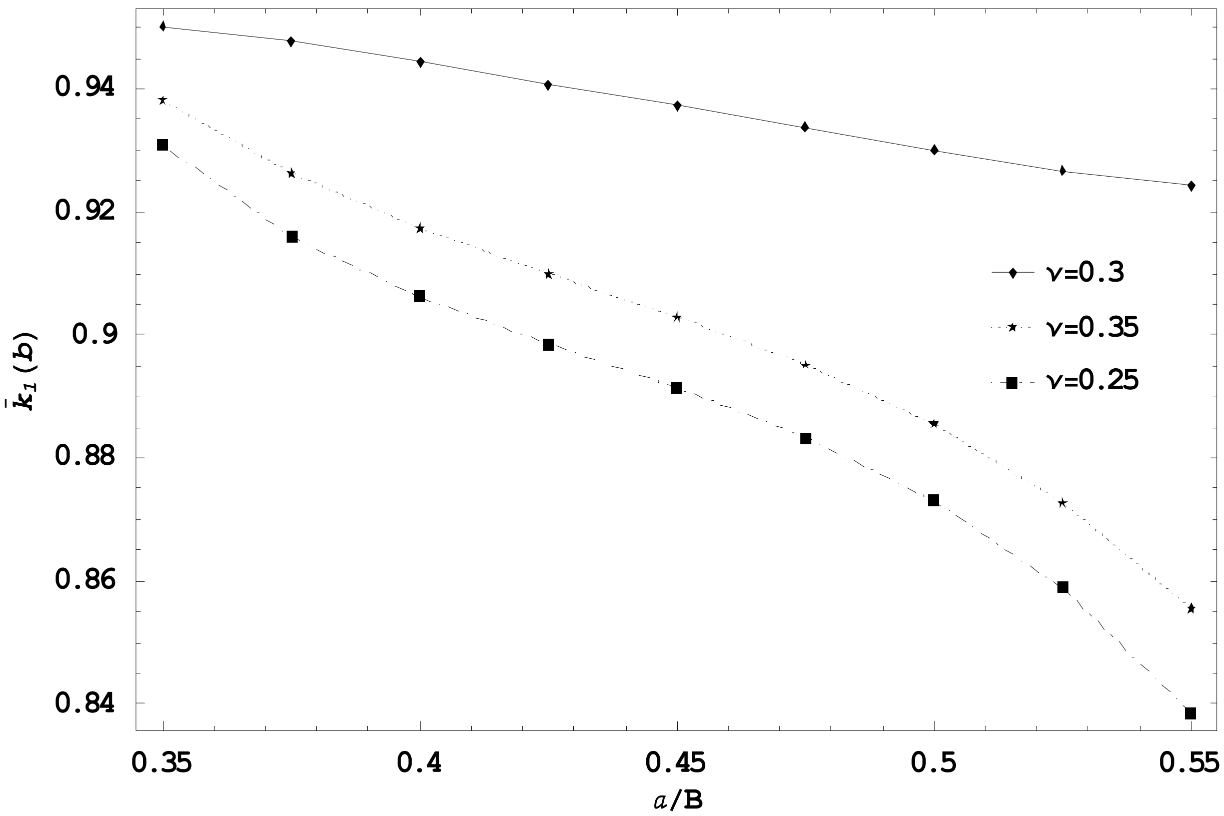

Figure 4.10. Variation of normalized SIFs, $\bar{k}_1(a)$ and $\bar{k}_1(b)$, for an internal crack in the thick walled cylinder ($\bar{k}_1(a) = k_1(a) / p_0\sqrt{(b-a)/2}$, $\bar{k}_1(b) = k_1(b) / p_0\sqrt{(b-a)/2}$, *A/B*=0.25, (b-a)/*B*=0.2)

CHAPTER 5

CONCLUSION

In this thesis, the stress intensity factors for an infinite hollow cylinder containing a ring-shaped crack is investigated. By using the procedures in literature the problem is defined and modeled in terms of a linear second order partial differential equation system with mixed boundary conditions. The integral transform techniques are used to solve this governing equations which are reduced to a singular integral equation. Solving the singular integral equation numerically, the normalized stress intensity factors at the tips of the crack, $\bar{k}_1(a)$ and $\bar{k}_1(b)$, are calculated for various geometric configurations and for different Poisson's ratio which is used as the material parameter. Numerical results are presented in tabular and graphical forms.

As the crack approaches the stress-free surface, $\bar{k}_1(a)$ increases. This is because the free lateral surface lets the crack open (Erdol and Erdogan 1978).

In the same way, as the crack approaches the rigid surface, $\bar{k}_1(b)$ decreases. This is because this rigid wall prevents the crack from opening. In other words, the strength of stress singularity decreases.

The results obtained in this study for the two special problems , *a finite crack in an infinite medium* and *a penny-shaped crack in a uniformly loaded infinite medium*, were in good agreement with those in the literature (discussed in Chapter 4).

As it can be seen from tables and figures given in Chapter 4, the values of $\bar{k}_1(a)$ are always greater than the values of $\bar{k}_1(b)$.

As a further study, the edge crack solution can be adapted to the present study by using different kernel and condition mentioned in Chapter 4.

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APPENDIX A

INTEGRATION FORMULAS

By using the integration formulas given below, equation (2.37) can be reduced to equation (2.38)

$$\int_0^\infty \frac{\alpha}{(\alpha^2+\lambda^2)} J_0(A\alpha)J_0(t\alpha)d\alpha = I_0(A\lambda)K_0(t\lambda), \qquad (A \le t)$$

$$\int_0^\infty \frac{\alpha^2}{(\alpha^2+\lambda^2)} J_1(A\alpha)J_0(t\alpha)d\alpha = -\lambda I_0(A\lambda)K_0(t\lambda), \qquad (A \le t)$$

$$\int_0^\infty \frac{\alpha}{(\alpha^2+\lambda^2)} J_1(A\alpha)J_1(t\alpha)d\alpha = I_1(A\lambda)K_1(t\lambda), \qquad (A \le t)$$

$$\int_0^\infty \frac{\alpha^2}{(\alpha^2+\lambda^2)} J_0(A\alpha)J_1(t\alpha)d\alpha = \lambda I_0(A\lambda)K_1(t\lambda), \qquad (A \le t)$$

$$\int_0^\infty \frac{\alpha}{(\alpha^2+\lambda^2)^2} J_0(A\alpha)J_0(t\alpha)d\alpha = \frac{1}{2\lambda}\left[-AI_1(A\lambda)K_0(t\lambda)+tI_0(A\lambda)K_1(t\lambda)\right], (A \le t)$$

$$\int_0^\infty \frac{\alpha^2}{(\alpha^2+\lambda^2)^2} J_1(A\alpha)J_0(t\alpha)d\alpha = \frac{1}{2\lambda}\left[\lambda AI_0(A\lambda)K_0(t\lambda)-\lambda tI_1(A\lambda)K_1(t\lambda)\right] (A \le t)$$

$$\int_0^\infty \frac{\alpha}{(\alpha^2+\lambda^2)^2} J_1(A\alpha)J_1(t\alpha)d\alpha = \frac{1}{2\lambda}\Big[-AI_0(A\lambda)K_1(t\lambda)+\frac{2}{\lambda}I_1(A\lambda)K_1(t\lambda)$$

$$+tI_1(A\lambda)K_0(t\lambda)], \qquad (A \le t)$$

$$\int_0^\infty \frac{\alpha^2}{(\alpha^2+\lambda^2)^2} J_0(A\alpha)J_1(t\alpha)d\alpha = \frac{1}{2\lambda}\left[-\lambda AI_1(A\lambda)K_1(t\lambda)+\lambda tI_0(A\lambda)K_0(t\lambda)\right].$$

$$\int_0^\infty \frac{\alpha}{(\alpha^2+\lambda^2)} J_0(B\alpha)J_0(t\alpha)d\alpha = K_0(B\lambda)I_0(t\lambda), \qquad (B \ge t)$$

$$\int_0^\infty \frac{\alpha^2}{(\alpha^2+\lambda^2)} J_1(B\alpha)J_0(t\alpha)d\alpha = \lambda K_1(B\lambda)I_0(t\lambda), \qquad (B \ge t)$$

$$\int_0^\infty \frac{\alpha}{\left(\alpha^2+\lambda^2\right)} J_1(B\alpha) J_1(t\alpha) d\alpha = K_1(B\lambda) I_1(t\lambda)\,, \qquad (B \geq t)$$

$$\int_0^\infty \frac{\alpha^2}{\left(\alpha^2+\lambda^2\right)} J_0(B\alpha) J_1(t\alpha) d\alpha = -\lambda K_0(B\lambda) I_1(t\lambda), \qquad (B \geq t)$$

$$\int_0^\infty \frac{\alpha}{\left(\alpha^2+\lambda^2\right)^2} J_0(B\alpha) J_0(t\alpha) d\alpha = \frac{1}{2\lambda}\left[BK_1(B\lambda) I_0(t\lambda) - tK_0(B\lambda) I_1(t\lambda)\right], \quad (B \geq t)$$

$$\int_0^\infty \frac{\alpha^2}{\left(\alpha^2+\lambda^2\right)^2} J_1(B\alpha) J_0(t\alpha) d\alpha = \frac{1}{2\lambda}\left[B\lambda K_0(B\lambda) I_0(t\lambda) - t\lambda K_1(B\lambda) I_1(t\lambda)\right], \ (B \geq t)$$

$$\int_0^\infty \frac{\alpha}{\left(\alpha^2+\lambda^2\right)^2} J_1(B\alpha) J_1(t\alpha) d\alpha = \frac{1}{2\lambda}\left[BK_0(B\lambda) I_1(t\lambda) + \frac{2}{\lambda} K_1(B\lambda) I_1(t\lambda)\right] - \frac{1}{2\lambda} tK_1(B\lambda) I_0(\lambda t), \qquad (B \geq t)$$

$$\int_0^\infty \frac{\alpha^2}{\left(\alpha^2+\lambda^2\right)^2} J_0(B\alpha) J_1(t\alpha) d\alpha = \frac{1}{2\lambda}\left[-B\lambda K_1(B\lambda) I_1(t\lambda) + t\lambda \kappa_0(B\lambda) I_0(t\lambda)\right], (B \geq t)$$

APPENDIX B

INTEGRAL FORMS

$S1$- $S4$ appearing in equation (2.39) are given as:

$$S1 = \frac{1}{(\kappa+1)}\int_a^b f(t)\,t\,g1\,dt \qquad S2 = \frac{1}{(\kappa+1)}\int_a^b f(t)\,t\,b1\,dt$$

$$S3 = \frac{1}{(\kappa+1)}\int_a^b f(t)\,t\,a1\,dt \qquad S4 = \frac{1}{(\kappa+1)}\int_a^b f(t)\,t\,d1\,dt$$

where

$$\mathtt{a1} = -4\,(t\,I_0(A\,\lambda)\,K_0(t\,\lambda) - A\,I_1(A\,\lambda)\,K_1(t\,\lambda))\,\lambda^2 + \frac{4\,(t\,I_1(A\,\lambda)\,K_0(t\,\lambda) - A\,I_0(A\,\lambda)\,K_1(t\,\lambda))\,\lambda}{A} + \frac{2\,(k+1)\,I_1(A\,\lambda)\,K_1(t\,\lambda)}{A}$$

$$\mathrm{b1} = -2\,t\lambda I_0(t\lambda)\,K_0(B\lambda) - (k+1)\,I_1(t\lambda)\,K_0(B\lambda) + 2\,B\lambda I_1(t\lambda)\,K_1(B\lambda)$$

$$\mathrm{d1} = \frac{1}{2}\,\lambda^2\,(A\,I_0(A\,\lambda)\,K_1(t\,\lambda) - t\,I_1(A\,\lambda)\,K_0(t\,\lambda))$$

$$\mathrm{g1} = (k+1)\,I_1(t\,\lambda)\,K_1(B\,\lambda) - 2\,\lambda\,(t\,I_0(t\,\lambda)\,K_1(B\,\lambda) - B\,I_1(t\,\lambda)\,K_0(B\,\lambda))$$

APPENDIX C

COEFFICIENTS

$\beta_{11} - \beta_{44}$ appearing in equation (3.4) are given as follows:

$$\beta_{11} = c_{11} - c_{31}(\kappa + 5),\ \beta_{12} = c_{12} - c_{32}(\kappa + 5),\ \beta_{13} = c_{13} - c_{33}(\kappa + 5),$$

$$\beta_{14} = c_{14} - c_{34}(\kappa + 5), \beta_{21} = c_{21} - c_{41}(\kappa + 5),\ \beta_{22} = c_{22} - c_{42}(\kappa + 5),$$

$$\beta_{23} = c_{23} - c_{43}(\kappa + 5),\ \beta_{24} = c_{24} - c_{44}(\kappa + 5),\ \beta_{31} = c_{31},\ \beta_{32} = c_{32},$$

$$\beta_{33} = c_{33},\ \beta_{34} = c_{34},\ \beta_{41} = c_{41},\ \beta_{42} = c_{42},\ \beta_{43} = c_{43},\ \beta_{44} = c_{44}$$

where the expressions for the coefficients c_{11}-c_{44} are

$$\begin{aligned} c_{11} = &\left(A\,B\,(K_0(B\,\lambda)\,H_{00} + K_1(B\,\lambda)\,H_{01}) - \frac{B\,K_1(A\,\lambda)}{\lambda}\right)\lambda^3 \\ &+ \left(A\,(\kappa + 1)\,K_1(B\,\lambda)\,H_{00} - \frac{1}{2}\,B\,(\kappa + 1)\,(K_0(B\,\lambda)\,H_{10} + K_1(B\,\lambda)\,H_{11})\right)\lambda^2 \\ &- \frac{1}{2}\,(\kappa + 1)^2\,K_1(B\,\lambda)\,H_{10}\,\lambda \end{aligned}$$

$$\begin{aligned} c_{12} = &\left(\frac{B\,K_0(A\,\lambda)}{\lambda} - A\,B\,(K_0(B\,\lambda)\,H_{10} + K_1(B\,\lambda)\,H_{11})\right)\lambda^3 \\ &+ \left(\frac{B\,K_1(A\,\lambda)}{A\,\lambda} + \frac{1}{2}\,B\,(\kappa - 1)\,(K_0(B\,\lambda)\,H_{10} + K_1(B\,\lambda)\,H_{01}) - A\,(\kappa + 1)\,K_1(B\,\lambda)\,H_{10}\right)\lambda^2 \\ &+ \frac{1}{2}\left(\kappa^2 - 1\right)K_1(B\,\lambda)\,H_{00}\,\lambda \end{aligned}$$

$$\begin{aligned} c_{13} = &\left(2\,A\,B(H_{00}\,K_0(A\lambda) - H_{10}\,K_1(A\lambda)) + \frac{2\,AK_1(B\lambda)}{\lambda}\right)\lambda^4 - \frac{B(\kappa + 1)\,K_0(B\lambda)\,\lambda^2}{A} \\ &+ \left(\frac{(1 - \kappa^2)\,K_1(B\lambda)}{2\,A\,\lambda} - \frac{BH_{10}\,(\kappa + 1)\,K_1(A\lambda)}{A}\right)\lambda^2 \end{aligned}$$

$$\begin{aligned} c_{14} = &\left(2\,A\,B(H_{01}\,K_0(A\lambda) - H_{11}\,K_1(A\lambda)) - \frac{2\,AK_0(B\lambda)}{\lambda}\right)\lambda^4 \\ &+ \left(2\,A(\kappa + 1)\,(H_{00}\,K_0(A\lambda) - H_{10}\,K_1(A\lambda)) + \frac{B(\kappa + 1)\,K_1(B\lambda)}{A\lambda}\right)\lambda^3 \\ &+ \left(-\frac{(\kappa + 1)\,(\kappa + 3)\,K_0(B\lambda)}{2\,A\,\lambda} - \frac{BH_{11}\,(\kappa + 1)\,K_1(A\lambda)}{A}\right)\lambda^2 - \frac{H_{10}\,(\kappa + 1)^2\,K_1(A\lambda)\,\lambda}{A} \end{aligned}$$

$$c_{21} = \left(-A\,BH_{00}\,I_0\,(B\lambda) - \frac{BI_1\,(A\lambda)}{\lambda} + A\,BH_{01}\,I_1\,(B\lambda) \right) \lambda^3 + \left(\frac{1}{2}\,BH_{10}\,(\kappa+1)\,I_0\,(B\lambda) \right.$$
$$\left. + AH_{00}\,(\kappa+1)\,I_1\,(B\lambda) - \frac{1}{2}\,BH_{11}\,(\kappa+1)\,I_1\,(B\lambda) \right) \lambda^2 - \frac{1}{2}\,BH_{10}\,(\kappa+1)\,I_1\,(B\lambda)\,\lambda$$

$$c_{22} = \left(-\frac{BI_0(A\lambda)}{\lambda} + A\,BH_{10}I_0(B\lambda) - A\,BH_{11}\,I_1(B\lambda) \right) \lambda^3$$
$$+ \left(-\frac{1}{2}\,BH_{00}\,(\kappa-1)\,I_0(B\lambda) + \frac{BI_1(A\lambda)}{A\lambda} + \frac{1}{2}\,BH_{01}\,(\kappa-1)\,I_1(B\lambda) \right.$$
$$\left. - AH_{10}\,(\kappa+1)\,I_1(B\lambda) \right) \lambda^2 + \frac{1}{2}\,H_{00}\left(\kappa^2-1\right) I_1(B\lambda)\,\lambda$$

$$c_{23} = \left(-2\,A\,BH_{00}I_0(A\lambda) - 2\,A\,BH_{10}\,I_1(A\lambda) + \frac{2\,AI_1(B\lambda)}{\lambda} \right) \lambda^4$$
$$+ \frac{B(\kappa+1)\,I_0(B\lambda)\,\lambda^2}{A} + \left(-\frac{BH_{10}\,I_1(A\lambda)}{A} - \frac{BH_{10}\kappa I_1(A\lambda)}{A} + \frac{(1-\kappa^2)\,I_1(B\lambda)}{2\,A\lambda} \right) \lambda^2$$

$$c_{24} = \left(-2\,ABH_{01}\,I_0\,(A\lambda) + \frac{2\,A\,I_0\,(B\lambda)}{\lambda} - 2\,A\,B\,H_{11}\,I_1\,(A\lambda) \right) \lambda^4$$
$$+ \left(-2\,AH_{00}\,I_0\,(A\lambda) - 2\,AH_{00}\,\kappa I_0\,(A\lambda) - 2\,AH_{10}\,I_1\,(A\lambda) \right.$$
$$\left. - 2\,AH_{10}\,\kappa I_1\,(A\lambda) + \frac{BI_1\,(B\lambda)}{A\lambda} + \frac{B\kappa I_1\,(B\lambda)}{A\lambda} \right) \lambda^3$$
$$+ \left(\frac{(\kappa^2+4\,\kappa+3)\,I_0\,(B\lambda)}{2\,A\lambda} - \frac{BH_{11}\,I_1\,(A\lambda)}{A} \right) \lambda^2$$
$$- \frac{H_{10}\,(\kappa+1)^2\,I_1\,(A\lambda)\,\lambda}{A}$$

$$c_{31} = \left(\frac{A\,K_0(A\,\lambda)}{2\,B\,\lambda} - \frac{1}{2}\,B\,H_{10}\,K_0(B\,\lambda) - \frac{1}{2}\,B\,H_{11}\,K_1(B\,\lambda) \right) \lambda^2$$
$$+ \frac{1}{4}\,(\kappa+1)\,(H_{11}\,K_0(B\,\lambda) - H_{10}\,K_1(B\,\lambda))\,\lambda$$

$$c_{32} = \left(\frac{1}{2}\,B\,H_{00}\,K_0(B\,\lambda) - \frac{A\,K_1(A\,\lambda)}{2\,B\,\lambda} + \frac{1}{2}\,B\,H_{01}\,K_1(B\,\lambda) \right) \lambda^2$$
$$+ \left(-\frac{K_0(A\,\lambda)}{4\,B\,\lambda} - \frac{1}{2}\,H_{01}\,K_0(B\,\lambda) + \frac{B\,H_{10}\,K_0(B\,\lambda)}{2\,A} - \frac{1}{4}\,H_{01}\,\kappa\,K_0(B\,\lambda) \right.$$
$$\left. + \frac{B\,H_{11}\,K_1(B\,\lambda)}{2\,A} + \frac{1}{4}\,H_{00}\,\kappa\,K_1(B\,\lambda) \right) \lambda - \frac{H_{11}\,(\kappa+1)\,K_0(B\,\lambda)}{2\,A}$$

$$c_{33} = \left(A\,H_{01}\,K_0(A\,\lambda) - \frac{B\,K_0(B\,\lambda)}{A\,\lambda} - A\,H_{11}\,K_1(A\,\lambda) \right) \lambda^3 - \frac{H_{11}\,(\kappa+1)\,K_1(A\,\lambda)\,\lambda}{2\,A}$$
$$- \frac{(\kappa+1)\,K_1(B\,\lambda)\,\lambda}{2\,A}$$

$$c_{34}=\left(A\,H_{00}\,K_0(A\,\lambda)-A\,H_{10}\,K_1(A\,\lambda)+\frac{B\,K_1(B\,\lambda)}{A\,\lambda}\right)\lambda^3-\frac{(\kappa+1)\,K_0(B\,\lambda)\,\lambda}{2\,A}$$
$$-\frac{H_{10}\,(\kappa+1)\,K_1(A\,\lambda)\,\lambda}{2\,A}$$

$$c_{41}=\left(-\frac{A\,I_0(A\,\lambda)}{2\,B\,\lambda}+\frac{1}{2}\,B\,H_{10}\,I_0(B\,\lambda)-\frac{1}{2}\,B\,H_{11}\,I_1(B\,\lambda)\right)\lambda^2$$
$$+\frac{1}{4}\,(-\kappa-1)\,(H_{11}\,I_0(B\,\lambda)+H_{10}\,I_1(B\,\lambda))\,\lambda$$

$$c_{42}=\left(-\frac{1}{2}\,B\,H_{00}\,I_0(B\,\lambda)-\frac{A\,I_1(A\,\lambda)}{2\,B\,\lambda}+\frac{1}{2}\,B\,H_{01}\,I_1(B\,\lambda)\right)\lambda^2$$
$$+\left(\frac{I_0(A\,\lambda)}{4\,B\,\lambda}+\frac{1}{2}\,H_{01}\,I_0(B\,\lambda)-\frac{B\,H_{10}\,I_0(B\,\lambda)}{2\,A}+\frac{1}{4}\,H_{01}\,\kappa\,I_0(B\,\lambda)\right.$$
$$\left.+\frac{B\,H_{11}\,I_1(B\,\lambda)}{2\,A}+\frac{1}{4}\,H_{00}\,\kappa\,I_1(B\,\lambda)\right)\lambda+\frac{H_{11}\,(\kappa+1)\,I_0(B\,\lambda)}{2\,A}$$

$$c_{43}=\left(-A\,H_{01}\,I_0(A\,\lambda)+\frac{B\,I_0(B\,\lambda)}{A\,\lambda}-A\,H_{11}\,I_1(A\,\lambda)\right)\lambda^3$$
$$-\frac{H_{11}\,(\kappa+1)\,I_1(A\,\lambda)\,\lambda}{2\,A}-\frac{I_1(B\,\lambda)\,\lambda}{2\,A}$$

$$c_{44}=\left(-A\,H_{00}\,I_0(A\,\lambda)-A\,H_{10}\,I_1(A\,\lambda)+\frac{B\,I_1(B\,\lambda)}{A\,\lambda}\right)\lambda^3-\frac{(\kappa+1)\,I_0(B\,\lambda)\,\lambda}{2\,A}$$
$$-\frac{H_{10}\,(\kappa+1)\,I_1(A\,\lambda)\,\lambda}{2\,A}$$

$$\boldsymbol{D}=ABI_0(B\lambda)^2K_0(A\lambda)^2\lambda^4-ABI_1(B\lambda)^2K_0(A\lambda)^2\lambda^4+ABI_0(A\lambda)^2K_0(B\lambda)^2\lambda^4$$
$$-ABI_1(A\lambda)^2K_0(B\lambda)^2\lambda^4-ABI_0(B\lambda)^2K_1(A\lambda)^2\lambda^4+ABI_1(B\lambda)^2K_1(A\lambda)^2\lambda^4$$
$$-ABI_0(A\lambda)^2K_1(B\lambda)^2\lambda^4+ABI_1(A\lambda)^2K_1(B\lambda)^2\lambda^4-2ABI_0(A\lambda)I_0(B\lambda)K_0(A\lambda)K_0(B\lambda)\lambda^4$$
$$-2ABI_0(B\lambda)I_1(A\lambda)K_0(B\lambda)K_1(A\lambda)\lambda^4-2ABI_0(A\lambda)I_1(B\lambda)K_0(A\lambda)K_1(B\lambda)\lambda^4$$
$$-2ABI_1(A\lambda)I_1(B\lambda)K_1(A\lambda)K_1(B\lambda)\lambda^4-AI_0(B\lambda)I_1(B\lambda)K_0(A\lambda)^2\lambda^3$$
$$-A\kappa I_0(B\lambda)I_1(B\lambda)K_0(A\lambda)^2\lambda^3+A\kappa I_0(B\lambda)I_1(B\lambda)K_1(A\lambda)^2\lambda^3$$
$$+AI_0(A\lambda)I_1(B\lambda)K_0(A\lambda)K_0(B\lambda)\lambda^3+A\kappa I_0(A\lambda)I_1(B\lambda)K_0(A\lambda)K_0(B\lambda)\lambda^3$$
$$-AI_1(A\lambda)I_1(B\lambda)K_0(A\lambda)K_0(B\lambda)\lambda^3-AI_0(B\lambda)I_1(B\lambda)K_0(A\lambda)K_1(A\lambda)\lambda^3$$
$$+A\kappa I_1(A\lambda)I_1(B\lambda)K_0(B\lambda)K_1(A\lambda)\lambda^3+AI_0(A\lambda)I_0(B\lambda)K_0(A\lambda)K_1(B\lambda)\lambda^3$$
$$-A\kappa I_0(A\lambda)I_0(B\lambda)K_0(A\lambda)K_1(B\lambda)\lambda^3-AI_0(A\lambda)^2K_0(B\lambda)K_1(B\lambda)\lambda^3$$

$$+A\kappa I_0(A\lambda)^2 K_0(B\lambda)K_1(B\lambda)\lambda^3 - AI_0(A\lambda)I_0(B\lambda)K_0(A\lambda)K_1(B\lambda)\lambda^3$$

$$+A\kappa I_1(A\lambda)^2 K_0(B\lambda)K_1(B\lambda)\lambda^3 + AI_0(A\lambda)I_1(A\lambda)K_0(B\lambda)K_1(B\lambda)\lambda^3$$

$$+AI_0(A\lambda)I_0(B\lambda)K_1(A\lambda)K_1(B\lambda)\lambda^3 - A\kappa I_0(B\lambda)I_1(A\lambda)K_1(A\lambda)K_1(B\lambda)\lambda^3$$

$$-\frac{BI_1(A\lambda)^2 K_0(B\lambda)^2\lambda^2}{2A} - \frac{B\kappa I_1(A\lambda)^2 K_0(B\lambda)^2\lambda^2}{2A} - \frac{BI_0(B\lambda)^2 K_1(A\lambda)^2\lambda^2}{2A}$$

$$-\frac{B\kappa I_0(B\lambda)^2 K_1(A\lambda)^2\lambda^2}{2A} + \frac{BI_1(B\lambda)^2 K_1(A\lambda)^2\lambda^2}{2A} + \frac{B\kappa I_1(B\lambda)^2 K_1(A\lambda)^2\lambda^2}{2A}$$

$$+\frac{BI_1(A\lambda)^2 K_1(B\lambda)^2\lambda^2}{2A} + \frac{B\kappa I_1(A\lambda)^2 K_1(B\lambda)^2\lambda^2}{2A} + \frac{B\lambda^2}{A} - \frac{BI_0(B\lambda)I_1(A\lambda)K_0(B\lambda)K_1(A\lambda)\lambda^2}{A}$$

$$-\frac{B\kappa I_0(B\lambda)I_1(A\lambda)K_0(B\lambda)K_1(A\lambda)\lambda^2}{A} - \frac{BI_1(A\lambda)I_1(B\lambda)K_1(A\lambda)K_1(B\lambda)\lambda^2}{A}$$

$$-\frac{B\kappa I_1(A\lambda)I_1(B\lambda)K_1(A\lambda)K_1(B\lambda)\lambda^2}{A} + \frac{A\lambda^2}{B} + \frac{\kappa^2 I_0(B\lambda)I_1(B\lambda)K_1(A\lambda)^2\lambda}{2A}$$

$$+\frac{\kappa I_0(B\lambda)I_1(B\lambda)K_1(A\lambda)^2\lambda}{A} + \frac{I_0(B\lambda)I_1(B\lambda)K_1(A\lambda)^2\lambda}{2A} + \frac{\kappa^2 I_1(A\lambda)I_1(B\lambda)K_0(B\lambda)K_1(A\lambda)\lambda}{2A}$$

$$+\frac{\kappa I_1(A\lambda)I_1(B\lambda)K_0(B\lambda)K_1(A\lambda)\lambda}{A} + \frac{I_1(A\lambda)I_1(B\lambda)K_0(B\lambda)K_1(A\lambda)\lambda}{2A}$$

$$-\frac{\kappa^2 I_1(A\lambda)^2 K_0(B\lambda)K_1(B\lambda)\lambda}{2A} - \frac{\kappa I_1(A\lambda)^2 K_0(B\lambda)K_1(B\lambda)\lambda}{A} - \frac{I_1(A\lambda)^2 K_0(B\lambda)K_1(B\lambda)\lambda}{2A}$$

$$-\frac{\kappa^2 I_0(B\lambda)I_1(A\lambda)K_1(A\lambda)K_1(B\lambda)\lambda}{2A} - \frac{\kappa I_0(B\lambda)I_1(A\lambda)K_1(A\lambda)K_1(B\lambda)\lambda}{A} - \frac{I_0(B\lambda)I_1(A\lambda)K_1(A\lambda)K_1(B\lambda)\lambda}{2A}$$

$$+\frac{\kappa^2}{4AB} + \frac{\kappa}{AB} + \frac{3}{4AB}$$

where

$$H_{ij}(\lambda A, \lambda B) = K_i(\lambda A)I_j(\lambda B) + (-1)^{i+j+1} I_i(\lambda A)K_j(\lambda B). \qquad (i,j=0,1)$$

APPENDIX D

ASYMPTOTIC EXPANSIONS

Asymptotic expansions for modified Bessel functions, I_0, K_0, I_1 and K_1 for $\lambda \to \infty$ (Abramowitz and Stegun 1965):

$$K_1(B\lambda) \sim \frac{e^{-\lambda B}\sqrt{\pi}}{\sqrt{2\lambda B}}\left(1+\frac{3}{8\lambda B}-\frac{15}{128\lambda^2 B^2}\right)$$

$$K_1(t\lambda) \sim \frac{e^{-\lambda t}\sqrt{\pi}}{\sqrt{2\lambda t}}\left(1+\frac{3}{8\lambda t}-\frac{15}{128\lambda^2 t^2}\right)$$

$$K_1(A\lambda) \sim \frac{e^{-\lambda A}\sqrt{\pi}}{\sqrt{2\lambda A}}\left(1+\frac{3}{8\lambda A}-\frac{15}{128\lambda^2 A^2}\right)$$

$$K_1(r\lambda) \sim \frac{e^{-\lambda r}\sqrt{\pi}}{\sqrt{2\lambda r}}\left(1+\frac{3}{8\lambda r}-\frac{15}{128\lambda^2 r^2}\right)$$

$$K_0(r\lambda) \sim \frac{e^{-\lambda r}\sqrt{\pi}}{\sqrt{2\lambda r}}\left(1-\frac{1}{8\lambda r}+\frac{9}{128\lambda^2 r^2}\right)$$

$$K_0(A\lambda) \sim \frac{e^{-\lambda A}\sqrt{\pi}}{\sqrt{2\lambda A}}\left(1-\frac{1}{8\lambda A}+\frac{9}{128\lambda^2 A^2}\right)$$

$$K_0(B\lambda) \sim \frac{e^{-\lambda B}\sqrt{\pi}}{\sqrt{2\lambda B}}\left(1-\frac{1}{8\lambda B}+\frac{9}{128\lambda^2 B^2}\right)$$

$$K_0(t\lambda) \sim \frac{e^{-\lambda t}\sqrt{\pi}}{\sqrt{2\lambda t}}\left(1-\frac{1}{8\lambda t}+\frac{9}{128\lambda^2 t^2}\right)$$

$$I_1(B\lambda) \sim \frac{e^{\lambda B}}{\sqrt{2\pi\lambda B}}\left(1-\frac{3}{8\lambda B}-\frac{15}{128\lambda^2 B^2}\right)$$

$$I_1(t\lambda) \sim \frac{e^{\lambda t}}{\sqrt{2\pi\lambda t}}\left(1-\frac{3}{8\lambda t}-\frac{15}{128\lambda^2 t^2}\right)$$

$$I_1(A\lambda) \sim \frac{e^{\lambda A}}{\sqrt{2\pi\lambda A}}\left(1-\frac{3}{8\lambda A}-\frac{15}{128\lambda^2 A^2}\right)$$

$$I_1(r\,\lambda) \sim \frac{e^{\lambda \mathrm{r}}}{\sqrt{2\,\pi \lambda \mathrm{r}}}\left(1 - \frac{3}{8\,\lambda \mathrm{r}} - \frac{15}{128\,\lambda^2\,r^2}\right)$$

$$I_0(r\,\lambda) \sim \frac{e^{\lambda \mathrm{r}}}{\sqrt{2\,\pi \lambda \mathrm{r}}}\left(1 + \frac{1}{8\,\lambda \mathrm{r}} + \frac{9}{128\,\lambda^2\,r^2}\right)$$

$$I_0(A\,\lambda) \sim \frac{e^{\lambda \mathrm{A}}}{\sqrt{2\,\pi \lambda \mathrm{A}}}\left(1 + \frac{1}{8\,\lambda \mathrm{A}} + \frac{9}{128\,\lambda^2\,A^2}\right)$$

$$I_0(B\,\lambda) \sim \frac{e^{\lambda \mathrm{B}}}{\sqrt{2\,\pi \lambda \mathrm{B}}}\left(1 + \frac{1}{8\,\lambda \mathrm{B}} + \frac{9}{128\,\lambda^2\,B^2}\right)$$

$$I_0(t\,\lambda) \sim \frac{e^{\lambda \mathrm{t}}}{\sqrt{2\,\pi \lambda \mathrm{t}}}\left(1 + \frac{1}{8\,\lambda \mathrm{t}} + \frac{9}{128\,\lambda^2\,t^2}\right)$$

APPENDIX E

ALGEBRAIC EQUALITIES

Some formulae used to obtain N_{11S} given in equation (3.13):

$$\frac{(B-r)(B-t)}{(2B-r-t)^3} = \frac{(B-r)}{(2B-r-t)^2} - \frac{(B-r)^2}{(2B-r-t)^3},$$

$$\frac{(B-t)}{(2B-r-t)^2} = \frac{1}{(2B-r-t)} - \frac{(B-r)}{(2B-r-t)^2},$$

$$\frac{(A-r)(A-t)}{(-2A+r+t)^3} = -\frac{(A-r)(A-t)}{(2A-r-t)^3} = -\frac{(A-r)}{(2A-r-t)^2} + \frac{(A-r)^2}{(2A-r-t)^3},$$

$$\frac{(A-t)}{(-2A+r+t)^2} = \frac{1}{2A-r-t} - \frac{(A-r)}{(2A-r-t)^2},$$

$$\frac{1}{(2B-r-t)^3} = -\frac{1}{(t+r-2B)^3} = -\frac{1}{2}\frac{d^2}{dr^2}\left[\frac{1}{(t+r-2B)}\right],$$

$$\frac{1}{(2B-r-t)^3} = -\frac{1}{(t+r-2B)^3} = -\frac{1}{2}\frac{d}{dr}\frac{1}{(t+r-2B)}.$$

APPENDIX F

GAUSS QUADRATURE

Gauss-Lobatto integration formula:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}\,w(t)dt = \sum_{i=1}^{n} C_i\, f(t_i)\, w(t_i)$$

where

$$t_i = \cos\left[(i-1)\pi/(n-1)\right], \qquad (\mathrm{i} = 1,2,\ldots\ldots\ldots,\mathrm{n}),$$

$$C_i = \frac{1}{n-1}, \qquad (\mathrm{i} = 2,3,\ldots.,\mathrm{n}-1), \qquad .$$

$$C_1 = C_n = \frac{1}{2(n-1)}.$$

Gauss-Laguerre integration formula:

$$\int_{0}^{\infty} f(t)\,dt = \int_{0}^{\infty} e^{-t}\left[e^{t} f(t)\right]dt \approx \sum_{i=1}^{n} w(t_i)e^{t} f(t_i)$$

where t_i are abscissas and $w(t_i)$ are weights of Laguerre integration.