THE LEAST PROPER CLASS CONTAINING WEAK SUPPLEMENTS

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ABSTRACT

THE LEAST PROPER CLASS CONTAINING WEAK SUPPLEMENTS

The main purpose of this thesis is to investigate the least proper class containing the class $\mathcal{WS}$ of R-modules determined by weak supplement submodules over a ring R, in particular, over hereditary rings. A submodule A of a module B has(is) weak supplement if and only if there exist a submodule V in B such that $A + V = B$ and the intersection of submodules of A and V is small in B. The class $\mathcal{WS}$ does not form a proper class, in general. By extending the class $\mathcal{WS}$, we obtained the least proper class containing the class $\mathcal{WS}$ of R-modules over hereditary rings. We investigate the homological objects of the least proper class. We determine the structure of elements of the proper class by submodules.

ÖZET

ZAYIF TÜMLEYENLERİ İÇEREN EN KÜÇÜK ÖZ SINIF

Bu tezde temel olarak, zayıf tümleyenler aracılığıyla tanımlanan $\mathcal{WS}$ sınıfını içeren en küçük öz sınıfın bir R halkası üzerinde, özel olarak, kalıtsal halkalar üzerinde incelenmesi amaçlanmıştır. Bir B modülünün A alt modülü, B'de zayıf tümleyendir ancak ve ancak B'nin bir V alt modülü için, $A + V = B$ ve, A ve V alt modüllerinin kesişimi B'de küçüktür. Genelde,$\mathcal{WS}$ sınıfı bir öz sınıf oluşturmaz. $\mathcal{WS}$ sınıfını genişleterek, kalıtsal halkalar üzerinde $\mathcal{WS}$ sınıfını içeren en küçük öz sınıfı elde ettik. Bu en küçük öz sınıfın homolojik nesneleri incelendi. Alt modüller yardımıyla bu öz sınıfın elemanların yapısı belirlendi.

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LIST of SYMBOLS AND ABBREVIATIONS

R	an associative ring with unit unless otherwise stated
$R_{\mathfrak{p}}$	the localization of a ring R at a prime ideal $\mathfrak{p}$ of R
$\mathbb{Z}$, $\mathbb{Z}^+$	the ring of integers, the set of all positive integers
$G[n]$	for a group G and integer n, $G[n] = \{g \in G \mid ng = 0\}$
G^1	the first Ulm subgroup of abelian group G: $G^1 = \bigcap_{n=1}^{\infty} nG$
$\mathbb{Q}$	the field of rational numbers
$\mathbb{Z}_{p^\infty}$	the Prüfer (divisible) group for the prime p (the p-primary part of the torsion group $\mathbb{Q}/\mathbb{Z}$)
R-module	*left* R-module
R-$\mathcal{M}od$	the category of *left* R-modules
$\mathcal{A}b = \mathbb{Z}$-$\mathcal{M}od$	the category of abelian groups ($\mathbb{Z}$-modules)
$\mathrm{Hom}_R(M, N)$	all R-module homomorphisms from M to N
$M \otimes_R N$	the tensor product of the *right* R-module M and the *left* R-module N
$\mathrm{Ker}\, f$	the kernel of the map f
$\mathrm{Im}\, f$	the image of the map f
$T(M)$	the torsion submodule of the module M: $T(M) = \{m \in M \mid rm = 0 \text{ for some } 0 \neq r \in R\}$
$\mathrm{Soc}\, M$	the socle of the R-module M
$\mathrm{Rad}\, M$	the radical of the R-module M
$\mathcal{T}_R$	the category of torsion R-modules
$\mathcal{B}$	the class of bounded R-modules
$\langle \mathcal{E} \rangle$	the smallest proper class containing the class $\mathcal{E}$ of short exact sequences
$\mathcal{P}$	a proper class of R-modules
$\hat{\mathcal{P}}$	the set $\{\mathbb{E} \mid r\mathbb{E} \in \mathcal{P} \text{ for some } 0 \neq r \in R\}$ for a proper class $\mathcal{P}$
$\pi(\mathcal{P})$	all $\mathcal{P}$-projective modules

$\pi^{-1}(\mathcal{M})$	the proper class of R-modules projectively generated by a class $\mathcal{M}$ of R-modules
$\iota(\mathcal{P})$	all $\mathcal{P}$-injective modules
$\iota^{-1}(\mathcal{M})$	the proper class of R-modules injectively generated by a class $\mathcal{M}$ of R-modules
$\tau(\mathcal{P})$	all $\mathcal{P}$-flat *right* R-modules
$\tau^{-1}(\mathcal{M})$	the proper class of R-modules flatly generated by a class $\mathcal{M}$ of *right* R-modules
$\overline{k}(\mathcal{M})$	the proper class coprojectively generated by a class $\mathcal{M}$ of R-modules
$\underline{k}(\mathcal{M})$	the proper class coinjectively generated by a class $\mathcal{M}$ of R-modules
$\mathrm{Ext}_R(C, A) = \mathrm{Ext}^1_R(C, A)$	the set of all equivalence classes of short exact sequences-starting with the R-module A and ending with the R-module C
$\mathrm{Text}_R(C, A)$	the set $\{\mathbb{E} \in \mathrm{Ext}(C, A) \mid r\mathbb{E} \equiv 0$ for some $0 \neq r \in R\}$ of equivalence classes of short exact sequences of R-modules
$\mathrm{Pext}(C, A)$	the set of all equivalence classes of pure-exact sequences-starting with the group A and ending with the group C
$\mathrm{Next}(C, A)$	the set of all equivalence classes of neat-exact sequences-starting with the group A and ending with the group C
$\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$	the proper class of pure-exact sequences of abelian groups
$\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$	the proper class of neat-exact sequences of abelian groups
$\mathcal{A}$	an abelian category (like R-$\mathcal{M}od$ or $\mathbb{Z}$-$\mathcal{M}od = \mathcal{A}b$)
	For a suitable abelian category $\mathcal{A}$ like R-$\mathcal{M}od$ or $\mathbb{Z}$-$\mathcal{M}od$, the following classes are defined:

$\mathcal{S}plit_{\mathcal{A}}$	the smallest proper class consisting of *only splitting* short exact sequences in the abelian category $\mathcal{A}$
$\mathcal{A}bs_{\mathcal{A}}$	the largest proper class consisting of *all* short exact sequences in the abelian category $\mathcal{A}$
$\mathcal{C}ompl_{\mathcal{A}}$	the proper class of complements in the abelian category $\mathcal{A}$
$\mathcal{S}uppl_{\mathcal{A}}$	the proper class of supplements in the abelian category $\mathcal{A}$
$\mathcal{N}eat_{\mathcal{A}}$	the proper class of neats in the abelian category $\mathcal{A}$
$\mathcal{C}o\text{-}\mathcal{N}eat_{\mathcal{A}}$	the proper class of coneats in the abelian category $\mathcal{A}$
$\mathcal{S}_{\mathcal{A}}$	the class of κ-exact sequences in the abelian category $\mathcal{A}$
$\mathcal{WS}_{\mathcal{A}}$	The proper class of weak supplements in the abelian category$\mathcal{A}$
$\mathcal{SB}_{\mathcal{A}}$	the class of β-exact sequences in the abelian category $\mathcal{A}$
$\leq$	submodule
$\ll$	small (=superfluous) submodule
$\trianglelefteq$	essential submodule
$\leq_c$	complement submodule (=closed submodule)
$\leq_s$	supplement submodule
$\leq_N$	neat submodule
$\leq_{cN}$	coneat submodule
$\subset^{\beta}$	$\mathcal{SB}$-submodule

CHAPTER 1

INTRODUCTION

In module theory, the problem of decomposition of a module into a direct sum of its submodules is a fundamental one, and a wide area of module theory is related with this problem. It is well known that a submodule of a module need not be a direct summand. Moreover, we can not state that a for every submodule U of M there is a submodule V satisfying $U + V = M$ that is minimal with respect to this property. If this is the case (that is there is no submodule $\widetilde{V}$ of V such that $\widetilde{V} \subsetneq V$ but still $U + \widetilde{V} = M$), V is called a *supplement* of U. Minimality of V is equivalent to $U \cap V \ll V$. Reducing the last condition to $U \cap V \ll M$, we get the definition of a weak supplement. Supplement submodules and weak supplement submodules are well-studied in the literature. For the definitions and related properties see (Wisbauer 1991). In series of papers from 1974, H. Zöschinger interested with supplement submodules (Zöschinger 1974a, 1974b, 1974c, 1978, 1980, 1981).

This thesis deals with the classes $\mathcal{S}mall$, $\mathcal{S}$ and $\mathcal{WS}$ of short exact sequence of R-modules determined by small, supplement and weak supplement submodules respectively, and the class $\overline{\mathcal{WS}}$ which is the least proper class contain all of them over a hereditary ring R. $\mathcal{S}mall$ is the class of all short exact sequences $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ where $\mathrm{Im}(\alpha) \ll B$, $\mathcal{WS}$ is the class of all short exact sequences $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ where $\mathrm{Im}(\alpha)$ has(is) a weak supplement in B. $\mathcal{S}$ is the class of all short exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ where $\mathrm{Im}(\alpha)$ has a supplement in B defined by Zöschinger may not form proper classes. The classes are different from other in general. On the other hand the proper classes generated by these classes, that is the least proper classes containing these classes are equivalent: $\langle \mathcal{S}mall \rangle = \langle \mathcal{S} \rangle = \langle \mathcal{WS} \rangle$ (The least proper class containing a class $\mathcal{A}$ is denoted by $\langle \mathcal{A} \rangle$). $\mathcal{WS}$-elements are preserved under $\mathrm{Ext}(g, f) : \mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C', A')$ with respect to the second variable, they are not preserved with respect to the first variable. We extend the class $\mathcal{WS}$ to the class $\overline{\mathcal{WS}}$, which consists of all images of $\mathcal{WS}$-elements of $\mathrm{Ext}(C, A')$

under $\mathrm{Ext}(f,\ 1_A) : \mathrm{Ext}(C',\ A) \longrightarrow \mathrm{Ext}(C,\ A)$ for all homomorphism $f : C \longrightarrow C'$.

In this chapter, we give a short summary about content of this thesis. In Section 2.1, we give some theoretical properties of $\mathrm{Ext}(C, A)$, its dependence upon the module A and C, its relation with known constructions. In Section 2.2, we give some properties and definitions about supplement submodules. In Section 2.3, we give some information about Dedekind domains and modules over Dedekind domains. In Section 2.4, we give some properties about neat, coneat and complement submodules. The definition and the properties of a proper class will be given in Chapter 3. The class $\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$ of pure-exact sequences of abelian groups is an important example of a proper class in the category of abelian groups. After, in Section 3.1, we deal with the structure of $\mathrm{Ext}_{\mathcal{P}}$ with respect to a proper class $\mathcal{P}$ and common methods to define a proper class. It is shown here that, if $\mathcal{M}$ is a given class of R-$\mathcal{M}od$ for an additive functor $T(M,\cdot) : R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$, the class of exact triples $\mathbb{E}$ such that $T(M,\mathbb{E})$ is exact form a proper class. This result is helpful in the definition of projectively, injectively generated proper classes. Finally, we give some theorems about coprojectively and coinjectively generated proper classes. In Chapter 4, we define the class $\overline{\mathcal{WS}}$ as the union of $\mathcal{WS}$-elements and the image of $\mathcal{WS}$-elements by with respect to the first variable and then we prove that $\overline{\mathcal{WS}}$ is a proper class and it is a least proper class which containing $\mathcal{S}mall$, $\mathcal{S}$ and $\mathcal{WS}$. In the last chapter, we investigate injective, projective, coinjective and coprojective modules with respect to $\overline{\mathcal{WS}}$. We also give all characterization of coinjective modules related with $\overline{\mathcal{WS}}$. Finally, we give structure of elements of the proper class $\overline{\mathcal{WS}}$ and some results related with this structure.

CHAPTER 2

PRELIMINARIES

This Chapter consists of a short summary of Chapter 3 from (Mac Lane 1963), some preliminary information about supplements in module theory and hereditary ring from (Wisbauer 1991) and (Cohn 2002). For further informations and missing proofs we refer to in (Fuchs 1970), (Vermani 2003) and (Mac Lane 1963) about group of extensions, in (Wisbauer 1991) about supplements, supplemented modules and in (Cohn 2002) about hereditary ring.

2.1. Module Extensions

Let A and C be modules over a fixed ring R. A short exact sequence

$$0 \longrightarrow A \xrightarrow{\mu} B \xrightarrow{\nu} C \longrightarrow 0\,, \tag{2.1}$$

of R-modules and R-module homomorphisms is an extension of A by C, where μ is an R-module monomorphism and ν is an R-module epimorphism with kernel $\mu(A)$. A morphism $\Gamma = \mathbb{E} \to \mathbb{E}'$ of extensions is a triple $\Gamma = (\alpha, \beta, \gamma)$ of module homomorphisms such that the diagram

$$\begin{tikzcd} \mathbb{E}: 0 \arrow[r] \arrow[d, "\Gamma"'] & A \arrow[r, "\mu"] \arrow[d, "\alpha"'] & B \arrow[r, "\nu"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ \mathbb{E}': 0 \arrow[r] & A' \arrow[r, "\mu'"] & B' \arrow[r, "\nu'"] & C' \arrow[r] & 0 \end{tikzcd} \tag{2.2}$$

is commutative. In particular, take $A = A'$ and $C = C'$; two extensions $\mathbb{E}$ and $\mathbb{E}'$ of A by C are said to be equivalent, denoted by $\mathbb{E} \equiv \mathbb{E}'$, if there is a morphism $(1_A, \beta, 1_C) : \mathbb{E} \to \mathbb{E}'$. In this case, $\beta : B \to B'$ is an isomorphism by the short Five Lemma. The set of all equivalence classes of extensions of A by C denoted by $\mathrm{Ext}_R(C, A)$.

Lemma 2.1 ((Mac Lane 1963),Lemma 1.2) *If $\mathbb{E}$ is an extension of an R-module A by an R-module C and if $\gamma : C' \to C$ is a module homomorphism, there exist an extension*

$\mathbb{E}'$ *of* A *by* C' *a morphism* $\Gamma = (1_A, \beta, \gamma) : \mathbb{E}' \to \mathbb{E}$. *The pair* $(\Gamma, \mathbb{E}')$ *is unique up to a equivalence of* $\mathbb{E}'$.

Lemma 2.2 ((Mac Lane 1963),Lemma 1.3) *Under the hypotheses of Lemma 2.1 each morphism* $\Gamma_1 = (\alpha_1, \beta_1, \gamma_1) : \mathbb{E}_1 \to \mathbb{E}$ *of extension with* $\gamma_1 = \gamma$ *can be written uniquely as a composite*

$$\mathbb{E}_1 \xrightarrow{(\alpha_1,\beta',1)} \mathbb{E}\gamma \xrightarrow{(1,\beta,\gamma)} \mathbb{E}\ . \tag{2.3}$$

More briefly, Γ_1 *can be "factored through"* $\Gamma : \mathbb{E}\gamma \to \mathbb{E}$.

Lemma 2.3 ((Mac Lane 1963),Lemma 1.4) *For* $\mathbb{E} \in \mathrm{Ext}(C, A)$ *and* $\alpha : A \to A'$ *there is an extension* $\mathbb{E}'$ *of* A' *by* C *and a morphism* $\Gamma = (\alpha, \beta, 1_C) : \mathbb{E} \to \mathbb{E}'$. *The pair* $(\Gamma, \mathbb{E}')$ *is unique up to a equivalence of* $\mathbb{E}'$.

Lemma 2.4 ((Mac Lane 1963),Lemma 1.5) *Under the hypotheses of Lemma 2.3, any morphism* $\Gamma_1 = (\alpha_1, \beta_1, \gamma_1) : \mathbb{E} \to \mathbb{E}_1$ *of extension with* $\alpha_1 = \alpha$ *can be written uniquely as a composite*

$$\mathbb{E} \xrightarrow{(\alpha,\beta,1)} \alpha\mathbb{E} \xrightarrow{(1,\beta',\gamma_1)} \mathbb{E}_1\ . \tag{2.4}$$

More briefly, Γ_1 *can be "factored through"* $\mathbb{E} \to \alpha\mathbb{E}$.

Lemma 2.5 ((Mac Lane 1963),Lemma 1.6) *For* α, γ *and* $\mathbb{E}$ *as in Lemma 2.1 and 2.3 there is a equivalence of extensions* $\alpha(E\gamma) \equiv (\alpha E)\gamma$.

Proposition 2.1 ((Mac Lane 1963),Proposition 1.8) *Any morphism* $\Gamma_1 = (\alpha, \beta, \gamma) : \mathbb{E} \to \mathbb{E}'$ *of extensions implies a equivalence* $\alpha E \equiv E'\gamma$.

The equivalence classes of extensions of A by C form a group.

Thus to portray the group operation of short exact sequence, we benefit from the diagonal map $\Delta_G : g \mapsto (g, g)$ and the codiagonal map $\nabla_G : (g_1, g_2) \mapsto g_1 + g_2$ of a module G. The maps Δ and ∇ may be used to rewrite the usual definition of the sum $f + g$ of two homomorphism $f, g : C \to A$ as

$$f + g = \nabla_A (f \oplus g) \Delta_C. \tag{2.5}$$

Given two extensions

$$\mathbb{E}_i : \qquad 0 \longrightarrow A_i \xrightarrow{\mu_i} B_i \xrightarrow{\nu_i} C_i \longrightarrow 0 \tag{2.6}$$

for $i = 1,\ 2$, we define their direct sum to be the extension

$$\mathbb{E}_1 \oplus \mathbb{E}_2 : 0 \longrightarrow A_1 \oplus A_2 \xrightarrow{\mu_1 \oplus \mu_2} B_1 \oplus B_2 \xrightarrow{\nu_1 \oplus \nu_2} C_1 \oplus C_2 \longrightarrow 0\ . \tag{2.7}$$

Theorem 2.1 ((Mac Lane 1963), Theorem 2.1) *For given R-modules A and C, the set* $\mathrm{Ext}_R(C, A)$ *of all equivalence classes of extensions of A by C is an abelian group under the binary operation which assigns to the equivalence classes of extensions* $\mathbb{E}_1$ *and* $\mathbb{E}_2$*, the equivalence class of the extension*

$$\mathbb{E}_1 + \mathbb{E}_2 = \nabla_A(\mathbb{E}_1 \oplus \mathbb{E}_2)\Delta_C. \tag{2.8}$$

The class of the split extension $0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0$ *is the zero element of this group, while the inverse of any* $\mathbb{E}$ *is the extension* $(-1_A)\mathbb{E}$*. For homomorphisms* $\alpha : A \longrightarrow A'$ *and* $\gamma : C' \longrightarrow C$ *one has*

$$\alpha(\mathbb{E}_1 + \mathbb{E}_2) \equiv \alpha\mathbb{E}_1 + \alpha\mathbb{E}_2, \qquad (\mathbb{E}_1 + \mathbb{E}_2)\gamma \equiv \mathbb{E}_1\gamma + \mathbb{E}_2\gamma, \tag{2.9}$$

$$(\alpha_1 + \alpha_2)\mathbb{E} \equiv \alpha_1\mathbb{E} + \alpha_2\mathbb{E}, \qquad \mathbb{E}(\gamma_1 + \gamma_2) \equiv \mathbb{E}\gamma_1 + \mathbb{E}\gamma_2. \tag{2.10}$$

2.8 is known as *Baer* sum; and the equivalences in 2.9 and 2.10 express that the maps $\alpha_* : \mathrm{Ext}(C, A) \to \mathrm{Ext}(C, A')$ and $\gamma^* : \mathrm{Ext}(C, A) \to \mathrm{Ext}(C', A)$ are group homomorphisms and that $(\alpha_1+\alpha_2)_* = (\alpha_1)_*+(\alpha_2)_*$ and $(\gamma_1+\gamma_2)^* = (\gamma_1)^*+(\gamma_2)^*$ for $\alpha_1,\ \alpha_2 : A \longrightarrow A'$ and $\gamma_1,\ \gamma_2 : C' \longrightarrow C$.

Theorem 2.2 ((Mac Lane and Eilenberg 1942), Lemma 1.6) Ext_R *is an additive bifunctor on* $R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od$ *to* $\mathcal{A}b$ *which is contravariant in the first and covariant in the second variable.*

In order to be consistent with the functorial notation for homomorphisms, we shall use the notation

$$\mathrm{Ext}_R(\gamma, \alpha) : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C', A') \tag{2.11}$$

instead of $\gamma^*\alpha_* = \alpha_*\gamma^*$; that is, $\mathrm{Ext}_R(\gamma, \alpha)$ acts as shown by

$$\mathrm{Ext}_R(\gamma, \alpha) : \mathbb{E} \mapsto \alpha\mathbb{E}\gamma. \tag{2.12}$$

Given an extension

$$\begin{tikzcd} & & & G \arrow[d, "\xi"] & \\ \mathbb{E} : 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\eta"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \\ & G & & & \end{tikzcd} \tag{2.13}$$

representing an element of $\mathrm{Ext}_R(C, A)$, and homomorphisms $\eta : A \to G$ and $\xi : G \to C$, we know that $\eta\mathbb{E}$ is an extension of G by C and $\mathbb{E}\xi$ is an extension of A by G, i.e., $\eta\mathbb{E}$ represents an element of $\mathrm{Ext}_R(C, G)$ and $\mathbb{E}\xi$ represents an element of $\mathrm{Ext}_R(G, A)$. In this way we obtain the maps

$$E^* : \mathrm{Hom}(A, G) \to \mathrm{Ext}_R(C, G) \tag{2.14}$$

$$E_* : \mathrm{Hom}(G, C) \to \mathrm{Ext}_R(G, A)$$

defined as

$$E^* : \eta \mapsto \eta\mathbb{E} \text{ and } E_* : \xi \mapsto \mathbb{E}\xi.$$

From 4.2 we can show that E^* and E_* are homomorphisms. If $\phi : G \to H$ is any homomorphism, as we have $(\phi\eta)\mathbb{E} \equiv \phi(\eta\mathbb{E})$ and $\mathbb{E}(\xi\phi) \equiv (\mathbb{E}\xi)\phi$, the diagrams

$$\begin{tikzcd} \mathrm{Hom}(A, G) \arrow[r] \arrow[d] & \mathrm{Ext}_R(C, G) \arrow[d] \\ \mathrm{Hom}(A, H) \arrow[r] & \mathrm{Ext}_R(C, H) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Hom}(H, C) \arrow[r] \arrow[d] & \mathrm{Ext}_R(H, A) \arrow[d] \\ \mathrm{Hom}(G, C) \arrow[r] & \mathrm{Ext}_R(G, A) \end{tikzcd} \tag{2.15}$$

with the obvious maps commute. E^* and E_* are called *connecting homomorphisms* for the short exact sequence 2.13.

Lemma 2.6 ((Mac Lane 1963), Proposition 1.7) *Given a diagram*

$$\mathbb{E}: \qquad \begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\eta"] & B \arrow[r, "\beta"] \arrow[dl, dashed, "\xi"] & C \arrow[r] & 0 \\ & G & & & \end{tikzcd} \tag{2.16}$$

with exact row, there exists a $\xi : B \to G$ making the triangle commute if and only if $\eta\mathbb{E}$ splits.

Lemma 2.7 ((Mac Lane 1963), Proposition 1.7) *If the diagram*

$$\mathbb{E}: \qquad \begin{tikzcd} & & & G \arrow[d, "\eta"] \arrow[dl, dashed, "\xi"'] & \\ 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd} \tag{2.17}$$

has exact row, then there is a $\xi : G \to B$ such that $\beta\xi = \eta$ if and only if $\mathbb{E}\eta$ splits.

With the aid of these lemmas, we have the following theorem which establishes a close connection between Hom and Ext_R.

Theorem 2.3 ((Mac Lane 1963), Theorem 3.4) *If* $: 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ *is an exact sequence, then the sequences*

$$0 \longrightarrow \mathrm{Hom}(C, G) \longrightarrow \mathrm{Hom}(B, G) \longrightarrow \mathrm{Hom}(A, G) \longrightarrow \tag{2.18}$$

$$\xrightarrow{E^*} \mathrm{Ext}_R(C, G) \xrightarrow{\beta^*} \mathrm{Ext}_R(B, G) \xrightarrow{\alpha^*} \mathrm{Ext}_R(A, G) \longrightarrow \cdots,$$

and

$$0 \longrightarrow \mathrm{Hom}(G, A) \longrightarrow \mathrm{Hom}(G, B) \longrightarrow \mathrm{Hom}(G, C) \longrightarrow \tag{2.19}$$

$$\xrightarrow{E_*} \mathrm{Ext}_R(G, A) \xrightarrow{\beta_*} \mathrm{Ext}_R(G, B) \xrightarrow{\alpha_*} \mathrm{Ext}_R(G, C) \longrightarrow \cdots$$

are exact for every module G.

If $\mathbb{E} : 0 \longrightarrow A \xrightarrow{\mu} B \xrightarrow{\nu} C \longrightarrow 0$ is an extension of A by C, and if $\alpha : A \to A$, $\gamma : C \to C$ are endomorphisms of A and C, respectively, then $\alpha\mathbb{E}$ and $\mathbb{E}\gamma$ will be extensions of A by C. The correspondences

$$\alpha_* : \mathbb{E} \mapsto \alpha\mathbb{E} \qquad \text{and} \qquad \gamma^* : \mathbb{E} \mapsto \mathbb{E}\gamma$$

are endomorphisms of $\mathrm{Ext}_R(C, A)$, which are called *induced endomorphisms* of Ext_R. The formulas $(\alpha_1 + \alpha_2)_* = (\alpha_1)_* + (\alpha_2)_*$ and $(\gamma_1 + \gamma_2)^* = (\gamma_1)^* + (\gamma_2)^*$ show that the endomorphism ring of A acts on $\mathrm{Ext}_R(C, A)$ and similarly the dual of the endomorphism ring C operates on $\mathrm{Ext}_R(C, A)$. These commute as is shown by $\alpha_*\gamma^* = \gamma^*\alpha_*$; hence $\mathrm{Ext}_R(C, A)$ is a (unital) bimodule over endomorphism rings of A and C, acting from the left and right, respectively.

2.2. Supplement Submodules

In this section, there are some definitions and some results about supplement submodules. See (Wisbauer 1991) and (Clark 2006) for more information about supplements and supplemented modules.

A submodule A of a module M is called small (superfluous) in M, written $A \ll M$, if for every $U \subseteq M$, the equality $A + U = M$ implies $U = M$. A submodule A of a module M is called large (essential) in M, written $A \trianglelefteq M$, if for every submodule $U \subseteq M$, the equality $A \cap U = 0$ implies $U = 0$.

Let A be a submodule of an R-module M. If there exists a submodule U which is minimal element in the set $\{U \mid U \subseteq M \text{ and } A + U = M\}$ then U is called a *supplement* of A in M.

Lemma 2.8 ((Wisbauer 1991), §41.1) *V is a supplement of U in M if and only if $U+V = M$ and $U \cap V \ll V$.*

Some properties of supplements are given in the following proposition.

Proposition 2.2 ((Wisbauer 1991), §41.1) *Let U, $V \subseteq M$ and V be a supplement of U in M.*

1. If $W + V = M$ for some $W \subseteq U$, then V is a supplement of W.

2. If M is finitely generated, then V is also finitely generated.

3. If U is a maximal submodule of M, then V is cyclic and $U \cap V = \mathrm{Rad}\, V$ is a (the unique) maximal submodule of V.

4. If $K \ll M$, then V is a supplement of $U + K$.

5. If $K \ll M$, then $V \cap K \ll V$ and $\operatorname{Rad} V = V \cap \operatorname{Rad} M$.

6. If $\operatorname{Rad} M \ll M$*, then U is contained in a maximal submodule of M.*

7. If $L \subseteq U$, $V + L/L$ is a supplement of U/L in M/L.

8. If $\operatorname{Rad} M \ll M$ *or* $\operatorname{Rad} M \subseteq U$ *and $p : M \longrightarrow M/\operatorname{Rad} M$ is the canonical epimorphism, then* $M/\operatorname{Rad} M = p(U) \oplus p(V)$.

Let M be a module. If every submodule of M has a supplement in M, then M is called a *supplemented module*. Artinian modules and semisimple modules are examples of supplemented modules while the ring $\mathbb{Z}$ of integers as a module over itself is an example which is not supplemented module.

Let U be a submodule of an R-module M. If there exists a submodule V of M such that $M = U + V$ and $U \cap V \ll M$ then U is called a *weak supplement* of V in M.

2.3. Hereditary Ring

A ring R is called hereditary if all submodules of projective modules over R are again projective. If all finitely generated submodules of projective modules over R are again projective, it is called semihereditary.

Principal ideal domains (PID) are hereditary. A commutative hereditary integral domain is called a Dedekind domain. A commutative semihereditary integral domain is called a Prüfer domain.

A Dedekind domain or Dedekind ring, is an integral domain in which every nonzero proper ideal factors into a product of prime ideals. A commutative ring which is not a field is a *valuation* ring, if its ideals are totally ordered by inclusion. Additionally, if R is an integral domain it is called a *valuation* domain. A PID with only one nonzero maximal ideals is called a *discrete valuation ring*, or DVR, and every discrete valuation ring is a valuation ring. A valuation ring is a PID if and only if it is a DVR or a field.

Let R be an integral domain and K be its field of fractions. An element of K is said to be *integral* over R if it is a root of a monic polynomial in $R[X]$. A commutative domain R is *integrally closed* if the elements of K which are integral over R are exactly the elements of R.

For an integral domain R which is not field, all of the following are equivalent:

1. Every nonzero proper ideal factors into prime ideals.

2. R is Noetherian, and the localization at each maximal ideal is DVR.

3. Every fractional ideal of R is invertible.

4. R is integrally closed , Noetherian domain with Krull dimension 1(i.e., all non-zero prime ideals of R are maximal).

So a Dedekind domain is a domain which satisfies any one, and hence all four, of (1) through (4).

The following lemma is well-known, we include it for completeness.

Lemma 2.9 *Let R be a commutative ring and Ω be the set of all maximal ideals of R. Then for an R-module M,* $\operatorname{Rad} M = \bigcap_{\mathfrak{p}\in\Omega} \mathfrak{p}M$.

Proof 2.1 *For a maximal ideal $\mathfrak{p}$, we can consider $M/\mathfrak{p}M$ as a module over $R/\mathfrak{p}$, so $M/\mathfrak{p}M$ is semisimple and therefore $\operatorname{Rad} M \subseteq \mathfrak{p}M$. Then we obtain $\operatorname{Rad} M \subseteq \bigcap_{\mathfrak{p}\in\Omega} \mathfrak{p}M$. Conversely, let $x \in M$ be such that $x \notin \operatorname{Rad} M$. Then there is a maximal submodule K in M such that $x \notin K$. M/K is a simple module, so $\mathfrak{q}M \subseteq K$ for some $\mathfrak{q} \in \Omega$. Then we obtain $x \notin \mathfrak{q}M$, hence $x \notin \bigcap_{\mathfrak{p}\in\Omega} \mathfrak{p}M$. This implies $\bigcap_{\mathfrak{p}\in\Omega} \mathfrak{p}M \subseteq \operatorname{Rad} M$.*

Theorem 2.4 ((Cohn 2002), Propositions 10.5.1, 10.5.4, 10.5.6) *For a commutative domain R, the following are equivalent.*

(i) R is a Dedekind domain.

(ii) Every ideal of R is projective.

(iii) R is Noetherian and the localization $R_\mathfrak{p}$ of R at $\mathfrak{p}$ is a DVR for all maximal ideals $\mathfrak{p}$ of R.

(iv) Every ideal of R can be expressed uniquely as a finite product of prime ideals.

Proposition 2.3 ((Sharpe and Vamos 1972), Proposition 2.10) *Every divisible module over a Dedekind domain is injective.*

Over a Dedekind domain R, by the use of Proposition 2.3 together with Lemma 2.9 we have that the conditions for an R-module M being divisible, injective and radical, i.e. $\operatorname{Rad} M = M$, are equivalent. For torsion R-modules, we have the following important result.

Proposition 2.4 ((Cohn 2002), Proposition 10.6.9) *Any torsion module M over a Dedekind domain is a direct sum of its primary parts, in a unique way:*

$$M = \oplus T_{\mathfrak{p}}(M)$$

and when M is finitely generated, only finitely many terms on the right are different from zero.

For more information about Dedekind domains and modules over a Dedekind domain see (Hazewinkel 2004) and (Sharpe and Vamos 1972).

CHAPTER 3

PROPER CLASSES

We will not see the general definition of proper classes in an abelian category as in Maclane (1963,Ch.12) since our main investigations are in the proper classes of modules. In Section 3.3., we review the definitions, which have been given in Section 3.1., for projectives, injectives, coprojectives, coinjectives with respect to a proper class, using diagrams and $\mathrm{Ext}_{\mathcal{P}}$ with respect to a proper class $\mathcal{P}$ mentioned in Section 3.2.. In the other sections of this chapter, we have summarized the results that we refer frequently for proper classes of R-modules which are projectively generated or injectively generated. In Section 3.6., coinjective and coprojective modules with respect to a projectively or injectively generated proper class is described. Our summary is from the survey (Sklyarenko 1978).

3.1. Proper Class

Let $\mathcal{P}$ be a class of short exact sequences of R-modules and R-module homomorphisms. If a short exact sequence

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{3.1}$$

belongs to $\mathcal{P}$, then f is said to be a *$\mathcal{P}$-monomorphism* and g is a *$\mathcal{P}$-epimorphism* (both are said to be *$\mathcal{P}$-proper* and the short exact sequence is said to be a *$\mathcal{P}$-proper* short exact sequence.). A short exact sequence $\mathbb{E}$ is determined by each of the monomorphism f and epimorphism g uniquely up to isomorphism.

Definition 3.1 *The class $\mathcal{P}$ is said to be* proper *(in the sense of Buchsbaum) if it satisfies the following conditions ((Buchsbaum 1959), (Mac Lane 1963), (Sklyarenko 1978)):*

P-1) If a short exact sequence $\mathbb{E}$ is in $\mathcal{P}$, then $\mathcal{P}$ contains every short exact sequence isomorphic to $\mathbb{E}$.

P-2) $\mathcal{P}$ contains all splitting short exact sequences.

P-3) *The composite of two $\mathcal{P}$-monomorphisms is a $\mathcal{P}$-monomorphism if this composite is defined.*

P-3') *The composite of two $\mathcal{P}$-epimorphisms is a $\mathcal{P}$-epimorphism if this composite is defined.*

P-4) *If g and f are monomorphisms, and $g \circ f$ is a $\mathcal{P}$-monomorphism, then f is a $\mathcal{P}$-monomorphism.*

P-4') *If g and f are epimorphisms, and $g \circ f$ is a $\mathcal{P}$-epimorphism, then g is a $\mathcal{P}$-epimorphism.*

The set $\text{Ext}_{\mathcal{P}}(C, A)$ of all short exact sequence of $\text{Ext}(C, A)$ that belongs to $\mathcal{P}$ is a subgroup of the group of the extensions $\text{Ext}^1_R(C, A)$.

$\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$ which is the proper class of all short exact sequence 3.1 of abelian group homomorphism such that $\text{Im}(f)$ is a pure subgroup of B, where a subgroup A of a group B is *pure* in B if $A \cap nB = nA$ for all integers n is an important example for proper classes in abelian groups (see (Fuchs 1970, §26-30) for the important notion of purity in abelian groups). The short exact sequences in $\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$ are called *pure-exact sequences* of abelian groups. The proper class $\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$ forms one of the origins of *relative* homological algebra; it is the reason why a proper class is also called *purity* (as in (Misina and Skornjakov 1960), (Generalov 1972), (Generalov 1972), (Generalov 1983)).

The smallest proper class of R-modules consists of only *splitting* short exact sequences of R-modules which we denote by $\mathcal{S}plit_{R\text{-}\mathcal{M}od}$.The largest proper class of R-modules consists of *all* short exact sequences of R-modules which we denote by $\mathcal{A}bs_{R\text{-}\mathcal{M}od}$ (*absolute purity*). Another example is the class $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$, consisting of all short exact sequences 3.1 such that $\text{Im} f$ is a supplement of some submodule K of B, is a proper class (see (Erdoğan 2004) or (Clark 2006) for a proof).

For a proper class $\mathcal{P}$ of R-modules, call a submodule A of a module B a $\mathcal{P}$-submodule of B, if the inclusion monomorphism $i_A : A \to B$, $i_A(a) = a$, $a \in A$, is a $\mathcal{P}$-monomorphism. We denote this by $A \subseteq_{\mathcal{P}} B$.

3.2. $\mathrm{Ext}_{\mathcal{P}}$ With Respect to a Proper Class $\mathcal{P}$

The functor Ext^n_R, $n \in \mathbb{Z}^+ \cup \{0\}$: In the proper class $\mathcal{A}bs_{R\text{-}\mathcal{M}od}$, there are enough injectives and enough projectives. So every module has a projective resolution and an injective resolution. Thus for given R-modules A, C we can take an injective resolution

$$0 \longrightarrow A \xrightarrow{\delta} E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} E_2 \longrightarrow \cdots \tag{3.2}$$

which is an exact sequence with all $E_0, E_1, E_2, \ldots$ injective and define for each $n \in \mathbb{Z}^+ \cup \{0\}$, $\mathrm{Ext}^n(C, A) = \mathrm{Ker}(\mathrm{Hom}(C, d_n)) / \mathrm{Im}(\mathrm{Hom}(C, d_{n-1}))$, that is $\mathrm{Ext}^n(C, -)$ is the n^{th}-right derived functor of the functor $\mathrm{Hom}(C, -) : R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$ (we set $d_{-1} = 0$, so that $\mathrm{Ext}^0(C, A) \cong \mathrm{Hom}(C, A)$). This group $\mathrm{Ext}^n(C, A)$ is well-defined, it is, up to isomorphism, independent of the choice of the injective resolution and in fact can also be defined using projective resolutions. The functor Ext remedies the inexactness of the functor Hom. See for example (Alizade and Pancar 1999), (Rotman 1979), (Mac Lane 1963) and (Cartan and Eilenberg 1956).

The functor Ext^1_R: There is an alternative definition of Ext^1_R using the so called Baer sum. Let A and C be R-modules. Two short exact sequences

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \quad \text{and} \quad \mathbb{E}' : 0 \longrightarrow A \xrightarrow{f'} B' \xrightarrow{g'} C \longrightarrow 0 \tag{3.3}$$

of R-modules and R-module homomorphisms starting with A and ending with C are said to be *equivalent* if we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "1_A"'] & B \arrow[r, "g"] \arrow[d, "\psi"'] & C \arrow[r] \arrow[d, "1_C"'] & 0 \\
0 \arrow[r] & A \arrow[r, "f'"] & B' \arrow[r, "g'"] & C \arrow[r] & 0
\end{tikzcd} \tag{3.4}$$

with some R-module homomorphism $\psi : B \longrightarrow B'$, where $1_A : A \longrightarrow A$ and $1_C : C \longrightarrow C$ are identity maps. Denote by $[\mathbb{E}]$ the equivalence class of the short exact sequence $\mathbb{E}$. $\mathrm{Ext}^1_R(C, A)$ consists of all equivalence classes of short exact sequences of R-modules and R-module homomorphisms starting with A and ending with C. The addition in $\mathrm{Ext}^1_R(C, A)$ is given by Baer sum. A bifunctor $Ext^1_R : R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$ is obtained along these lines. Denote Ext_R^1 shortly by Ext_R. For more information see Mac Lane(1963) Chapter III.

Let A, C be R-modules. $\mathbb{E} \in\in \mathrm{Ext}_R(C, A)$ means that $\mathbb{E}$ is an element of an element of the group $\mathrm{Ext}_R(C, A)$, that is the equivalence class $[\mathbb{E}] \in \mathrm{Ext}_R(C, A)$, so it just means that $\mathbb{E}$ is a short exact sequence of R-modules starting with A and ending with C. If the underlying ring R is fixed, we just write $\mathrm{Ext}(C, A)$ instead of $\mathrm{Ext}_R(C, A)$ when there is no ambiguity.

Note that when the ring R is *commutative*, $\mathrm{Ext}_R(C, A)$ has a natural R-module structure for R-modules A, C. So, we have in this case a bifunctor $Ext_R^1 : R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \longrightarrow R\text{-}\mathcal{M}od$.

The functor $\mathrm{Ext}_{\mathcal{P}}^1$: In a proper class $\mathcal{P}$, we may not have enough injectives and enough projectives, so it is not possible in this case to use derived functors to give relative versions of Ext. But the alternative definition of Ext_R^1 may be used in this case.

For a proper class $\mathcal{P}$ and R-modules A, C, denote by $\mathrm{Ext}_{\mathcal{P}}^1(C, A)$ or shortly by $\mathrm{Ext}_{\mathcal{P}}(C, A)$, the equivalence classes of all short exact sequences *in* $\mathcal{P}$ which start with A and end with C. This turns out to be a subgroup of $\mathrm{Ext}_R(C, A)$ and a bifunctor $\mathrm{Ext}_{\mathcal{P}}^1 : R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$ is obtained which is a subfunctor of Ext_R^1. See (Mac Lane 1963, Ch. 12, §4-5). Alternatively, using such a subfunctor will help to define a proper class.

The functor $\mathrm{Ext}_{\mathcal{P}}^n$, $n \in \mathbb{Z}^+ \cup \{0\}$: Similar to the construction for $\mathrm{Ext}_{\mathcal{P}}^1$, by considering long extensions

$$0 \longrightarrow A \longrightarrow B_1 \longrightarrow B_2 \longrightarrow \cdots \longrightarrow B_n \longrightarrow C \longrightarrow 0 \tag{3.5}$$

with a suitable equivalence relation and addition gives us a bifunctor $\mathrm{Ext}_{\mathcal{P}}^n : R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$. See (Mac Lane 1963, Ch.12, §4-5).

Furthermore, the functor $\mathrm{Ext}_{\mathcal{P}}^1$ is a subfunctor of Ext_R^1 and it is called an *E-functor* see (Butler and Horrocks 1961). By (Nunke 1963, Theorem 1.1), an *E-functor* $\mathrm{Ext}_{\mathcal{P}}^1$ of Ext_R^1 gives a proper class if it satisfies one of the properties *P-3)* and *P-3')*. This result enables us to define a proper class in terms of subfunctors of Ext_R^1.

Let $T(M, B)$ be an additive functor in the argument B (covariant or contravariant), left or right exact and depending on an R-module M from $R\text{-}\mathcal{M}od$. If $\mathcal{M}$ is a given class of modules of this category, we denote by $t^{-1}(\mathcal{M})$ the class $\mathcal{P}$ of short exact sequences $\mathbb{E}$ such that $T(M, \mathbb{E})$ is exact for all $M \in \mathcal{M}$.

Lemma 3.1 ((Sklyarenko 1978), Lemma 0.1) *$\mathcal{P} = t^{-1}(\mathcal{M})$ is a proper class.*

Let $t(\mathcal{P})$ be the class of all objects M for which the triples $T(M, \mathbb{E})$, $\mathbb{E} \in \mathcal{P}$ are exact (we assume that for object 0 the functor $T(0, B)$ is exact).

Lemma 3.2 ((Sklyarenko 1978), Lemma 0.2) *We have the relations $\mathcal{P} \subseteq t^{-1}(t(\mathcal{P}))$, $\mathcal{M} \subseteq t(t^{-1}(\mathcal{M}))$, $t(\mathcal{P}) = t(t^{-1}(t(\mathcal{P})))$ and $t^{-1}(\mathcal{M}) = t^{-1}(t(t^{-1}(\mathcal{M})))$, and also a bijection between the classes of the form $t^{-1}(M)$ and $t(\mathcal{P})$.*

For a proper class $\mathcal{P}$ over an integral domain R, we denote by $\hat{\mathcal{P}}$ the class of the short exact sequences $\mathbb{E} : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules such that $r\mathbb{E} \in \mathcal{P}$ for some $0 \neq r \in R$ where r also denotes the multiplication homomorphism by $r \in R$. Thus

$$\hat{\mathcal{P}} = \{\mathbb{E} \mid r\mathbb{E} \in \mathcal{P} \text{ for some } 0 \neq r \in R\}.$$

In case of abelian groups the class $\hat{\mathcal{P}}$ is studied in (Walker 1964), (Alizade 1986) and (Alizade and Pancar 1999) for $\mathcal{P} = \mathcal{S}plit$ where it was denoted by *Text* since $\text{Ext}^1_{\hat{\mathcal{S}plit}}(C, A) = T(\text{Ext}(C, A))$ the torsion part of $\text{Ext}(C, A)$.

Theorem 3.1 ((Alizade 1986)) *In case of abelian groups, $\hat{\mathcal{P}}$ is proper class for every proper class $\mathcal{P}$.*

Let $\mathcal{E}$ be a class of short exact sequences. The smallest proper class containing $\mathcal{E}$ is said to be *generated by* $\mathcal{E}$ and denoted by $< \mathcal{E} >$ see (Pancar 1997).

Since the intersection of any family of proper classes is proper, for a class $\mathcal{E}$ of short exact sequences

$$< \mathcal{E} >= \bigcap \{\mathcal{P} : \mathcal{E} \subseteq \mathcal{P} ; \mathcal{P} \text{ is a proper class } \}.$$

For more information about proper classes generated by a class of short exact sequences see (Pancar 1997).

3.3. Projectives, Injectives, Coprojectives and Coinjectives with Respect to a Proper Class

Take a short exact sequence

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules and R-module homomorphisms.

An R-module M is said to be *projective with respect to the short exact sequence* $\mathbb{E}$, or *with respect to the epimorphism* g if any of the following equivalent conditions holds:

1. Every diagram

$$\begin{tikzcd} \mathbb{E}: & 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \\ & & & & M \arrow[u, "\gamma"] \arrow[ul, dashed, "\tilde{\gamma}"] & \end{tikzcd} \tag{3.6}$$

   where the first row is $\mathbb{E}$ and $\gamma : M \longrightarrow C$ is an R-module homomorphism can be embedded in a commutative diagram by choosing an R-module homomorphism $\tilde{\gamma} : M \longrightarrow B$; that is, for every homomorphism $\gamma : M \longrightarrow C$, there exits a homomorphism $\tilde{\gamma} : M \longrightarrow B$ such that $g \circ \tilde{\gamma} = \gamma$.

2. The sequence

$$\mathrm{Hom}(M, \mathbb{E}) : \quad 0 \longrightarrow \mathrm{Hom}(M, A) \xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C) \longrightarrow 0 \tag{3.7}$$

   is exact.

Dually, an R-module M is said to be *injective with respect to the short exact sequence* $\mathbb{E}$, or *with respect to the monomorphism* f if any of the following equivalent conditions holds:

1. Every diagram

$$\begin{tikzcd} \mathbb{E}: & 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[dl, dashed, "\tilde{\alpha}"] & C \arrow[r] & 0 \\ & & M & & & \end{tikzcd} \tag{3.8}$$

where the first row is $\mathbb{E}$ and $\alpha : A \longrightarrow M$ is an R-module homomorphism can be embedded in a commutative diagram by choosing an R-module homomorphism $\tilde{\alpha} : B \longrightarrow M$; that is, for every homomorphism $\alpha : A \longrightarrow M$, there exists a homomorphism $\tilde{\alpha} : B \longrightarrow M$ such that $\tilde{\alpha} \circ f = \alpha$.

2. The sequence

$$\mathrm{Hom}(\mathbb{E}, M): \quad 0 \longrightarrow \mathrm{Hom}(C, M) \xrightarrow{g^*} \mathrm{Hom}(B, M) \xrightarrow{f^*} \mathrm{Hom}(A, M) \longrightarrow 0 \tag{3.9}$$

is exact.

Denote by $\mathcal{P}$ a proper class of R-modules.

The following definitions have been given in Section 3.1.. An R-module M is said to be *$\mathcal{P}$-projective* [*$\mathcal{P}$-injective*] if it is projective [injective] with respect to all short exact sequences in $\mathcal{P}$. Denote all $\mathcal{P}$-projective [$\mathcal{P}$-injective] modules by $\pi(\mathcal{P})$ [$\iota(\mathcal{P})$]. An R-module C is said to be *$\mathcal{P}$-coprojective* if *every* short exact sequence of R-modules and R-module homomorphisms of the form

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

ending with C is in the proper class $\mathcal{P}$. An R-module A is said to be *$\mathcal{P}$-coinjective* if *every* short exact sequence of R-modules and R-module homomorphisms of the form

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

starting with A is in the proper class $\mathcal{P}$.

Using the functor $\mathrm{Ext}_{\mathcal{P}}$, the $\mathcal{P}$-projectives, $\mathcal{P}$-injectives, $\mathcal{P}$-coprojectives, $\mathcal{P}$-coinjectives are simply described as extreme ends for the subgroup $\mathrm{Ext}_{\mathcal{P}}(C, A) \leq \mathrm{Ext}_R(C, A)$ being 0 or the whole of $\mathrm{Ext}_R(C, A)$:

1. An R -module C is *$\mathcal{P}$-projective* if and only if

$$\mathrm{Ext}_{\mathcal{P}}(C, A) = 0 \text{ for all } R\text{-modules } A.$$

2. An R -module C is *$\mathcal{P}$-coprojective* if and only if

$$\mathrm{Ext}_{\mathcal{P}}(C, A) = \mathrm{Ext}_R(C, A) \text{ for all } R\text{-modules } A.$$

3. An R -module A is *$\mathcal{P}$-injective* if and only if

$$\mathrm{Ext}_{\mathcal{P}}(C, A) = 0 \text{ for all } R\text{-modules } C.$$

4. An R -module A is *$\mathcal{P}$-coinjective* if and only if

$$\mathrm{Ext}_{\mathcal{P}}(C, A) = \mathrm{Ext}_R(C, A) \text{ for all } R\text{-modules } C.$$

A class $\mathcal{P}$ of R-modules is said to have *enough projectives* if for every module A we can find a $\mathcal{P}$-epimorhism from some $\mathcal{P}$-projective module P to A. A class $\mathcal{P}$ of R-modules is said to have *enough injectives* if for every module B we can find a $\mathcal{P}$-monomorphism from B to some $\mathcal{P}$-injective module J. A proper class $\mathcal{P}$ of R-modules with enough projectives [enough injectives] is also said to be a *projective proper class* [resp. *injective proper class*].

The following propositions give the relation between projective (resp. injective) modules with respect to a class $\mathcal{E}$ of short exact sequences and with respect to the proper class $< \mathcal{E} >$ generated by $\mathcal{E}$.

Proposition 3.1 ((Pancar 1997), Propositions 2.3 and 2.4)

(a) $\pi(\mathcal{E}) = \pi(< \mathcal{E} >)$

(b) $\iota(\mathcal{E}) = \iota(< \mathcal{E} >)$.

Proposition 3.2 ((Misina and Skornjakov 1960), Propositions 1.9 and 1.14) *If in the short exact sequence* $0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0$*, the modules M and K are $\mathcal{P}$-coprojective ($\mathcal{P}$-coinjective), then N is $\mathcal{P}$-coprojective ($\mathcal{P}$-coinjective).*

Proof Let A be an R-module. Suppose that M and K are $\mathcal{P}$-coprojective. Then $0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0 \in \mathcal{P}$. We have the following exact sequences;

$$
\begin{array}{c}
0 \longrightarrow \mathrm{Hom}(K,A) \longrightarrow \mathrm{Hom}(N,A) \longrightarrow \mathrm{Hom}(M,A) \longrightarrow \\
\longrightarrow \mathrm{Ext}^1_{\mathcal{P}}(K,A) \longrightarrow \mathrm{Ext}^1_{\mathcal{P}}(N,A) \longrightarrow \mathrm{Ext}^1_{\mathcal{P}}(M,A) \longrightarrow \cdots \\
0 \longrightarrow \mathrm{Hom}(K,A) \longrightarrow \mathrm{Hom}(N,A) \longrightarrow \mathrm{Hom}(M,A) \longrightarrow \\
\longrightarrow \mathrm{Ext}^1_{R}(K,A) \longrightarrow \mathrm{Ext}^1_{R}(N,A) \longrightarrow \mathrm{Ext}^1_{R}(M,A) \longrightarrow \cdots
\end{array}
\quad . \tag{3.10}
$$

Since M and K are $\mathcal{P}$-coprojective, we have the equalities and an inclusion map α in the following diagram.

$$
\begin{tikzcd}
\mathrm{Ext}^1_{\mathcal{P}}(K,A) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_{\mathcal{P}}(N,A) \arrow[r] \arrow[d, "\alpha"'] & \mathrm{Ext}^1_{\mathcal{P}}(M,A) \arrow[d, equal] \\
\mathrm{Ext}^1_{R}(K,A) \arrow[r] & \mathrm{Ext}^1_{R}(N,A) \arrow[r] & \mathrm{Ext}^1_{R}(M,A)
\end{tikzcd} \tag{3.11}
$$

Then $\mathrm{Ext}^1_{\mathcal{P}}(N,A) = \mathrm{Ext}^1_{R}(N,A)$ for every R-module A, which shows that N is $\mathcal{P}$-coprojective.

For $\mathcal{P}$-coinjectives, the proof can be done by using the functor $\mathrm{Hom}(B,\cdot)$ for an R-module B. □

Proposition 3.3 ((Misina and Skornjakov 1960), Proposition 1.12) *An R-module M is $\mathcal{P}$-coprojective if and only if there is a $\mathcal{P}$-epimorphism from a projective R-module P to M.*

Proof (⇒) Take any epimorphism $\gamma : P \longrightarrow M$ from a projective R-module P to M. Since M is $\mathcal{P}$-coprojective, γ is a $\mathcal{P}$-epimorphism.

(⇐) Let $\gamma : P \longrightarrow M$ be a $\mathcal{P}$-epimorphism and $K = \mathrm{Ker}\,\gamma$. Then the short exact sequence $0 \longrightarrow K \longrightarrow P \xrightarrow{\gamma} M \longrightarrow 0$ is in $\mathcal{P}$. For every R-module A, we have the following commutative diagram with exact rows and inclusion map α:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Hom}(K,A) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_{\mathcal{P}}(M,A) \arrow[r] \arrow[d, "\alpha"] & \mathrm{Ext}^1_{\mathcal{P}}(P,A) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}(K,A) \arrow[r] & \mathrm{Ext}^1_R(M,A) \arrow[r] & \mathrm{Ext}^1_R(P,A) \arrow[r] & \cdots
\end{tikzcd} \tag{3.12}$$

where the equality $\mathrm{Ext}^1_{\mathcal{P}}(P,A) = \mathrm{Ext}^1_R(P,A) = 0$ holds, since P is projective. Then $\mathrm{Ext}^1_{\mathcal{P}}(M,A) = \mathrm{Ext}^1_R(M,A)$, hence M is $\mathcal{P}$-coprojective. □

Corollary 3.1 ((Misina and Skornjakov 1960), Proposition 1.13)
If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *is a short exact sequence in a proper class* $\mathcal{P}$ *and* B *is* $\mathcal{P}$*-coprojective, then* C *is also* $\mathcal{P}$*-coprojective.*

Proof Take any epimorphism $\gamma : P \to C$ from a projective R-module P to C. We have the following diagram with the exact columns and rows:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & Ker\gamma \arrow[r, equal] \arrow[d] & Ker\gamma \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B' \arrow[r, "\alpha'"] \arrow[d, "\gamma'"] & P \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r, "\alpha"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since B is $\mathcal{P}$-coprojective, γ' is $\mathcal{P}$-epimorphism. Since α is $\mathcal{P}$-epimorphism, $\gamma \circ \alpha' = \alpha \circ \gamma'$ is $\mathcal{P}$-epimorphism by $(P-3')$. By $(P-4')$, γ is $\mathcal{P}$-epimorphism. Therefore C is $\mathcal{P}$-coprojective by Proposition 3.3. □

Dually, for $\mathcal{P}$-coinjective modules we have the following proposition:

Proposition 3.4 ((Misina and Skornjakov 1960), Proposition 1.7) *An R-module* N *is* $\mathcal{P}$*-coinjective if and only if there is* $\mathcal{P}$*-monomorphism from* N *to an injective module* I*.*

Proof ($\Rightarrow$) Take any monomorphism $\alpha : N \longrightarrow I$ from N to an injective R-module I. Since N is $\mathcal{P}$-coinjective, α is a $\mathcal{P}$-monomorphism.

($\Leftarrow$) Let $\alpha : N \longrightarrow I$ be a $\mathcal{P}$-monomorphism and $L = I/\operatorname{Im}\alpha$. Then the short exact sequence $0 \longrightarrow N \xrightarrow{\alpha} I \longrightarrow L \longrightarrow 0$ is in $\mathcal{P}$. For every R-module B, we have the following exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Hom}(B,L) \arrow[r] \arrow[d, equal] & \operatorname{Ext}^1_{\mathcal{P}}(B,N) \arrow[r] \arrow[d] & \operatorname{Ext}^1_{\mathcal{P}}(B,I) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \operatorname{Hom}(B,L) \arrow[r] & \operatorname{Ext}^1_{R}(B,N) \arrow[r] & \operatorname{Ext}^1_{R}(B,I) \arrow[r] & \cdots
\end{tikzcd} \tag{3.13}$$

where the equality $\operatorname{Ext}^1_{\mathcal{P}}(B,I) = \operatorname{Ext}^1_{R}(B,I)=0$ holds, since I is injective. Then $\operatorname{Ext}^1_{\mathcal{P}}(B,N) = \operatorname{Ext}^1_{R}(B,N)$, i.e. N is $\mathcal{P}$-coinjective. □

Corollary 3.2 ((Misina and Skornjakov 1960), Proposition 1.8)
If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence in a proper class $\mathcal{P}$ and B is $\mathcal{P}$-coinjective, then A is also $\mathcal{P}$-coinjective.

3.4. Projectively Generated Proper Classes

For a given class $\mathcal{M}$ of modules, denote by $\pi^{-1}(\mathcal{M})$ the class of all short exact sequences $\mathbb{E}$ of R-modules and R-module homomorphisms such that $\operatorname{Hom}(M,\mathbb{E})$ is exact for all $M \in \mathcal{M}$, that is,

$$\pi^{-1}(\mathcal{M}) = \{\mathbb{E} \in \mathcal{A}bs_{R\text{-}\mathcal{M}od} | \operatorname{Hom}(M,\mathbb{E}) \text{ is exact for all } M \in \mathcal{M}\}.$$

$\pi^{-1}(\mathcal{M})$ is a proper class by Lemma 3.1 if we take $T(M,\cdot) = \operatorname{Hom}(M,\cdot)$. In fact $\pi^{-1}(\mathcal{M})$ is the largest proper class $\mathcal{P}$ for which each $M \in \mathcal{M}$ is $\mathcal{P}$-projective and it is called the proper class *projectively generated* by $\mathcal{M}$.

Taking $T(M,\cdot) = \operatorname{Hom}(M,\cdot)$, we obtain also the following consequence of Lemma 3.2.

Proposition 3.5 *Let $\mathcal{P}$ be a proper class and $\mathcal{M}$ be a class of modules. Then we have*

1. $\mathcal{P} \subseteq \pi^{-1}(\pi(\mathcal{P}))$,

2. $\mathcal{M} \subseteq \pi(\pi^{-1}(\mathcal{M}))$,

3. $\pi(\mathcal{P}) = \pi(\pi^{-1}(\pi(\mathcal{P})))$,

4. $\pi^{-1}(\mathcal{M}) = \pi^{-1}(\pi(\pi^{-1}(\mathcal{M})))$.

For a proper class $\mathcal{P}$, the *projective closure* of $\mathcal{P}$ is the proper class $\pi^{-1}(\pi(\mathcal{P}))$ which contains $\mathcal{P}$. If the projective closure of $\mathcal{P}$ is equal to $\mathcal{P}$ itself, then it is said to be *projectively closed*, and that occurs if and only if it is projectively generated.

Proposition 3.6 ((Sklyarenko 1978),Proposition 1.1) *Every projective proper class is projectively generated.*

Let $\mathcal{P}$ be a proper class of R-modules. Direct sums of $\mathcal{P}$-projective modules are $\mathcal{P}$-projective. Direct summand of an $\mathcal{P}$-projective module is $\mathcal{P}$-projective.

A proper class $\mathcal{P}$ is called $\prod$-*closed* if for *every* collection $\{\mathbb{E}_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{P}$, the product $\mathbb{E} = \prod_{\lambda\in\Lambda} \mathbb{E}_\lambda$ is in $\mathcal{P}$, too.

Proposition 3.7 ((Sklyarenko 1978),Proposition 1.2) *Every projectively generated proper class is* $\prod$*-closed.*

A subclass $\mathcal{M}$ of a class $\overline{\mathcal{M}}$ of modules is called a *projective basis* for $\overline{\mathcal{M}}$ if every module in $\overline{\mathcal{M}}$ is a direct summand of a direct sum of modules in $\mathcal{M}$ and of free modules.

Proposition 3.8 ((Sklyarenko 1978),Proposition 2.1) *If* $\mathcal{M}$ *is a set, then the proper class* $\pi^{-1}(\mathcal{M})$ *is projective, and* $\mathcal{M}$ *is a projective basis for the class of all* $\mathcal{P}$*-projective modules.*

There are some criteria for $\pi^{-1}(\mathcal{M})$ to be projective even when $\mathcal{M}$ is not a set.

Proposition 3.9 ((Sklyarenko 1978),Proposition 2.3) *If* $\mathcal{M}$ *is a class of modules closed under passage to factor modules, then the proper class* $\pi^{-1}(\mathcal{M})$ *is projective, and* $\mathcal{M}$ *is a projective basis for the class of all* $\mathcal{P}$*-projective modules.*

Theorem 3.2 ((Sklyarenko 1978), Theorem 1.2) *Let* $\mathcal{M}$ *be a class of modules. Consider the class* $\mathcal{R}$*, defined as:*

$$\mathrm{Ext}^1_{\mathcal{R}}(C, A) = \bigcap_{M,f} \mathrm{Ker}\{f^1 : \mathrm{Ext}^1_R(C, A) \to \mathrm{Ext}^1_R(M, A)\}$$

over all $M \in \mathcal{M}$ and all homomorphisms $f : M \longrightarrow C$. Then exact triples $0 \longrightarrow A \longrightarrow X \longrightarrow C \longrightarrow 0$ belonging to $\mathrm{Ext}_{\mathcal{R}}(C, A)$*, form a proper class and $\mathcal{R}$ coincides with $\pi^{-1}(\mathcal{M})$.*

3.5. Injectively Generated Proper Classes

For a given class $\mathcal{M}$ of modules, denote by $\iota^{-1}(\mathcal{M})$ the class of all short exact sequences $\mathbb{E}$ of R-modules and R-module homomorphisms such that $\mathrm{Hom}(\mathbb{E}, M)$ is exact for all $M \in \mathcal{M}$, that is,

$$\iota^{-1}(\mathcal{M}) = \{\mathbb{E} \in \mathcal{A}bs_{R\text{-}\mathcal{M}od} | \,\mathrm{Hom}(\mathbb{E}, M) \text{ is exact for all } M \in \mathcal{M}\}. \tag{3.14}$$

$\iota^{-1}(\mathcal{M})$ is a proper class by Lemma 3.1 if we take $T(M, \cdot) = \mathrm{Hom}(\cdot, M)$. In fact $\iota^{-1}(\mathcal{M})$ is the largest proper class $\mathcal{P}$ for which each $M \in \mathcal{M}$ is $\mathcal{P}$-injective which is called the proper class *injectively generated* by $\mathcal{M}$.

For a proper class $\mathcal{P}$, the *injective closure* of $\mathcal{P}$ is the proper class $\iota^{-1}(\iota(\mathcal{P}))$ which contains $\mathcal{P}$. If the injective closure of $\mathcal{P}$ is equal to $\mathcal{P}$ itself, then it is said to be *injectively closed*, and that occurs if and only if it is injectively generated.

Proposition 3.10 ((Sklyarenko 1978),Proposition 3.1) *Every injective proper class is injectively generated.*

Let $\mathcal{P}$ be a proper class of R-modules. Direct product of $\mathcal{P}$-injective modules is $\mathcal{P}$-injective. Direct summand of an $\mathcal{P}$-injective module is $\mathcal{P}$-injective.

A proper class $\mathcal{P}$ is called $\oplus$*-closed* if for *every* collection $\{\mathbb{E}_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{P}$, the direct sum $\mathbb{E} = \bigoplus_{\lambda\in\Lambda} \mathbb{E}_\lambda$ is in $\mathcal{P}$, too.

Proposition 3.11 ((Sklyarenko 1978),Proposition 1.2) *Every injectively generated proper class is $\oplus$-closed.*

An injective module is called *elementary* if it coincides with the injective envelope of some *cyclic* submodule. Such modules form a set and every injective module can be embedded in a direct product of elementary injective modules (Sklyarenko 1978, Lemma 3.1).

A subclass $\mathcal{M}$ of a class $\overline{\mathcal{M}}$ of modules is called an *injective basis* for $\overline{\mathcal{M}}$ if every module in $\overline{\mathcal{M}}$ is a direct summand of a direct product of modules in $\mathcal{M}$ and of certain elementary injective modules.

Proposition 3.12 ((Sklyarenko 1978),Proposition 3.3) *If $\mathcal{M}$ is a set, then the proper class $\iota^{-1}(\mathcal{M})$ is injective, and $\mathcal{M}$ is an injective basis for the class of all $\mathcal{P}$-injective modules.*

Even when $\mathcal{M}$ is not a set but:

Proposition 3.13 ((Sklyarenko 1978), Proposition 3.4) *If $\mathcal{M}$ is a class of modules closed under taking submodules, then the proper class $\iota^{-1}(\mathcal{M})$ is injective, and $\mathcal{M}$ is an injective basis for the class of all $\mathcal{P}$-injective modules.*

Theorem 3.3 ((Sklyarenko 1978), Theorem 3.2) *Let $\mathcal{M}$ be a class of modules. Consider the class $\mathcal{R}$, defined as:*

$$\mathrm{Ext}^1_{\mathcal{R}}(C,A) = \bigcap_{M,f} \mathrm{Ker}\{f_1 : \mathrm{Ext}^1_R(C,A) \to \mathrm{Ext}^1_R(C,M)\}$$

over all $M \in \mathcal{M}$ and all homomorphisms $f : A \longrightarrow M$. Then exact triples $0 \longrightarrow A \longrightarrow X \longrightarrow C \longrightarrow 0$ belonging to $\mathrm{Ext}_{\mathcal{R}}(C,A)$, form a proper class and $\mathcal{R}$ coincides with $\iota^{-1}(\mathcal{M})$.

3.6. Coinjective and Coprojective Modules with Respect to a Projectively or Injectively Generated Proper Class

Throughout this section let $\mathcal{P}$ be a proper class of R-modules.

Proposition 3.14 ((Sklyarenko 1978),Proposition 9.1) *The intersection of the classes of all $\mathcal{P}$-projective modules and $\mathcal{P}$-coprojective modules coincides with the class of all projective R-modules.*

Proposition 3.15 ((Sklyarenko 1978),Proposition 9.2) *The intersection of the classes of all $\mathcal{P}$-injective modules and $\mathcal{P}$-coinjective modules is the class of all injective R-modules.*

Proposition 3.16 ((Sklyarenko 1978),Proposition 9.3)

1. *If $\mathcal{P}$ is injectively closed, then every direct sum of $\mathcal{P}$-coinjective modules is $\mathcal{P}$-coinjective.*

2. *If $\mathcal{P}$ is $\prod$-closed, then every product of $\mathcal{P}$-coinjective modules is $\mathcal{P}$-coinjective.*

3. *If $\mathcal{P}$ is $\oplus$-closed, then every direct sum of $\mathcal{P}$-coprojective modules is $\mathcal{P}$-coprojective.*

Proposition 3.17 ((Sklyarenko 1978), Proposition 9.4) *If $\mathcal{P}$ is injectively generated, then for an R-module C, the condition* $\mathrm{Ext}^1_R(C, J) = 0$ *for all $\mathcal{P}$-injective J is equivalent to C being $\mathcal{P}$-coprojective.*

Moreover:

Proposition 3.18 *If $\mathcal{P} = \iota^{-1}(\mathcal{M})$ for a class $\mathcal{M}$ of modules, then for an R-module C, the condition* $\mathrm{Ext}^1_R(C, M) = 0$ *for all $M \in \mathcal{M}$ is equivalent to C being $\mathcal{P}$-coprojective.*

Proof Suppose C is a $\mathcal{P}$-coprojective module. Let $M \in \mathcal{M}$. Take an element $[\mathbb{E}] \in \mathrm{Ext}^1_R(C, M)$:

$$\mathbb{E} : 0 \longrightarrow M \longrightarrow B \longrightarrow C \longrightarrow 0$$

Since C is $\mathcal{P}$-coprojective, $\mathbb{E} \in \mathcal{P}$. Then $\mathbb{E}$ splits because M, being an element of $\mathcal{M}$, is $\mathcal{P}$-injective as $\mathcal{P} = \iota^{-1}(\mathcal{M})$. Hence $[\mathbb{E}] = 0$ as required. Thus $\mathrm{Ext}^1_R(C, M) = 0$.

Conversely, suppose for an R-module C, $\mathrm{Ext}^1_R(C, M) = 0$ for all $M \in \mathcal{M}$. Take any short exact sequence $\mathbb{E}$ of R-modules ending with C:

$$\mathbb{E} : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Applying $\mathrm{Hom}(-, M)$, we obtain the following exact sequence by the long exact sequence connecting Hom and Ext:

$$0 \longrightarrow \mathrm{Hom}(C, M) \longrightarrow \mathrm{Hom}(B, M) \longrightarrow \mathrm{Hom}(A, M) \longrightarrow \mathrm{Ext}^1_R(C, M) = 0$$

So $\mathrm{Hom}(\mathbb{E}, M)$ is exact for every $M \in \mathcal{M}$. This means $\mathbb{E} \in \iota^{-1}(\mathcal{M}) = \mathcal{P}$. □

Proposition 3.19 ((Sklyarenko 1978), Proposition 9.5) *If $\mathcal{P}$ is projectively generated, then for an R-module A, the condition* $\operatorname{Ext}^1_R(P, A) = 0$ *for all $\mathcal{P}$-projective P is equivalent to A being $\mathcal{P}$-coinjective.*

Moreover:

Proposition 3.20 *If $\mathcal{P} = \pi^{-1}(\mathcal{M})$ for a class $\mathcal{M}$ of modules, then for an R-module A, the condition* $\operatorname{Ext}^1_R(M, A) = 0$ *for all $M \in \mathcal{M}$ is equivalent to A being $\mathcal{P}$-coinjective.*

Proof Suppose A is a $\mathcal{P}$-coinjective module. Let $M \in \mathcal{M}$. Take an element $[\mathbb{E}] \in \operatorname{Ext}^1_R(M, A)$:

$$\mathbb{E} : 0 \longrightarrow A \longrightarrow B \longrightarrow M \longrightarrow 0$$

Since A is $\mathcal{P}$-coinjective, $\mathbb{E} \in \mathcal{P}$. Then $\mathbb{E}$ splits because M, being an element of $\mathcal{M}$, is $\mathcal{P}$-projective as $\mathcal{P} = \pi^{-1}(\mathcal{M})$. Hence $[\mathbb{E}] = 0$ as required. Thus $\operatorname{Ext}^1_R(M, A) = 0$.

Conversely, suppose for an R-module A, $\operatorname{Ext}^1_R(M, A) = 0$ for all $M \in \mathcal{M}$. Take any short exact sequence $\mathbb{E}$ of R-modules starting with A:

$$\mathbb{E} : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Applying $\operatorname{Hom}(M, -)$, we obtain the following exact sequence by the long exact sequence connecting Hom and Ext:

$$0 \longrightarrow \operatorname{Hom}(M, A) \longrightarrow \operatorname{Hom}(M, B) \longrightarrow \operatorname{Hom}(M, C) \longrightarrow \operatorname{Ext}^1_R(M, A) = 0$$

So $\operatorname{Hom}(M, \mathbb{E})$ is exact for every $M \in \mathcal{M}$. This means $\mathbb{E} \in \pi^{-1}(\mathcal{M}) = \mathcal{P}$. □

3.7. Coprojectively and Coinjectively Generated Proper Classes

Let $\mathcal{M}$ and $\mathcal{J}$ be classes of modules over some ring R. The smallest proper class $\overline{k}(\mathcal{M})$ (resp. $\underline{k}(\mathcal{J})$) for which all modules in $\mathcal{M}$ (resp. $\mathcal{J}$) are coprojective (resp. coinjective) is said to be coprojectively (resp. coinjectively) generated by $\mathcal{M}$ (resp. $\mathcal{J}$).

Theorem 3.4 ((Alizade 1985a), Theorem 2) *Let $\mathcal{J}$ be a class of modules closed under* *extensions. Consider the class $\mathcal{R}$ of exact triples, defined as:*

$$\mathrm{Ext}_{\mathcal{R}}(C, A) = \bigcup_{I,\alpha} \mathrm{Im}\{\mathrm{Ext}(C, I) \xrightarrow{\alpha_*} \mathrm{Ext}(C, A)\}$$

over all $I \in \mathcal{J}$ and all homomorphisms $\alpha : I \longrightarrow A$. Then exact triples $0 \longrightarrow A \longrightarrow X \longrightarrow C \longrightarrow 0$ belonging to $\mathrm{Ext}_{\mathcal{R}}(C, A)$, form a proper class and $\mathcal{R}$ coincides with $\underline{k}(\mathcal{J})$.

Theorem 3.5 ((Alizade 1985a), Theorem 2) *Let $\mathcal{M}$ be a class of modules closed under extensions. Consider the class $\mathcal{R}$ of exact triples, defined as:*

$$\mathrm{Ext}_{\mathcal{R}}(C, A) = \bigcup_{M,\alpha} \mathrm{Im}\{\mathrm{Ext}(M, A) \xrightarrow{\alpha^*} \mathrm{Ext}(C, A)\}$$

over all $M \in \mathcal{M}$ and all homomorphisms $\alpha : C \longrightarrow M$. $\mathcal{R}$ is a proper class and coincides with $\overline{k}(\mathcal{M})$.

Definition 3.2 *For a proper class $\mathcal{P}$ of short exact sequences of R-modules, the global dimension of $\mathcal{P}$ is defined as*

$$gl.dim\mathcal{P} = inf\{n : Ext^{n+1}(C, A) = 0 \textit{for all A and C in R-modules}\}.$$

If there is no such n, then $gl.dim\mathcal{P} = \infty$.

Definition 3.3 *For a proper class $\mathcal{P}$ of short exact sequences of R-modules, the injective dimension of a module A with respect to $\mathcal{P}$ is defined by the formula*

$$inj.dimA = inf\{n : Ext^{n+1}(C, A) = 0 \textit{for all C in R-modules}\}.$$

Proposition 3.21 ((Alizade 1985b)) *If R is a hereditary ring, then $inj.dimA \leq 1$ for every proper class $\mathcal{P}$ and $\mathcal{P}$-coinjective module A.*

Proposition 3.22 ((Alizade 1985b)) *If $\underline{k}(\mathcal{J})$ is closed under extensions, then $gl.dim\underline{k}(\mathcal{J}) \leq gl.dimR$ for every coinjectively generated class $\underline{k}(\mathcal{J})$.*

Corollary 3.3 ((Alizade 1985b)) *If R is a hereditary ring, then $inj.dim\underline{k}(\mathcal{J}) \leq 1$ for every coinjectively generated class $\underline{k}(\mathcal{J})$.*

For more information about coprojectively and coinjectively generated proper classes see (Alizade 1985a),(Alizade 1985b) and (Alizade 1986).

CHAPTER 4

THE LEAST PROPER CLASS CONTAINING $\mathcal{WS}$

In this chapter, we investigate the class of short exact sequences related to weak supplements and the least proper class containing this class. In Section 4.1.,we give some definitions about some classes of short exact sequences and some relations about these classes. We give an example in order to show that the class of short exact sequences related to weak supplements need not be a proper class. In Section 4.2. , we define a new class of short exact sequences and then we show that it is a proper class and also it is the least proper class containing the class of short exact sequences related to weak supplements.

4.1. The $\mathcal{WS}$-Elements of $\mathrm{Ext}(C, A)$

A short exact sequence

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{4.1}$$

is called *κ-exact* if $\operatorname{Im} f$ has a supplement in B, i.e. a minimal element in the set $\{V \subset B \mid V + \operatorname{Im} f = B\}$. In this case we say that $\mathbb{E} \in \mathrm{Ext}(C, A)$ is a *κ-element* and the class of all *κ-exact* short exact sequences will be denoted by $\mathcal{S}$.

We denote by $\mathcal{WS}$ the class of short exact sequences 4.1, where $\operatorname{Im} f$ has (is) a weak supplement in B, i.e. there is a submodule K of B such that $\operatorname{Im} f + K = B$ and $\operatorname{Im} f \cap K \ll B$. We denote by $\mathcal{S}mall$ the class of short exact sequences 4.1 where $\operatorname{Im} f \ll B$.

$\mathcal{WS}$ need not be a proper class in general.

Example 4.1 *Let $R = \mathbb{Z}$ and consider the composition $\beta \circ \alpha$ of the monomorphisms $\alpha : 2\mathbb{Z} \longrightarrow \mathbb{Z}$ and $\beta : \mathbb{Z} \longrightarrow \mathbb{Q}$ where α and β are the corresponding inclusions. Then we have $0 \longrightarrow 2\mathbb{Z} \xrightarrow{\beta\circ\alpha} \mathbb{Q} \longrightarrow \mathbb{Q}/2\mathbb{Z} \longrightarrow 0$ is a $\mathcal{WS}$-element, but*

$0 \longrightarrow 2\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ *is not a* $\mathcal{WS}$*-element as* $2\mathbb{Z}$ *have not a weak supplement in* $\mathbb{Z}$*.*

If X is a $\mathcal{S}mall$-submodule of an R-module Y, then Y is a supplement of X in Y, so X is $\mathcal{S}$-submodule of Y. If U is a $\mathcal{S}$-submodule of an R-module Z, then a supplement V of U in Z is also a weak supplement, therefore U is a $\mathcal{WS}$-submodule of Z. These arguments give us the relation $\mathcal{S}mall \subseteq \mathcal{S} \subseteq \mathcal{WS}$ for any ring R.

4.2. The $\overline{\mathcal{WS}}$-Elements of $\mathrm{Ext}(C, A)$

The main problem with the investigation of the $\mathcal{WS}$-elements in $\mathrm{Ext}(C, A)$ is that they do not form a subgroup. The reason is the fact that while $\mathcal{WS}$-elements are preserved under $\mathrm{Ext}(g, f) : \mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C', A')$ with respect to the second variable, they are not preserved with respect to the first variable. We extend the class $\mathcal{WS}$ to the class $\overline{\mathcal{WS}}$, which consists of all images of $\mathcal{WS}$-elements of $\mathrm{Ext}(C', A)$ under $\mathrm{Ext}(f, 1_A) : \mathrm{Ext}(C', A) \longrightarrow \mathrm{Ext}(C, A)$ for all homomorphism $f : C \longrightarrow C'$. We will prove in this chapter that $\overline{\mathcal{WS}}$ is the least proper class containing $\mathcal{WS}$. To prove that $\overline{\mathcal{WS}}$ is a proper class we will use the result of (Nunke 1963, Theorem 1.1) that states that a class $\mathcal{P}$ of short exact sequences is proper if $\mathrm{Ext}_{\mathcal{P}}(C, A)$ is a subfunctor of $\mathrm{Ext}_R(C, A)$, then $\mathrm{Ext}_{\mathcal{P}}(C, A)$ is a subgroup of $\mathrm{Ext}_R(C, A)$ for every R-modules A, C and the composition of two $\mathcal{P}$-monomorphism(epimorphism) is a $\mathcal{P}$-monomorphism(epimorphism).

Definition 4.1 *A short exact sequence* $\mathbb{E} : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *is said to be extended weak supplement if there is a short exact sequence* $\mathbb{E}' : 0 \xrightarrow{f} A \longrightarrow B' \longrightarrow C' \longrightarrow 0$ *such that* $Im(f)$ *has(is) a weak supplement and there is a homomorphism* $g : C \longrightarrow C'$ *such that* $\mathbb{E} = g^*(\mathbb{E}')$*, that is, there is a commutative diagram:*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "g"] & 0 : \mathbb{E} \\
0 \arrow[r] & A \arrow[r, "f"] & B' \arrow[r] & C' \arrow[r] & 0 : \mathbb{E}'
\end{tikzcd}$$

The class of all extended weak supplement short exact sequences will be denoted by

$\overline{\mathcal{WS}}$. *So* $\operatorname{Ext}_{\overline{\mathcal{WS}}}(C, A) = \{\, \mathbb{E} : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \ \mid \mathbb{E} = g^*(\mathbb{E}')$ *for some* $\mathbb{E}' : 0 \longrightarrow A \longrightarrow B \longrightarrow C' \longrightarrow 0 \ \in \mathcal{WS}$ *and* $g : C \to C' \,\}$.

Lemma 4.1 *If* $f : A \longrightarrow A'$*, then* $f_* : \operatorname{Ext}(C, A) \longrightarrow \operatorname{Ext}(C, A')$ *preserves* $\mathcal{WS}$*-elements.*

Proof Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence in the class $\mathcal{WS}$ and $f : A \longrightarrow A'$ be an arbitrary homomorphism. We have the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "f'"'] & C \arrow[r] \arrow[d, equal] & 0 : E \\
0 \arrow[r] & A' \arrow[r, "\alpha'"] & B' \arrow[r] & C \arrow[r] & 0 : E_1
\end{tikzcd}$$

where $E_1 = f_*(E)$.

If V is a weak supplement of $Im\alpha$ in B, then $Im\alpha + V = B$ and $Im\alpha \cap V \ll B$. Then $f'(V) + Im\alpha' = B'$ by push out diagram and $f'(V) \cap Im\alpha' = f'(Im\alpha \cap V) \ll f'(B) \subseteq B'$. So $E_1 \in \mathcal{WS}$. □

Lemma 4.2 *If* $f : A \longrightarrow A'$*, then* $f_* : \operatorname{Ext}(C, A) \longrightarrow \operatorname{Ext}(C, A')$ *preserves* $\overline{\mathcal{WS}}$*-elements.*

Proof Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence in the class $\overline{\mathcal{WS}}$ and $f : A \longrightarrow A'$ be an arbitrary homomorphism. Then there is $E' : 0 \longrightarrow A \longrightarrow B_1 \longrightarrow C_1 \longrightarrow 0 \ \in \mathcal{WS}$ and a homomorphism $g : C \longrightarrow C_1$ such that $E = g^*(E')$. We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[rr] & & A \arrow[rr] \arrow[dl, "f"'] \arrow[dd, equal] & & B \arrow[rr] \arrow[dl] \arrow[dd] & & C \arrow[rr] \arrow[dl, equal] \arrow[dd, "g"] & & 0 : E \\
0 \arrow[rr] & & A' \arrow[rr] \arrow[dd, equal] & & B' \arrow[rr] \arrow[dd, "\alpha"] & & C \arrow[rr] \arrow[dd, "g"] & & 0 : E_1 & \\
 & 0 \arrow[rr] & & A \arrow[rr] \arrow[dl, "f"'] & & B_1 \arrow[rr] \arrow[dl, "f'"] & & C_1 \arrow[rr] \arrow[dl, equal] & & 0 : E' \\
0 \arrow[rr] & & A' \arrow[rr, "\alpha'"] & & B_2 \arrow[rr] & & C_1 \arrow[rr] & & 0 : E'_1 &
\end{tikzcd}$$

where $E'_1 = f_*(E')$.

Then $E_1 = f_*(E) = f_* \circ g^*(E') = g^* \circ f_*(E') = g^*(E'_1)$. Since $E' \in \mathcal{WS}$, $E'_1 = f_*(E') \in \mathcal{WS}$, and so $g^*(E'_1) = E_1 \in \overline{\mathcal{WS}}$. □

Lemma 4.3 *If* $g : C' \longrightarrow C$, *then* $g^* : \mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C', A)$ *preserves* $\overline{\mathcal{WS}}$-*elements.*

Proof Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence in the class $\overline{\mathcal{WS}}$ and $g : C' \longrightarrow C$ be an arbitrary homomorphism.Then there is $E_1 : 0 \longrightarrow A \longrightarrow B_1 \longrightarrow C_1 \longrightarrow 0 \in \mathcal{WS}$ and a homomorphism $f : C \longrightarrow C_1$ such that $E = f^*(E_1)$. We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d, "g"'] & 0 : E' \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "f"'] & 0 : E \\
0 \arrow[r] & A \arrow[r] & B_1 \arrow[r] & C_1 \arrow[r] & 0 : E_1
\end{tikzcd}$$

where $E' = g^*(E)$.

$E' = g^* \circ f^*(E_1) = (f \circ g)^*(E_1)$. Since $E_1 \in \mathcal{WS}$, $E' \in \overline{\mathcal{WS}}$. □

Corollary 4.1 *Every multiple of a* $\overline{\mathcal{WS}}$-*element of* $\mathrm{Ext}(C, A)$ *is again a* $\overline{\mathcal{WS}}$-*element.*

Proposition 4.1 *If* $E_1, E_2 \in \mathrm{Ext}_{\mathcal{WS}}(C, A)$, *then* $E_1 \oplus E_2 \in \mathrm{Ext}_{\mathcal{WS}}(C \oplus C, A \oplus A)$.

Proof Let $E_1, E_2 \in \mathrm{Ext}_{\mathcal{WS}}(C, A)$, then there exist a submodule V_i in B_i such that $V_i + A = B_i$ and $V_i \cap A \ll B_i$, $i = 1, 2$. Then

$$E_1 \oplus E_2 : 0 \longrightarrow A \oplus A \longrightarrow B_1 \oplus B_2 \longrightarrow C \oplus C \longrightarrow 0 \in \mathcal{WS}$$

since $(A \oplus A) + (V_1 \oplus V_2) = B_1 \oplus B_2$ and $(A \oplus A) \cap (V_1 \oplus V_2) = (V_1 \cap A) \oplus (V_2 \cap A) \ll B_1 \oplus B_2$. □

Corollary 4.2 *The* $\overline{\mathcal{WS}}$-*elements of* $\mathrm{Ext}(C, A)$ *form a subgroup.*

Proof Let $E_1, E_2 \in \mathrm{Ext}_{\overline{\mathcal{WS}}}(C, A)$. We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
E_1 \oplus E_2 : 0 \arrow[r] & A \oplus A \arrow[r] \arrow[d, equal] & B_1 \oplus B_2 \arrow[r] \arrow[d] & C \oplus C \arrow[r] \arrow[d] & 0 \\
E_1' \oplus E_2' : 0 \arrow[r] & A \oplus A \arrow[r] & B_1' \oplus B_2' \arrow[r] & C_1' \oplus C_2' \arrow[r] & 0
\end{tikzcd}$$

where E_1 and E_2 are the image of short exact sequences E_1' and E_2' from $\mathcal{WS}$ respectively. $E_1' \oplus E_2'$ is $\mathcal{WS}$-element by Proposition 4.1 and so $E_1 \oplus E_2$ is $\overline{\mathcal{WS}}$-element. By Theorem 2.1, $E_1 + E_2 = \nabla_A(E_1 \oplus E_2)\Delta_C$ where the diagonal map $\Delta_C : c \mapsto (c, c)$ and the codiagonal map $\nabla_A : (a_1, a_2) \mapsto a_1 + a_2$. So we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \oplus A \arrow[r] \arrow[d, "\nabla_A"'] & B_1 \oplus B_2 \arrow[r] \arrow[d] & C \oplus C \arrow[r] \arrow[d, equal] & 0 : E_1 \oplus E_2 \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & X \arrow[r] & C \oplus C \arrow[r] & 0 : E \\
0 \arrow[r] & A \arrow[r] & Y \arrow[r] \arrow[u] & C \arrow[r] \arrow[u, "\Delta_C"'] & 0 : E_1 + E_2
\end{tikzcd}$$

Then E is in $\overline{\mathcal{WS}}$ by Lemma 4.2, $E_1 + E_2$ is in $\overline{\mathcal{WS}}$ by Lemma 4.3. □

Now by (Nunke 1963, Theorem 1.1) to prove that $\overline{\mathcal{WS}}$ class is a proper class it remains only to show that the composition of two $\overline{\mathcal{WS}}$-monomorphisms(or epimorphisms) is $\overline{\mathcal{WS}}$-monomorphisms(or epimorphisms). Firstly we prove some useful results.

Lemma 4.4 *Let $A \subseteq B \subseteq C$ be R-modules. If A is direct summand in B and B has a weak supplement in C, then the short exact sequence* $0 \longrightarrow A \longrightarrow C \longrightarrow C/A \longrightarrow 0$ *is in* $\overline{\mathcal{WS}}$.

Proof Let $B = A \oplus B'$. We have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
E_1 : 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & A \oplus B' \arrow[r] \arrow[d, "f"] & B' \arrow[r] \arrow[d] & 0 \\
E_2 : 0 \arrow[r] & A \arrow[r] & C \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & 0 \\
 & & C' \arrow[r, equal] \arrow[d] & C' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

E_3

By the codiagonal map $\nabla_C : (c_1, c_2) \mapsto c_1 + c_2$ and the monomorphism $f_A \oplus f_{B'}$: $(a, b') \mapsto (f(a), f(b'))$, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
E_1' : 0 \arrow[r] & A \oplus B' \arrow[r, "f_A \oplus f_{B'}"] \arrow[d, equal] & C \oplus C \arrow[r] \arrow[d, "\nabla_C"] & B_1 \oplus D \arrow[r] \arrow[d] & 0 \\
E_3 : 0 \arrow[r] & A \oplus B' \arrow[r, "f"] & C \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

Since E_3 is in $\mathcal{WS}$, E_1' is in $\overline{\mathcal{WS}}$. By the monomorphisms $f_A \oplus 1_{B'} : (a, b') \mapsto (f(a), b')$ and $1_C \oplus f_{B'} : (c, b') \mapsto (c, f(b'))$, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
E_2' : 0 \arrow[r] & A \oplus B' \arrow[r, "f_A \oplus 1_{B'}"] \arrow[d, equal] & C \oplus B' \arrow[r] \arrow[d, "1_C \oplus f_{B'}"] & B_1 \arrow[r] \arrow[d] & 0 \\
E_1' : 0 \arrow[r] & A \oplus B' \arrow[r, "f_A \oplus f_{B'}"] & C \oplus C \arrow[r] & B_1 \oplus D \arrow[r] & 0
\end{tikzcd}$$

E'_2 is in $\overline{\mathcal{WS}}$, by Lemma 4.3. Finally, the following diagram is commutative with exact rows and by Lemma 4.2, E_2 is in $\overline{\mathcal{WS}}$.

$$\begin{tikzcd}
E'_2 : 0 \arrow[r] & A \oplus B' \arrow[r, "f_A \oplus 1_{B'}"] \arrow[d, "1_A \oplus 0_{B'}"] & C \oplus B' \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d, equal] & 0 \\
E_2 : 0 \arrow[r] & A \arrow[r] & C \arrow[r] & B_1 \arrow[r] & 0
\end{tikzcd}$$

□

Lemma 4.5 *The composition of an Small-epimorphism and a $\mathcal{WS}$-epimorphism is a $\mathcal{WS}$-epimorphism.*

Proof Let $f : B \to B'$ be a small epimorphism and $h : B' \to C$ be a $\mathcal{WS}$-epimorphism; i.e. we have a commutative exact diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r, "h \circ f"] \arrow[d, "f"'] & C \arrow[r] \arrow[d, equal] & 0 : E \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r, "h"] \arrow[d] & C \arrow[r] & 0 : E_1 \\
 & 0 & 0 & &
\end{tikzcd}$$

E_2

with $E_2 \in Small$ and $E_1 \in \mathcal{WS}$. Then without of loss generality we can assume that $K \ll B$ and A/K has a weak supplement in B/K. So there is a submodule D/K of B/K such that $D/K + A/K = B/K$ and $(D \cap A)/K \ll B/K$. Therefore we have $A + D = B$ and $A \cap D \ll B$, i.e. A has a weak supplement in B. □

Lemma 4.6 *Let R be hereditary ring. For a $\overline{\mathcal{WS}}$ class of short exact sequences of R modules, the composition of an Small-epimorphism and a $\overline{\mathcal{WS}}$-epimorphism is a $\overline{\mathcal{WS}}$-epimorphism.*

Proof Let $f : B \to B'$ be a small epimorphism and $h : B' \to C$ be a $\overline{\mathcal{WS}}$-epimorphism; i.e. we have a commutative exact diagram:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d, "g"] & B \arrow[r, "h\circ f"] \arrow[d, "f"'] & C \arrow[r] \arrow[d, equal] & 0 : E \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r, "h"] \arrow[d] & C \arrow[r] & 0 : E_1 \\
 & 0 & 0 & &
\end{tikzcd}
$$

E_2

with $E_2 \in Small$ and $E_1 \in \overline{\mathcal{WS}}$. Then there is a commutative diagram with exact rows and with $E_3 \in \mathcal{WS}$:

$$
\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, equal] & B' \arrow[r, "h"] \arrow[d] & C \arrow[r] \arrow[d] & 0 : E_1 \\
0 \arrow[r] & A' \arrow[r] & B_1 \arrow[r] & C_1 \arrow[r] & 0 : E_3
\end{tikzcd}
$$

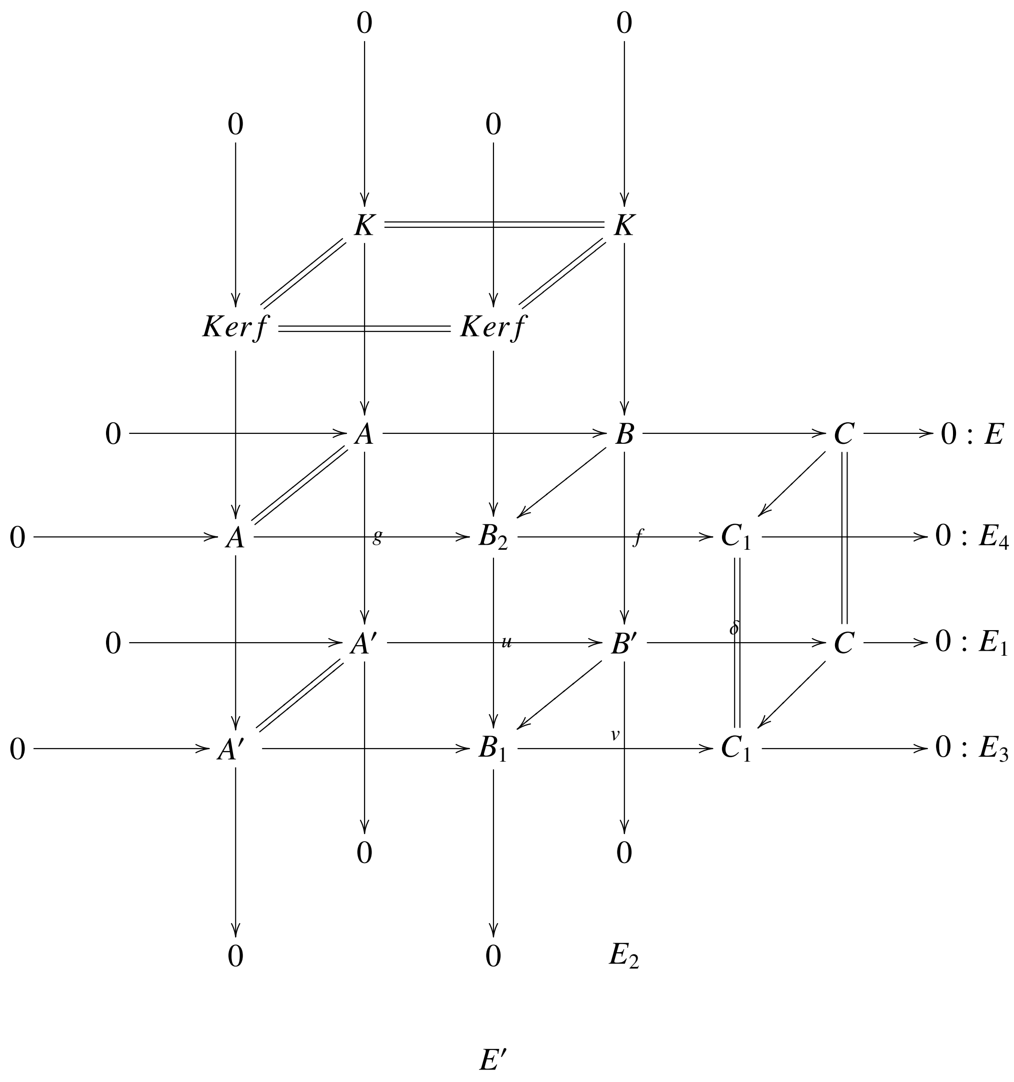
Since R is hereditary the homomorphism

$$Ext^1(1_{C_1}, g) : Ext^1(C_1, A) \to Ext^1(C_1, A')$$

is an epimorphism therefore

$$E_3 = Ext^1(1_{C_1}, g)(E_4)$$

for some $E_4 : 0 \longrightarrow A \longrightarrow B_2 \longrightarrow C_1 \longrightarrow 0$. Then we have the following commutative exact diagram:



Since $K = Kerf \ll B$, u is $Small$ epimorphism. Therefore $v \circ u$ is a $\mathcal{WS}$-epimorphism by Lemma 4.5, i.e. $E_4 \in \mathcal{WS}$. Then $E \in \overline{\mathcal{WS}}$. $\square$

Theorem 4.1 *If R is a hereditary ring, $\overline{\mathcal{WS}}$ is a proper class.*

Proof By Lemma 4.2, Lemma 4.3, Corollary 4.2, $Ext_{\overline{\mathcal{WS}}}(C, A)$ is an E-functor in the sense Buttler and Horrocks (1961). By (Nunke 1963, Theorem 1.1), it is sufficient to show that the composition of two $\overline{\mathcal{WS}}$ monomorphism is a $\overline{\mathcal{WS}}$ monomorphism. Let $f : A \to B$ and $g : B \to C$ be $\overline{\mathcal{WS}}$-monomorphisms. Then for the short exact sequence $E_2 : 0 \longrightarrow B \xrightarrow{g} C \longrightarrow F \longrightarrow 0 \in \overline{\mathcal{WS}}$ we have $E_2 = h^*(E_2')$ for some

$E_2' : 0 \longrightarrow B \longrightarrow C' \longrightarrow F' \longrightarrow 0 \in \mathcal{WS}$ and homomorphism $h : F \to F'$. Therefore we have a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & & & & 0 \arrow[dd] & & 0 \arrow[dd] & & \\
 & & & & 0 \arrow[dd] & & 0 \arrow[dd] & & & \\
 & 0 \arrow[rr] & & A \arrow[rr] \arrow[dd, equal] & & B \arrow[rr] \arrow[dd] & & D \arrow[rr] \arrow[dd] & & 0 : E_1 \\
0 \arrow[rr] & & A \arrow[rr, "f"] \arrow[ur, equal] \arrow[dd, equal] & & B \arrow[rr] \arrow[ur, equal] \arrow[dd] & & D \arrow[rr] \arrow[ur, equal] \arrow[dd] & & 0 : E_1 & \\
 & 0 \arrow[rr] & & A \arrow[rr, "g"] & & C' \arrow[rr] \arrow[dd] & & D_1 \arrow[rr] \arrow[dd] & & 0 : E' \\
0 \arrow[rr] & & A \arrow[rr] \arrow[ur, equal] & & C \arrow[rr] \arrow[ur] \arrow[dd] & & D' \arrow[rr] \arrow[ur, "x"] \arrow[dd] & & 0 : E & \\
 & & & & & F' \arrow[rr, equal] \arrow[dd] & & F' \arrow[dd] & & \\
 & & & & F \arrow[rr, equal] \arrow[ur, "h"] \arrow[dd] & & F \arrow[ur, "h"] \arrow[dd] & & & \\
 & & & & & 0 & & 0 & & \\
 & & & & 0 & E_2' & 0 & E_3' & & \\
 & & & & E_2 & & E_3 & & &
\end{tikzcd}$$

where E_2 and E_3 are images of E_2' and E_3' respectively under the first variable. Now for the short exact sequence $E_1 : 0 \longrightarrow A \xrightarrow{f} B \longrightarrow D \longrightarrow 0 \in \overline{\mathcal{WS}}$ we have $E_1 = u^*(E_1')$ for some $E_1' : 0 \longrightarrow A \longrightarrow B_1 \longrightarrow D_2 \longrightarrow 0 \in \mathcal{WS}$ and homomorphism $u : D \to D_2$.

Therefore we have a commutative diagram with exact rows and columns:

$$\begin{tikzcd}[column sep=small]
 & & & & & 0 \arrow[dd] & & 0 \arrow[dd] & & \\
 & & & & 0 \arrow[dd] & & 0 \arrow[dd] & & & \\
 & 0 \arrow[rr] & & A \arrow[rr] \arrow[dl, equal] \arrow[dd, equal] & & B_1 \arrow[rr] \arrow[dd] & & D_2 \arrow[rr] \arrow[dd] & & 0 : E'_1 \\
0 \arrow[rr] & & A \arrow[rr] \arrow[dd, equal] & & B \arrow[rr] \arrow[ur, "v"] \arrow[dd] & & D \arrow[rr] \arrow[ur, "u"] \arrow[dd] & & 0 : E_1 & \\
 & 0 \arrow[rr] & & A \arrow[rr] \arrow[dl, equal] & & C_1 \arrow[rr] \arrow[dd] & & D_3 \arrow[rr] \arrow[dd] & & 0 : E'' \\
0 \arrow[rr] & & A \arrow[rr] & & C' \arrow[rr] \arrow[ur] \arrow[dd] & & D_1 \arrow[rr] \arrow[ur, "y"] \arrow[dd] & & 0 : E' & \\
 & & & & & F' \arrow[rr, equal] \arrow[dl, equal] \arrow[dd] & & F' \arrow[dl, equal] \arrow[dd] & & \\
 & & & & F' \arrow[rr, equal] \arrow[dd] & & F' \arrow[dd] & & & \\
 & & & & & 0 & & 0 & & \\
 & & & & 0 & E''_2 & 0 & E''_3 & & \\
 & & & & E'_2 & & E'_3 & & &
\end{tikzcd}$$

where $E''_2 = v_*(E'_2)$, $E''_3 = u_*(E'_3)$. Without lost of generality we can assume that $A \leq B_1 \leq C_1$. Since $E'_1 \in \mathcal{WS}$, there is a submodule K of B_1 such that $A + K = B_1$ and $A \cap K \ll B_1$. Then $A/(A \cap K) \oplus K/(A \cap K) = B_1/A \cap K$, that is, $A/(A \cap K)$ is direct summand in

$B_1/(A \cap K)$. Then we have the following diagram with exact rows and columns:

$$
\begin{tikzcd}
 & & & & & 0 \arrow[dd] & & 0 \arrow[dd] & \\
 & & & & 0 \arrow[dd] & & 0 \arrow[dd] & & \\
 & 0 \arrow[rr] & & \frac{A}{A\cap K} \arrow[rr] \arrow[dd, equal] & & \frac{B_1}{A\cap K} \arrow[rr] \arrow[dd] & & D_2 \arrow[r] \arrow[dd] & 0 : E''_1 \\
0 \arrow[rr] & & A \arrow[rr] \arrow[ur, "\sigma^1"] \arrow[dd, equal] & & B_1 \arrow[rr] \arrow[ur, "\sigma^2"] \arrow[dd] & & D_2 \arrow[rr] \arrow[ur, equal] \arrow[dd] & & 0 : E'_1 \\
 & 0 \arrow[rr] & & \frac{A}{A\cap K} \arrow[rr] & & E \arrow[rr] \arrow[dd] & & D_3 \arrow[r] \arrow[dd] & 0 : E''' \\
0 \arrow[rr] & & A \arrow[rr] \arrow[ur] & & C_1 \arrow[rr] \arrow[ur, "w"] \arrow[dd] & & D_3 \arrow[rr] \arrow[ur, equal] \arrow[dd] & & 0 : E'' \\
 & & & & & F' \arrow[rr, equal] \arrow[dd] & & F' \arrow[dd] & \\
 & & & & F' \arrow[rr, equal] \arrow[ur, equal] \arrow[dd] & & F' \arrow[ur, equal] \arrow[dd] & & \\
 & & & & & 0 & & 0 & \\
 & & & & 0 & E'''_2 & 0 & & \\
 & & & & E''_2 & & & &
\end{tikzcd}
$$

where $\sigma^1 : A \to A/(A \cap K)$ and $\sigma^2 : B_1 \to B_1/(A \cap K)$ are canonical epimorphisms, $E''_1 = \sigma^1_*(E'_1)$, $E'''_2 = \sigma^2_*(E''_2)$. Since $E'_2 \in \mathcal{WS}$, E''_2 and E'''_2 are in $\mathcal{WS}$. By Lemma 4.4, $E''' \in \mathcal{WS}$. By 3×3 Lemma $Kerw = Ker\sigma^2 = A \cap K \ll C_1$. Therefore by Lemma 4.6 $E'' \in \overline{\mathcal{WS}}$. Now $E = (y \circ x)^*(E''') \in \mathcal{WS}$ by Lemma 4.3. □

Corollary 4.3 *If R is hereditary, then $\langle \mathcal{S}mall \rangle = \langle \mathcal{S} \rangle = \langle \mathcal{WS} \rangle = \overline{\mathcal{WS}}$.*

Proof The equivalence $\langle \mathcal{S}mall \rangle = \langle \mathcal{S} \rangle = \langle \mathcal{WS} \rangle$ had been proved in (Demirci 2008). Since $\langle \mathcal{WS} \rangle$ is the least proper class containing $\mathcal{WS}$ and $\mathcal{WS}$ is contained in the proper class $\overline{\mathcal{WS}}$, $\langle \mathcal{WS} \rangle \subseteq \overline{\mathcal{WS}}$. Conversely, let $\mathbb{E} : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \in \overline{\mathcal{WS}}$. Then there exists a short exact sequence $\mathbb{E}'$ in $\mathcal{WS}$ such that the following diagram is commutative.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 : \mathbb{E} \\
0 \arrow[r] & A \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 : \mathbb{E}'
\end{tikzcd}$$

Then $\mathbb{E}' \in \langle \mathcal{WS} \rangle$ and since $\langle \mathcal{WS} \rangle$ is proper class, $\mathbb{E} \in \langle \mathcal{WS} \rangle$ and we have that $\overline{\mathcal{WS}} \subseteq \langle \mathcal{WS} \rangle$. This completes the proof. □

CHAPTER 5

HOMOLOGICAL OBJECTS OF $\overline{\mathcal{WS}}$

In this Chapter, R denotes a Dedekind domain which is not a field and K denotes its field of fractions, we will denote the set of maximal ideals of R by Ω.

5.1. Coinjective Submodules with Respect to $\overline{\mathcal{WS}}$

Lemma 5.1 *Let R be a Dedekind ring. For an R-module M the following are equivalent:*

(i) A is $\overline{\mathcal{WS}}$-coinjective.

(ii) There is a submodule N of A such that N is small in the injective hull $\hat{A}$ of A and A/N is injective.

(iii) A has a weak supplement in its injective hull $\hat{A}$.

Proof $(i \Rightarrow ii)$ Let E be $\overline{\mathcal{WS}}$-element. By definition of $\overline{\mathcal{WS}}$, E is an image of a $\mathcal{WS}$-element, say E_4, such that $g^*(E_4) = E$. Then, there exist a submodule V of B such that $A + V = B$ and $A \cap V \ll B$. Since epimorphic image of a injective module is injective, $A/A \cap V$ which is direct summand of a epimorphic image of $\hat{A}$ is injective. And since A is essential in its injective hull $\hat{A}$, α is a monomorphism. So $\hat{A}$ is an injective submodule of B' and, $\hat{A}$ is a direct summand of B', and so $A \cap V \ll \hat{A}$. Then we obtain the following commutative diagram where $E', E_2 \in Small$ and $E_1, E_3 \in \mathcal{S}plit$.

$$
\begin{tikzcd}
 & & & 0 \arrow[dd] & & 0 \arrow[dd] & & & \\
 & & 0 \arrow[dd] & & 0 \arrow[dd] & & & & \\
 & & & A \cap V \arrow[rr, equal] \arrow[dl, equal] \arrow[dd] & & A \cap V \arrow[dl, equal] \arrow[dd] & & & \\
 & & A \cap V \arrow[rr, equal] \arrow[dd] & & A \cap V \arrow[dd] & & & & \\
 & 0 \arrow[rr] & & A \arrow[rr] \arrow[dl, equal] \arrow[dd] & & \hat{A} \arrow[rr, "f"] \arrow[dl, "\alpha"'] \arrow[dd] & & C \arrow[r] \arrow[dl, "g"] \arrow[dd, equal] & 0 : E \\
0 \arrow[rr] & & A \arrow[rr] \arrow[dd] & & B \arrow[rr, "\gamma"] \arrow[dd] & & C_1 \arrow[rr] \arrow[dd, equal] & & 0 : E_4 \\
 & 0 \arrow[rr] & & A' \arrow[rr] \arrow[dl, equal] \arrow[dd] & & B' \arrow[rr, "\delta"] \arrow[dl] \arrow[dd] & & C \arrow[r] \arrow[dl] & 0 : E_1 \\
0 \arrow[rr] & & A' \arrow[rr] \arrow[dd] & & B_1 \arrow[rr] \arrow[dd] & & C_1 \arrow[rr] & & 0 : E_3 \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 0 & E_2 & & & \\
 & & & & E' & & & &
\end{tikzcd}
$$

($ii \Rightarrow iii$) By the hypothesis, we obtain the following diagram where $E \in \mathcal{S}mall$ and $E_1 \in \mathcal{S}plit$.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & N \arrow[r, equal] \arrow[d] & N \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & \hat{A} \arrow[r, "f"] \arrow[d, "\gamma"'] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r, "\delta"] \arrow[d] & C \arrow[r] & 0 : E_1 \\
 & 0 & 0 & &
\end{tikzcd}
$$

$$E$$

Then γ is a $\mathcal{S}mall$-epimorphism and δ is a $\mathcal{S}plit$-epimorphism. So $f = \delta \circ \gamma$ is $\mathcal{WS}$-epimorphism by Lemma 4.5.

($iii \Rightarrow i$) By Proposition 3.4, since every $\mathcal{WS}$-element is an $\overline{\mathcal{WS}}$-element. □

Definition 5.1 *A module M is said to be coatomic if $Rad(M/U) \neq M/U$ for every proper submodule U of M or equivalently every proper submodule of M is contained in a maximal submodule of M.*

Lemma 5.2 ((Zöschinger 1978b), Lemma 2.1) *For an R-module M the following are equivalent:*

(i) M has a weak supplement in its injective hull $\hat{M}$.

(ii) There is an injective module I containing M such that M has a supplement in I.

(iii) There is an extension N of M, such that M is a direct summand in N and N has a supplement in its injective hull $\hat{N}$.

(iv) M has a dense coatomic submodule.

Proposition 5.1 ((Zöschinger 1974c),Proof of Lemma 3.3) *Let A, B be R-modules and $A \subseteq B$. Then $A \ll B$ if and only if A is coatomic and $A \subseteq \mathrm{Rad}\, B$.*

Proposition 5.2 *If there is a Small-monomorphism from a module A to any module A', then A is a $\overline{\mathcal{WS}}$-coinjective module.*

Proof Without of loss generality we can assume that $A \ll A'$. Then A is small in injective hull A'. Thus A is $\overline{\mathcal{WS}}$-coinjective by Proposition 3.4. □

Corollary 5.1 *Every coatomic module is a $\overline{\mathcal{WS}}$-coinjective.*

Proof Every coatomic submodule is small in its injective hull by Proposition 5.1. Then by Proposition 5.2, every coatomic module is a $\overline{\mathcal{WS}}$-coinjective. □

The converse of Corollary 5.1 is not true in general. For example the $\mathbb{Z}$-module $\mathbb{Q}$ is a weakly supplemented module and every submodule of $\mathbb{Q}$ is $\overline{\mathcal{WS}}$-coinjective. If we assume that every proper submodule of $\mathbb{Q}$ is coatomic, then we come to the conclusion that $\mathbb{Q}$ is hollow. But $\mathbb{Q}$ is not hollow and so $\mathbb{Q}$ has a $\overline{\mathcal{WS}}$-coinjective proper submodule which is not coataomic. And also the group of p-adic numbers, J_p, is $\overline{\mathcal{WS}}$-coinjective but not coatomic.

Proposition 5.3 *Let R be a domain. Then every bounded R-module is $\overline{\mathcal{WS}}$-coinjective.*

Proof Let B be a bounded R-module and I be an injective hull of B. We will show that $B \ll I$. Suppose $B + X = I$ for some $X \subset I$. Since B is bounded, there exists $0 \neq r \in R$ such that $rB = 0$. Then $I = rI = rB + rX = rX$, since I is divisible. Therefore $X = I$ and so $B \ll I$. I is $\langle\mathcal{S}mall\rangle$-coinjective, since it is injective. Then B is $\langle\mathcal{S}mall\rangle$-coinjective by Corollary 3.2. □

Lemma 5.3 ((Demirci 2008), Lemma 4.5) *Let S be a DVR, A be a reduced torsion S-module and B be a bounded submodule of A. If A/B is divisible, then A is also bounded.*

Lemma 5.4 *Let M is torsion and reduced module over a Discrete Valuation Ring . Then M is $\overline{\mathcal{WS}}$-coinjective iff M is coatomic.*

Proof ($\Rightarrow$)Since M is $\overline{\mathcal{WS}}$-coinjective, M has a dense coatomic submodule N by Lemma 5.2. Since M is torsion , N is torsion. Since N is coatomic, $N = B + R^n$ with $p^m B = 0$ for some $n \in N$ (Zöschinger 1974b). Since N is torsion $R^n = 0$ and N is bounded. By Lemma 5.3, M is bounded and so it is coatomic.

($\Leftarrow$)Since any coatomic module is small in its injective hull, it is $\langle\mathcal{S}mall\rangle$-coinjective and also it is $\overline{\mathcal{WS}}$-coinjective. □

Definition 5.2 *A module M is called radical-supplemented, if $Rad(M)$ has a supplement in M.*

Zöschinger proved that If M has a weak supplement in its injective hull, then $T(M)$ is radical-supplemented and there exists $n \geq 0$ with $\mathbf{p} - Rank(M/T(M)) \leq n$ for all maximal ideals $\mathbf{p}$ in (Zöschinger 1978b). From this we obtain the following Corollary by Proposition 3.4.

Corollary 5.2 *If M is a $\overline{\mathcal{WS}}$-coinjective, then $T(M)$ is radical-supplemented and there exists $n \geq 0$ with $\mathbf{p} - Rank(M/T(M)) \leq n$ for all maximal ideals $\mathbf{p}$.*

Zöschinger proved that the class of R-modules, which have a weak supplement in their injective hull is closed under factor modules and group extensions. This class contains all torsion-free modules with finite rank in (Zöschinger 1978b). From this we obtain the following Corollary by Proposition 3.4.

Corollary 5.3 *The class of R-modules, which $\overline{\mathcal{WS}}$-coinjective is closed under factor modules and group extensions. This class contains torsion-free modules with finite rank.*

Corollary 5.4 *Every finitely generated module is $\overline{\mathcal{WS}}$-coinjective.*

Proof Every finitely generated module is small in its injective hull. □

Theorem 5.1 *Let $\mathcal{J}$ be a class of modules which $\overline{\mathcal{WS}}$-coinjective. Then, $\underline{k}(\mathcal{J}) = \overline{\mathcal{WS}}$.*

Proof (⊇) Let E_1 be a $\overline{\mathcal{WS}}$-element. Then, there is a $\mathcal{WS}$-element E_2 such that the following diagram is commutative.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "g"'] & 0 : E_1 \\
0 \arrow[r] & A \arrow[r] & B_1 \arrow[r] & C_1 \arrow[r] & 0 : E_2
\end{tikzcd}$$

There exist a submodule V of B_1 such that $A + V = B_1$ and $A \cap V \ll B_1$. So, we obtain the following diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & A \cap V \arrow[r, equal] \arrow[d, "f"] & A \cap V \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d, "f'"] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d, equal] & 0 : E_2 \\
0 \arrow[r] & A/A \cap V \arrow[r] \arrow[d] & B_1/A \cap V \arrow[r] \arrow[d] & C_1 \arrow[r] & 0 \in \mathcal{S}plit \\
 & 0 & 0 & &
\end{tikzcd}$$

If we apply the functor $Hom(C_1, \ )$, we obtain the following

$$0 \longrightarrow \mathrm{Hom}(C_1, A \cap V) \longrightarrow \mathrm{Hom}(C_1, A) \longrightarrow \mathrm{Hom}(C_1, A/A \cap V) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}(C_1, A \cap V) \xrightarrow{f_*} \mathrm{Ext}(C_1, A) \xrightarrow{f'_*} \mathrm{Ext}(C_1, A/A \cap V) = 0$$

Then, f_* is epimorphism and so there exist $E_3 \in \mathrm{Ext}(C_1, A \cap V)$ such that $f_*(E_3) = E_2$. Since the following square is commutative:

$$\begin{tikzcd}
\mathrm{Ext}(C_1, A \cap V) \arrow[r, "g^*"] \arrow[d, "f_*"] & \mathrm{Ext}(C, A \cap V) \arrow[d, "f_*"] \\
\mathrm{Ext}(C_1, A) \arrow[r, "g^*"] & \mathrm{Ext}(C, A)
\end{tikzcd}$$

$g^* \circ f_*(E_3) = E_1 = f_* \circ g^*(E_3)$. Hence, we obtain the following diagram.

$$\begin{tikzcd}
 & 0 \arrow[rr] & & A \cap V \arrow[rr] \arrow[dl, equal] \arrow[dd, "f"] & & B \arrow[rr] \arrow[dl] \arrow[dd] & & C \arrow[r] \arrow[dl, "g"] \arrow[dd, equal] & 0 : E \\
0 \arrow[rr] & & A \cap V \arrow[rr] \arrow[dd, "f"] & & B_2 \arrow[rr] \arrow[dd] & & C_1 \arrow[rr] \arrow[dd, equal] & & 0 : E_3 \\
 & 0 \arrow[rr] & & A \arrow[rr] \arrow[dl, equal] & & B \arrow[rr] \arrow[dl] & & C \arrow[r] \arrow[dl, "g"] & 0 : E_1 \\
0 \arrow[rr] & & A \arrow[rr] & & B_1 \arrow[rr] & & C_1 \arrow[rr] & & 0 : E_2
\end{tikzcd}$$

Since $A \cap V \ll B_1$, $A \cap V$ is $\overline{\mathcal{WS}}$-coinjective by Proposition 5.2. Then $E \in \underline{k}(\mathcal{J})$ and since $\underline{k}(\mathcal{J})$ is subfunctor, $E_1 \in \underline{k}(\mathcal{J})$.

($\subseteq$) $\underline{k}(\mathcal{J}) \subseteq \overline{\mathcal{WS}}$ is trival. □

By the Propositions 3.21 and 3.22, we obtain that the following Corollaries:

Corollary 5.5 *The* global dimension*of* $\overline{\mathcal{WS}}$ *is gl.dim*$\overline{\mathcal{WS}} \leq 1$.

Corollary 5.6 *inj.dimA* ≤ 1 *for every* $\overline{\mathcal{WS}}$*-coinjective module A.*

5.2. Injective Submodules with Respect to $\overline{\mathcal{WS}}$

Corollary 5.7 *Over a Dedekind domain which R,* $\overline{\mathcal{WS}}$*-injective modules are only the injective R-modules.*

Proof Let M be a $\overline{\mathcal{WS}}$-injective module and I be any ideal of Dedekind domain R. Since R is Dedekind domain, R is noetherian ring and so I is finitely generated. $\mathbb{E} : 0 \longrightarrow I \xrightarrow{f} R \longrightarrow R/I \longrightarrow 0$ in $\overline{\mathcal{WS}}$ by Corollary 5.4. Since M is $\overline{\mathcal{WS}}$-injective module; for every homomorphism $\alpha : I \longrightarrow M$, there exists a homomorphism $\tilde{\alpha} : R \longrightarrow M$ such that $\tilde{\alpha} \circ f = \alpha$. We have the following commutative diagram,

$$\begin{tikzcd}
\mathbb{E} : & 0 \arrow[r] & I \arrow[r, "f"] \arrow[d, "\alpha"] & R \arrow[r] \arrow[dl, dashed, "\tilde{\alpha}"] & R/I \arrow[r] & 0 \\
 & & M & & &
\end{tikzcd} \tag{5.1}$$

Since for any left ideal I of R-homomorphism: $I \to M$ can be extended to an R-homomorphism: $R \to M$, then M is injective R-module by Baer's criterion (**??**, Theorem 3.3.5). □

We obtain the following Corollary by using Proposition 3.1 from Corollary 5.7.

Corollary 5.8 *$\mathcal{WS}$-injective modules are only the injective R-modules.*

5.3. Projective and Coprojective Submodules with Respect to $\overline{\mathcal{WS}}$

For $\overline{\mathcal{WS}}$-projective modules, we obtain the following criteria:

Lemma 5.5 *If C is any module such that $Ext_R(C, A') = 0$ for every coatomic module A', then C is an $\overline{\mathcal{WS}}$-projective module.*

Proof An R -module C is *$\mathcal{P}$-projective* if and only if $\mathrm{Ext}_{\mathcal{P}}(C, A) = 0$ for all R-modules A. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence in the class $\overline{\mathcal{WS}}$. In Proof of Theorem 5.1, it was shown that every elements of $\overline{\mathcal{WS}}$ is an image of a short exact sequence with starting a coatomic module such as

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "f"] & B_1 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 : E_1 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 : E \end{tikzcd}$$

where f is a monomorphism from a coatomic module A' to A.

Since A' is coatomic module, E_1 is in $\mathcal{S}plit$ with respect to our assumption. Then $E = f_*(E_1) = 0$. This completes the proof. □

Corollary 5.9 *Every finitely presented module is $\overline{\mathcal{WS}}$-coprojective.*

Proof Let a finitely presented module F. There is a epimorphism from a projective module P to F, $f : P \to F$. Since F is finitely presented, P and $Kerf$ is finitely generated. Thus $Kerf$ is $\overline{\mathcal{WS}}$-coinjective by Corollary 5.4. Then F is $\overline{\mathcal{WS}}$-coprojective by Proposition 3.3. □

5.4. Coinjective Submodules with Respect to $\overline{\mathcal{WS}}$ over DVR

In the following part R is always a discrete valuation ring with quotient field $K \neq R$ and the maximal ideal (p).

Corollary 5.10 *If $M/Rad(M)$ is simple, M is $\overline{\mathcal{WS}}$-coinjective.*

Proof Zöschinger proved that if $M/Rad(M)$ is simple, then M has a supplement in every extension N with N/M is torsion in (Zöschinger 1974c). Since every module is essential in its injective hull, M is essential in $E(M)$ and also $E(M)/M$ is torsion. So M has a supplement in its injective hull. Then M is $\overline{\mathcal{WS}}$-coinjective by Proposition 3.4. □

Theorem 5.2 ((Zöschinger 1974c), Theorem 3.1) *For an R-module M the following are equivalent:*

(a) M is radical-supplemented.

(b) $Rad^n(M) = Rad^{n+1}(M)$ is finitely generated for some $n \geq 0$.

(c) The basic-submodule of M is coatomic.

(d) $M = T(M) \oplus X$ where the reduced part of $T(M)$ is bounded and $X/Rad(X)$ is finitely generated.

Lemma 5.6 ((Zöschinger 1974c), Lemma 3.2) *(a) The class of radical-supplemented R-modules is closed under factor modules, pure submodules and extensions.*

(b) If M is radical-supplemented and M/U is reduced, then U is also radical-supplemented.

(c) Every submodule of M is radical-supplemented if and only if $T(M)$ is supplemented and $M/T(M)$ has finite rank.

By Lemma 5.2, Theorem 5.2 and Lemma 5.6, we obtain the following Corollary.

Corollary 5.11 *For an R-module M the following are equivalent:*

(a) M *is* $\overline{\mathcal{WS}}$*-coinjective.*

(b) M *is radical-supplemented.*

(c) $M = T(M) \oplus X$ *where the reduced part of* $T(M)$ *is bounded and* $X/Rad(X)$ *is finitely generated.*

(d) The class of $\overline{\mathcal{WS}}$*-coinjective R-modules is closed under factor modules, pure submodules and extensions.*

(e) Every submodule of M *is* $\overline{\mathcal{WS}}$*-coinjective if and only if* $T(M)$ *is supplemented and* $M/T(M)$ *has finite rank.*

CHAPTER 6

COATOMİC SUPPLEMENT SUBMODULES

Throughout this chapter all rings are hereditary rings, unless otherwise stated. In this chapter, we define the notion ”coatomic supplement” and give some results about the relation between coatomic supplement and supplement submodules.

6.1. Coatomic Supplement Submodules

Let U be a submodule of an R-module M. If there exists a submodule V of M such that $M = U + V$ and $U \cap V$ is coatomic then U is called a *coatomic supplement* of V in M.

We study the class Σ of σ-exact sequences where an element $\mathbb{E} : 0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ of $\mathrm{Ext}_R(C, A)$ is called σ-exact if $\mathrm{Im}\,\alpha$ has a coatomic supplement in B.

Lemma 6.1 *If $f : A \longrightarrow A'$, then $f_* : \mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C, A')$ preserves σ -element.*

Proof Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence in $\mathrm{Ext}(C, A)$ and $f : A \longrightarrow A'$ be an arbitrary homomorphism. The following diagram is commutative with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"'] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "f'"'] & C \arrow[r] \arrow[d, equal] & 0 : E \\
0 \arrow[r] & A' \arrow[r, "\alpha'"'] & B' \arrow[r] & C \arrow[r] & 0 : E_1
\end{tikzcd}$$

where $f_*(E) = E_1$. If V is a coatomic supplement of $Im\alpha$ in B, then $Im\alpha + V = B$ and $V \cap Im\alpha$ is coatomic. Then $f'(V) + Im\alpha' = B'$ by pushout diagram and $f'(V) \cap Im\alpha' = f'(V \cap Im\alpha)$ is coatomic, since $V \cap Im\alpha$ is coatomic and homomorphic image of a coatomic module is coatomic. □

Lemma 6.2 *If $g : C' \longrightarrow C$, then $g^* : \mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C', A)$ preserves σ-elements.*

Proof Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence in $\mathrm{Ext}(C, A)$ and $g : C' \longrightarrow C$ be an arbitrary homomorphism. The following diagram is commutative with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha'"'] \arrow[d, equal] & B' \arrow[r, "\beta'"'] \arrow[d, "g'"'] & C' \arrow[r] \arrow[d, "g"'] & 0 : E_1 \\
0 \arrow[r] & A \arrow[r, "\alpha"'] & B \arrow[r, "\beta"'] & C \arrow[r] & 0 : E
\end{tikzcd}$$

where $g^*(E) = E_1$.

Let V be a coatomic supplement of $Ker\beta$ in B, i.e. $Ker\beta + V = B$ and $V \cap Ker\beta$ is coatomic. Then $g'^{-1}(V) + Ker\beta' = B'$ by pullback diagram. Since g' induces an isomorphism between $D' = g'^{-1}(V) \cap Ker\beta'$ and $D = V \cap Ker\beta$ and epimorphic image of coatomic module coatomic, D' is coatomic. □

Corollary 6.1 *Every multiple of a σ-element of* $\mathrm{Ext}(C, A)$ *is again a σ-element.*

Theorem 6.1 *The class Σ of σ-elements coincide with the class $\overline{\mathcal{WS}}$ of $\overline{\mathcal{WS}}$-elements.*

Proof Assume that A has a coatomic supplement in B, then there exists a submodule V of B such that $B = A + V$ and $A \cap V$ is coatomic. So, the following diagram is commutative with exact columns and rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & A \cap V \arrow[r, equal] \arrow[d] & A \cap V \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "\gamma"'] & C \arrow[r] \arrow[d, equal] & 0 : E \\
0 \arrow[r] & A/A \cap V \arrow[r] \arrow[d] & B/A \cap V \arrow[r, "\alpha"] \arrow[d] & C \arrow[r] & 0 : E_1 \\
 & 0 & 0 & &
\end{tikzcd}$$

Clearly α is $\mathcal{S}plit$-epimorphism and since coatomic module is $\overline{\mathcal{WS}}$-coinjective, γ is $\overline{\mathcal{WS}}$-epimorphism. Then, the composition $\alpha \circ \gamma$ is an $\overline{\mathcal{WS}}$-epimorphism. So, E is a $\overline{\mathcal{WS}}$-element. To prove the converse, let $E \in \overline{\mathcal{WS}}$, then there is E_1 in the class $\mathcal{WS}$ such that the following diagram is commutative with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 : E \\
0 \arrow[r] & A \arrow[r, "\alpha'"] & B' \arrow[r] & C' \arrow[r] & 0 : E_1
\end{tikzcd}$$

If V is weak supplement of $Im\alpha'$ in B', then $Im\alpha' + V = B'$ and $Im\alpha' \cap V \ll B'$ and so $Im\alpha' \cap V$ is coatomic by Proposition 5.1. By Lemma 6.2, E is σ-element. □

Let R be a discrete valuation ring with quotient field $K \neq R$ and the maximal ideal (p). If A is a coatomic submodule of B, then it does not need to be small in B, but, since $B/Rad(B)$ semisimple, from

$$X/Rad(B) \oplus (A + Rad(B))/Rad(B) = B/Rad(B)$$

nevertheless follows that $X + A = B$ with $X \cap A$ small in B. So, every coatomic submodule has a weak supplement in every extension.

Lemma 6.3 *$\mathcal{WS}$ form a proper class over the Discrete Valuation Ring .*

Proof Assume that A has a coatomic supplement in B, then there exists a submodule V of B such that $B = A + V$ and $A \cap V$ is coatomic. So, the following diagram is commutative with exact column and rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & A \cap V \arrow[r, equal] \arrow[d] & A \cap V \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "\gamma"] & C \arrow[r] \arrow[d, equal] & 0 : E \\
0 \arrow[r] & A/A \cap V \arrow[r] \arrow[d] & B/A \cap V \arrow[r, "\alpha"] \arrow[d] & C \arrow[r] & 0 : E_1 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $A \cap V$ is coatomic, γ is $\mathcal{WS}$-epimorphism. Then, the composition $\alpha \circ \gamma$ is $\mathcal{WS}$-epimorphism. So, E is $\mathcal{WS}$-element. □

Let us consider the short exact sequence

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$$

in which $V + \operatorname{Im}\alpha = B$ for some $V \subset B$, where $V \cap \operatorname{Im}\alpha \ll V$ and $V \cap \operatorname{Im}\alpha$ is bounded, i.e. V is a supplement of $\operatorname{Im}\alpha$ in B with $V \cap \operatorname{Im}\alpha$ is bounded.We will call such sequences β-exact and denote $\operatorname{Im}\alpha \subset^{\beta} B$ as in Zöschinger. In this case we say that $\mathbb{E} \in \operatorname{Ext}(C, A)$ is a *β-element*. Over a Dedekind domain, any β-element of $\operatorname{Ext}_R(C, A)$ is a κ-element as well as a torsion element (Zöschinger 1978a). Let us denote the β-elements of $\operatorname{Ext}_R(C, A)$ by $\mathcal{SB}$. In order to show that every κ-element need not be a β-element, we give an example over $R = \mathbb{Z}$.

Example 6.1 *Consider the inclusion homomorphism* $f : \bigoplus_p \mathbb{Z}_p \longrightarrow \bigoplus_p \mathbb{Z}_{p^\infty}$ *where p ranges over all prime numbers in $\mathbb{Z}$. Then* $\operatorname{Im} f = \bigoplus_p \mathbb{Z}_p$ *is small in* $\bigoplus_p \mathbb{Z}_{p^\infty}$*, so f is a $\mathcal{S}$-monomorphism.* $\bigoplus_p \mathbb{Z}_{p^\infty}$ *itself is the only supplement of* $\operatorname{Im} f$ *in* $\bigoplus_p \mathbb{Z}_{p^\infty}$. $\operatorname{Im} f = \bigoplus_p \mathbb{Z}_p$ *is not bounded, hence f is not an $\mathcal{SB}$-monomorphism.*

Lemma 6.4 ((Zöschinger 1978a), Lemma 1.2) *If A,C are torsion, then*

$$Ext_\beta(C, A) = Ext(C, A)_\kappa \cap T(Ext(C, A)).$$

If A,C are torsion, $Ext(C, A)_{\overline{\mathcal{WS}}} = Ext(C, A)_\kappa$ and so the following corollary obtained from Lemma 6.4.

Corollary 6.2 *If A,C are torsion, then*

$$Ext(C, A)_\beta = Ext(C, A)_{\overline{\mathcal{WS}}} \cap T(Ext(C, A)).$$

6.2. The Relations Between the Class $\overline{\mathcal{WS}}$ and the Related Other Classes

In this section, we deal with complements (closed submodules) and supplements in unital R-modules for an associative ring R with unity using relative homological algebra via the known two dual proper classes of short exact sequences of R-modules and R-module homomorphisms, $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ and $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$, and related other proper classes like $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ and $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$.

If A is a submodule of B such that $K \cap A = 0$ (that is the above second condition for direct sum holds) and A is *maximal* with respect to this property (that is there is *no* submodule $\tilde{A}$ of B such that $\tilde{A} \supsetneqq A$ but still $K \cap \tilde{A} = 0$), then A is called a *complement of K in B* and K is said to *have a complement in B*. By Zorn's Lemma, it is seen that K always has a complement in B (unlike the case for supplements). In fact, by Zorn's Lemma, we know that if we have a submodule A' of B such that $A' \cap K = 0$, then there exists a complement A of K in B such that $A \supseteq A'$.

A subgroup A of a group B is said to be neat in B if $A \cap pB = pA$ for all prime numbers p. The criterion for being a coneat submodule is like being a supplement in the following weaker sense:

Proposition 6.1 ((Mermut 2004), Propositions 3.4.2) *For a submodule A of a module B, the following are equivalent:*

1. *A is coneat in B,*

2. *There exists a submodule $K \leq B$ such that ($K \geq \operatorname{Rad} A$ and,)*

$$A + K = B \quad \textit{and} \quad A \cap K = \operatorname{Rad} A.$$

3. *There exists a submodule $K \leq B$ such that*

$$A + K = B \quad \textit{and} \quad A \cap K \leq \operatorname{Rad} A.$$

The class $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ [$\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$] consists of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{6.1}$$

of R-modules and R-module homomorphisms such that $\mathrm{Im}(f)$ is a complement [resp. supplement] in B. $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ [$\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$] consists of all short exact sequences 6.1 of R-modules and R-module homomorphisms such that $\mathrm{Im}(f)$ is a neat [resp. coneat] in B.

The proper class $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ is an injectively generated proper class containing $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$

Proposition 6.2 ((Mermut 2004), Proposition 3.4.1) *For any ring R,*

$$\mathcal{S}uppl_{R\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} \subseteq \iota^{-1}(\{\textit{all (semi-)simple R-modules}\})$$

We have,

$$\begin{aligned} \mathcal{N}eat_{R\text{-}\mathcal{M}od} &= \pi^{-1}(\{\text{all semisimple } R\text{-modules}\}) \\ &= \pi^{-1}(\{M \mid \operatorname{Soc} M = M,\ M \text{ an } R\text{-module}\}), \end{aligned}$$

where $\operatorname{Soc} M$ is the socle of M, that is the sum of all simple submodules of M. Dualizing this,

$$\begin{aligned} \mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} &= \iota^{-1}(\{\text{all } R\text{-modules with zero radical}\}) \\ &= \iota^{-1}(\{M \mid \operatorname{Rad} M = 0,\ M \text{ an } R\text{-module}\}). \end{aligned}$$

If A is a $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-submodule of an R-module B, denote this by $A \leq_{cN} B$ and say that A is a *coneat submodule* of B, or that the submodule A of the module B is *coneat in B*.

Proposition 6.3 ((Mermut 2004), Proposition 5.2.6) *For a Dedekind domain W,*

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} \subseteq \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}.$$

Theorem 6.2 ((Mermut 2004), Theorem 5.4.6) *Let W be a Dedekind domain which is not a field.*

1. *If* $\operatorname{Rad} W = 0$*, then*

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subsetneq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} \subsetneq \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}.$$

2. *If* $\operatorname{Rad} W \neq 0$*, then*

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subsetneq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}.$$

Theorem 6.3 ((Mermut 2004), Theorem 5.2.3) *For a Dedekind domain* W*, and* W*-modules* A, C*,*

$$\operatorname{Ext}_{\mathcal{C}ompl_{W\text{-}\mathcal{M}od}}(C, A) = \operatorname{Ext}_{\mathcal{N}eat_{W\text{-}\mathcal{M}od}}(C, A) = \operatorname{Rad}(\operatorname{Ext}_{W\text{-}\mathcal{M}od}(C, A)).$$

Lemma 6.5 *For a Discrete Valuation Ring* $\mathbb{R}$*,*

$$\mathcal{S}uppl_{\mathbb{R}\text{-}\mathcal{M}od} = \overline{\mathcal{WS}} \cap \mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{R}\text{-}\mathcal{M}od} = \overline{\mathcal{WS}} \cap \mathcal{N}eat_{\mathbb{R}\text{-}\mathcal{M}od} = \overline{\mathcal{WS}} \cap \mathcal{C}ompl_{\mathbb{R}\text{-}\mathcal{M}od}.$$

Proof We have the relation $\mathcal{S}uppl_{\mathbb{R}\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{R}\text{-}\mathcal{M}od} \subseteq \mathcal{N}eat_{\mathbb{R}\text{-}\mathcal{M}od} = \mathcal{C}ompl_{\mathbb{R}\text{-}\mathcal{M}od}$ by Proposition 6.3. Hence, $\mathcal{S}uppl_{\mathbb{R}\text{-}\mathcal{M}od} \subseteq \mathcal{C}ompl_{\mathbb{R}\text{-}\mathcal{M}od} \cap \overline{\mathcal{WS}}$. Conversely, let $\mathbb{E}_1 \in\in \operatorname{Ext}_{\mathcal{C}ompl}(C, A) \cap \operatorname{Ext}_{\overline{\mathcal{WS}}}(C, A)$:

$$\mathbb{E}_1 : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

For simplicity, suppose A is a submodule of B and f is the inclusion homomorphism. A has a weak supplement in B by Lemma 6.3 and so there exits a submodule K of B such that

$$A + K = B \quad \text{and} \quad A \cap K \ll B.$$

Since the classes of complements and coclosed submodules are the same, A is closed in B and so $A \cap K \ll A$. Then, $\mathcal{C}ompl_{\mathbb{R}\text{-}\mathcal{M}od} \cap \overline{\mathcal{WS}} \subseteq \mathcal{S}uppl_{\mathbb{R}\text{-}\mathcal{M}od}$. □

Lemma 6.6 *Let R be a Discrete Valuation Ring which is not a field, then*

$$\mathcal{S}uppl_{\mathbb{R}\text{-}\mathcal{M}od} = \overline{\mathcal{WS}} \cap \mathrm{Rad}(\mathrm{Ext}_{\mathbb{R}}(C, A)).$$

Proof It follows from Theorem 6.2, Theorem 6.3 and Lemma 6.5. □

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