ENTANGLEMENT AND TOPOLOGICAL SOLITON STRUCTURES IN HEISENBERG SPIN MODELS

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To My Father...

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ABSTRACT

ENTANGLEMENT AND TOPOLOGICAL SOLITON STRUCTURES IN HEISENBERG SPIN MODELS

Quantum entanglement and topological soliton characteristics of spin models are studied. By identifying spin states with qubits as a unit of quantum information, quantum information characteristic as entanglement is considered in terms of concurrence. Eigenvalues, eigenstates, density matrix and concurrence of two qubit Hamiltonian of XYZ, pure DM, Ising, XY, XX, XXX and XXZ models with Dzialoshinskii- Moriya DM interaction are constructed. For time evolution of two qubit states, periodic and quasi-periodic evolution of entanglement are found. Entangled two qubit states with exchange interaction depending on distance $J(R)$ between spins and influence of this distance on entanglement of the system are considered. Different exchange interactions in the form of Calogero- Moser type I, II, III and Herring-Flicker potential which applicable to interaction of Hydrogen molecule are used.

For geometric quantum computations, the geometric (Berry) phase in a two qubit XX model under the DM interaction in an applied magnetic field is calculated. Classical topological spin model in continuum media under holomorphic reduction is studied and static N soliton and soliton lattice configurations are constructed. The holomorphic time dependent Schrödinger equation for description of evolution in Ishimori model is derived. The influence of harmonic potential and bound state of solitons are studied. Relation of integrable soliton dynamics with multi particle problem of Calogero-Moser type is established and N soliton and N soliton lattice motion are found. Special reduction of Abelian Chern-Simons theory to complex Burgers' hierarchy, the Galilean group, dynamical symmetry and Negative Burgers' hierarchy are found.

ÖZET

HEISENBERG SPİN MODELLERİNDE DOLAŞIKLIK VE TOPOLOJİK SOLİTON YAPILARI

Bu tezde spin modellerinin kuantum dolaşıklığı ve topolojik soliton özellikleri çalışılmıştır. Spin durumları kübitler aracılığıyla tanımlanarak dolaşıklık uyum ölçümü kullanılarak kuantum enformasyon özelliği olarak ele alınmıştır. DM etkileşimli XYZ, DM, Ising, XY, XX, XXX ve XXZ modellerinin iki kübitli Hamiltonyeninin özdeğer, özvektör, yoğunluk matrisi ve uyumu hesaplanmıştır. İki kübitli durumların zamanla değişimi ile dolaşıklığın periyodik ve kuasiperiyodik evrimi bulunmuştur. Uzaklığa bağlı takas etkileşimli dolaşık iki kübit durumları ve uzaklığın bu durumların dolaşıklığına etkisi incelenmiştir. Calogero- Moser I, II, III tipinde ve Hidrojen moleküllerinin etkileşimine uygulaması olan Herring-Flicker potansiyeli gibi farklı takas etkileşimleri kullanılmıştır.

Geometrik kuantum hesaplamaları için iki kübitli manyetik alan içindeki DM etkileşimli XX modeli geometrik (Berry) faz hesaplanmıştır. Analitik indirgemeli topolojik spin modelleri çalışılmıştır. Statik N-soliton ve N-soliton kafes konfigürasyonları kurulmuştur. Ishimori modelinde evrimi tanımlamak için zamana bağlı analitik Schrödinger denklemi çıkarılmıştır. Harmonik potansiyelin etkisi ve solitonların sınır durumları çalışılmıştır. Integrallenebilir soliton dinamiği ile Calogero-Moser tipindeki çok parçacıklı problem arasındaki bağıntı kurulmuş ve N-soliton ile N-soliton latis hareketleri bulunmuştur. Abelyan Chern Simons teorisinin karmaşık Burgers' hiyerarşisine özel indirgenmesi, Gali- lean grup, dinamik simetri ve negatif Burgers' hiyerarşisi bulunmuştur.

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CHAPTER 1

INTRODUCTION

In the information age, computers have become an indispensable part of our lives so that the computer industry has been growing enormously and the size of integrated circuits has been decreasing very rapidly. Thanks to the miniaturization, computational power of modern computers increase. Observing these facts, Gordon Moore in 1965 proposed the so called Moore's law (Moore, 1965) which indicates that the number of transistors on a single chip doubles approximately every 18 to 24 months. This evolution law requires miniaturization in memory and processor units. According to these estimates, the exponential growth has not been reached yet. In the near future, quantum switches, devices on the nanoscale length (10^{-9}), will substitute silicon based transistors. As a result, near future computers will work on laws of quantum mechanics rather than the classical ones. Richard Feynman (Feynman, 1982) and David Deutsch (Deutsch, 1985) were the first who proposed new type of modern computers based on the laws of quantum mechanics. According to Feynman, some quantum mechanical calculations could be implemented more efficiently on a quantum computer rather than on a classical computer. Later, in 1994 Peter Shor (Shor, 1994) proposed a quantum algorithm that solves efficiently the prime factorization problem, which is a crucial problem in computer science. This algorithm provides an exponential improvement in computational speed when compared to classical ones. Some cryptographic systems such as RSA cryptosystem are based on the conjecture that no efficient algorithms exist for solving the prime factorization problem. RSA cryptosystem would be broken if Shor's algorithm is implemented on a quantum computer. As a next breakthrough in quantum algorithms in 1997, Lov Grover (Grover, 1997) found a fast algorithm for searching databases and it requires only efforts that grow as the square root of the number of entries. Aside from these algorithms, the original ideas of Feynman, using a quantum computer for the simulation of quantum problems have become increasingly interesting today. In brief, a quantum computer is a machine that is based on quantum logic in contrast to the classical computer.

The unit of information in a classical computer is called a bit and it takes two values 0 or 1 while unit of information in a quantum computer is called a quantum bit (qubit). The values of a bit correspond to the status of a switch (1 = on, 0 = off) on an electronic device. Unlike a bit, a qubit is a two-level quantum system, described by a two-dimensional complex Hilbert space. In contrast to classical states which are related to the basis in this space, any superposition of these states also represents a state of a qubit. There are several ways to realize qubits as an atom, nuclear spin, or a polarized photon. It turns out that mathematical description of qubit is equivalent to Pauli formalism describing spin angular momentum in quantum mechanics. According to modern high energy physics, spin of a particle is a fundamental intrinsic characteristic property of all elementary particles, namely, the same kind of elementary particles has the same spin quantum number.

In 1924 Wolfgang Pauli proposed the concept of spin ((Pauli, 1925)); as a ”two-valued quantum degree of freedom” associated with the electron in the outermost shell. This allowed him to formulate the Pauli Exclusion Principle stating that no two electrons can share the same quantum state at the same time. In 1927, Pauli formalized the theory of spin using the modern theory of quantum mechanics discovered by Schrödinger and Heisenberg. He pioneered the use of 2x2 matrices which are known as Pauli matrices, representing the spin operators, and introduced a two-component spinor wave-function. Pauli's theory of spin is non-relativistic. However, in 1928, Paul Dirac (Dirac, 1928) discovered the Dirac equation which describes the relativistic electron. In the Dirac equation, a four-component spinor (known as a ”Dirac spinor”) is used for the electron wave-function. It turns out that in non-relativistic limit, the Dirac equation reduces to the Schrödinger-Pauli equation as a descriptive of non-relativistic electron with spin. As first direct experimental evidence of the electron spin, the correct explanation of the Stern-Gerlach experiment (Stern & Gerlach, 1922a) and (Stern & Gerlach, 1922b) was only given in 1927.

Spin plays a crucial role in magnetic properties of many materials like ferromagnetic and antiferromagnetic materials. First it was proposed by Dirac and Heisenberg (Dirac, 1926), (Heisenberg, 1926), the magnetic Hamiltonian H to be proportional to $JS_i \cdot S_j$ where constant J is called the exchange interaction. For example based on this Hamiltonian, the Heisenberg ferromagnetic model is

$$H = \sum_{ij} J\vec{S}_i \cdot \vec{S}_j \tag{1.1}$$

where $J < 0$, in the case $J > 0$ this Hamiltonian refers to antiferromagnetic model. Related with this Heisenberg model several anisotropic modifications were proposed. The XXZ generalization which appear when some anisotropy like easy axis or easy plane anisotropy take place. Particular cases of these models like Ising Model or XY model has variety of applications in statistical physics. The one dimensional version of then for N spin chain model it can be solved by the Bethe Ansatz (Bethe, 1931) . Exact solvability from one side and wide field of applications from another side, shows the importance of these models and many researchers work on these models. In these cases, Ising type of models appears effectively in two level systems. Heisenberg type spin models are very important models. From one side magnetic properties of materials lead to the nonlinear magnetic systems where magnetic properties of materials described by domain walls, solitons and vortices. Topological characteristic playing essential role in description of materials. This is why, studying exactly solvable topological spin models with domain walls, soliton and soliton solutions has become actual problem of study. From another side, treatment of spin models as two level quantum systems make them an important tool for quantum computation and information. Every two level quantum system plays the role of qubit as a unit of information and the interaction of spins then plays the role of qubit gates (2 and higher). For performing quantum computations, nonlocal property of quantum systems as entanglement becomes important tool to study. Being motivated by these two directions, in this thesis we study spin models as nonlinear dynamical systems with nontrivial topological solutions and as qubit systems with entanglement property.

Thesis consists of two parts: First part devoted to entanglement property of spin models while the second part is devoted to topological properties of spin models (formulation of classical spin models, exact solutions and topological properties, reduction of the model, relation with Complex Burgers' equation).

In Chapter 2 we introduce the concept of qubit and establish the relation between spin and qubit. Entanglement property and concurrence as a measure of entanglement become subject of Section 2.5. Heisenberg spin models and effective spin models are introduced in Section 2.6 and 2.7. In Section 2.8 we formulate the concurrence as a measure of thermal entanglement.

In Chapter 3 we consider two qubit entanglement in Heisenberg spin models. We study the influence of DM interaction on entanglement of two qubits in all particular magnetic spin models Ising , XY, XX, XXX, XXZ and the most general XYZ model.

In Chapter 4 we consider time evolution of two qubit states. In Section 4.1 we analyze the periodic and quasiperiodic behavior of entanglement. The concept of fidelity introduced and time evolution found in Section 4.2. In Section 4.3 we established the link between time evolution of states and SWAP gate.

In Chapter 5 we study entanglement dependence on distance between interacting qubits. In Section 5.1 we discuss Ising model in transverse magnetic field with distance dependent exchange interaction in the form of Calogero-Moser type I, II and III. Herring-Flicker type distance dependence is the subject of Section 5.2

In Chapter 6 we present briefly geometric quantum computation. Dynamic and geometric phases are formulated in Section 6.1. The influence of Dzialohinskii-Moriya interaction on Berry's phase is discussed in Section 6.2.

In Chapter 7 we consider the problem of magnetic solitons in a magnetic fluid model. We formulate the topological magnet model in Section 7.1 and its stereographic projection representation in Section 7.2. The anti-holomorphic reduction of topological magnetic system to the linear complex Schrödinger equation in considered in Section 7.3. In Section 7.4 we study special form of topological magnet as the Ishimori model. Applying all results on integrable soliton dynamics in the complex Burgers' equation to the magnetic soliton evolution, we construct N magnetic solitons in Section 7.5, and study their dynamics in Section 7.6. By time dependent Schrödinger problem in harmonic potential, Section 7.7 , we construct the bound state of N solitons in Section 7.8.

In Chapter 8 we establish relation of N soliton equations with the Calogero-Moser multiparticle systems, Section 6.1, showing integrability and the Hamiltonian structure for N soliton, Section 6.2 and N-soliton lattices, Section 6.3.

In Chapter 9 we consider the Abelian Chern-Simons Gauge Field Theory in 2+1 dimensions and its relation with holomorphic Burgers' Hierarchy. Complex Galilean group and soliton generations are studied in Section 9.1. In Section 9.2 we show that the anti-holomorphic Burgers' hierarchy appears in the Chern-Simons gauge field theory. Complex Galilean group hierarchy and soliton solutions are studied in Section 9.3. The holomorphic Schrödinger hierarchy and corresponding Burgers' hierarchy are discussed in Section 9.4.

In conclusions we summarize main results in the thesis. In appendices we show Lax representation we derived system of equations describing evolution of N solitons.

The main results presented in this thesis were published in the following papers.

- Pashaev O.K., Gurkan Z.N., 2007: Abelian Chern-Simons solitons and holomorphic Burgers' hierarchy, *Theor. Math. Phys.* , **152**, 1, 1017-1029.

- Gurkan Z. N., Pashaev O. K., 2008: Integrable soliton dynamics in anisotropic planar spin liquid model, *Chaos, Solitons and Fractals*, **38** , 238- 253.

- Kwan M. K., Gurkan Z. N., and Kwek L. C., 2008: Berry's phase under the Dzyaloshinskii Moriya interaction, *Physical Review A* , **77**, 062311.

- Gurkan Z. N. , Pashaev O., 2010: Entanglement in two qubit magnetic models with DM antisymmetric anisotropic exchange interaction, *International Journal of Modern Physics B*, **24**, 8, 943-965.

PART I

QUBIT ENTANGLEMENT IN SPIN MODELS

CHAPTER 2

SPIN, QUBIT AND ENTANGLEMENT

Quantum computers based on quantum logic and they process information and performs logic operations by laws of quantum mechanics.

2.1. Qubit

A quantum bit or a qubit is a two-level quantum system, described by a two-dimensional complex Hilbert space, generated by a pair of normalized and mutually orthogonal quantum states. Two possible states for a qubit

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.1}$$

form the computational basis and correspond to the values of 0 and 1 of the classical bit. From the superposition principle of quantum mechanics, arbitrary state of the qubit may be written as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{2.2}$$

where the amplitudes α and β are complex numbers, constrained by the normalization condition $|\alpha|^2 + |\beta|^2 = 1$. The state vectors are defined only up to a global phase of no physical significance, this is why without loss of generality, one may choose α to be real and positive. Then generic state of a qubit maybe written as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle = \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix} \tag{2.3}$$

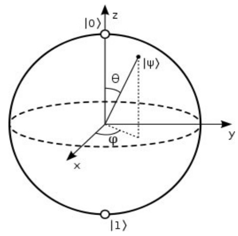
where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. If the state of the qubit is described by (2.3), as a result of the measurement one obtains $|0\rangle$ or $|1\rangle$ states with probabilities

$$p_0 = |\langle 0|\psi\rangle|^2 = \cos^2\frac{\theta}{2}, \quad p_1 = |\langle 1|\psi\rangle|^2 = \sin^2\frac{\theta}{2} \tag{2.4}$$

and $p_1 + p_2 = 1$. Using the normalization condition $|\alpha|^2 + |\beta|^2 = 1$ and the global phase freedom, the qubit's state can be represented by a point on a two dimensional sphere of unit radius, called the Bloch sphere. This sphere can be embedded in a three-dimensional space of Cartesian coordinates $x = \cos\phi\sin\theta, y = \sin\phi\sin\theta, z = \cos\theta$ so that $x^2 + y^2 + z^2 = 1$. Thus, the qubit state (2.3) can be written in terms of these coordinates as

$$|\psi\rangle = \begin{pmatrix} \sqrt{\frac{1+z}{2}} \\ \frac{x+iy}{\sqrt{2(1+z)}} \end{pmatrix}. \tag{2.5}$$

A Bloch vector $\vec{r}$ is a vector whose components $\vec{r}(x, y, z)$ single out a point on the Bloch sphere. Therefore, each Bloch vector must satisfy the normalization condition $x^2+y^2+z^2 = 1$. It can also be defined in terms of angles θ and ϕ. Eq.(2.5) gives a relation



between Bloch vector and the qubit state so that any Bloch vector determines a qubit state as well as qubit states can be associated with corresponding Bloch vector. Another useful representation of the state (2.3) is obtained by means of the projector operator $\hat{P} = |\psi\rangle\langle\psi|$, $\hat{P}^2 = \hat{P}$. The matrix representation of the operator $\hat{P}$ in the computational basis $\{|0\rangle\langle 0|, |0\rangle\langle 1|, |1\rangle\langle 0|, |1\rangle\langle 1|$. is given by

$$\begin{aligned} P &= \frac{1+z}{2}|0\rangle\langle 0| + \frac{1-z}{2}|1\rangle\langle 1| + \frac{x-iy}{2}|0\rangle\langle 1| + \frac{x+iy}{2}|1\rangle\langle 0| \\ &= \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \end{aligned} \tag{2.6}$$

The state of a qubit

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle, \ (0 \leq \theta \leq \pi, 0 \leq \phi \, 2\pi) \tag{2.7}$$

can be measured using the Pauli operators,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.8}$$

so that the following expectation values for the state $|\psi\rangle$ obtained

$$\langle\psi|\sigma_x|\psi\rangle = \sin\theta\cos\phi = x \tag{2.9}$$

$$\langle\psi|\sigma_y|\psi\rangle = \sin\theta\sin\phi = y \tag{2.10}$$

$$\langle\psi|\sigma_z|\psi\rangle = \cos\theta = z. \tag{2.11}$$

The coordinates (x, y, z) can be obtained with arbitrary accuracy by means of standard projective measurements on the computational basis, that is, measuring σ_z. From eq. (2.4) we obtain

$$p_0 - p_1 = |\langle 0|\psi\rangle|^2 - |\langle 1|\psi\rangle|^2 = \cos\frac{\theta}{2}^2 - \sin\frac{\theta}{2}^2 = \cos\theta = z \tag{2.12}$$

Thus, the coordinate z is given by difference of the probabilities to obtain outcomes 0 or 1 from a measurement of σ_z. If we have a large number N of systems identically prepared in the state (2.3), we can estimate z as $\frac{N_0}{N} - \frac{N_1}{N}$, where N_0 and N_1 count the number of outcomes 0 and 1. Therefore, z can be measured to any required accuracy, provided we measure a sufficiently large number of states. The coordinates x and y can be obtained by using the possibility to operate a unitary transformation on the qubit. If the unitary transformation described by the matrix

$$U_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{2.13}$$

is applied to the state (2.3), we obtain the state $|\psi^{(1)}\rangle = U_1|\psi\rangle$. A projective measurement in the computational basis then gives outcome 0 or 1 with probabilities $p_0^{(1)} = |\langle 0|\psi^{(1)}\rangle|^2$ and $p_1^{(1)} = |\langle 1|\psi^{(1)}\rangle|^2$, respectively. Therefore, we obtain

$$p_0^{(1)} - p_1^{(1)} = |\langle 0|\Psi_1\rangle|^2 - |\langle 1|\Psi_1\rangle|^2 = \cos\phi\sin\theta = x \tag{2.14}$$

In the same way, if the state (2.3) is transformed by means of the matrix

$$U_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \tag{2.15}$$

we obtain the state $|\psi^{(2)}\rangle = U_2|\psi\rangle$. Therefore,

$$p_0^{(2)} - p_1^{(2)} = |\langle 0|\psi_2\rangle|^2 - |\langle 1|\psi_2\rangle|^2 = \sin\phi\sin\theta = y, \tag{2.16}$$

where $p_0^{(2)} = |\langle 0|\psi^{(2)}\rangle|^2$ and $p_1^{(2)} = |\langle 1|\psi^{(2)}\rangle|^2$ give the probabilities to obtain outcome 0 or 1 from the measurement of the qubit polarization along z (Benenti et al., 2004).

Spin $1/2$ states can be interpreted as qubit states. For spin operator $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$, spin states are superposition of spin up $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and spin down $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ states. These states can be identified with computational basis $|\uparrow\rangle \equiv |0\rangle$ and $|\downarrow\rangle \equiv |1\rangle$ as

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle. \tag{2.17}$$

The spin operators generate the $SU(2)$ algebra and Pauli matrices satisfy

$$[\sigma_i, \sigma_j] = 2\epsilon_{ijk}\sigma_k \tag{2.18}$$

where ϵ_{ijk} is Levi-Civita antisymmetric tensor.

2.2. Qubit and $SU(2)$ Coherent State

The $SU(2)$ coherent states are defined for states generated with angular momentum raising and lowering operators

$$\hat{S}^{\pm} = \hat{S}_x \pm i\hat{S}_y \tag{2.19}$$

(Radcliffe, 1971) and (Arrechi et al., 1972), where relevant spin operators $\hat{S}_i$, $i = 1, 2, 3$ have $SU(2)$ commutation relations

$$[\hat{S}_i, \hat{S}_j] = i\epsilon_{ijk}\hat{S}_k. \tag{2.20}$$

The $SU(2)$ coherent states are generated from vacuum state $|0\rangle$ as

$$|\psi\rangle = \frac{e^{\hat{S^+}\psi}}{\sqrt{1+|\psi|^2}}|0\rangle = \frac{|0\rangle + \psi|1\rangle}{\sqrt{1+|\psi|^2}} \tag{2.21}$$

where $S^-|0\rangle = |1\rangle$. The eigenstates of $\hat{S}_z$ are $|0\rangle$ and $|1\rangle$, they generate $2D$ Hilbert space and represent a single qubit. An $SU(2)$ coherent state is an arbitrary pure qubit state. Indeed if

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad |\psi_1|^2 + |\psi_2|^2 = 1 \tag{2.22}$$

then, in terms of homogeneous coordinate $\psi = \psi_2/\psi_1$ we have

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \psi_1 \begin{pmatrix} 1 \\ \psi \end{pmatrix}. \tag{2.23}$$

We fix ψ_1 by normalization condition $\langle\psi|\psi\rangle = 1$, so that up to the global phase we have for the qubit state

$$|\psi\rangle = \frac{1}{\sqrt{1+|\psi|^2}} \begin{pmatrix} 1 \\ \psi \end{pmatrix} \tag{2.24}$$

This state coincides with the spin $1/2$ generalized coherent state (2.21).

From another side, solving normalization condition in (2.22) we have one qubit state

$$|\theta, \varphi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\varphi}|1\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\varphi} \end{pmatrix} \tag{2.25}$$

which is determined by point (θ, φ) on the Bloch sphere. In this parametrization the homogeneous variable is

$$\psi = \frac{\psi_2}{\psi_1} = \tan\frac{\theta}{2}e^{i\phi}. \tag{2.26}$$

This determines the stereographic projection of a point $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ on the unit sphere to the complex plane ψ. Therefore the Bloch sphere considered as a Riemann sphere for the extended complex plane ψ by the stereographic projection, determines the $SU(2)$ or the spin coherent state

$$|\psi\rangle = \frac{|0\rangle + \psi|1\rangle}{\sqrt{1+|\psi|^2}}. \tag{2.27}$$

Then the computational basis states $|0\rangle = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in this coherent state representation are just points in extended complex plane $(\Re\psi, \Im\psi) \cup \{\infty\}$, as $\psi = 0$ and $\psi = \infty$ respectively. These points are symmetrical points under the unit circle at the origin.

2.3. Multiple Qubit States

Tensor product of qubits

$$|\psi_1\psi_2...\psi_n\rangle \equiv |\psi_1\rangle \otimes |\psi_2\rangle \otimes ... \otimes |\psi_n\rangle \tag{2.28}$$

gives the multi qubit state. Quantum gates act on this state as unitary operators transforming the multiqubit state $|\uparrow\downarrow\downarrow\downarrow ... \uparrow ...\rangle = |0111...0...\rangle$ to another multiqubit state. It turns out that the multispin states or spin complexes can be interpreted as n- qubit states. In particular as ferromagnetic ground state

$$\begin{aligned} |\Phi^{\uparrow}\rangle &= |\uparrow\uparrow\uparrow\uparrow ... \uparrow\rangle = S_2^+ S_3^+ ... S_n^+ |11...1\rangle = |00...0\rangle & (2.29) \\ |\Phi^{\downarrow}\rangle &= |\downarrow\downarrow\downarrow\downarrow ... \downarrow\rangle = S_2^- S_3^- ... S_n^- |00...0\rangle = |11...1\rangle & (2.30) \end{aligned}$$

and Neel state or anti-ferromagnetic ground state

$$|\Phi\rangle = |\uparrow\downarrow\uparrow\downarrow ... \downarrow\rangle = S_2^+ S_3^- S_4 ... S_n^- |1010...1\rangle = |0101...0\rangle \tag{2.31}$$

where

$$S_i^+ |1\rangle_i = |0\rangle_i, \quad S_i^- |0\rangle = |1\rangle_i \tag{2.32}$$

If we have a finite spin chain so that, at every site of the chain we have spin states then the total spin state of the chain forms a spin complex.

2.4. Quantum Gates

Like the classical computer consisting of an electrical circuit containing logic gates, a quantum computer is built from a quantum circuit containing elementary quantum gates to manipulate the quantum information. Single qubit gates can be described by 2×2 unitary matrices. Rotation of the Bloch sphere about an arbitrary axis is a unitary transformation

$$R_n(\delta) = \cos\frac{\delta}{2} I - i \sin\frac{\delta}{2}(\vec{n}\cdot\vec{\sigma}) \tag{2.33}$$

where the unit vector $\vec{n} = (n_x, n_y, n_z)$. ***Phase shift gate*** is represented as

$$R_z(\delta) = \cos\frac{\delta}{2} I - i \sin\frac{\delta}{2}\sigma_z = e^{-i\frac{\delta}{2}} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{pmatrix}. \tag{2.34}$$

Applying the phase shift gate to generic vector ψ we have

$$R_z(\delta)|\psi\rangle = \cos\frac{\delta}{2}|0\rangle + e^{i(\phi+\delta)} \sin\frac{\delta}{2}|1\rangle. \tag{2.35}$$

Hadamard gate is one of the most important single qubit gates and corresponds to rotations and reflections of the sphere. Rotation through an angle $\delta = \pi$ about the axis $\vec{n} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$ gives the so called Hadamard gate

$$H = \frac{1}{\sqrt{2}}(\sigma_z + \sigma_x) \tag{2.36}$$

H performs the unitary transformation which is Hadamard transform:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \equiv |+\rangle \tag{2.37}$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \equiv |-\rangle \tag{2.38}$$

Hadamard gate can be expressed as a matrix in the computational basis $\{|0\rangle, |1\rangle\}$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2.39}$$

Single qubit gates do not promise much in computations since interaction of qubits are needed. The most common two qubit gate is Controlled-NOT (CNOT) gate. This gate acts on the states of the computational basis, $\{|i_1 i_0\rangle = |00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ as the classical XOR gate:

$$CNOT(|x\rangle|y\rangle) = |x\rangle|x \oplus y\rangle \tag{2.40}$$

where $x, y = 0, 1$ and $\oplus$ indicating addition modulo 2.

$$U_{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.41}$$

Quantum gates are universal. Any unitary operation in Hilbert space of $n-$ qubits can be decomposed into one-qubit and two-qubit $CNOT$ gates. The generic state can be reached starting from $|0\rangle$ as

$$R_z(\frac{\pi}{2} + \phi)HR_z(\theta)H|0\rangle = e^{i\frac{\theta}{2}}(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle) \tag{2.42}$$

(Benenti et al., 2004).

2.5. Entanglement and Concurrence

The entanglement property has been discussed at the early years of quantum mechanics as a specifical quantum mechanical nonlocal correlation (Schrödinger, 1935)-(Bell, 1964) and recently it becomes a key point of the quantum information theory (Bennet, 2000). We can write the generic two-qubit state in the computational basis as

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle \tag{2.43}$$

where c_{00}, c_{01}, c_{10} and c_{11} are complex coefficients. Because the state is defined up to a global phase factor and the normalization condition

$$|c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2 = 1 \tag{2.44}$$

non-separable(entangled) two qubit states have 6 real degrees of freedom while a separable state

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_0\rangle \tag{2.45}$$

has only 4 real degrees of freedom. The complexity of entanglement grows exponentially with the number of qubits. A separable state of n qubits depends only on $2n$ real parameters while the most general entangled state has $2(2^n - 1)$ degrees of freedom. For entangled subsystems the whole state vector cannot be separated into a product of the subsystem states. This is why these subsystems are no longer independent, even if they are far separated spatially. A measurement on one subsystem not only gives information about the other subsystem, but also provides possibility of manipulating it. Therefore entanglement becomes the main tool in quantum computations and information processing, quantum cryptography, teleportation and etc., (Angelakis et al., 2006). Due to the intrinsic pairwise character of the entanglement, entangled qubit pairs play crucial role in such computations. If the state is separable

$$\begin{aligned} |\psi\rangle &= (\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle) && (2.46)\\ &= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle && (2.47) \end{aligned}$$

then

$$C = |\alpha_1\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\beta_1\alpha_2| = 0 \tag{2.48}$$

If $C \neq 0$ then the state is not separable. It is called entangled state. Determinant (2.48) is called the ***concurrence*** and it can be considered as a measure of entanglement. One qubit gates can not generate entanglement so to transform separable states to nonseparable states we need two qubit gate, for example $CNOT$ gate. Applying $CNOT$ gate to separable states

$$CNOT(\alpha|0\rangle + \beta|1\rangle)|0\rangle = \alpha|00\rangle + \beta|11\rangle \tag{2.49}$$

for $\alpha, \beta \neq 0$ we obtain non-separable states. To study entangled states we need to introduce mixed states which is derived in terms of density operator.

2.5.1. Density Operator

If $A|i\rangle = i|i\rangle$ and we know the system is in a state $|\psi\rangle$, we can say measurement of an observable A will give a value i with probability $|\langle i|\psi\rangle|^2$. If a state of the system can be represented by a state vector $|\psi\rangle$, the system is said to be in a pure state. In such state, expectation value of A is given by

$$\langle A\rangle = \frac{\langle\psi|A|\psi\rangle}{\langle\psi|\psi\rangle} \tag{2.50}$$

If the state of the system is not known completely the system is in mixed state. An ensemble must then be formed with elements in different possible states, weighted according to any available partial knowledge about the state of the system, so that

$$\langle A\rangle = \sum_i p_i \langle\psi_i|A|\psi_i\rangle \tag{2.51}$$

where $\langle\psi_i|\psi_i\rangle = 1$ and $\sum_i p_i = 1$. In order to describe a system in which the probability that it is in the state $|\psi_i\rangle$ is p_i, we introduce density operator

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{2.52}$$

which is Hermitian $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i| = (\sum_i p_i|\psi_i\rangle\langle\psi_i|)^\dagger = \rho^\dagger$, has unit trace $Tr(\rho) = 1$ and is the positive operator

$$\langle\phi_i|\rho|\phi_i\rangle = \sum_i p_i\langle\phi_i|\psi_i\rangle\langle\psi_i|\phi_i\rangle = \sum_i p_i|\langle\psi_i|\phi_i\rangle|^2 \geq 0 \tag{2.53}$$

where $|\phi_i\rangle$ be any ket and p_i is real and positive. If the state of the system is known then it is in a pure state. The density operator of a pure state is

$$\rho = |\psi\rangle\langle\psi|. \tag{2.54}$$

Since $\rho^2 = \rho$ it is also a projector. The trace of an operator ρ is $Tr(\rho) = \langle\psi|\psi\rangle = 1$ and $Tr(\rho^2) = 1$ for a pure state . The density operator of a mixed state is

$$\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|. \tag{2.55}$$

The trace of an operator ρ is $Tr(\rho) = \sum_i p_i = 1$ and

$$Tr(\rho^2) = \sum_i p_i^2 < 1 \tag{2.56}$$

for mixed state.

Density matrix for a qubit can be written in the next form

$$\begin{aligned} \rho &= \frac{1}{2}(I + S_x\sigma_x + S_y\sigma_y + S_z\sigma_z) \equiv \frac{1}{2}(I + \vec{S}\cdot\vec{\sigma}) && (2.57) \\ &= \frac{1}{2}\begin{pmatrix} 1+S_z & S_x - iS_y \\ S_x + iS_y & 1 - S_z \end{pmatrix} && (2.58) \end{aligned}$$

$$Tr(\rho) = 1, \qquad Tr(\rho^2) = \frac{1+\vec{S}^2}{2} \tag{2.59}$$

If the state is pure $Tr(\rho^2) = 1$, and $|\vec{S}| = 1$, so it represents a ***Bloch sphere***. If the state is mixed $Tr(\rho^2) < 1$ and as follows $|\vec{S}| < 1$, then it represents a ***Bloch ball***.

2.5.2. Reduced Density Matrix

We consider the two systems A and B that can be described by a density operator, ρ^{AB}. The reduced density operator for system A is defined by

$$\rho^A \equiv tr_B(\rho^{AB}), \tag{2.60}$$

where tr_B is a map of operators known as the partial trace over system B. The partial trace is defined by

$$tr_B(|\psi_1\rangle\langle\psi_2| \otimes |\phi_1\rangle\langle\phi_2|) \equiv |\psi_1\rangle\langle\psi_2| tr(|\phi_1\rangle\langle\phi_2|) \tag{2.61}$$

where $|\psi_1\rangle$ and $|\psi_2\rangle$ are any two vectors in the state space of A, and $|\phi_1\rangle$ and $|\phi_2\rangle$ are any two vectors in the state space of B. As an example let us consider the Bell state $|\psi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ which has the density operator

$$\rho = \frac{|00\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 11|}{2} \tag{2.62}$$

Tracing out the second qubit, we find the reduced density matrix for the first qubit

$$\rho^A = tr_B(\rho) \tag{2.63}$$
$$= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \frac{I}{2}. \tag{2.64}$$

Since $tr(\rho^A)^2 = \frac{1}{2} \leq 1$ this state is a mixed state. The state of the joint system of two qubits is a pure state, that is, it is known exactly: however, the first qubit is in a mixed state, that is, a state about which we apparently do not have maximal knowledge. This strange property, that the joint state of a system can be completely known, yet a subsystem be in mixed states, is another hallmark of quantum entanglement (Nielsen & Chuang).

Lemma 2.5.2.1 ***(Singular Value Decomposition:)*** *Any complex $d_A \times d_B$ matrix A can be written as*

$$A = UDV^\dagger \tag{2.65}$$

where U and V are unitary, and $D = \begin{pmatrix} c_1 & & \\ & . & \\ & & c_n \end{pmatrix}$ *is diagonal with* $c_i > 0$ *and* $\sum_i c_k^2 = 1$.

Proof 2.5.2.2

$$A = UDV^\dagger, \quad A^\dagger = VDU^\dagger \tag{2.66}$$

$$AA^\dagger U = UD^2, \quad A^\dagger AV = VD^2 \tag{2.67}$$

Eigenvectors of $AA^\dagger$ gives columns of U and eigenvectors of $A^\dagger A$ gives columns of V. Square roots of eigenvalues of $AA^\dagger$ or $A^\dagger A$ gives diagonal elements of D.

Lemma 2.5.2.3 *(**Schmidt Decomposition**:) Every pure state in the Hilbert space $\mathcal{H} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$ with dimension $d = d_A \cdot d_B$, can be expressed in the form*

$$|\psi\rangle = \sum_i^r c_i |e_i\rangle_A \otimes |f_i\rangle_B \tag{2.68}$$

where $\{|e_i\rangle\}$ is an orthonormal basis for $\mathcal{H}_\mathcal{A}$, $\{|f_i\rangle\}$ is an orthonormal basis for $\mathcal{H}_\mathcal{B}$ with c_i real, $c_i > 0$ and $\sum_i c_k^2 = 1$.

Proof 2.5.2.4

$$\begin{aligned} |\psi\rangle = \sum_{ij} t_{ij}|ij\rangle &= \sum_{ijk} U_{ik} c_k V^\dagger_{kj} |ij\rangle && (2.69) \\ &= \sum_{ijk} c_k \underbrace{U_{ik}|i\rangle}_{|e_k\rangle_A} \otimes \underbrace{V^\dagger_{kj}|j\rangle}_{|f_k\rangle_B} && (2.70) \\ &= \sum_k c_k |e_k\rangle_A \otimes |f_k\rangle_B && (2.71) \end{aligned}$$

where c_i real, $c_i > 0$ and $\sum_i c_k^2 = 1$.

2.6. Concurrence for Pure State

Density matrix written in Schmidt basis is

$$\rho = |\psi\rangle\langle\psi| = \sum_i c_i^2 |e_i\rangle_A \otimes |f_i\rangle_B \langle e_i|_A \otimes \langle f_i|_B \tag{2.72}$$

Then the reduced density matrix ρ_A can be expressed as

$$\rho_A = Tr_B(\rho_{AB}) = \sum_i c_i^2 |e_i\rangle_A \langle e_i|_A \tag{2.73}$$

$c_i = \sqrt{\lambda_i}$ where λ_i are eigenvalues of ρ_A (or ρ_B). When $Tr(\rho_A) = 1$ the state is ***separable*** (unentangled) and when $Tr(\rho_A) < 1$ the state is ***non-separable*** (entangled). This allows us to characterize level of entanglement in terms of concurrence. We have

$$\begin{aligned} Tr(\rho_A) &= c_1^2 + c_2^2 = 1 && (2.74) \\ Tr(\rho_A^2) &= c_1^4 + c_2^4 = (c_1^2 + c_2^2)^2 - 2c_1^2 c_2^2 = 1 - 2c_1^2 c_2^2. && (2.75) \end{aligned}$$

If $c_1 = 0 \ or \ c_2 = 0$, $Tr(\rho_A^2) = 1$, then the state is a pure state and if $c_1 c_2 \neq 0$, $Tr(\rho_A^2) < 1$, the state is a mixed state. Then we have

$$\sqrt{2(1 - Tr(\rho_A^2))} = 2c_1 c_2 = C \tag{2.76}$$

where C is called the concurrence. Relation

$$Tr(\rho_A) = c_1^2 + c_2^2 = 1 \tag{2.77}$$

implies $c_1 = \cos\alpha, \ c_2 = \sin\alpha$ where $0 < \alpha < \frac{\pi}{2}$, c_1 and c_2 are positive. Then the concurrence

$$C = 2c_1 c_2 = \sin 2\alpha. \tag{2.78}$$

Since $0 \leq \sin 2\alpha \leq 1$, we have restrictions on values of the concurrence $0 \leq C \leq 1$. Now we can apply the definition of concurrence (2.76) to pure state. For $T = t_{ij}$ and $D = c_{ij}$ we have $T = UDV^{\dagger}$ and $D = U^{\dagger}TV$

$$\begin{aligned} 2|\det D| &= 2|\det U^{\dagger}||\det T||\det V| && (2.79) \\ &= 2|e^{i\tau}||\det T||e^{i\theta}| && (2.80) \\ &= 2|\det T| && (2.81) \\ 2|c_1 c_2| &= 2|\det t_{ij}| = C && (2.82) \end{aligned}$$

Then for pure state $|\psi\rangle = \sum_{ij} t_{ij}|ij\rangle = t_{00}|00\rangle + t_{01}|01\rangle + t_{10}|10\rangle + t_{11}|11\rangle$ the concurrence is

$$C = 2|t_{00}t_{11} - t_{01}t_{10}| \tag{2.83}$$

This formula coincides with the determinant definition of entanglement introduced in (2.48). For example, let us consider the Bell State

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{2.84}$$

Then the concurrence is

$$\begin{aligned} C &= 2|t_{00}t_{11} - t_{01}t_{10}| && (2.85) \\ &= 2|\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} - 0| = 1 && (2.86) \end{aligned}$$

and the state is maximally entangled.

2.7. Heisenberg Spin Models

As we have seen in Section 2.5 two qubit gate, $CNOT$ gate, can generate entanglement of qubits. This gate can be considered as an interaction between two qubits. Realizing qubits as spins we can describe generic two qubit gate as an interaction between spins. The simplest example of two spins $\vec{S}_i$ and $\vec{S}_j$ interaction is given by next exchange interaction Hamiltonian

$$H = J \ \vec{S}_i \cdot \vec{S}_j, \tag{2.87}$$

where parameter J is called the exchange interaction and

$$\vec{S}_i \cdot \vec{S}_j = |\vec{S}_i| \cdot |\vec{S}_j| \cos\theta_{ij}. \tag{2.88}$$

For the chain of N spins,

$$H = J \sum_{i=1}^{N} \vec{S}_i \cdot \vec{S}_{i+1} \tag{2.89}$$

it gives the Heisenberg spin chain with the nearest neighbor interaction. If $J < 0$ then minimum energy or the ground state of the system is

$$H = -|J| \sum_i |\vec{S}_i| \cdot |\vec{S}_{i+1}| \underbrace{\cos\theta_{i\ i+1}}_{\theta_{i\ i+1}=0} \tag{2.90}$$

for $\theta_{ii+1} = 0$ which corresponds to the ferromagnetic ground state.

$$|\Phi^{\uparrow}\rangle \ = \ |\uparrow\uparrow\uparrow\uparrow \ ... \uparrow\rangle = |00000...0\rangle \tag{2.91}$$

$$|\Phi^{\downarrow}\rangle \ = \ |\downarrow\downarrow\downarrow\downarrow \ ... \downarrow\rangle = |11111...1\rangle \tag{2.92}$$

If $J > 0$ then minimum energy or the ground state of the system is

$$H = |J| \sum_i |\vec{S}_i| \cdot |\vec{S}_{i+1}| \underbrace{\cos\theta_{i\ i+1}}_{\theta_{i\ i+1}=\pi} \tag{2.93}$$

the anti-ferromagnetic ground state.

$$|\Phi\rangle = |\uparrow\downarrow\uparrow\downarrow ... \downarrow\rangle = |0101...0\rangle \tag{2.94}$$

Generalization of the Heisenberg model (2.89) can be written in the form

$$H = J\sum_{i=1}^{n}(J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z) \tag{2.95}$$

is known as the XYZ model. When $J_x = J_y = J_z$ (2.95) reduces to the Heisenberg model (2.89) or XXX model. Another reductions are known as the Ising model $(J_x = J_y = 0)$

$$H = J\sum_{i=1}^{n} S_i^z S_{i+1}^z, \tag{2.96}$$

XX model if $J_x = J_y = J$ and $J_z = 0$

$$H = J\sum_{i=1}^{n}(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y), \tag{2.97}$$

XY model if $J_z = 0$

$$H = \sum_{i=1}^{n}(J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y), \tag{2.98}$$

XXZ model if $J_x = J_y$- easy axis or easy plane anisotropic Heisenberg model

$$H = \sum_{i=1}^{n}[J(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + J_z S_i^z S_{i+1}^z] \tag{2.99}$$

2.8. Effective Spin Models

In this section we like to show universality of general Heisenberg spin models. Indeed the model which appeared first as description of spin interaction, then explored intensively as models of studying phase transitions in quantum systems. Moreover, recently the models appear as effective models in description of nuclear spins and in the description of electron correlations of Hydrogen molecule. Below we give two physical examples effectively described by the Ising model.

2.8.1. Ising Model for Two Nuclear Spins

Recently two nuclear spins were considered in a model with weak Heisenberg type interaction in a constant longitudinal magnetic field along z direction (Tong & Tao, 2006)

$$H = H_z + H_{xy} \tag{2.100}$$

$$H_z = -\frac{1}{2}(\omega_1\sigma_1^z + \omega_2\sigma_2^z + J\sigma_1^z\sigma_2^z) \tag{2.101}$$

$$H_{xy} = -\frac{1}{2}(J\sigma_1^x\sigma_2^x + J\sigma_1^y\sigma_2^y) \tag{2.102}$$

where the isotropic form for the spin coupling J is assumed, and $\omega_{1,2} \equiv (B \mp b)$ are the Larmor frequencies of two nuclear spins, $\hbar = 1$. In the experiments, two different nuclear spins are selected, $\omega_1 \neq \omega_2$ (we assume $\omega_1 > \omega_2$), and the longitudinal constant magnetic field is in the order of $1THz$, so that ω_1, ω_2 are much larger than J and $\eta = \frac{J}{(\omega_1-\omega_2)} \ll 1$. H_{xy} is non-diagonal in σ_z representation and due to quantum fluctuations of order η^2, can be ignored. Thus, the Ising part H_z of the Hamiltonian is a well precise approximation (Tong & Tao, 2006). However as will see in next section, for the Ising model with external magnetic fields no entanglement occurs, this is why two nuclear spins in this model are unentangled for any ω_1 and ω_2.

2.8.2. Ising Model for Electron Correlations

To understand the entanglement behavior for H_2 molecule using quantum chemistry methods, the entanglement for a simpler two-electron model system was calculated (Huang & Kais, 2005). This is a model of two spin $1/2$ electrons with an exchange coupling constant J in an effective transverse magnetic field of strength B. In order to describe the environment of the electrons in a molecule, we simply introduce a small effective external magnetic field B. The general Hamiltonian for such a system is given by

$$H = -J\sigma_1^x \otimes \sigma_2^x - B\sigma_1^z \otimes I_2 - BI_1 \otimes \sigma_2^z \tag{2.103}$$

This Hamiltonian has the form of effective Ising model which describes electron correlations in molecular systems.

2.9. Thermal Entanglement and Wootters Concurrence

In realistic situation quantum computers will work in environment with nonzero temperature, so important question is to see the influence of temperature on entanglement property. Since by increasing temperature generically entanglement should decrease and at some critical value of temperature it disappears. Starting from this critical temperature quantum computers will not work. Our goal in the next chapter is to study influence on critical temperature of different physical parameters like exchange constants, magnetic fields, anisotropic exchange DM interaction, distance between qubits, to find the way to increase the critical temperature and so far the entanglement. We are not going to work with real physical systems but with some models, studying basic principles and influence of these parameters on entanglement. This will allow us to see in which situations critical temperature can be increased. According to quantum statistics (Landau & Lifshitz, 1980) the state of the quantum system at thermal equilibrium is determined by the density matrix

$$\rho(T) = \frac{e^{-H/kT}}{Tr[e^{-H/kT}]} = \frac{e^{-H/kT}}{Z}, \tag{2.104}$$

where $Z = Tr[e^{-H/kT}]$ is the partition function, k is the Boltzmann constant and T is the temperature. As $\rho(T)$ represents a thermal state, the entanglement in this state is called the ***thermal entanglement***. The degree of entanglement could be characterized by the concurrence C_{12}, which is defined as (Wootters, 1998), (Hill & Wootters, 1997)

$$C_{12} = max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \tag{2.105}$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 > 0$ are the ordered square roots of eigenvalues of the operator

$$\rho_{12} = \rho(\sigma^y \otimes \sigma^y)\rho^*(\sigma^y \otimes \sigma^y). \tag{2.106}$$

The concurrence is bounded function $0 \leq C_{12} \leq 1$, so that when $C_{12} = 0$, the states are unentangled, while for $C_{12} = 1$, the states are maximally entangled.

CHAPTER 3

TWO QUBIT ENTANGLEMENT IN SPIN MODELS

The results shown in this chapter appeared in (Gurkan & Pashaev, 2010) and presented in (Gurkan & Pashaev, 2010). It is clear that single qubit gates are unable to generate entanglement in an N qubit separable system, and to prepare an entangled state one needs an inter qubit interaction, which is a two qubit gate. The simplest two qubit interaction is described by the Ising Hamiltonian in the form of $J\sigma_1^z\sigma_2^z$. More general interaction between two qubits is given by the Heisenberg magnetic spin interaction models. These models have been extensively studied during several decades, experimentally in condensed matter systems (Baryakhtar et al., 1998) and theoretically as exactly solvable many body problems (Bethe, 1931), (Lieb & Mattis, 1966), (Baxter, 1982). Now they become promising to realize quantum computation and information processing, by generating entangled qubits and constructing quantum gates (Zheng & Guo, 2000), (Imamoglu et al., 1999) in a more general context than the magnetic chains.

Recently in this way interaction of two nuclear spins having the Heisenberg form were considered (Tong & Tao, 2006). The nuclear spins from one side are well isolated from the environment and their decoherence time is sufficiently long. From another side nuclei with spin $1/2$ are natural representatives of qubits in quantum information processing, which can realize quantum computational algorithms by using NMR (Yusa et al., 2005), (Chuang et al., 1998), (Vandersypen et al., 2001).

Very recently entanglement of two qubits (Wootters, 1998) and its dependence on external magnetic fields, anisotropy and temperature have been considered in several Heisenberg models: the Ising model (Gunlycke et al., 2001), (Terzis & Paspalakis, 2004), (Childs et al., 2003); the XX and XY models (Zheng & Guo, 2000), (Wang, 2002), (Wang, 2001a), (Kamta & Starace, 2002), (Xi & Liu, 2007), (Hamieh & Katsnelson, 2005), (Sun et al., 2003); the XXX model (Arnesen et al., 2001); the XXZ model (Wang, 2001c); and the XYZ model (Xi et al., 2002), (Rigolin, 2004), (Zhou et al., 2003). Particularly dependence of entanglement on the type of spin ordering, was shown, so that in the isotropic Heisenberg spin chain (the XXX model) spin states are unentangled in the ferromagnetic case $J < 0$, while for the antiferromagnetic case $J > 0$ entanglement occurs for sufficiently small temperature $T < T_c = \frac{2J}{k \ln 3}$. Significant point in the study of such models is how to increase entanglement in situation when it already exists or to create entanglement in situation when it does not exist. Certainly this can be expected from a generalization of bilinear spin-spin interaction of the Heisenberg form. Around 50 years ago explaining weak ferromagnetism of antiferromagnetic crystals ($\alpha - Fe_2O_3, MnCO_3$ and CrF_3), has been controversial problem for a decade, Dzialoshinskii (Dzialoshinski, 1958) from phenomenological arguments, and Moriya (Moriya, 1960) from microscopic grounds, have introduced anisotropic antisymmetric exchange interaction, the Dzialoshinskii-Moriya (DM) interaction, expressed by

$$\vec{D} \cdot [\vec{S}_1 \times \vec{S}_2]. \tag{3.1}$$

This interaction arising from extension of the Anderson superexchange interaction theory by including the spin orbit coupling effect (Moriya, 1960), is important not only for the weak ferromagnetism but also for the spin arrangement in antiferromagnets of low symmetry. In contrast to the Heisenberg interaction which tends to render neighbor spins parallel, the DM interaction has the effect of turning them perpendicular to one another. As we will see in the present thesis it turns out that such spin arrangements are likely to increase entanglement. In most materials with weak ferromagnetism and the DM coupling, parameter D is small compared to J. The values reported in the literature range from $\frac{D}{J} \approx 0.02$ to 0.07 (see (Aristov & Maleyev, 2000) and references therein). However in some compounds the DM interaction can attain a sizeable value in comparison with the usual symmetric superexchange J. Depending on compound its value varies between $\frac{D}{J} \approx 0.05$ to 0.2. Moreover, recently the DM interaction was found to be present in a number of quasi-one-dimensional magnets (Pires & Goueva, 2000). Even it was found that the compound $RbCoCl_3.2H_2O$ is described as a pure DM chain (Elearney & Merchant, 1999). The low-temperature magnetic behavior of this compound gives strong evidence that the material consists of weakly interacting linear chains with predominant DM interaction. In addition, study of the DM interaction influence on dynamics of the one dimensional quantum antiferromagnet shows the big difference in the behavior, depending on whether the coupling D is smaller or larger than the exchange interaction J (Pires & Goueva, 2000). All these results imply that a study of spin models with DM interaction could have realistic applications. Then for applications in quantum computations it poses the problem to find the entanglement dependence on this interaction.

In the present chapter we study the influence of the Dzialoshinskii-Moriya interaction on entanglement of two qubits in all particular magnetic spin models, including the most general XYZ model (Gurkan & Pashaev, 2010). We find that in all cases, inclusion of the DM interaction creates, when it does not exist, or strengthens, when it exists, entanglement. For example, we show that in the case of isotropic Heisenberg XXX model discussed above, inclusion of this term increases entanglement for antiferromagnetic case and for sufficiently strong coupling

$$D > (kTsinh^{-1}e^{|J|/kT} - J^2)^{1/2} \tag{3.2}$$

it creates entanglement even in ferromagnetic case. We give detailed physical explanations of these results by studying ground state of the system at $T = 0$. In this state we find nonanalytic dependence of concurrence on the DM interaction and establish its relation with the quantum phase transition. These results indicate that spin models with DM coupling have some potential applications in quantum computations, and DM interaction could be an efficient control parameter of entanglement.

3.1. Spectrum and Density Matrix

We start our consideration with the most general XYZ model, by inclusion of homogeneous B and nonhomogeneous b magnetic fields, and choosing the DM interaction (3.1) in the form $\frac{\vec{D}}{2} = \frac{D}{2} \cdot \vec{z}$. Then for two qubits we have Hamiltonian

$$H = \frac{1}{2}[J_x\, \sigma_1^x \sigma_2^x + J_y\, \sigma_1^y \sigma_2^y + J_z\, \sigma_1^z \sigma_2^z + B_+\, \sigma_1^z + B_-\, \sigma_2^z + D(\sigma_1^x \sigma_2^y - \sigma_1^y \sigma_2^x)], \tag{3.3}$$

where product of σ_1^i and σ_2^j

$$\sigma_1^i \sigma_2^j \equiv \sigma_1^i \otimes \sigma_2^j \tag{3.4}$$

is a tensor product which we will frequently skip $\otimes$ and $B_+ \equiv B + b, B_- \equiv B - b$ and $\sigma_i^x, \sigma_i^y, \sigma_i^z$, $i = 1, 2$ denote Pauli matrices related with the first and the second qubits.

To study the thermal entanglement in this system firstly we need to obtain all eigenvalues and eigenstates of the Hamiltonian (3.3):

$$H|\Psi_i\rangle = E_i|\Psi_i\rangle, \quad i = 1, 2, 3, 4. \tag{3.5}$$

Simple calculations show that the energy levels are:

$$E_{1,2} = \frac{J_z}{2} \mp \mu, \quad E_{3,4} = -\frac{J_z}{2} \mp \nu \tag{3.6}$$

where $\mu \equiv \sqrt{B^2 + J_-^2}$, $\nu \equiv \sqrt{b^2 + J_+^2 + D^2}$, $J_\pm \equiv \frac{J_x \pm J_y}{2}$, and the corresponding wave functions are (for details see Appendix A)

$$|\Psi_{1,2}\rangle = \frac{1}{\sqrt{2\mu(\mu \pm B)}} \begin{bmatrix} J_- \\ 0 \\ 0 \\ -(B \pm \mu) \end{bmatrix}, \quad |\Psi_{3,4}\rangle = \frac{1}{\sqrt{2\nu(\nu \mp b)}} \begin{bmatrix} 0 \\ (b \mp \nu) \\ J_+ - iD \\ 0 \end{bmatrix}. \tag{3.7}$$

For $B = 0,\ b = 0,\ D = 0$ these wave functions reduce to the maximally entangled Bell states

$$|\Psi_{2,1}\rangle \longrightarrow |B_{0,3}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \tag{3.8}$$

$$|\Psi_{4,3}\rangle \longrightarrow |B_{1,2}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle \tag{3.9}$$

State of the system at thermal equilibrium is determined by the density matrix (for details Appendix B)

$$\rho(T) = \frac{e^{-H/kT}}{Tr[e^{-H/kT}]} = \frac{e^{-H/kT}}{Z}, \tag{3.10}$$

where $Z = Tr[e^{-H/kT}]$ is the partition function, k is the Boltzmann constant and T is the temperature. Then by exponentiation of Hamiltonian (3.3) we find

$$e^{-H/kT} = \begin{bmatrix} A_{11} & 0 & 0 & A_{14} \\ 0 & A_{22} & A_{23} & 0 \\ 0 & A_{32} & A_{33} & 0 \\ A_{41} & 0 & 0 & A_{44} \end{bmatrix} \tag{3.11}$$

where

$$
\begin{aligned}
A_{11} &= e^{\frac{-J_z}{2kT}}\left[\cosh\frac{\mu}{kT}-\frac{B}{\mu}\sinh\frac{\mu}{kT}\right] && (3.12)\\
A_{44} &= e^{-\frac{J_z}{2kT}}\left[\cosh\frac{\mu}{kT}+\frac{B}{\mu}\sinh\frac{\mu}{kT}\right] && (3.13)\\
A_{14} &= -e^{-\frac{J_z}{2kT}}\frac{J_-}{\mu}\sinh\frac{\mu}{kT} && (3.14)\\
A_{41} &= -e^{-\frac{J_z}{2kT}}\frac{J_-}{\mu}\sinh\frac{\mu}{kT} && (3.15)\\
A_{22} &= e^{\frac{J_z}{2kT}}\left[\cosh\frac{\nu}{kT}-\frac{b}{\nu}\sinh\frac{\nu}{kT}\right] && (3.16)\\
A_{33} &= e^{\frac{J_z}{2kT}}\left[\cosh\frac{\nu}{kT}+\frac{b}{\nu}\sinh\frac{\nu}{kT}\right] && (3.17)\\
A_{23} &= -e^{\frac{J_z}{2kT}}\frac{J_+ + iD}{\nu}\sinh\frac{\nu}{kT} && (3.18)\\
A_{32} &= -e^{\frac{J_z}{2kT}}\frac{J_+ - iD}{\nu}\sinh\frac{\nu}{kT} && (3.19)
\end{aligned}
$$

and

$$Z = Tr[e^{-H/kT}] = 2\left[e^{\frac{-J_z}{2kT}}\cosh\frac{\mu}{kT} + e^{\frac{J_z}{2kT}}\cosh\frac{\nu}{kT}\right]. \tag{3.20}$$

As $\rho(T)$ represents a thermal state, the entanglement in this state is called the ***thermal entanglement***. The degree of entanglement could be characterized by the concurrence C_{12}, which is defined as (Wootters, 1998), (Hill & Wootters, 1997)

$$C_{12} = max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \tag{3.21}$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 > 0$ are the ordered square roots of eigenvalues of the operator

$$\rho_{12} = \rho(\sigma^y \otimes \sigma^y)\rho^*(\sigma^y \otimes \sigma^y). \tag{3.22}$$

The concurrence is bounded function $0 \leq C_{12} \leq 1$, so that when $C_{12} = 0$, the states are unentangled, while for $C_{12} = 1$, the states are maximally entangled.

For the general Hamiltonian (3.3) we find :

$$\lambda_{1,2} = \frac{e^{\frac{-J_z}{2kT}}}{Z}\left|\sqrt{1+\frac{J_-^2}{\mu^2}\sinh^2\frac{\mu}{kT}} \mp \frac{J_-}{\mu}\sinh\frac{\mu}{kT}\right| \tag{3.23}$$

$$\lambda_{3,4} = \frac{e^{\frac{J_z}{2kT}}}{Z}\left|\sqrt{1+\frac{J_+^2+D^2}{\nu^2}\sinh^2\frac{\nu}{kT}} \mp \frac{\sqrt{J_+^2+D^2}}{\nu}\sinh\frac{\nu}{kT}\right|. \tag{3.24}$$

Then, to calculate the concurrence we need to order these eigenvalues. Since they depend on several parameters, before studying the most general case, it is useful to treat all particular cases separately to clarify the influence of the DM interaction on the entanglement. Starting from pure DM model we study various Heisenberg models, including the general XYZ case.

Before this, we like just to stress here the general observation on the concurrence (3.21). If the biggest eigenvalue say λ_1 is degenerate, then its positive contribution would be compensated by the another degenerate one, so that $C_{12} = 0$ and states are always unentangled. We will encounter this situation in several cases and it has a simple physical explanation. The degenerate biggest eigenvalues of the density matrix correspond to the minimal values of the energy, so that the ground state of the system becomes degenerate and no entanglement occurs.

3.2. Pure DM Model

For pure DM model $J_x = J_y = J_z = 0$ and $B = b = 0, D \neq 0$ the Hamiltonian is in the next form

$$H = \frac{D}{2}(\sigma^x \otimes \sigma^y - \sigma^y \otimes \sigma^x). \tag{3.25}$$

As we discussed in introduction some realistic quasi-one dimensional compounds with predominance of DM interaction can be described as a pure DM model (Elearney & Merchant, 1999). Here we consider the main characteristic properties of the DM coupling interaction interaction between two qubits and its influence on the entanglement. If in Hamiltonian (3.3) we put $J_x = J_y = J_z = 0$ and $B = b = 0$ then the model is determined completely by the DM term (3.1). In this case the first two eigenstates become degenerate

$E_1 = E_2 = 0$ and $E_{3,4} = \pm D$. For definiteness we choose $D > 0$, then for $T = 0$ the ground state of the system with energy $E_4 = -D$ is an entangled state $|10\rangle - i|01\rangle$. When temperature increases this state becomes mixed with the higher states and entanglement decreases. But for sufficiently large value of D the ground state can be alienated so that entanglement increases. This shows that for a given D there exists

$$kT_c = \frac{D}{\ln(1+\sqrt{2})} \tag{3.26}$$

so that for the under critical case $T < T_c$ the states become entangled and the concurrence is

$$C_{12} = \frac{\sinh \frac{D}{kT} - 1}{\cosh \frac{D}{kT} + 1} . \tag{3.27}$$

As it can be seen in Fig. 3.1 for $T = 0$ the concurrence $C_{12} = 1$ and the ground state is maximally entangled.

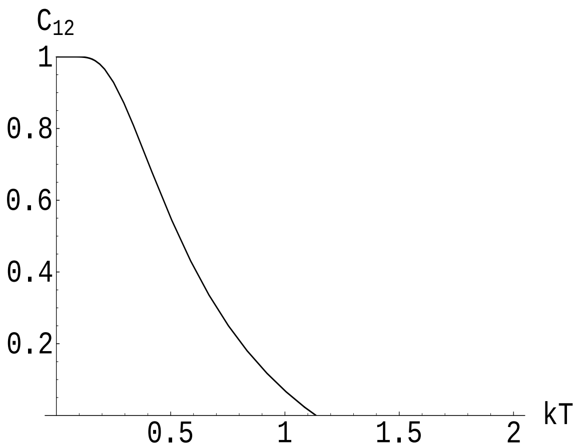


Figure 3.1. Concurrence versus temperature for $D = 1$ and $T_c = 1.136$

3.3. Ising Model

For $J_x = J_y = 0$, $J_z \neq 0$ and $B = b = 0, D = 0$ the Hamiltonian

$$H = \frac{J_z}{2}\sigma_1^z \otimes \sigma_2^z \tag{3.28}$$

or in the matrix form

$$H = \begin{bmatrix} \frac{J_z}{2} & 0 & 0 & 0 \\ 0 & -\frac{J_z}{2} & 0 & 0 \\ 0 & 0 & -\frac{J_z}{2} & 0 \\ 0 & 0 & 0 & \frac{J_z}{2} \end{bmatrix} \tag{3.29}$$

describes the Ising model . When both anti-ferromagnetic and ferromagnetic cases analyzed in the anti-ferromagnetic case ($J_z > 0$), the ordered eigenvalues are

$$\lambda_1 = \lambda_2 \frac{e^{J_z/2kT}}{Z} > \lambda_3 = \lambda_4 = \frac{e^{-J_z/2kT}}{Z}. \tag{3.30}$$

where $Z = 4\cosh\frac{J_z}{2kT}$ and the concurrence is

$$C_{12} = max\{\frac{-e^{J_z/2kT}}{2\cosh\frac{J_z}{2kT}}, 0\} = 0. \tag{3.31}$$

In the ferromagnetic case ($J_z < 0$), the ordered eigenvalues are

$$\lambda_1 = \lambda_2 \frac{e^{|J_z|/2kT}}{Z} > \lambda_3 = \lambda_4 = \frac{e^{-|J_z|/2kT}}{Z}. \tag{3.32}$$

where $Z = 4\cosh\frac{|J_z|}{2kT}$ and the concurrence is

$$C_{12} = max\{\frac{-e^{-|J_z|/2kT}}{2\cosh\frac{|J_z|}{2kT}}, 0\} = 0 \tag{3.33}$$

It was observed for both cases the concurrence is zero and the states are always unentangled (Gunlycke et al., 2001), (Terzis & Paspalakis, 2004), (Childs et al., 2003). The physical insight of such behavior is easy to understand. When $J_- = J_+ = 0$ the density matrix ρ (3.10) is diagonal in the standard basis which implies the absence of quantum correlations. Despite of having four maximally entangled states as the eigenvectors, the states $|\Psi_{1,2}\rangle$ and $|\Psi_{3,4}\rangle$ are degenerated, so that the Ising thermal state has no entanglement. The situation does not change if one includes homogeneous B or non-homogeneous b magnetic fields, because the density matrix ρ is still diagonal and no entanglement occurs .

3.3.1. Ising Model with DM Interaction

With addition of the DM interaction we have the next Hamiltonian

$$H = \frac{1}{2}[J_z\ \sigma_1^z\sigma_2^z + D(\sigma_1^x\sigma_2^y - \sigma_1^y\sigma_2^x)] \tag{3.34}$$

or in the matrix form

$$H = \begin{bmatrix} \frac{J_z}{2} & 0 & 0 & 0 \\ 0 & -\frac{J_z}{2} & iD & 0 \\ 0 & -iD & -\frac{J_z}{2} & 0 \\ 0 & 0 & 0 & \frac{J_z}{2} \end{bmatrix}. \tag{3.35}$$

The eigenvalues are

$$\lambda_1 = \frac{e^{(J_z+2D)/2kT}}{Z}, \quad \lambda_2 = \frac{e^{(J_z-2D)/2kT}}{Z}, \quad \lambda_3 = \lambda_4 = \frac{e^{-J_z/2kT}}{Z}. \tag{3.36}$$

where $Z = 2(e^{J_z/2kT}\cosh\frac{D}{kT} + e^{-J_z/2kT})$. In contrast to magnetic fields, which does not create entanglement, inclusion of the DM interaction contributes to the nondiagonal elements of ρ and creates entanglement. In the anti-ferromagnetic case, the addition of the DM interaction to the Ising model splits the degenerate ground state with $E_3 = E_4 = -\frac{J_z}{2}$ so that it becomes a singlet with $E_3 = -\frac{|J_z|}{2} - D$, for $D > 0$ or $E_4 = -\frac{|J_z|}{2} + D$, for

$D < 0$. Ordering the eigenvalues $\lambda_1 > \lambda_2 > \lambda_3 = \lambda_4$ we have the concurrence

$$C_{12} = max\{\frac{\sinh \frac{|D|}{kT} - e^{-J_z/kT}}{\cosh \frac{|D|}{kT} + e^{-J_z/kT}}, 0\} \tag{3.37}$$

Then $C_{12} = 0$ and no entanglement occurs. If $\sinh \frac{|D|}{kT} \leq e^{-J_z/kT}$. When $\sinh \frac{|D|}{kT} > e^{-J_z/kT}$ the states are entangled

$$C_{12} = \frac{\sinh \frac{|D|}{kT} - e^{-J_z/kT}}{\cosh \frac{|D|}{kT} + e^{-J_z/kT}}. \tag{3.38}$$

Moreover states become more entangled for low temperatures (maximally entangled for any D and $T = 0$).

$$\lim_{kT \to 0} C_{12} = 1 \tag{3.39}$$

and for stronger DM interaction

$$\lim_{D \to \infty} C_{12} = 1. \tag{3.40}$$

When temperature increases the maximally entangled ground state becomes mixed with the higher eigenstates and the entanglement decreases. However, for a given temperature by increasing the DM interaction $D > D_c$, where $D_c = kT \sinh^{-1} e^{-J_z/kT}$, we can decrease this mixture and increase entanglement. In the ferromagnetic case the ground state for small D at $T = 0$ is also a doublet and no entanglement occurs. However, with growing D the eigenstate $E_3 = \frac{|J_z|}{2} - D$ is lowering so that at critical value $D_c = |J_z|$ the ground state becomes triplet. With weak DM interaction $|D| < |J_z|$, the ordered eigenvalues are $\lambda_3 = \lambda_4 > \lambda_1 > \lambda_2$ and we obtain the concurrence as

$$C_{12} = max\{\frac{-\cosh \frac{|D|}{kT}}{\cosh \frac{|D|}{kT} e^{-|J_z|/kT} + e^{|J_z|/kT}}, 0\} = 0 \tag{3.41}$$

no entanglement occurs. With strong DM interaction $|D| > |J_z|$, the ordered eigenvalues are $\lambda_1 > \lambda_3 = \lambda_4 > \lambda_2$ and obtain the concurrence as

$$C_{12} = max\{\frac{\sinh\frac{|D|}{kT} - e^{|J_z|/2kT}}{\cosh\frac{|D|}{kT} + e^{|J_z|/2kT}}, 0\} \tag{3.42}$$

When $D > D_c$ the ground state E_3 is maximally entangled singlet. With growing temperature, a mixture of this state with the higher states decreases entanglement. For given temperature T, there exist the critical value

$$D_c = |J_z| + \frac{kT}{2}\ln(1 + e^{-2|J_z|/kT}) \tag{3.43}$$

so that for $D > D_c$ the concurrence is

$$C_{12} = \frac{\sinh\frac{|D|}{kT} - e^{|J_z|/kT}}{\cosh\frac{|D|}{kT} + e^{|J_z|/kT}}.. \tag{3.44}$$

Moreover states become more entangled for low temperatures

$$\lim_{kT \to 0} C_{12} = 1 \tag{3.45}$$

and for stronger DM interaction

$$\lim_{D \to \infty} C_{12} = 1\,. \tag{3.46}$$

There is entanglement even in ferromagnetic case with strong spin-orbit coupling. Comparing eqs. (3.38) and (3.44) we can see that in anti-ferromagnetic case, states can be more easily entangled then in the ferromagnetic one

3.3.2. Ising Model for Two Nuclear Spins with DM Interaction

As an application of the above calculations here we discuss entanglement of two nuclear spins. Recently two nuclear spins were considered in a model with weak Heisenberg type interaction in a constant longitudinal magnetic field along z direction (Tong & Tao, 2006)

$$\begin{aligned} H &= H_z + H_{xy} && (3.47) \\ H_z &= -\frac{1}{2}(\omega_1\sigma_1^z + \omega_2\sigma_2^z + J\sigma_1^z\sigma_2^z) && (3.48) \\ H_{xy} &= -\frac{1}{2}(J\sigma_1^x\sigma_2^x + J\sigma_1^y\sigma_2^y) && (3.49) \end{aligned}$$

where the isotropic form for the spin coupling J is assumed, and $\omega_{1,2} \equiv (B \mp b)$ are the Larmor frequencies of two nuclear spins, $\hbar = 1$. In the experiments, two different nuclear spins are selected, $\omega_1 \neq \omega_2$ (we assume $\omega_1 > \omega_2$), and the longitudinal constant magnetic field is in the order of $1THz$, so that ω_1, ω_2 are much larger than J and $\eta = \frac{J}{(\omega_1 - \omega_2)} \ll 1$. H_{xy} is non-diagonal in σ_z representation and due to quantum fluctuations of order η^2, can be ignored. Thus, the Ising part H_z of the Hamiltonian is a well precise approximation (Tong & Tao, 2006). However as we have seen above, for the Ising model with external magnetic fields no entanglement occurs, this is why two nuclear spins in this model are unentangled for any ω_1 and ω_2. From another side, as follows from our consideration in Sec.3.1 the addition of an interaction between qubits in the form of the DM coupling could make them entangled. Now by adding the DM interaction to two nuclear spin Hamiltonian (3.48) we get the Ising model with homogeneous magnetic field B, nonhomogeneous magnetic field b and the DM interaction D. In the antiferromagnetic and the ferromagnetic cases, when $J_z = \pm|J_z|$ respectively, for sufficiently strong $D > D_c$, where

$$\frac{D_c}{\sqrt{D_c^2 + b^2}} \sinh \frac{\sqrt{D_c^2 + b^2}}{kT} = e^{\mp \frac{|J_z|}{kT}}\ , \tag{3.50}$$

the states become entangled and the concurrence is

$$C_{12} = \frac{\frac{D}{\nu} \sinh \frac{\nu}{kT} - e^{\mp \frac{|J_z|}{kT}}}{\cosh \frac{\nu}{kT} + \cosh \frac{B}{kT} e^{\mp \frac{|J_z|}{kT}}} \tag{3.51}$$

where $B = (\omega_1 + \omega_2)/2$, $b = (\omega_1 - \omega_2)/2$ and $\nu = \sqrt{\frac{(\omega_2-\omega_1)^2}{4} + D^2}$. It is worth to note that the homogeneous magnetic field B does not change critical value for the entanglement, but could change level of the entanglement. Moreover, increasing magnetic field decreases value of the entanglement. It turns out that for the system at $T = 0$, the concurrence becomes nonanalytic when $D = D_c$

$$C_{12} = \begin{cases} \frac{D}{\nu}, & \nu > B \mp |J_z|; \\ \\ \frac{D}{2\nu}, & \nu = B \mp |J_z|; \\ \\ 0, & \nu < B \mp |J_z|, \end{cases} \tag{3.52}$$

which implies quantum phase transitions at the critical value $D_c = (B \mp |J_z|)^2 - b^2$.

3.4. XY Heisenberg Model

In pure XY Heisenberg Model $J_z = 0, J_x \neq J_y$ and $B = 0, b = 0, D = 0$ with the Hamiltonian

$$H = \frac{1}{2}[J_x \sigma_1^x \sigma_2^x + J_y \sigma_1^y \sigma_2^y] \tag{3.53}$$

or in the matrix form

$$H = \begin{bmatrix} 0 & 0 & 0 & J_- \\ 0 & 0 & J_+ & 0 \\ 0 & J_+ & 0 & 0 \\ J_- & 0 & 0 & 0 \end{bmatrix}, \tag{3.54}$$

the eigenvalues are calculated as

$$\lambda_1 = \frac{e^{J_-/kT}}{Z}, \quad \lambda_2 = \frac{e^{-J_-/kT}}{Z}, \quad \lambda_3 = \frac{e^{J_+/kT}}{Z}, \quad \lambda_4 = \frac{e^{-J_+/kT}}{Z}, \tag{3.55}$$

where

$$Z = 2(\cosh \frac{J_-}{kT} + \cosh \frac{J_+}{kT}). \tag{3.56}$$

Next we will analyze anti-ferromagnetic and ferromagnetic cases. In the anti-ferromagnetic case, the ordered eigenvalues are $\lambda_3 > \lambda_1 > \lambda_2 > \lambda_4$ and the concurrence is found as

$$C_{12} = max\{\frac{\sinh \frac{J_+}{kT} - \cosh \frac{J_-}{kT}}{\cosh \frac{J_-}{kT} + \cosh \frac{J_+}{kT}}, 0\}. \tag{3.57}$$

For $\sinh \frac{J_+}{kT} > \cosh \frac{J_-}{kT}$ the concurrence is

$$C_{12} = \frac{\sinh \frac{J_+}{kT} - \cosh \frac{J_-}{kT}}{\cosh \frac{J_-}{kT} + \cosh \frac{J_+}{kT}} \tag{3.58}$$

and for

$$\lim_{T \to 0} C_{12} = 1 \tag{3.59}$$

we have maximally entangled state. For $\sinh \frac{J_+}{kT} \leq \cosh \frac{J_-}{kT}$ the concurrence is $C_{12} = 0$ and no entanglement occurs. In the ferromagnetic case the concurrence for the ferromagnetic case is

$$C_{12} = max\{\frac{\sinh \frac{|J_-|}{kT} - \cosh \frac{J_+}{kT}}{\cosh \frac{|J_-|}{kT} + \cosh \frac{J_+}{kT}}, 0\}. \tag{3.60}$$

According to (3.60) the entanglement occurs only when $\sinh \frac{|J_-|}{kT} > \cosh \frac{J_+}{kT}$ with concurrence

$$C_{12} = \frac{\sinh \frac{J_+}{kT} - \cosh \frac{|J_-|}{kT}}{\cosh \frac{|J_-|}{kT} + \cosh \frac{J_+}{kT}} \tag{3.61}$$

$$\lim_{T\to 0} C_{12} = 1 \tag{3.62}$$

and no entanglement for $\sinh \frac{|J_-|}{kT} \leq \cosh \frac{J_+}{kT}$ and $C_{12} = 0$ (Wang, 2001a), (Hamieh & Katsnelson, 2005) .

Kamta and Starace investigated the thermal entanglement of a two- qubit Heisenberg XY chain in the presence of an external magnetic field along the z- axis in (Kamta & Starace, 2002) with the Hamiltonian

$$H = \frac{1}{2}[J_x\, \sigma_1^x \sigma_2^x + J_y\; \sigma_1^y \sigma_2^y + B\,(\sigma_1^z +\, \sigma_2^z)] \tag{3.63}$$

and eigenvalues

$$\lambda_{1,2} \;=\; \frac{1}{Z}\sqrt{1+\frac{2J_-^2}{\mu^2}\sinh^2\frac{\mu}{kT} \mp \frac{2J_-}{\mu}\sinh\frac{\mu}{kT}\sqrt{1+\frac{J_-^2}{\mu^2}\sinh^2\frac{\mu}{kT}}} \tag{3.64}$$

$$\lambda_3 \;=\; \frac{e^{J_+/kT}}{Z}, \quad \lambda_4 = \frac{e^{J_-/kT}}{Z} \tag{3.65}$$

where

$$Z = 2(\cosh\frac{\mu}{kT} + \cosh\frac{|J_+|}{kT}). \tag{3.66}$$

They showed that by adjusting the magnetic field strength, entangled states are produced for any finite temperature.

Sun et al. (Sun et al., 2003) extended later the work reported in (Kamta & Starace, 2002) by introducing a non-uniform magnetic field. The Hamiltonian is

$$H = \frac{1}{2}[J_x\, \sigma_1^x \sigma_2^x + J_y\; \sigma_1^y \sigma_2^y + (B+b)\, \sigma_1^z + (B-b)\, \sigma_2^z]. \tag{3.67}$$

Comparing to the uniform field case, they showed that entanglement can be more effectively controlled via a non-uniform magnetic field. Comparing to uniform field case entanglement can be more effectively controlled via non-uniform magnetic field.

3.4.1. XY Heisenberg Model with DM Interaction

By addition of the DM coupling the Hamiltonian becomes

$$H = \frac{1}{2}[J_x \, \sigma_1^x \sigma_2^x + J_y \ \sigma_1^y \sigma_2^y + D(\sigma_1^x \sigma_2^y - \sigma_1^y \sigma_2^x)] \tag{3.68}$$

and the eigenvalues are calculated as

$$\lambda_{1,2} = \frac{e^{\pm J_- /kT}}{Z}, \ \ \lambda_{3,4} = \frac{e^{\pm \sqrt{J_+^2 + D^2}/kT}}{Z} \tag{3.69}$$

where $Z = 2\left[\cosh \frac{|J_-|}{kT} + \cosh \frac{\sqrt{J_+^2+D^2}}{kT}\right]$. For the anti-ferromagnetic case the concurrence is in the next form

$$C_{12} = max\{ \frac{\sinh \frac{\sqrt{J_+^2+D^2}}{kT} - \cosh \frac{J_-}{kT}}{\cosh \frac{\sqrt{J_+^2+D^2}}{kT} + \cosh \frac{J_-}{kT}}, 0\}. \tag{3.70}$$

It shows that for any temperature T we can adjust sufficiently strong DM interaction D to have entanglement. For $\sinh \frac{\sqrt{J_+^2+D^2}}{kT} > \cosh \frac{J_-}{kT}$ the concurrence is calculated as

$$C_{12} = \frac{\sinh \frac{\sqrt{J_+^2+D^2}}{kT} - \cosh \frac{J_-}{kT}}{\cosh \frac{\sqrt{J_+^2+D^2}}{kT} + \cosh \frac{J_-}{kT}}. \tag{3.71}$$

In the $T \to 0$ limit

$$\lim_{T \to 0} C_{12} = 1 \tag{3.72}$$

the states are maximally entangled.

3.5. XX Heisenberg Model

In this section we consider isotropic XY Model so called XX model. For $J_z = 0, J_x = J_y \equiv J$ and $B = 0, b = 0, D = 0$. The Hamiltonian of this system is

$$H = \frac{J}{2}(\sigma_1^x \otimes \sigma_2^x + \sigma_1^y \otimes \sigma_2^y) \tag{3.73}$$

or in the matrix form

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & J & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.74}$$

the eigenvalues are

$$\lambda_1 = \frac{e^{J/kT}}{Z}, \quad \lambda_2 = \lambda_3 = \frac{1}{Z}, \quad \lambda_4 = \frac{e^{-J/kT}}{Z}. \tag{3.75}$$

In the anti-ferromagnetic case the ordered eigenvalues are $\lambda_1 > \lambda_2 = \lambda_3 > \lambda_4$ and the concurrence is

$$C_{12} = max\{\frac{\sinh \frac{J}{kT} - 1}{\cosh \frac{J}{kT} + 1}, 0\}. \tag{3.76}$$

For $\sinh \frac{J}{kT} > 1$ the concurrence is

$$C_{12} = \frac{\sinh \frac{J}{kT} - 1}{\cosh \frac{J}{kT} + 1} \tag{3.77}$$

with

$$\lim_{T \to 0} C_{12} = 1 \tag{3.78}$$

shows the states are maximally entangled and for $\sinh \frac{J}{kT} \leq 1, \;\; C_{12} = 0$ no entanglement occurs. In the ferromagnetic case the eigenvalues are

$$\lambda_1 = \frac{e^{-|J|/kT}}{Z}, \quad \lambda_2 = \lambda_3 = \frac{1}{Z}, \quad \lambda_4 = \frac{e^{|J|/kT}}{Z} \tag{3.79}$$

and the concurrence is in the next form

$$C_{12} = max\{\frac{\sinh \frac{|J|}{kT} - 1}{\cosh \frac{|J|}{kT} + 1}, 0\}. \tag{3.80}$$

For $\sinh \frac{|J|}{kT} > 1$ the concurrence is

$$C_{12} = \frac{\sinh \frac{|J|}{kT} - 1}{\cosh \frac{|J|}{kT} + 1} \tag{3.81}$$

with

$$\lim_{T \to 0} C_{12} = 1 \tag{3.82}$$

and for $\sinh \frac{|J|}{kT} \leq 1, \;\; C_{12} = 0$. In both antiferromagnetic and ferromagnetic cases the states become entangled at sufficiently small temperature

$$T < T_c = \frac{|J|}{k \, sinh^{-1} 1} \, . \tag{3.83}$$

As was shown in (Zheng & Guo, 2000), (Imamoglu et al., 1999), (Wang, 2002), (Xi et al., 2002) inclusion of the magnetic field does not change this critical temperature.

3.5.1. XX Model with DM Interaction

For $B = b = 0,\ D \neq 0$ the Hamiltonian is

$$H = \frac{J}{2}[J(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y) + D(\sigma_1^x\sigma_2^y - \sigma_1^y\sigma_2^x)] \tag{3.84}$$

and the ordered eigenvalues are

$$\lambda_4 = \frac{e^{\sqrt{J^2+D^2}/kT}}{Z} > \lambda_3 = \frac{e^{-\sqrt{J^2+D^2}/kT}}{Z} > \lambda_{1,2} = \frac{1}{Z}, \tag{3.85}$$

where partition function is $Z = 2(1 + \cosh\frac{\nu}{kT})$. Then the entanglement occurs when $\sinh\frac{\sqrt{J^2+D^2}}{kT} > 1$ and the concurrence is

$$C_{12} = \frac{\sinh\frac{\sqrt{J^2+D^2}}{kT} - 1}{\cosh\frac{\sqrt{J^2+D^2}}{kT} + 1}. \tag{3.86}$$

Comparison with the pure XX model (3.83) shows that the critical temperature

$$T_c = \frac{\sqrt{J^2+D^2}}{k\,sinh^{-1}1} \tag{3.87}$$

in this case increases with growing D. For $D = 0$ $|\Psi_3\rangle$ in (3.7) is the ground state with eigenvalue $E_3 = -|J_+|$, which is maximally entangled Bell state, so that the concurrence $C_{12} = 1$. As T increases the concurrence decreases due to the mixing of other states with this maximally entangled one[1] .

[1] In ref. (Wang, 2001c) entanglement in XX model with DM coupling was derived but not in the general XXZ case as it is claimed in the paper.

3.5.2. Ising Model in Transverse Magnetic Field

As a particular case of the general XY model now we consider the transverse Ising model, when $J_y = 0$, with external magnetic field B in $z-$ direction (Kamta & Starace, 2002), and with addition of DM interaction:

$$H = \frac{1}{2}[J_x(\sigma_1^x \sigma_2^x) + B(\sigma_1^z + \sigma_2^z) + D(\sigma_1^x \sigma_2^y - \sigma_1^y \sigma_2^x)]. \tag{3.88}$$

The corresponding eigenvalues and the partition function Z can be written as follows

$$\lambda_{1,2} = \frac{1}{Z}\left|\sqrt{1 + \frac{J^2}{B^2+J^2}\sinh^2\frac{\sqrt{B^2+J^2}}{kT}} \mp \frac{J}{\sqrt{B^2+J^2}}\sinh\frac{\sqrt{B^2+J^2}}{kT}\right| \tag{3.89}$$

$$\lambda_{3,4} = \frac{1}{Z}e^{\mp\frac{\sqrt{J^2+D^2}}{kT}}, \tag{3.90}$$

with the partition function

$$Z = 2\left[\cosh\frac{\sqrt{B^2+J^2}}{kT} + \cosh\frac{\sqrt{D^2+J^2}}{kT}\right]. \tag{3.91}$$

To find the maximal eigenvalue we compare the difference of λ_4 and λ_2 as a function of B, D and T, $\lambda_4 - \lambda_2 \equiv f(B, D, T)$:

$$f = e^{\frac{\sqrt{J^2+D^2}}{kT}} - \sqrt{1 + \frac{J^2}{B^2+J^2}\sinh^2\frac{\sqrt{B^2+J^2}}{kT}} - \frac{J}{\sqrt{B^2+J^2}}\sinh\frac{\sqrt{B^2+J^2}}{kT} \tag{3.92}$$

When $f(B, D, T) = 0$ we find the critical $D = D_c(B, T)$ as

$$D_c(B,T) = \sqrt{-J^2 + T^2\left(\ln\left[\sqrt{1 + \frac{J^2}{B^2+J^2}\sinh^2\frac{\sqrt{B^2+J^2}}{kT}} + \frac{J}{\sqrt{B^2+J^2}}\sinh\frac{\sqrt{B^2+J^2}}{kT}\right]\right)^2} \tag{3.93}$$

In Fig. 3.2 we plot D_c as a function of T for different values of magnetic field $B = 0.05, 0.5, 0.7, 1$ ($J = 1, k = 1$). The 3D plot of D_c as a function of B and T for the same values of parameters is given in Fig. 3.3.

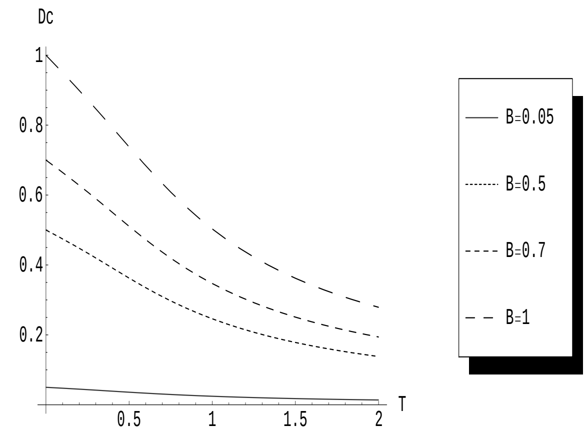


Figure 3.2. D_c versus T for $B = 0.05, 0.5, 0.7, 1$

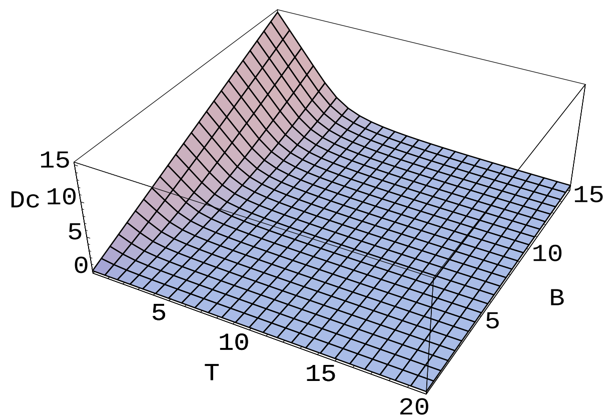


Figure 3.3. 3D plot D_c versus B and T

For critical $D = D_c$, the eigenvalues are degenerate $\lambda_2 = \lambda_4$ and as a result the concurrence $C_{12}(B, D_c, T) = 0$. However the value of concurrence is different for the under critical and the over critical cases. In under critical case when $D < D_c$ the maximal eigenvalue is λ_2 and for the concurrence we have

$$C_{12} = max\{\frac{\frac{J}{\sqrt{B^2+J^2}}\sinh\frac{\sqrt{B^2+J^2}}{kT} - \cosh\frac{\sqrt{D^2+J^2}}{kT}}{\cosh\frac{\sqrt{B^2+J^2}}{kT} + \cosh\frac{\sqrt{D^2+J^2}}{kT}}, 0\}, \tag{3.94}$$

while in the over critical case, when $D > D_c$, the maximum eigenvalue is λ_4 and the concurrence is

$$C_{12} = max\{\frac{\sinh\frac{\sqrt{D^2+J^2}}{kT} - \sqrt{1 + \frac{J^2}{B^2+J^2}\sinh^2\frac{\sqrt{B^2+J^2}}{kT}}}{\cosh\frac{\sqrt{B^2+J^2}}{kT} + \cosh\frac{\sqrt{D^2+J^2}}{kT}}, 0\}. \tag{3.95}$$

In pure Ising model when $B = 0$ and $D = 0$ as we can see from (3.92) we have $f(0, 0, T) = 0$ and no entanglement occurs. But as reported in (Kamta & Starace, 2002) an addition of the transverse magnetic field to the Ising model could create entanglement. Now we can generalize these results by analyzing in addition the influence of DM interaction on entanglement in the Ising model with the magnetic field. When $B = 0$ the addition of solely DM term creates entanglement at sufficiently strong D, and this value of D becomes bigger for higher temperatures. If we have both terms $B \neq 0$ and $D \neq 0$, then with increasing D the behavior of entanglement becomes nontrivial. In Figs. 3.4.a, 3.4.b, 3.4.c we show behavior of entanglement as a function of D for different temperatures. When $T = 0$ entanglement is nonanalytic function of D, given by the step function

$$C_{12}(D) = \begin{cases} \frac{J}{\sqrt{J^2+B^2}}, & D < D_c\,; \\ 0, & D = D_c\,; \\ 1, & D > D_c, \end{cases} \tag{3.96}$$

where $D_c = B$ (see Fig. 3.4-a). This nonanalytic behavior signals on the quantum phase transition (Sachdev, 1999) appearing at $D = D_c = 1$. In Fig. 3.4-b at temperature $T = 0.5$ the entanglement as a function of D decreases down to zero and at $D_c \approx 0.75$ reaches its nondifferentiable minima. After this, it increases monotonically with growing

D. For higher temperature $T = 1$ in Fig. 3.4-c, the entanglement is zero until D becomes sufficiently strong at $D = D_c$, where entanglement appears and monotonically grows with growing D.

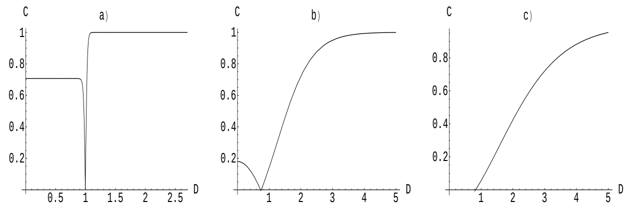


Figure 3.4. Concurrence of Ising model in transverse magnetic field versus D, when $B = 1$ and $T = 0.01,\ 0.5,\ 1$

3.6. XXX Heisenberg Model

In pure XXX model $J_x = J_y = J_z \equiv J$ and $B = b = D = 0$ with the Hamiltonian

$$H = \frac{1}{2}[J(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y + \sigma_1^z\sigma_2^z)]\,, \tag{3.97}$$

entanglement behavior for the ferromagnetic and the antiferromagnetic cases is different. For $J_x = J_y = J_z \equiv J$ and $B = 0, b = 0, D = 0$, the eigenvalues are

$$\lambda_{1,2} = \frac{e^{-J/2kT}}{Z},\ \ \lambda_3 = \frac{e^{-J/2kT}}{Z},\ \ \lambda_4 = \frac{e^{3J/2kT}}{Z} \tag{3.98}$$

where the partition function is

$$Z = 2(e^{-J/2kT} + e^{J/2kT}\cosh\frac{J}{kT})\,. \tag{3.99}$$

It was observed before (Arnesen et al., 2001) that for the ferromagnetic case ($J < 0$) the concurrence

$$C_{12} = max\{\frac{-\cosh\frac{|J|}{kT}}{\cosh\frac{|J|}{kT} + e^{|J|/kT}}, 0\} = 0 \tag{3.100}$$

is zero and the states are always unentangled. It happens because when $J < 0$, the ground state of the system is an equal mixture of the triplet states with energy, $E_1 = E_2 = E_4 = -\frac{|J|}{2}$. The density matrix ρ is diagonal and inclusion of magnetic field does not change the result. Increasing temperature T just increases the singlet mixture with the triplet, which can only decrease entanglement (Arnesen et al., 2001), (Nielsen, 2000).

In the anti-ferromagnetic case the situation is different. In this case the ground state is the maximally entangled singlet state with $E_3 = -\frac{3J}{2}$. The concurrence is

$$C_{12} = max\{\frac{\sinh\frac{J}{kT} - e^{-J/kT}}{e^{-J/kT} + \cosh\frac{J}{2kT}}, 0\} \tag{3.101}$$

For $\sinh\frac{J}{kT} > e^{-J/kT}$ the concurrence has the form

$$C_{12} = \frac{\sinh\frac{J}{kT} - e^{-J/kT}}{e^{-J/2kT} + \cosh\frac{J}{2kT}} \tag{3.102}$$

It decreases with T due to mixing of the triplet higher states with the singlet ground state. For a given coupling constant J entanglement occurs at temperature $T < \frac{2J}{k\ln 3}$ (Wang, 2001c)

$$\lim_{T\to 0} C_{12} = 1 \tag{3.103}$$

For $\sinh\frac{J}{kT} \leq e^{-J/kT}$ the concurrence is $C_{12} = 0$ and no entanglement occurs.

3.6.1. XXX Heisenberg Model with Magnetic Field

With inclusion of magnetic field B the eigenvalues of XXX Heisenberg model are calculated as

$$\lambda_{1,2} = \frac{e^{-J/2kT}}{Z}, \;\; \lambda_3 = \frac{e^{-J/2kT}}{Z}, \;\; \lambda_4 = \frac{e^{3J/2kT}}{Z} \tag{3.104}$$

and the partition function is

$$Z = 2(e^{-J/2kT}\cosh\frac{B}{kT} + e^{J/2kT}\cosh\frac{J}{kT})\,, \tag{3.105}$$

In the anti-ferromagnetic case the concurrence is found as

$$C_{12} = max\{\frac{\sinh\frac{J}{kT} - e^{-J/kT}}{e^{-J/kT}\cosh\frac{B}{kT} + \cosh\frac{J}{2kT}}, 0\}. \tag{3.106}$$

For $\sinh\frac{J}{kT} > e^{-J/kT}$ the concurrence is

$$C_{12} = \frac{\sinh\frac{J}{kT} - e^{-J/kT}}{e^{-J/2kT}\cosh\frac{B}{kT} + \cosh\frac{J}{2kT}} \tag{3.107}$$

and for sufficiently small temperature $T < \frac{2J}{k\ln 3}$ entanglement occurs and

$$\lim_{T\to 0} C_{12} = 1 \tag{3.108}$$

Comparing eqs. (3.102) and (3.107) we can see that the inclusion of magnetic field B does not change critical value but decrease entanglement. While for $\sinh\frac{J}{kT} < e^{-J/kT}$ no entanglement occurs $C_{12} = 0$. In the ferromagnetic case the concurrence is calculated as

$$C_{12} = max\{\frac{-\cosh\frac{|J|}{kT}}{\cosh\frac{|J|}{kT} + e^{|J|/kT}\cosh\frac{B}{kT}},\, 0\} = 0. \tag{3.109}$$

In this case no entanglement occurs. As a result inclusion of magnetic field does not change the result, for ferromagnets spins are always disentangled and for anti-ferromagnets entanglement is observed.

3.6.2. XXX Heisenberg Model with DM Interaction

Now by adding DM coupling for the antiferromagnetic and the ferromagnetic cases we have the Hamiltonian is in the form

$$H = \frac{1}{2}[J(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y + \sigma_1^z\sigma_2^z) + D(\sigma_1^x\sigma_2^y - \sigma_1^y\sigma_2^x)] \tag{3.110}$$

or in the matrix form

$$H = \begin{bmatrix} \frac{J}{2} & 0 & 0 & 0 \\ 0 & -\frac{J}{2} & J+iD & 0 \\ 0 & J-iD & -\frac{J}{2} & 0 \\ 0 & 0 & 0 & \frac{J}{2} \end{bmatrix}, \tag{3.111}$$

The eigenvalues are

$$\lambda_{1,2} = \frac{e^{-J/2kT}}{Z}, \quad \lambda_3 = \frac{e^{(J-2\sqrt{J^2+D^2})/2kT}}{Z}, \quad \lambda_3 = \frac{e^{(J+2\sqrt{J^2+D^2})/2kT}}{Z} \tag{3.112}$$

where the partition function is

$$Z = 2(e^{-J/2kT} + e^{J/2kT}\cosh\frac{\sqrt{J^2+D^2}}{kT}) \tag{3.113}$$

In the anti-ferromagnetic case, the concurrence is

$$C_{12} = max\{\frac{\sinh\frac{\sqrt{J^2+D^2}}{kT} - e^{-J/kT}}{e^{-J/kT} + \cosh\frac{\sqrt{J^2+D^2}}{kT}}, 0\} \tag{3.114}$$

For given temperature when $D > D_c = \sqrt{kTsinh^{-1}e^{-J/kT} - J^2}$ there is entanglement with concurrence

$$C_{12} = \frac{\sinh \frac{\sqrt{J^2+D^2}}{kT} - e^{-J/kT}}{e^{-J/kT} + \cosh \frac{\sqrt{J^2+D^2}}{kT}} \,. \tag{3.115}$$

In this case the ground state of the system remains singlet with energy $E_3 = -\frac{|J|}{2} - \sqrt{J^2 + D^2}$, while from degenerate excited triplet state one of the energy levels $E_4 = -\frac{|J|}{2} + \sqrt{J^2 + D^2}$ is splitting up. With increasing coupling D the gap between ground state and the first excited doublet state is increasing, this is why the system becomes more entangled. As we can see inclusion of the DM coupling, in the XXX model, increases entanglement in the antiferromagnetic case and creates entanglement even in the ferromagnetic case. In the ferromagnetic the concurrence is

$$C_{12} = max\{\frac{\sinh \frac{\sqrt{J^2+D^2}}{kT} - e^{|J|/kT}}{e^{|J|/kT} + \cosh \frac{\sqrt{J^2+D^2}}{kT}}, 0\} \tag{3.116}$$

For given temperature when $D > D_c = \sqrt{kTsinh^{-1}e^{|J|/kT} - J^2}$ there is entanglement with concurrence

$$C_{12} = \frac{\sinh \frac{\sqrt{J^2+D^2}}{kT} - e^{|J|/kT}}{e^{|J|/kT} + \cosh \frac{\sqrt{J^2+D^2}}{kT}} \tag{3.117}$$

and for $\sinh \frac{\sqrt{J^2+D^2}}{kT} < e^{|J|/kT}$ no entanglement occurs $C_{12} = 0$. In this case from unentangled triplet ground state one of the states splits with the energy $E_3 = \frac{|J|}{2} - \sqrt{J^2 + D^2}$. Then at temperature zero this state becomes maximally entangled ground state. This way the DM interaction creates entanglement in the ferromagnetic case. With increasing D the gap between singlet ground state and the first doublet state increases, this is why entanglement in the ferromagnetic case increases. Inclusion of spin- orbit coupling D increase entanglement.

3.7. XXZ Heisenberg Model

When $J_x = J_y = J \neq J_z$ the Hamiltonian (3.3) becomes

$$H = \frac{1}{2}[J(\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y + \Delta\, \sigma_1^z \sigma_2^z) + B_+\, \sigma_1^z + B_-\, \sigma_2^z + D(\sigma_1^x \sigma_2^y - \sigma_1^y \sigma_2^x)], \quad (3.118)$$

where $\Delta \equiv J_z/J$.

In a pure XXZ ferromagnetic model when $J_z < 0$ and $-|J_z| < J < |J_z|$ or $|\Delta| > 1$, we have the degenerate maximal eigenvalues $\lambda_1 = \lambda_2$ and no entanglement occurs. This happens since the ground state of the system is doublet with eigenvalues $E_1 = E_2 = -\frac{|J_z|}{2}$. In particular case $|\Delta| = 1$ or $|J| = |J_z|$ we have reduction to the XXX model, where the energy level E_3 merges to the ground state, and the last one becomes triplet state, as we discussed above in Sec. 3.6. For $J > 0$ and $\Delta > -1$ the maximal eigenvalue is λ_3 and the states are entangled when $\sinh \frac{J}{kT} > e^{-J_z/kT}$ with the concurrence

$$C_{12} = \frac{\sinh \frac{J}{kT} - e^{-J_z/kT}}{\cosh \frac{J}{kT} + e^{-J_z/kT}}. \quad (3.119)$$

For $J < 0$ and $\Delta < 1$ the maximal eigenvalue is λ_4 and the states are entangled for $\sinh \frac{|J|}{kT} > e^{-J_z/kT}$ with the concurrence

$$C_{12} = \frac{\sinh \frac{|J|}{kT} - e^{-J_z/kT}}{\cosh \frac{|J|}{kT} + e^{-J_z/kT}}. \quad (3.120)$$

3.7.1. XXZ Heisenberg Model with DM Interaction

With addition of the DM coupling we have the eigenvalues

$$\lambda_{1,2} = \frac{1}{2\left[1 + e^{J_z/kT} \cosh \frac{\sqrt{J^2+D^2}}{kT}\right]}, \quad \lambda_{3,4} = \frac{e^{\mp\sqrt{J^2+D^2}/kT}}{2\left[e^{-J_z/kT} + \cosh \frac{\sqrt{J^2+D^2}}{kT}\right]}. \quad (3.121)$$

Then for $J_z < 0$ and $|J_z| > |J|$, there exists critical value $D_c = \sqrt{J_z^2 - J^2}$ so that for $D > D_c$ and $\sinh \frac{\sqrt{J^2+D^2}}{kT} > e^{-J_z/kT}$ the states are entangled with the concurrence

$$C_{12} = \frac{\sinh \frac{\sqrt{J^2+D^2}}{kT} - e^{|J_z|/kT}}{\cosh \frac{\sqrt{J^2+D^2}}{kT} + e^{|J_z|/kT}}. \tag{3.122}$$

This happens because for $J_z < 0$, $|J_z| > |J|$ and $D = 0$, the ground state is doublet with $E_1 = E_2 = -\frac{|J_z|}{2}$, and by increasing D so that $D > D_c$, the higher energy level E_3 lowers to the singlet ground state which is maximally entangled. Comparison of (3.122) with (3.120) shows that with growing D entanglement increases. It is worth to note that the concurrence (3.122) for both signs of J is the same. Moreover, as easy to see in (3.122) parameters J and D appear symmetrically. It means that the concurrence could be increased by growing J with fixed D either by growing D with fixed J. This reflects the known result (Wreszinski & Alcaraz, 1990) on equivalence of the Heisenberg XXZ model with DM coupling to pure XXZ model with modified anisotropy parameter and a certain type of boundary conditions. In fact comparing entanglement in our formulas for pure antiferromagnetic case (3.120) with the one including the DM interaction (3.122), we can see that the concurrences are connected by the replacement $J \to J\sqrt{1+\frac{D^2}{J^2}}$, which corresponds to the substitution for the anisotropy parameter in the pure XXZ model as $\Delta \to \frac{\Delta}{\sqrt{1+\frac{\Delta^2 D^2}{J_z^2}}}$.

3.7.2. XXZ Heisenberg Model with DM Interaction and Magnetic Field

If we take into account the DM interaction D and magnetic field B simultaneously, the above results for critical value of the DM coupling are still valid, but the level of entanglement decreases according to

$$C_{12} = \frac{\sinh \frac{\sqrt{J^2+D^2}}{kT} - e^{-J_z/kT}}{\cosh \frac{\sqrt{J^2+D^2}}{kT} + e^{-J_z/kT} \cosh \frac{B}{kT}}. \tag{3.123}$$

For $T = 0$ and $J_z > 0$ we have nonanalytic behavior

$$C_{12} = \begin{cases} 1, & \sqrt{D^2 + J^2} > B - J_z; \\ \frac{1}{2}, & \sqrt{D^2 + J^2} = B - J_z; \\ 0, & \sqrt{D^2 + J^2} < B - J_z, \end{cases} \tag{3.124}$$

which signals appearance of quantum phase transitions. The concurrence versus temperature for different values of coupling D is shown in Fig. 5, where $J = 1$, $J_z = 0.5$ and magnetic field $B = 2$. As we can see in general the entanglement decreases with growing temperature. However we like to emphasize that for $D < D_c$ in Fig. 5a, when $D = 0.1$, the entanglement is increasing with growing temperature. This phenomena can be explained by the fact that for such values of the parameters at $T = 0$ the ground state is the separable state with energy $E_1 = \frac{J_z}{2} - B = -1.75$, and the concurrence is zero (see the last case in eqn. (3.124)). When temperature increases the entangled state with energy $E_3 = \frac{-J_z}{2} \mp \sqrt{J^2 + D^2} = -1.255$ becomes involved into the mixture and entanglement is increasing.

When $D = D_c$ the entanglement decreases smoothly from $C_{12} = 0.5$ (Fig. 3.5b, $D_c = 1.118$). By increasing D ($D = 1.19$), first it gives sharp decrease from $C_{12} = 1$ (Fig. 3.5 c) and then it vanishes slowly. When D becomes bigger ($D = 3$) entanglement decreases slowly from $C_{12} = 1$ (Fig. 3.5 d).

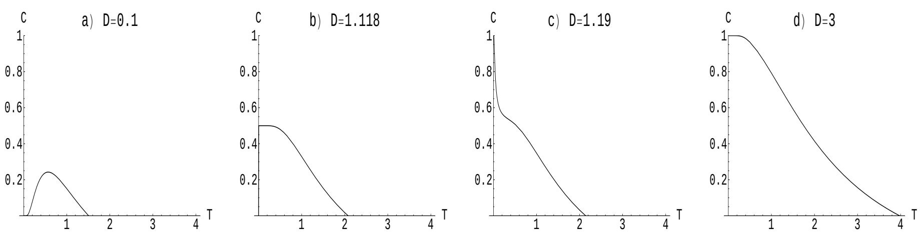


Figure 3.5. Concurrence in XXZ model versus temperature for $B = 2$ and a) $D = 0.1$, b) $D = 1.118$, c) $D = 1.19$, d) $D = 3$

We compare the concurrence versus magnetic field for different temperatures, when $D = 0$ (Fig. 3.6) and when $D = 2$ (Fig.3.7). In both cases at $T = 0$ the entanglement vanishes abruptly as B crosses critical value $B_c = \sqrt{B^2 + J^2} + J_z$. This special point $T = 0$, $B = B_c$ at which entanglement becomes nonanalytic function of B, is the point of quantum phase transition. Comparison of figures 3.6 and 3.7 shows that the critical value of B at which entanglement disappears suddenly is growing with increasing coupling D: in Fig.3.6, $B_c = 2$ and in Fig.3.7, $B_c = 3.3$. It shows again that increasing DM coupling improves entanglement.

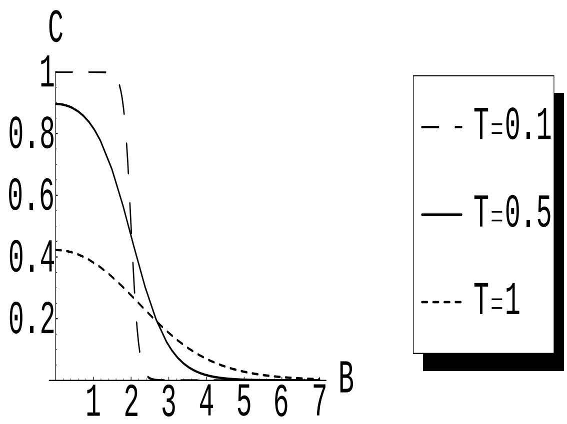


Figure 3.6. Concurrence versus magnetic field B for $D = 0$ and $T = 0.1, 0.5, 1$.

3.8. Pure XYZ Heisenberg Model

In the antiferromagnetic case, we start from the pure XYZ model, where for determinacy we chose $J_z > J_y > J_x > 0$ implying $J_+ > |J_-| > 0, J_- = -|J_-| < 0$. Eigenstates of the Hamiltonian (3.3) are

$$E_{1,2} = \frac{|J_z|}{2} \pm |J_-|, \;\; E_{3,4} = -\frac{|J_z|}{2} \mp |J_+|. \tag{3.125}$$

For zero temperature the ground state is maximally entangled Bell state $|01\rangle - |10\rangle$ with the energy

$$E_3 = -\frac{|J_z|}{2} - |J_+|. \tag{3.126}$$

When temperature increases, the state mixes with higher states decreasing entanglement. Using the highest eigenvalue

$$\lambda_4 = \frac{1}{Z}\exp\frac{|J_z| + 2|J_+|}{2kT} \tag{3.127}$$

the concurrence is calculated as

$$C_{12} = max\{\frac{\sinh\frac{J_+}{kT} - \cosh\frac{J_-}{kT}e^{-J_z/kT}}{\cosh\frac{J_+}{kT} + \cosh\frac{J_-}{kT}e^{-J_z/kT}}, 0\}. \tag{3.128}$$

Then entanglement occurs when

$$\sinh\frac{J_+}{kT} > \cosh\frac{J_-}{kT}e^{-J_z/kT}. \tag{3.129}$$

It shows that entanglement depends essentially on the anisotropy, and grows with J_+ and decreases with J_- (Rigolin, 2004).

In the ferromagnetic case, let $J_z < J_y < J_x < 0$ then $J_+ = -|J_+|$, $J_- = |J_-| > 0$ and $J_z = -|J_z|$. For pure XYZ model, eigenstates of the Hamiltonian are $E_{1,2} = -\frac{|J_z|}{2} \mp |J_-|$ and $E_{3,4} = \frac{|J_z|}{2} \pm |J_+|$. For zero temperature the ground state is maximally entangled Bell state $|00\rangle - |11\rangle$ with the energy $E_1 = -\frac{|J_z|}{2} - |J_-|$. With increasing temperature this state mixes with other states and entanglement decreases so that the concurrence is

$$C_{12} = \frac{\sinh\frac{|J_-|}{kT} - \cosh\frac{|J_+|}{kT}e^{-|J_z|/kT}}{\cosh\frac{|J_-|}{kT} + \cosh\frac{|J_+|}{kT}e^{-|J_z|/kT}}. \tag{3.130}$$

When temperature reaches the critical value $T = T_c$, given by a solution of the following transcendental equation

$$\sinh\frac{|J_-|}{kT_c} = \cosh\frac{|J_+|}{kT_c}e^{-|J_z|/kT}, \tag{3.131}$$

the concurrence vanishes and state becomes unentangled.

3.9. XYZ Model with DM Interaction

In the anti-ferromagnetic case inclusion of the DM coupling, remains the energy levels E_1 and E_2 the same as above, while $E_{3,4} = -\frac{|J_z|}{2} \mp \sqrt{J_+^2 + D^2}$. In this case the ground state continues to be entangled state but with the energy E_3. With growing temperature, mixing of this state with the higher states decreases the entanglement. If we consider the difference between two lower states $E_4 - E_3 = \sqrt{J_+^2 + D^2}$, then by increasing the coupling D, it can be made arbitrary large, so that the entanglement will increase. For $D >> |J_+|$ the state would be maximally entangled.

At $T = 0$ the concurrence

$$C_{12} = \begin{cases} 1, & \sqrt{D^2 + J_+^2} > J_- - J_z; \\ 0, & \sqrt{D^2 + J_+^2} = J_- - J_z; \\ 1, & \sqrt{D^2 + J_+^2} < J_- - J_z, \end{cases} \tag{3.132}$$

is nonanalytic function in D, and it signals about the quantum phase transition at $D = D_c$ where $\sqrt{D_c^2 + J_+^2} = J_- - J_z$. When the temperature increases, entanglement occurs for

$$\sinh \frac{\sqrt{J_+^2 + D^2}}{kT} > e^{-J_z/kT} \cosh \frac{J_-}{kT}, \tag{3.133}$$

and the concurrence

$$C_{12} = \frac{\sinh \frac{\nu}{kT} - e^{-J_z/kT} \cosh \frac{J_-}{kT}}{\cosh \frac{\nu}{kT} + e^{-J_z/kT} \cosh \frac{J_-}{kT}}, \tag{3.134}$$

increases with growing anisotropy J_+ and the coupling D.

In the ferromagnetic case, with inclusion of the DM coupling, the first couple of energy levels is the same $E_{1,2} = \frac{-|J_z|}{2} \mp |J_-|$ while the second couple becomes $E_{3,4} = \frac{|J_z|}{2} \mp \sqrt{J_+^2 + D^2}$. For $D < D_c$ where D_c satisfies the equation $\sqrt{D_c^2 + J_+^2} = |J_z| + |J_-|$, the ground state of the system is the maximally entangled Bell state $|00\rangle - |11\rangle$. If we increase D, the difference between energy levels E_1 and E_3 decreases, so that at $D = D_c$ the ground state becomes degenerate and entanglement vanishes. When $D > D_c$ the ground state E_3 becomes entangled again.

Due to the mixture of states by increasing temperature the entanglement decreases, so that, in the under critical region $D < D_c$ the concurrence is

$$C_{12} = max\{\frac{\sinh\frac{|J_-|}{kT} - \cosh\frac{\sqrt{J_+^2+D^2}}{kT}e^{-|J_z|/kT}}{\cosh\frac{|J_-|}{kT} + \cosh\frac{\sqrt{J_+^2+D^2}}{kT}e^{-|J_z|/kT}}, 0\}, \tag{3.135}$$

while in the over critical region $D > D_c$ it is

$$C_{12} = max\{\frac{\sinh\frac{\sqrt{J_+^2+D^2}}{kT} - e^{|J_z|/kT}\cosh\frac{|J_-|}{kT}}{\cosh\frac{\sqrt{J_+^2+D^2}}{kT} + e^{|J_z|/kT}\cosh\frac{|J_-|}{kT}}, 0\}. \tag{3.136}$$

For $D = D_c$, due to $\lambda_1 = \lambda_3$, the entanglement vanishes for any temperature. The entanglement dependence on T and D is shown in Figs. 3.8 and 3.9. For $T = 0$ the figures show nonanalyticity at $D = D_c$ which signals a quantum phase transition. The entanglement behavior in the under and the over critical regions is qualitatively different. For the under critical case with fixed temperature the entanglement decreases with growing D, and the level of entanglement quickly decreases with temperature. From another side, for fixed temperature in the over critical region the entanglement increases, and the level of entanglement decreases with temperature quite slowly. In addition if at $T = 0$ we have only one critical point $D = D_c$ in which entanglement is zero, for $T > 0$ entanglement vanishes at some interval which includes D_c and this interval extends with growing temperature. This is a result of ground state mixture with higher states. However by increasing D we can always lower the level of our ground state to decrease this mixture and increase entanglement.

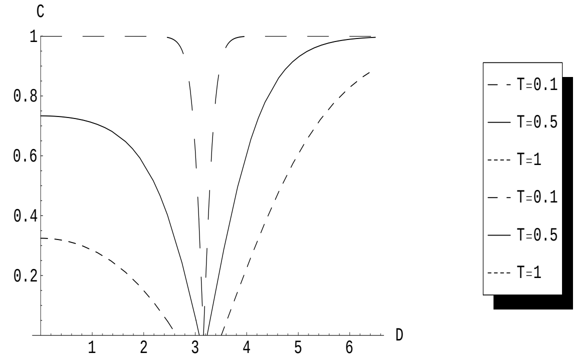


Figure 3.7. Concurrence in ferromagnetic XYZ model versus coupling D at temperature $T = 0.1,\ 0.5,\ 1$

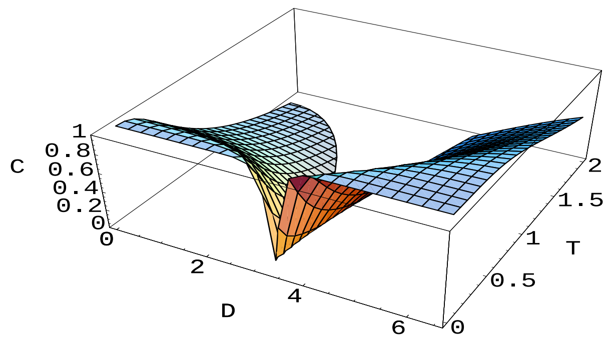


Figure 3.8. 3D plot of concurrence in ferromagnetic XYZ model versus coupling D and temperature T

CHAPTER 4

TIME EVOLUTION OF ENTANGLEMENT

During the evolution of a generic two qubit state, entanglement of the system could be changed. So that even starting from a separable state with $C = 0$, the time evolution of quantum state can produce entangled and even maximally entangled state and vice versa. In this chapter we study evolution of two qubit entanglement in Heisenberg XYZ model with the Dzialoshinskii-Moriya (DM) interaction. These results presented in (Gurkan & Pashaev, 2009). First we consider the evolution operator

$$U(t) = \exp[-\frac{i}{\hbar}Ht] \tag{4.1}$$

with the Hamiltonian

$$H = \frac{1}{2}[J_x\,\sigma_1^x\sigma_2^x + J_y\;\sigma_1^y\sigma_2^y + J_z\;\sigma_1^z\sigma_2^z + B_+\,\sigma_1^z + B_-\,\sigma_2^z + D(\sigma_1^x\sigma_2^y - \sigma_1^y\sigma_2^x)]. \tag{4.2}$$

The matrix form of this operator is

$$U(t) = \begin{bmatrix} A_{11} & 0 & 0 & A_{14} \\ 0 & A_{22} & A_{23} & 0 \\ 0 & A_{32} & A_{33} & 0 \\ A_{41} & 0 & 0 & A_{44} \end{bmatrix} \tag{4.3}$$

where

$$
\begin{aligned}
A_{11} &= e^{-\frac{iJ_z t}{2\hbar}}\left[\cos\frac{\mu t}{\hbar} - \frac{iB}{\mu}\sin\frac{\mu t}{\hbar}\right] = e^{-\frac{iw_3 t}{4}}\left[\cos w_1 t - \frac{iB}{w_1}\sin w_1 t\right] && (4.4)\\
A_{44} &= e^{-\frac{iJ_z t}{2\hbar}}\left[\cos\frac{\mu t}{\hbar} + \frac{iB}{\mu}\sin\frac{\mu t}{\hbar}\right] = e^{-\frac{iw_3 t}{4}}\left[\cos w_1 t + \frac{iB}{w_1}\sin w_1 t\right] && (4.5)\\
A_{14} &= -ie^{-\frac{iJ_z t}{2\hbar}}\frac{J_-}{\mu}\sin\frac{\mu t}{\hbar} = -ie^{-\frac{iw_3 t}{4}}\frac{J_-}{w_1}\sin w_1 t && (4.6)\\
A_{41} &= -ie^{-\frac{iJ_z t}{2\hbar}}\frac{J_-}{\mu}\sin\frac{\mu t}{\hbar} = -ie^{-\frac{iw_3 t}{4}}\frac{J_-}{w_1}\sin w_1 t && (4.7)\\
A_{22} &= e^{\frac{iJ_z t}{2\hbar}}\left[\cos\frac{\nu t}{\hbar} - \frac{ib}{\nu}\sin\frac{\nu t}{\hbar}\right] = e^{\frac{iw_3 t}{4}}\left[\cos w_2 t - \frac{ib}{w_2}\sin w_2 t\right] && (4.8)\\
A_{33} &= e^{\frac{iJ_z t}{2\hbar}}\left[\cos\frac{\nu t}{\hbar} + i\frac{b}{\nu}\sin\frac{\nu t}{\hbar}\right] = e^{\frac{iw_3 t}{4}}\left[\cos w_2 t + \frac{ib}{w_2}\sin w_2 t\right] && (4.9)\\
A_{23} &= e^{\frac{iJ_z t}{2\hbar}}\frac{D - iJ_+}{\nu}\sin\frac{\nu t}{\hbar} = e^{\frac{iw_3 t}{4}}\frac{D - iJ_+}{w_2}\sin w_2 t && (4.10)\\
A_{32} &= e^{\frac{iJ_z t}{2\hbar}}\frac{-D - iJ_+}{\nu}\sin\frac{\nu t}{\hbar} = e^{\frac{iw_3 t}{4}}\frac{-D - iJ_+}{w_2}\sin w_2 t && (4.11)
\end{aligned}
$$

with parameters $\nu = \sqrt{b^2 + J_+^2 + D^2}$, $\mu = \sqrt{J_-^2 + B^2}$, and frequencies

$$
\begin{aligned}
\omega_1 &= \frac{\mu}{\hbar} = \frac{\sqrt{J_-^2 + B^2}}{\hbar} && (4.12)\\
\omega_2 &= \frac{\nu}{\hbar} = \frac{\sqrt{b^2 + J_+^2 + D^2}}{\hbar} && (4.13)\\
\omega_3 &= \frac{2J_z}{\hbar}. && (4.14)
\end{aligned}
$$

Acting by the evolution operator (4.1) to the initial state $|\Psi(0)\rangle = \sum_{ij} c_{ij}(0)|ij\rangle$ of the system we get the time dependent wave function in the form

$$
|\Psi(t)\rangle = \sum_{ij} c_{ij}(t)|ij\rangle = U(t)|\Psi(0)\rangle \tag{4.15}
$$

where $c_{ij}(t) = \langle ij|\Psi(t)\rangle$.

4.1. Time Dependent Concurrence

Entanglement of (4.15) depends on time. To find how it changes with time we use the concurrence characteristic in the determinant form (2.48).

$$C(t) = 2|c_{00}(t)c_{11}(t) - c_{01}(t)c_{10}(t)| \tag{4.16}$$

or

$$C(t) = 2\begin{vmatrix} A_{11}c_{00}(0) + A_{14}c_{11}(0) & A_{22}c_{01}(0) + A_{23}c_{10}(0) \\ A_{33}c_{10}(0) + A_{32}c_{01}(0) & A_{44}c_{11}(0) + A_{41}c_{00}(0) \end{vmatrix}. \tag{4.17}$$

Here we restrict our consideration to a specific case of XY model, but our analysis can be easily extended to other cases. For this particular model, $J_z = 0$, and as follows $\omega_3 = 0$, this is why only two characteristic frequencies ω_1 (4.12) and ω_2 (4.13) remain. Then, time dependence of the entanglement would be determined by ratio of these frequencies. For specific values of parameters $J_+ = 1, J_- = \sqrt{3}$, $B = 1, b = 0$ and $D = 0$, we get commensurable frequencies of motion $\omega_1 = 2$ and $\omega_2 = 1$. So that in this case, the concurrence

$$C(t) = \frac{1}{16}[31 - 4(1 + 2\sqrt{3}) \cos 2t - 12 \cos 4t + (-3 + 2\sqrt{3}) \cos 6t - 3 \cos 8t] \tag{4.18}$$

is a periodic function of time with period $T = \pi$, oscillating between $C = 0$ and $C = 1$ states (see Fig. 4.1).

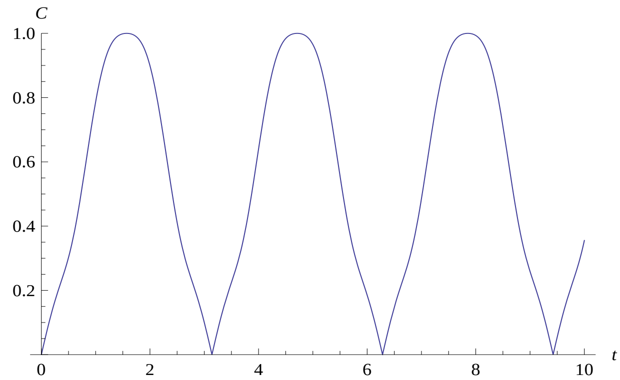


Figure 4.1. Concurrence versus time for B=1, b=0, D=0, $\omega_1 = 2$, $\omega_2 = 1$

To see evolution of concurrence we will use phase portrait in $(C, \dot{C})$ plane, (see Fig. 4.2) then we can see that the phase portrait represents a closed orbit of motion for commensurable ratio of frequencies ω_1 and ω_2.

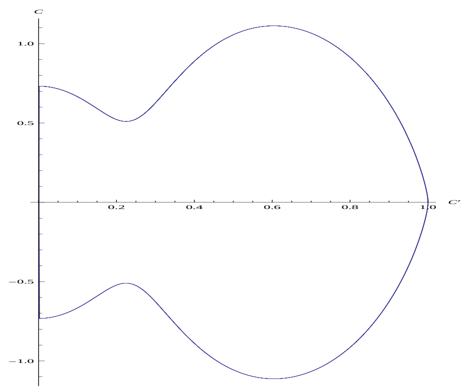


Figure 4.2. Phase portrait B=1, b=0, D=0, $\omega_1 = 2$, $\omega_2 = 1$

For $J_+ = 1, J_- = \sqrt{2}, B = 1, b = 0, D = 0$ the frequencies $\omega_1 = \sqrt{3}$ and $\omega_2 = 1$ are incommensurable and the concurrence for these values

$$C(t) = \frac{1}{18}(1 - 3\cos 2t + 2\cos 2\sqrt{3}t)^2 + \frac{1}{2}(\sin 2t - \sqrt{2}\sin 2\sqrt{6}t)^2. \tag{4.19}$$

In this case the entanglement evolves as a quasi-periodic function of time (Besicovitch,

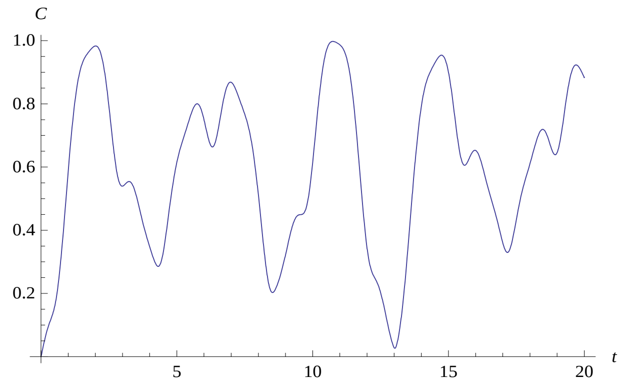


Figure 4.3. Concurrence versus time for B=1, b=0, D=0, $\omega_1 = \sqrt{3},\ \omega_2 = 1$

1954)(see Fig. 4.3). So that entangled and non-entangled states appear in time without any type of regularity. In the phase plane $(C, \dot{C})$ the phase curve is irregular and nonclosed (Fig. 4.4).

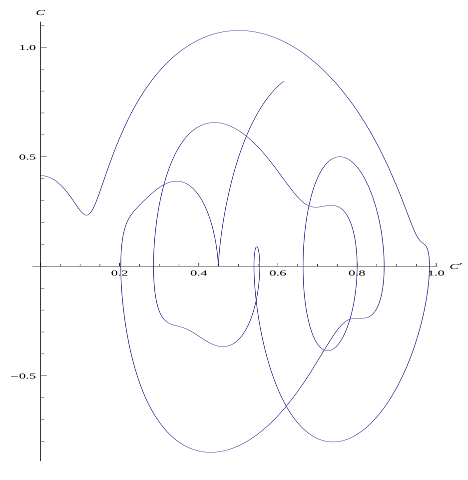


Figure 4.4. Phase portrait B=1,b=0,D=0,$\omega_1 = \sqrt{3},\ \omega_2 = 1$

4.2. Fidelity

In addition to entanglement property of two qubit states exist another characteristic of qubit states which is called the fidelity. It is determined by

$$F = |\langle\psi_1|\psi_2\rangle|^2, \tag{4.20}$$

and characterize closeness of two states $|\psi_1\rangle$ and $|\psi_2\rangle$ in the Hilbert space. In this section we are going to calculate fidelity evolution with time, showing closeness of evolved state $|\psi(t)\rangle$ to initial state $|\psi(0)\rangle$.

$$F(t) = |\langle\Psi(0)|\Psi(t)\rangle|^2 \tag{4.21}$$

By evolution operator

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle \Longrightarrow F(t) = |\langle\Psi(0)|U(t)|\Psi(0)\rangle|^2 \tag{4.22}$$

For generic two qubit state

$$|\Psi(t)\rangle = \sum_{i,j} c_{ij}(t)|ij\rangle \tag{4.23}$$

we have

$$|\Psi(0)\rangle = \sum_{k,l} c_{kl}(t)|ij\rangle \Longrightarrow F(t) = \left|\sum_{i,j} \bar{c}_{ij}(0)c_{ij}(t)\right|^2. \tag{4.24}$$

Then

$$
\begin{aligned}
F(t) &= |\langle\Psi(0)|U(t)|\Psi(0)\rangle|^2 && (4.25)\\
&= ||c_{00}|^2 A_{11} + |c_{01}|^2 A_{22} + |c_{10}|^2 A_{33} + |c_{11}|^2 A_{44} \\
&+ \bar{c}_{01}c_{10}A_{23} + \bar{c}_{10}c_{01}A_{32} + \bar{c}_{00}c_{11}A_{14} + \bar{c}_{11}c_{00}A_{41}|^2 && (4.26)
\end{aligned}
$$

For the maximally entangled Bell state as an initial state

$$\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{4.27}$$

$c_{00} = 1/\sqrt{2}$, $c_{11} = 1/\sqrt{2}$ and fidelity is oscillating in time

$$
\begin{aligned}
F(t) &= \frac{1}{4}|A_{11} + A_{44} + 2A_{14}|^2 && (4.28)\\
&= \cos^2\frac{\mu t}{\hbar} + \frac{J_-^2}{\mu^2}\sin^2\frac{\mu t}{\hbar} && (4.29)
\end{aligned}
$$

with frequency $\omega = \frac{\mu}{\hbar}$ where $\mu = \sqrt{J_-^2 + B^2}$.

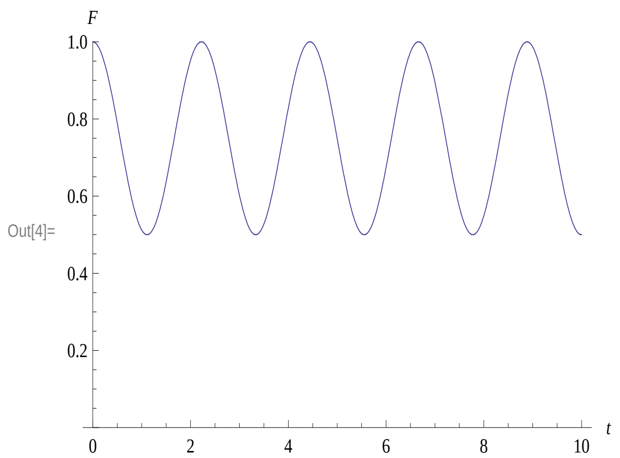


Figure 4.5. Fidelity versus time for $J_- = 1, B = 1, \mu = \sqrt{2}$

In Fig. 4.5 we plot time evolution of initially maximally entangled state. As we can see fidelity in this case is oscillating in time between maximally entangled state $C = 1$ and state with concurrence $C = 0.5$. So during the evolution the state never leave below this value. Concurrence is

$$C = \left| \frac{J_-^2 - B^2}{J_-^2 + B^2} \right| \tag{4.30}$$

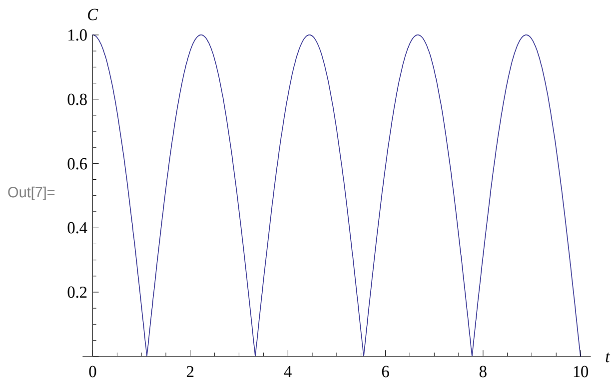


Figure 4.6. Concurrence versus time for $J_- = 1, B = 1, \mu = \sqrt{2}$

4.3. SWAP Gate

Here we like to show the direct relationship between XYZ model with DM coupling, $B = 0$, $b = 0$ and quantum gates. Then evolution of the standard basis is given by

$$U(t)|00\rangle \rightarrow e^{\frac{-iJ_z t}{2\hbar}} \left[\cos\frac{tJ_-}{\hbar}|00\rangle - i\sin\frac{tJ_-}{\hbar}|11\rangle\right], \tag{4.31}$$

$$U(t)|11\rangle \rightarrow e^{\frac{-iJ_z t}{2\hbar}} \left[\cos\frac{tJ_-}{\hbar}|11\rangle - i\sin\frac{tJ_-}{\hbar}|00\rangle\right], \tag{4.32}$$

$$U(t)|01\rangle \rightarrow e^{\frac{iJ_z t}{2\hbar}} \left[\cos\frac{t\nu}{\hbar}|01\rangle - i\frac{J_+ - iD}{\nu}\sin\frac{t\nu}{\hbar}|10\rangle\right], \tag{4.33}$$

$$U(t)|10\rangle \rightarrow e^{\frac{iJ_z t}{2\hbar}} \left[\cos\frac{t\nu}{\hbar}|10\rangle - i\frac{J_+ + iD}{\nu}\sin\frac{t\nu}{\hbar}|01\rangle\right] \tag{4.34}$$

where $\nu = \sqrt{J_+^2 + D^2}$.

If we consider particular case of pure DM model when $J_i = 0$ the we have

$$U(t)|00\rangle \rightarrow |00\rangle, \tag{4.35}$$

$$U(t)|11\rangle \rightarrow |11\rangle, \tag{4.36}$$

$$U(t)|01\rangle \rightarrow \cos\frac{tD}{\hbar}|01\rangle - \sin\frac{tD}{\hbar}|10\rangle, \tag{4.37}$$

$$U(t)|10\rangle \rightarrow \cos\frac{tD}{\hbar}|01\rangle + \sin\frac{tD}{\hbar}|01\rangle \tag{4.38}$$

For time moments $t = \frac{\hbar\pi}{2D}$ we have

$$U(\frac{\pi\hbar}{2D})|00\rangle = |00\rangle, \quad U(\frac{\pi\hbar}{2D})|11\rangle = |11\rangle \tag{4.39}$$

$$U(\frac{\pi\hbar}{2D})|01\rangle = -|10\rangle, \quad U(\frac{\pi\hbar}{2D})|10\rangle = |01\rangle. \tag{4.40}$$

Therefore we can see that the operator $U(\frac{\pi\hbar}{2D})$ acts as the SWAP gate. Moreover at time

$t = \pi\hbar/4D$ the states $|01\rangle$ and $|10\rangle$ becomes maximally entangled Bell states.

$$U(\frac{\pi\hbar}{4D})|01\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{4.41}$$

$$U(\frac{\pi\hbar}{4D})|10\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \tag{4.42}$$

CHAPTER 5

ENTANGLEMENT DEPENDENCE ON DISTANCE BETWEEN INTERACTING QUBITS

In realistic spin lattice, position of spins could oscillate by producing phonons. In this case the exchange integrals are function of position and depend on distance between spins. In the present chapter we study entangled two qubit states with exchange interaction depending on distance $J(R)$ between spins and influence of this distance on entanglement of the system. We analyze the concurrence and its dependence on distance for various values of magnetic field B. These results presented in (Gurkan & Pashaev, 2010).

First we consider the Ising model in transverse magnetic field with the Hamiltonian

$$H = \frac{1}{2}[J(R)\,\sigma_1^x\sigma_2^x + B(\sigma_1^z + \sigma_2^z)]. \tag{5.1}$$

The eigenvalues are

$$E_{1,2} = \frac{\mp\sqrt{4B^2 + J(R)^2}}{2} \tag{5.2}$$

$$E_{3,4} = \frac{\mp J(R)}{2} \tag{5.3}$$

and the corresponding eigenvectors

$$|\psi_{1,2}\rangle = \begin{pmatrix} \frac{2B\mp\sqrt{4B^2+J(R)^2}}{J} \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.4}$$

$$|\psi_{3,4}\rangle = \begin{pmatrix} 0 \\ \mp 1 \\ 1 \\ 0 \end{pmatrix}. \tag{5.5}$$

Performing calculation of concurrence according to the determinant form of the concurrence (2.48) for the ground state $|\psi_1\rangle$, we find the concurrence depending on R as

$$C(R) = \frac{|J(R)|}{\sqrt{J(R)^2 + 4B^2}}. \tag{5.6}$$

5.1. Calogero-Moser Model Type I

First we consider $J(R)$ in the form of the Calogero-Moser type I model, where the two particle potential is

$$J(R) = \frac{1}{R^2}\,. \tag{5.7}$$

It is monotonically decreasing function of polynomial type displayed in Fig. 5.1.

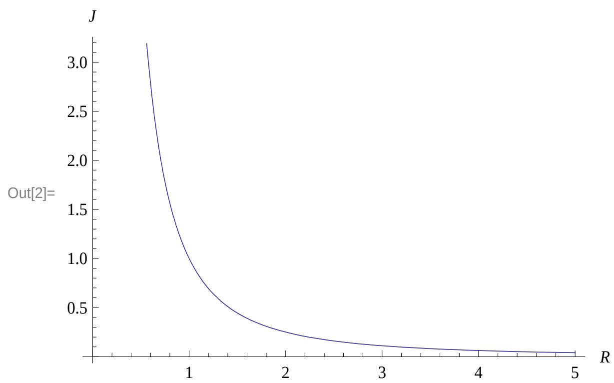


Figure 5.1. Potential $J(R) = 1/R^2$ versus R

We consider the two qubit Ising model with exchange interaction given by (5.7)

$$H = \frac{1}{2R^2}\sigma_1^x\sigma_2^x + \frac{B}{2}(\sigma_1^z + \sigma_2^z). \tag{5.8}$$

Performing calculation for the concurrence we get

$$C = \frac{1}{\sqrt{1 + 4B^2R^4}} \,. \tag{5.9}$$

In Fig. 5.2 we plot concurrence as a function of the distance between two qubits for different values of magnetic field B. It can be observed that the concurrence is zero at the limit $R \to \infty$.

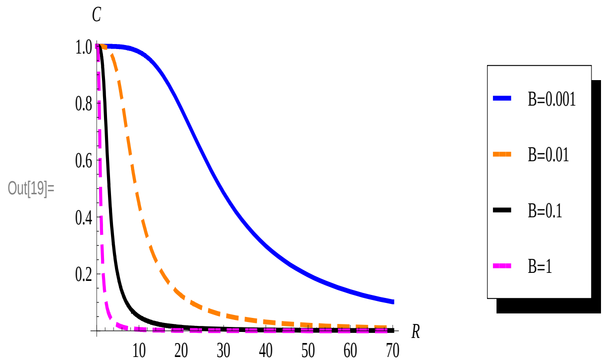


Figure 5.2. Concurrence as a function of distance R and magnetic field $B = 0.001, B = 0.01, B = 0.1, B = 1$ for potential $J(R) = 1/R^2$

In Fig. 5.3 we plot 3D plot of concurrence as a function of distance between two qubits for different values of magnetic field B.

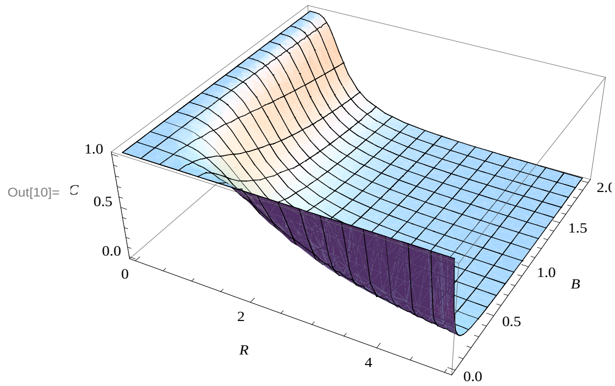


Figure 5.3. Concurrence as a function of distance R and magnetic field B for potential $J(R) = 1/R^2$

5.2. Calogero-Moser Model Type III

Next we consider Calogero-Moser type III model with exchange interaction

$$J(R) = \frac{1}{\sin^2 R} \tag{5.10}$$

as a periodic function of R as seen in Fig. 5.4.

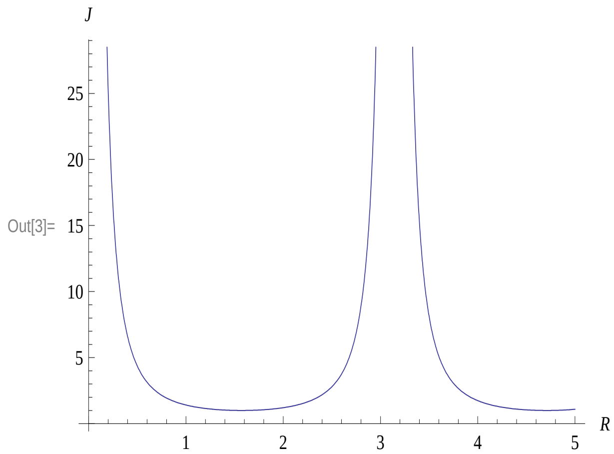


Figure 5.4. Potential $J(R) = 1/\sin^2 R$ versus R

Corresponding spin model with Hamiltonian

$$H = \frac{1}{2\sin^2 R}\sigma_1^x\sigma_2^x + \frac{B}{2}(\sigma_1^z + \sigma_2^z) \tag{5.11}$$

is called Haldane-Shastry Model (Haldane, 1988), (Shastry,1988). The concurrence calculated as

$$C = \frac{1}{\sqrt{1 + 4B^2\sin^4 R}} \tag{5.12}$$

which is also a periodic function of R with the same period, taking maximal value $C = 1$ at $R = 0$ (mod π) and minimal value $C \approx 0.25$ at $R = \pi/2$ (mod π.)

Concurrence as a function of distance R is shown in Fig. 5.5 and C is also plotted in Fig. 5.6 depend on R and B.

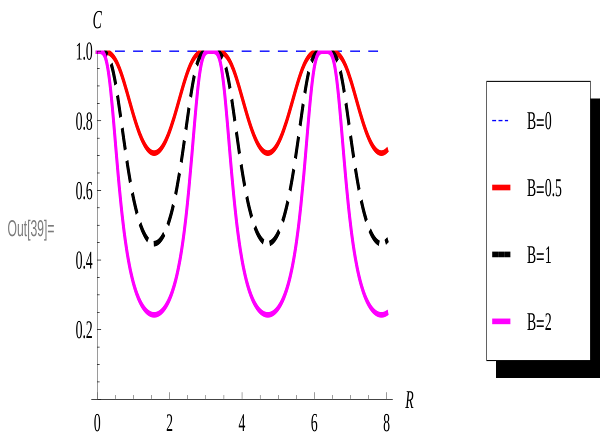


Figure 5.5. Concurrence as a function of distance R and magnetic field $B = 0.001, B = 0.01, B = 0.1, B = 1$ for potential $J(R) = 1/\sin^2 R$

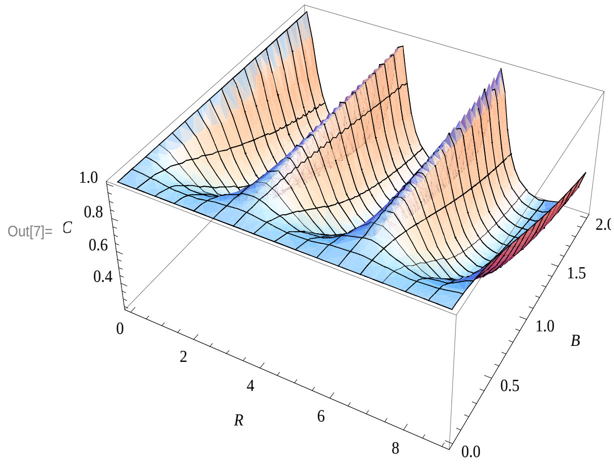


Figure 5.6. Concurrence as a function of distance R and magnetic field B for potential $J(R) = 1/\sin^2 R$

5.3. Calogero-Moser Model Type II

Finally we consider Calogero-Moser type II model which is a hyperbolic version of Haldane-Shastry model with the exchange interaction

$$J(R) = \frac{1}{\sinh^2 R} \tag{5.13}$$

exponentially decreasing with R displayed in Fig. (5.7).

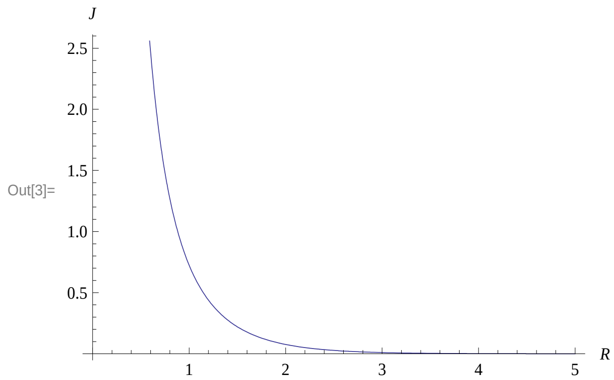


Figure 5.7. Potential $J(R) = 1/\sinh^2 R$ versus R

The corresponding Hamiltonian is written as

$$H = \frac{1}{2\sinh^2 R}\sigma_1^x\sigma_2^x + \frac{B}{2}(\sigma_1^z + \sigma_2^z)\,. \tag{5.14}$$

and the concurrence is calculated as

$$C = \frac{1}{\sqrt{1 + 4B^2 \sinh^4 R}}\,. \tag{5.15}$$

Concurrence for Hamiltonian (5.13) It is shown in Fig. 5.8 for various values of magnetic field B.

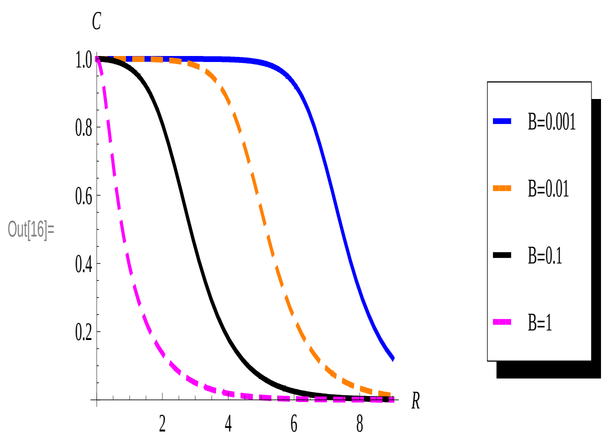


Figure 5.8. Concurrence as a function of distance R and magnetic field $B = 0.001, B = 0.01, B = 0.1, B = 1$ for potential $J(R) = 1/\sinh^2 R$

5.4. Herring-Flicker Potential

In a recent paper (Huang & Kais, 2005) a relation between entanglement and the electron correlation energy in H_2 molecule has been analyzed and it was shown that the entanglement can be used as an alternative measure of the electron correlation in quantum chemistry calculations. Despite of the standard definition of electron correlation as the difference between the Hartree-Fock energy and the exact solution of the nonrelativistic Schrodinger equation, it is found that entanglement can be used as an alternative measure of electron correlations. In these calculations following Herring-Flicker, the exchange coupling constant J for H_2 molecule has been approximated as a function of the interatomic distance R: $J(R) = -0.821R^{5/2}e^{-2R} + O(R^2e^{-2R})$. This is why, as a next example we consider concurrence for exchange interaction in the form

$$J(R) = -0.821R^{5/2}e^{-2R} + O(R^2e^{-2R}). \tag{5.16}$$

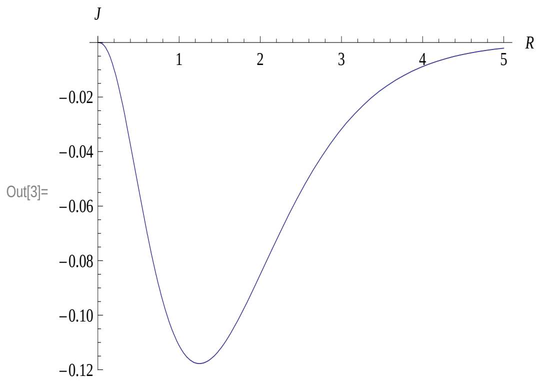


Figure 5.9. Exchange interaction $J(R)$ as a function of distance R

In Fig. 5.9 we display this function with extreme minimal value at $R \approx 1.2$, exponentially approaching the horizontal asymptotes $J = 0$. Corresponding concurrence for various magnetic fields is shown in Fig. 5.10. It has characteristic maxima at $R \approx 1.2$ and $B = 1$. With growing B the region of maximal C is extending to almost all characteristic region $0 < R < 5$.

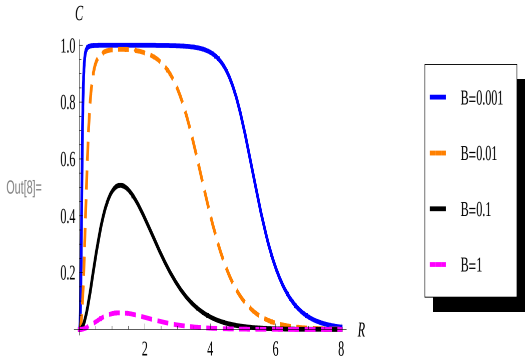


Figure 5.10. Concurrence C versus distance R for $B = 0.001, B = 0.01, B = 0.1, B = 1$ respectively

CHAPTER 6

GEOMETRIC QUANTUM COMPUTATION

The results presented in this chapter partially appeared in (Kwan et al., 2008). One of the recently proposed perspective direction in quantum computation is related with geometric quantum computation, based on geometric phase in quantum mechanics. When a quantum mechanical system undergoes a cyclic evolution, a phase of the wave function is acquired as a result of the geometrical properties of the parameter space of the Hamiltonian. This geometric phase or the Berry phase is a purely geometric effect that only depends on the area covered by the motion of the system. Pancharatnam was the first to introduce the concept of geometric phase in 1956 (Pancharatnam, 1956). Then Michael Berry in 1984 realized that geometric (Berry) phase is a generic feature of quantum mechanics (Berry, 1984). Existence of Berry phases have been demonstrated in a variety of quantum systems (Shapera & Wilczek, 1989), NMR (Suter et al., 1987), (Goldman et al., 1996), optical systems (Tomita & Chiao, 1986), experimental (Jones et al., 2000). Very recently (Ekert et al., 2000) proposed geometric phases have the potential of performing quantum computations.

6.1. Dynamic and Geometric Phase

A particle which starts out in the n^{th} eigenstate of $H(0)$ remains, in the n^{th} eigenstate of $H(t)$, picking up only a time dependent phase factor with the wave function

$$|\Psi(t)\rangle = e^{i(\theta(t)+\gamma(t))}|\psi(t)\rangle. \tag{6.1}$$

The time evolution of a quantum system is governed by the Schrödinger equation

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle. \tag{6.2}$$

Substituting (6.1) in (6.2) we have

$$\dot{\theta}(t) + \dot{\gamma}(t) = -\frac{1}{\hbar} \langle \Psi(t)|H(t)|\Psi(t)\rangle + i \langle \psi(t)|\frac{d}{dt}|\psi(t)\rangle \tag{6.3}$$

after the integration of (6.3) we find the dynamical phase $\theta(t)$ as

$$\theta(t) = -\frac{1}{\hbar} \int_0^\tau \langle \Psi(t)|H(t)|\Psi(t)\rangle \, dt \tag{6.4}$$

and the geometric phase $\gamma(t)$ as

$$\gamma(t) = i \int_0^\tau \langle \psi(t)|\frac{d}{dt}|\psi(t)\rangle \, dt \tag{6.5}$$

In the present chapter we are going to calculate Berry phase for two qubit XX Heisenberg Hamiltonian with DM interaction term and external magnetic field B. The purpose is to find dependence of geometric phase on the parameters of the system.

6.2. Berry's Phase under Dzialoshinskii-Moriya Interaction

In this section, we consider an XX chain with DM interaction in an applied magnetic field of the form

$$H = \sum_{\langle i,j\rangle} \left[J\left(S_i^x \cdot S_j^x + S_i^y \cdot S_j^y\right) + \vec{D}_{ij} \cdot \vec{S}_i \times \vec{S}_j \right] + \vec{B} \cdot \vec{S}_1 \tag{6.6}$$

where the sum is taken over the nearest neighbor sites, the spin operator $\vec{S} \equiv (S^x, S^y, S^z)$, the vector $\vec{D}_{ij}$ is the DM vector and $\vec{B}$ is the orientation of the magnetic field, which is applied only to the first site as in (Yi et al., 2004). For simplicity, we shall choose DM vector so that it is aligned to the z-component, parameterize the vector

$$\vec{B} = B_0(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{6.7}$$

and set the spin-spin coupling term J =1. We consider two sites under the Hamiltonian in Eq (6.6). There are four eigenstates $|E_i\rangle, i = 1, 2, 3, 4$,

$$|E_i\rangle = \frac{1}{\sqrt{N}}[a_i|00\rangle + b_i|01\rangle + c_i|10\rangle + d_i|11\rangle] \tag{6.8}$$

where

$$\begin{aligned} a_i &= \frac{e^{-2i\phi}}{\sin^2\theta}\frac{(E_i - \cos\theta)(E_i^2 - g_1 g_2 - 1)}{g_1} && (6.9) \\ b_i &= \frac{e^{-i\phi}}{\sin\theta}(E_i + \cos\theta) && (6.10) \\ c_i &= \frac{e^{-i\phi}}{\sin\theta}\frac{(E_i^2 - 1)}{g_1} && (6.11) \\ d_i &= 1, && (6.12) \end{aligned}$$

$$N = |a_i|^2 + |b_i|^2 + |c_i|^2 + |d_i|^2 \tag{6.13}$$

and $g_1 = \frac{2J+2iD}{B_0}$, $g_2 = \frac{2J-2iD}{B_0}$. Corresponding eigenvalues are

$$\begin{aligned} E_1 &= -E_2 = -\frac{\sqrt{2 + g_1 g_2 - \sqrt{g_1 g_2 (2 + g_1 g_2 - 2\cos 2\theta)}}}{\sqrt{2}} \\ E_3 &= -E_4 = -\frac{\sqrt{2 + g_1 g_2 + \sqrt{g_1 g_2 (2 + g_1 g_2 - 2\cos 2\theta)}}}{\sqrt{2}} && (6.14) \end{aligned}$$

Note that $2 + g_1 g_2 \geq \sqrt{g_1 g_2 (2 + g_1 g_2 - 2\cos 2\theta)}$ and that $E_1 \leq E_2 \leq E_3 \leq E_4$. Thus, E_1 corresponds to the ground state, $|E_2\rangle$ corresponds to the first excited state, and so forth.

For each eigenvector $|E_i\rangle$, we consider situation in which the external magnetic field undergoes adiabatic evolution in the azimuthal angle ϕ for closed loop at a fixed polar angle θ. The dynamical phase of the system is zero, and the total phase of the system is equal to the geometric (Berry) phase. Thus the geometric (Berry) phase is given by

$$\gamma = i\int_0^{2\pi} \langle E_i|\frac{d}{d\phi}|E_i\rangle d\phi \tag{6.15}$$

$$\langle E_i|\frac{d}{d\phi}|E_i\rangle = \frac{-2i\bar{a}_j\dot{a}_j - i\bar{b}_j\dot{b}_j - i\bar{c}_j\dot{c}_j}{|a_j|^2 + |b_j|^2 + |c_j|^2 + |d_j|^2} \tag{6.16}$$

$$= f(\theta) \tag{6.17}$$

We can calculate the Berry Phase as

$$\gamma = i\int_0^{2\pi}\langle E_i|\frac{d}{d\phi}|E_i\rangle d\phi = i\int_0^{2\pi} f(\theta)d\phi = 2\pi i f(\theta). \tag{6.18}$$

Due to the symmetry inherent in the eigenstates, it turns out that the eigenstates $|E_1\rangle$ and $|E_4\rangle$ (and $|E_2\rangle$ and $|E_3\rangle$) yields the same Berry phase as one adiabatically evolves the parameter ϕ around a closed path. The graph of the Berry phase against the polar angle θ for the eigenstate $|E_1\rangle$ (or $|E_4\rangle$) for different values of B field and with the DM interaction set to unity is shown in Fig. 6.1. As shown in Fig. (6.1), an increase

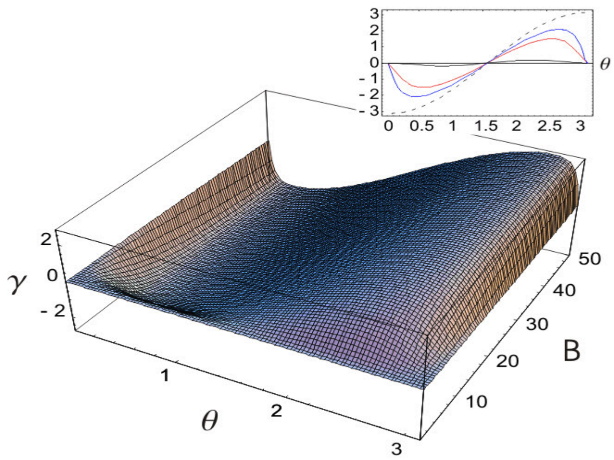


Figure 6.1. Geometric phase for the ground state $|E_1\rangle$ (or the highest excited state, $|E_4\rangle$) with different values of the external magnetic field and with constant DM interaction, $D = 1$. The inset shows the cross-sectional plots for different values of B. The dashed plot in the inset is the limit of the variation of Berry phase with θ for $B \to \infty$.

in the external magnetic field can substantially increase the amount of the Berry phase.

Moreover, in the large B limit, i.e. $B \to \infty$, the values of $g_1, g_2 \to 0$ so

$$E_1 = -1 \tag{6.19}$$

and

$$|E_1\rangle = \begin{pmatrix} 0 \\ -e^{-i\phi} \sin \frac{\theta}{2} \\ 0 \\ \cos \frac{\theta}{2} \end{pmatrix} \tag{6.20}$$

Calculating the Berry phase we have

$$\gamma = i \int_0^{2\pi} \langle E_i | \frac{d}{d\phi} | E_i \rangle d\phi = i \int_0^{2\pi} (-i) \sin^2 \frac{\theta}{2} d\phi = \pi(1 - \cos\theta) \tag{6.21}$$

the Berry phase assumes the value of $-\pi \cos\theta$ in the large B limit, i.e. $B \to \infty$ independent of the value of D. Unlike the case of the ground state (or highest excited state), the Berry phase could be non-trivial for low magnetic field if one confines the evolution to polar angle near $\theta = \dfrac{\pi}{2}$.

PART II

TOPOLOGICAL SOLITONS IN SPIN MODELS

CHAPTER 7

CLASSICAL SPIN MODELS IN CONTINUOUS MEDIA

The results presented in this chapter appeared in (Gurkan & Pashaev, 2008). Magnetic materials could be arranged as spin chains and as spin lattices. By identifying spins with qubits, information characteristics of these materials maybe represented by chain of qubits or lattice of qubits. In the linear chain case, quantum states are represented by spin complexes, and every spin complex is a N qubit computational basis state. The ground state of the system depends on magnetic order, and for ferromagnetic spin chain the ground state is one of the states

$$|00...0\rangle = |\uparrow\uparrow ... \uparrow\rangle, \tag{7.1}$$

$$|11...1\rangle = |\downarrow\downarrow ... \downarrow\rangle. \tag{7.2}$$

Then excitations in the chain appear as flipping. Propagation of these excitations in linear approximation are described by magnons. Another type of excitations for spin chains, corresponds to the domain wall, separating spin up and spin down states. All these states appearing as a computational basis are involved in information characteristic of one dimensional magnetic materials.

In two dimensional lattice case with ferromagnetic order, the ground state corresponds to orientation of all spins in up or down directions. If in the plane with ferromagnetic ground state at finite point suppose at origin, we have spin flips then it appear as a magnetic soliton configuration. Depending on flipping spins in the lattice, simple soliton or multi soliton configurations can appear. These configurations are characterized by winding number or the topological charge. Under time evolution these solitons in general can move and interact with each other. In the present chapter we consider continuous distribution of qubits in the plane as $2 + 1$ dimensional spin field. By using spin coherent states, this field can be described by a classical unit vector attached to every point of the plane. Evolution of this vector field is determined by classical continuous Heisenberg model.

7.1. Topological Magnet Model

The classical Heisenberg spin model (Makhankov & Pashaev, 1992) determines evolution of the classical spin vector

$$\vec{S} = (S_1(x,y,t), S_2(x,y,t), S_3(x,y,t)) \tag{7.3}$$

valued on two dimensional sphere S^2,

$$S_1^2 + S_2^2 + S_3^2 = 1 \tag{7.4}$$

according to the Landau-Lifshitz equation

$$\vec{S}_t = \vec{S} \times \Delta\vec{S}. \tag{7.5}$$

In the spin liquid (ferromagnetic fluid) one have in addition to magnetic variables $\vec{S} = \vec{S}(x,y,t)$ the hydrodynamic variable $\vec{v}(x,y,t)$ (Volovik, 1987) and time derivative $\partial/\partial t$ would be replaced by the material derivative (Martina et al., 1994a)

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{v}\nabla). \tag{7.6}$$

Between hydrodynamic and spin variables exists relation called the Mermin-Ho relation (Ho & Mermin, 1980), (Mermin & Ho, 1976). It relates vorticity of the flow with the topological charge density (or winding number),

$$rot\,\vec{v} = \vec{S}\cdot(\partial_x\vec{S} \times \partial_y\vec{S}). \tag{7.7}$$

Then we have a simple model of ferromagnetic fluid - the so called Topological Magnet model (Martina et al., 1994b),

$$\vec{S}_t + \upsilon^a \partial_a \vec{S} = \vec{S} \times \partial^a \partial_a \vec{S} \tag{7.8}$$
$$\partial_a \upsilon_b - \partial_b \upsilon_a = 2\vec{S}(\partial_a \vec{S} \times \partial_b \vec{S}) \tag{7.9}$$

where the scalar product $A^a B_a = A^a g_{ab} B^b$, $a = 1, 2$ is determined by the metric tensor $g_{ab} = diag(1, \alpha^2)$, $\alpha^2 = \pm 1$. For particular case of the metric $g_{ab} = (1, -1)$ we have the system

$$\vec{S}_t + \upsilon_1 \partial_1 \vec{S} - \upsilon_2 \partial_2 \vec{S} = \vec{S} \times (\partial_1^2 - \partial_2^2) \vec{S} \tag{7.10}$$
$$\partial_1 \upsilon_2 - \partial_2 \upsilon_1 = 2\vec{S}(\partial_1 \vec{S} \times \partial_2 \vec{S}). \tag{7.11}$$

For this system we have the next lemmas

Lemma 7.1.0.1 *The following identities hold*

$$-v_1 \partial_1^2 \vec{S} \cdot \partial_1 \vec{S} = -\frac{1}{2} \partial_1 [v_1 (\partial_1 \vec{S})^2] + \frac{1}{2} (\partial_1 v_1)(\partial_1 \vec{S})^2 \tag{7.12}$$
$$v_2 \partial_2^2 \vec{S} \cdot \partial_2 \vec{S} = +\frac{1}{2} \partial_2 [v_2 (\partial_2 \vec{S})^2] - \frac{1}{2} (\partial_2 v_2)(\partial_2 \vec{S})^2 \tag{7.13}$$

Proof 7.1.0.2 *Proof of Eqn. (7.12) is given as*

$$-v_1 \partial_1^2 \vec{S} \cdot \partial_1 \vec{S} = -\partial_1 [v_1 (\partial_1 \vec{S})^2] + \partial_1 \vec{S} \cdot \partial_1 (v_1 \partial_1 \vec{S}) \tag{7.14}$$
$$= -\partial_1 [v_1 (\partial_1 \vec{S})^2] + (\partial_1 \vec{S})^2 \partial_1 v_1 + v_1 \partial_1 \vec{S} \cdot \partial_1^2 \vec{S} \tag{7.15}$$
$$-2v_1 \partial_1^2 \vec{S} \cdot \partial_1 \vec{S} = -\partial_1 [v_1 (\partial_1 \vec{S})^2] + (\partial_1 \vec{S})^2 \partial_1 v_1 \,. \tag{7.16}$$

Proof of Eqn. (7.13) is given by

$$-v_2 \partial_2^2 \vec{S} \cdot \partial_2 \vec{S} = -\partial_2 [v_2 (\partial_2 \vec{S})^2] + \partial_2 \vec{S} \cdot \partial_2 (v_2 \partial_2 \vec{S}) \tag{7.17}$$
$$= -\partial_2 [v_2 (\partial_1 \vec{S})^2] + (\partial_2 \vec{S})^2 \partial_2 v_2 + v_2 \partial_2 \vec{S} \cdot \partial_2^2 \vec{S} \tag{7.18}$$
$$-2v_2 \partial_2^2 \vec{S} \cdot \partial_2 \vec{S} = -\partial_2 [v_2 (\partial_2 \vec{S})^2] + (\partial_2 \vec{S})^2 \partial_2 v_2 \,. \tag{7.19}$$

Lemma 7.1.0.3 *The following identities hold*

$$-v_1\partial_2\vec{S}\cdot\partial_1\partial_2\vec{S} = \frac{1}{2}\partial_1[v_1(\partial_2\vec{S})^2] - \frac{1}{2}\partial_1 v_1(\partial_2\vec{S})^2 \tag{7.20}$$

$$-v_2\partial_1\vec{S}\cdot\partial_1\partial_2\vec{S} = -\frac{1}{2}\partial_2[v_2(\partial_1\vec{S})^2] + \frac{1}{2}\partial_2 v_2(\partial_1\vec{S})^2 \tag{7.21}$$

Proof 7.1.0.4 *Proof of Eqn. (7.20) can be written as*

$$\begin{aligned} -v_1\partial_2\vec{S}\cdot\partial_1\partial_2\vec{S} &= \partial_1[v_1(\partial_2\vec{S})^2] - \partial_1 v_1(\partial_2\vec{S})^2 - v_1\partial_1\partial_2\vec{S}\cdot\partial_2\vec{S} && (7.22)\\ &= \frac{1}{2}\partial_1[v_1(\partial_2\vec{S})^2] - \frac{1}{2}\partial_1 v_1(\partial_2\vec{S})^2\,. && (7.23)\end{aligned}$$

Proof of Eqn. (7.21) is as follows

$$\begin{aligned} -v_2\partial_1\vec{S}\cdot\partial_1\partial_2\vec{S} &= -\partial_2[v_2(\partial_1\vec{S})^2] + \partial_2 v_2(\partial_1\vec{S})^2 + v_2\partial_1\partial_2\vec{S}\cdot\partial_1\vec{S} && (7.24)\\ &= -\frac{1}{2}\partial_2[v_2(\partial_1\vec{S})^2] + \frac{1}{2}\partial_2 v_2(\partial_1\vec{S})^2\,. && (7.25)\end{aligned}$$

Theorem 7.1.0.5 *For the system which has been defined with Eqs. (7.10)and (7.11) with the flow constrained by the incompressibility condition is given by*

$$\partial_1 v_1 + \partial_2 v_2 = 0, \tag{7.26}$$

and the conservation law is given by

$$\partial_t J_0 + \partial_2 J_2 - \partial_1 J_1 = 0\,. \tag{7.27}$$

where

$$\begin{aligned} J_0 &= (\partial_1\vec{S})^2 + (\partial_2\vec{S})^2, && (7.28)\\ J_1 &= -2\partial_1\vec{S}\cdot\vec{S}\times(\partial_1^2-\partial_2^2)\vec{S} + v_1 J_0 + 2\vec{S}\cdot(\partial_1\vec{S}\times\partial_2^2\vec{S} - \partial_1\partial_2\vec{S}\times\partial_2\vec{S}) && (7.29)\\ J_2 &= 2\partial_2\vec{S}\cdot\vec{S}\times(\partial_1^2-\partial_2^2)\vec{S} + v_2 J_0 - 2\vec{S}\cdot(\partial_1^2\vec{S}\times\partial_1\partial_2\vec{S} - \partial_1\vec{S}\times\partial_2\vec{S})\,. && (7.30)\end{aligned}$$

Proof 7.1.0.6

$$\begin{aligned}
\partial_t J_0 &= \partial_t[(\partial_1\vec{S})^2 - \alpha^2(\partial_2\vec{S})^2] && (7.31)\\
&= 2[\partial_1\vec{S}\cdot\partial_1\partial_t\vec{S} + \partial_2\vec{S}\cdot\partial_2\partial_t\vec{S}] && (7.32)\\
&= 2[\partial_1(\partial_1\vec{S}\cdot\partial_t\vec{S}) - \partial_1^2\vec{S}\cdot\partial_t\vec{S} + \partial_2^2\vec{S}\cdot\partial_t\vec{S} - \partial_2(\partial_2\vec{S}\cdot\partial_t\vec{S})] && (7.33)
\end{aligned}$$

Eq. (7.33) can be reorganized as

$$\partial_t J_0 - 2\partial_1(\partial_1\vec{S}\cdot\partial_t\vec{S}) - 2\partial_2(\partial_2\vec{S}\cdot\partial_t\vec{S}) = -2[\partial_1^2\vec{S}\cdot\partial_t\vec{S} + \partial_2^2\vec{S}\cdot\partial_t\vec{S}] \qquad (7.34)$$

Using equations of motion (7.10) we estimate expression in the r.h.s. of Eqn. (7.34)

$$\begin{aligned}
\partial_1^2\vec{S}\cdot\partial_t\vec{S} + \partial_2^2\vec{S}\cdot\partial_t\vec{S} &= (\partial_1^2\vec{S} + \partial_2^2\vec{S})[-v_1\partial_1\vec{S} + v_2\partial_2\vec{S}]\\
&+ (\partial_1^2\vec{S} + \partial_2^2\vec{S})[\vec{S}\times(\partial_1^2 - \partial_2^2)\vec{S}] && (7.35)\\
&= -v_1\partial_1^2\vec{S}\cdot\partial_1\vec{S} - v_1\partial_2^2\vec{S}\cdot\partial_1\vec{S} + v_2\partial_1^2\vec{S}\cdot\partial_2\vec{S}\\
&+ v_2\partial_2^2\vec{S}\cdot\partial_2\vec{S} - (\partial_1^2\vec{S} + \partial_2^2\vec{S})\cdot(\vec{S}\times(\partial_1^2 - \partial_2^2)\vec{S}) && (7.36)\\
&= -\frac{1}{2}\partial_1[v_1(\partial_1\vec{S})^2] - \frac{1}{2}(\partial_1 v_1)(\partial_1\vec{S})^2 + \frac{1}{2}\partial_2[v_2(\partial_2\vec{S})^2]\\
&+ \frac{1}{2}(\partial_2 v_2)(\partial_2\vec{S})^2 + \partial_1[v_2\partial_1\vec{S}\partial_2\vec{S}] - \partial_2[v_1\partial_1\vec{S}\partial_2\vec{S}]\\
&- 2\vec{S}(\partial_1\vec{S}\times\partial_2\vec{S})(\partial_1\vec{S}\cdot\partial_2\vec{S}) + \frac{1}{2}\partial_1(v_1(\partial_2\vec{S})^2)\\
&- \frac{1}{2}\partial_2(v_2(\partial_1\vec{S})^2) - \frac{1}{2}\partial_1 v_1((\partial_2\vec{S})^2) + \frac{1}{2}\partial_2 v_2((\partial_1\vec{S})^2)\\
&- (\partial_1^2\vec{S} + \partial_2^2\vec{S})\cdot(\vec{S}\times(\partial_1^2 - \partial_2^2)\vec{S}) && (7.37)
\end{aligned}$$

and we find

$$\begin{aligned}
\partial_t J_0 &= 2\partial_1(\partial_1\vec{S}\cdot\partial_t\vec{S}) + 2\partial_2(\partial_2\vec{S}\cdot\partial_t\vec{S})\\
&- 2\partial_1[-\frac{1}{2}v_1[(\partial_1\vec{S})^2 - (\partial_2\vec{S})^2] + v_2\partial_1\vec{S}\cdot\partial_2\vec{S} + (\partial_2\vec{S})^2\cdot(\vec{S}\times\partial_1\vec{S})\\
&+ \partial_1\partial_2\vec{S}\cdot(\vec{S}\times\partial_2\vec{S})]\\
&- 2\partial_2[\frac{1}{2}v_2[(\partial_1\vec{S})^2 - (\partial_2\vec{S})^2] - v_1\partial_1\vec{S}\cdot\partial_2\vec{S} + (\partial_1\vec{S})^2\cdot(\vec{S}\times\partial_2\vec{S})]\\
&- \partial_1\partial_2\vec{S}\cdot(\vec{S}\times\partial_1\vec{S})] \quad . && (7.38)
\end{aligned}$$

Due to the above theorem 7.1.0.5 the energy functional is written as

$$E = \int\int J_0 d^2x\,, \tag{7.39}$$

or it is given by

$$E = \int\int \{(\partial_1 \vec{S})^2 + (\partial_2 \vec{S})^2\} d^2x\,. \tag{7.40}$$

Here the energy is conserved quantity. From another side, there exist another integral of motion, the topological charge of a spin configuration, defined as

$$Q = \frac{1}{4\pi}\int\int \vec{S}\cdot(\partial_1 \vec{S} \times \partial_2 \vec{S}) d^2x\,. \tag{7.41}$$

These two conserved quantities are related by the Bogomolnyi Inequality

$$E \geq |Q| \tag{7.42}$$

which means that the energy is bounded below by topological charge (Makhankov & Pashaev, 1992).

To find Bogomolnyi inequality we do several transformations of the evident inequality

$$\int\int (\partial_i \vec{S} \pm \epsilon_{ij} \vec{S} \times \partial_j \vec{S})^2 d^2x \geq 0, \quad i,j = 1,2 \tag{7.43}$$

writing explicitly

$$\begin{aligned}
\int\int (\partial_i \vec{S} \pm \epsilon_{ij}(\vec{S} \times \partial_j \vec{S}) \cdot (\partial_i \vec{S} \pm \epsilon_{ik} \vec{S} \times \partial_k \vec{S}) d^2x \quad &\geq \quad 0 & (7.44) \\
\int\int [(\partial_i \vec{S})^2 + \epsilon_{ij}\epsilon_{ik}(\vec{S} \times \partial_j \vec{S}) \cdot (\vec{S} \times \partial_k \vec{S}) & \\
\pm \epsilon_{ij}(\vec{S} \times \partial_j \vec{S})\partial_i \vec{S} \pm \epsilon_{ik}\partial_i(\vec{S}\vec{S} \times \partial_k \vec{S})] d^2x \quad &\geq \quad 0 & (7.45)
\end{aligned}$$

we have

$$\int\int[(\partial_i\vec{S})^2+\delta_{jk}\partial_j\vec{S}\cdot\partial_k\vec{S}\pm\epsilon_{ij}\partial_i\vec{S}(\vec{S}\times\partial_j\vec{S})\pm\epsilon_{ik}\partial_i\vec{S}(\vec{S}\times\partial_k\vec{S})]d^2x\geq 0 \quad (7.46)$$

where $\epsilon_{ij}\epsilon_{ik}=\delta_{jk}$

$$\int\int[(\partial_i\vec{S})^2+(\partial_j\vec{S})^2\pm\epsilon_{ij}\partial_i\vec{S}(\vec{S}\times\partial_j\vec{S})\pm\epsilon_{ik}\partial_i\vec{S}(\vec{S}\times\partial_k\vec{S})]d^2x\geq 0. \quad (7.47)$$

By cyclic permutation

$$\int\int[(\partial_i\vec{S})^2+(\partial_j\vec{S})^2\pm\epsilon_{ij}\vec{S}(\partial_j\vec{S}\times\partial_i\vec{S})\pm\epsilon_{ik}\vec{S}(\partial_k\vec{S}\times\partial_i\vec{S})]d^2x\geq 0 \quad (7.48)$$

$$\underbrace{\int\int 2[(\partial_1\vec{S})^2+(\partial_2\vec{S})^2]d^2x}_{2E}\mp\underbrace{\int\int 4\vec{S}\cdot(\partial_1\vec{S}\times\partial_2\vec{S})d^2x}_{16\pi Q}\geq 0 \quad (7.49)$$

finally we have

$$E\mp 8\pi Q\geq 0\Longrightarrow E\geq\pm 8\pi Q \quad (7.50)$$

For $Q>0$ we obtain

$$E\geq 8\pi Q=8\pi|Q| \quad (7.51)$$

while for $Q<0$ we obtain

$$E\geq -8\pi Q=8\pi|Q| \quad (7.52)$$

Combining together Eqns. (7.50) and (7.51) we have

$$E \geq 8\pi|Q|\,. \tag{7.53}$$

This inequality is saturated for spin configurations satisfying the first order system (Martina et al., 1994a)

$$\partial_i \vec{S} \pm \epsilon_{ij} \vec{S} \times \partial_j \vec{S} = 0 \tag{7.54}$$

called the Belavin Polyakov self-duality equations (Belavin & Polyakov, 1975).

7.2. Self Duality and Stereographic Projection Representation

If we consider the spin phase space, the 2-dimensional sphere, as a Riemann sphere for a complex plane, we can project points on this sphere to that plane. The stereographic projections are given by formulas

$$S_1 + iS_2 = S_+ = \frac{2\zeta}{1+|\zeta|^2}, \quad S_3 = \frac{1-|\zeta|^2}{1+|\zeta|^2} \tag{7.55}$$

where ζ is complex valued function. Now we will rewrite the self-duality equations (7.54) in the stereographic projection form: For $i = j$, where $i, j = 1, 2$, Eqn. 7.54 is written as

$$\begin{aligned} \partial_1 \vec{S} \pm \vec{S} \times \partial_2 \vec{S} &= 0 & (7.56) \\ \partial_2 \vec{S} \mp \vec{S} \times \partial_1 \vec{S} &= 0\,. & (7.57) \end{aligned}$$

Here we can write

$$\partial_z = \frac{1}{2}(\partial_1 - i\partial_2) \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)\,. \tag{7.58}$$

Multiplying (7.57) by i and then adding to (7.56) we have

$$\partial_{\bar{z}}\vec{S} \mp i\vec{S} \times \partial_{\bar{z}}\vec{S} = 0 \tag{7.59}$$

$$\partial_{z}\vec{S} \pm i\vec{S} \times \partial_{z}\vec{S} = 0 \tag{7.60}$$

Eqn. (7.59)

$$\partial_{\bar{z}}\vec{S} - i\vec{S} \times \partial_{\bar{z}}\vec{S} = 0 \tag{7.61}$$

can be written explicitly

$$\partial_{\bar{z}}S_1 - i(\vec{S} \times \partial_{\bar{z}}\vec{S})_1 = \partial_{\bar{z}}S_1 - i(S_2\partial_{\bar{z}}S_3 - S_3\partial_{\bar{z}}S_2) = 0 \tag{7.62}$$

$$\partial_{\bar{z}}S_2 - i(\vec{S} \times \partial_{\bar{z}}\vec{S})_2 = \partial_{\bar{z}}S_2 - i(S_3\partial_{\bar{z}}S_1 - S_1\partial_{\bar{z}}S_3) = 0 \tag{7.63}$$

Multiplying (7.63) by i and then adding to (7.62) we have

$$\partial_{\bar{z}}S_+ + [S_3\partial_{\bar{z}}S_+ - \partial_{\bar{z}}S_3S_+] = 0 \tag{7.64}$$

Substituting S_3 and S_+ in (7.55) we have the analyticity condition:$\zeta_{\bar{z}} = 0$ Eqn. (7.60)

$$\partial_{z}\vec{S} - i\vec{S} \times \partial_{z}\vec{S} = 0 \tag{7.65}$$

can be written explicitly

$$\partial_{z}S_1 - i(\vec{S} \times \partial_{z}\vec{S})_1 = \partial_{z}S_1 - i(S_2\partial_{z}S_3 - S_3\partial_{z}S_2) = 0 \tag{7.66}$$

$$\partial_{z}S_2 - i(\vec{S} \times \partial_{z}\vec{S})_2 = \partial_{z}S_2 - i(S_3\partial_{z}S_1 - S_1\partial_{z}S_3) = 0 \tag{7.67}$$

Multiplying (7.67) by i and then adding to (7.66) we have

$$\partial_z S_+ + [S_3 \partial_z S_+ - \partial_z S_3 S_+] = 0 \tag{7.68}$$

Substituting S_3 and S_+ in (7.68)we have the anti-analyticity condition:

$$\zeta_z = 0\,. \tag{7.69}$$

The above consideration shows that the self- duality equations in the stereographic projection form are just the analyticity conditions while for the anti-self-duality equations they are anti-analyticity conditions. In both cases the energy (7.40) reaches its minima.

7.3. Anti-Holomorphic Reduction and Topological Magnet

As we have seen analytic/anti-analytic configurations saturate Bogomolny inequality and have minimal energy. This suggest to search solutions of topological magnet (7.10) and (7.11) with holomorphic/anti-holomorphic stereographic projections. For this reason we first rewrite equations in the stereographic form

$$i(\zeta_t + v_1 \partial_1 \zeta - v_2 \partial_2 \zeta) + \partial_1^2 \zeta - \partial_2^2 \zeta - 2\frac{(\partial_1 \zeta)^2 - (\partial_2 \zeta)^2}{1 + |\zeta|^2}\bar{\zeta} = 0 \tag{7.70}$$

$$\partial_1 v_2 - \partial_2 v_1 = -4i \frac{\partial_1 \bar{\zeta} \partial_2 \zeta - \partial_2 \bar{\zeta} \partial_1 \zeta}{(1 + |\zeta|^2)^2}. \tag{7.71}$$

In complex coordinates we have

$$\begin{aligned} i\zeta_t + iv_1(\zeta_z + \zeta_{\bar{z}}) + v_2(\zeta_z - \zeta_{\bar{z}}) &+ (\partial_z + \partial_{\bar{z}})^2 \zeta + (\partial_z - \partial_{\bar{z}})^2 \zeta \\ &- 2\frac{(\zeta_z + \zeta_{\bar{z}})^2 - (\zeta_z - \zeta_{\bar{z}})^2}{1 + |\zeta|^2}\bar{\zeta} = 0\,. \end{aligned} \tag{7.72}$$

For $v_+ = v_1 + iv_2$ and $v_- = v_1 - iv_2$ (7.72) becomes

$$i(\zeta_t + v_-\zeta_z + v_+\zeta_{\bar{z}}) + 2(\partial_z^2\zeta + \partial_{\bar{z}}^2\zeta) - 4\frac{\bar{\zeta}}{1+|\zeta|^2}(\zeta_z^2 + \zeta_{\bar{z}}^2) = 0 \tag{7.73}$$

Eqn. (7.71) is written in the form

$$\begin{aligned}
\partial_1 v_2 - \partial_2 v_1 &= (\partial_z + \partial_{\bar{z}})v_2 - i(\partial_z - \partial_{\bar{z}})v_1 \\
&= \partial_z(v_2 - iv_1) + \partial_{\bar{z}}(v_2 + iv_1) \\
&= i[-\partial_z(v_1 + iv_2) + \partial_{\bar{z}}(v_1 - iv_2)] \\
&= i[\partial_{\bar{z}}v_- - \partial_z v_+] \qquad (7.74)
\end{aligned}$$

or in complex coordinates

$$\begin{aligned}
i[\partial_{\bar{z}}v_- - \partial_z v_+] &= \frac{-4i}{(1+|\zeta|^2)^2}(\partial_1\bar{\zeta}\partial_2\zeta - \partial_2\bar{\zeta}\partial_1\zeta) \\
&= \frac{8}{(1+|\zeta|^2)^2}[\zeta_z\bar{\zeta_z} - \bar{\zeta}_z\zeta_{\bar{z}}] \qquad (7.75)
\end{aligned}$$

If ζ is anti-holomorphic $\zeta_z = 0$, then the system defined with (7.73) and (7.75) is reduced to

$$i\zeta_t + iv_+\zeta_{\bar{z}} + 2\zeta_{\bar{z}\bar{z}} - 4\frac{\zeta_{\bar{z}}^2}{1+|\zeta|^2}\bar{\zeta} = 0 \tag{7.76}$$

and

$$\partial_z v_+ - \partial_{\bar{z}}v_- = \frac{-8i}{(1+|\zeta|^2)^2}\bar{\zeta}_z\zeta_{\bar{z}}\,. \tag{7.77}$$

To be consistent, the anti-holomorphicity constraint must be compatible with the time evolution. So that

$$\frac{\partial\zeta(\bar{z},t)}{\partial z} = 0 \Longrightarrow \forall t' \;\; \frac{\partial\zeta(\bar{z},t')}{\partial z} = 0 \tag{7.78}$$

$$\begin{aligned}
\frac{\partial\zeta(\bar{z},t+dt)}{\partial z} &= \frac{\partial}{\partial z}[\zeta(\bar{z},t+dt)] \\
&= \frac{\partial}{\partial z}[\zeta(\bar{z},t)+\frac{\partial\zeta}{\partial t}dt] \\
&= \frac{\partial\zeta(\bar{z},t)}{\partial z}+\frac{\partial}{\partial z}\frac{\partial\zeta}{\partial t}dt \\
&= \frac{\partial}{\partial z}\frac{\partial\zeta}{\partial t} \\
&= 0\,. \qquad (7.79)
\end{aligned}$$

Proposition 7.3.0.7 *For incompressible flow*

$$v_{1x}+v_{2y}=0 \Longrightarrow div\,\vec{v}=0 \qquad (7.80)$$

the anti-holomorphic constraint $\zeta_z=0$ *is compatible with the time evolution*

$$\frac{\partial}{\partial t}\zeta_z=0. \qquad (7.81)$$

Proof 7.3.0.8 *Differentiating (7.76) with respect to* z

$$\frac{\partial}{\partial z}\left(i\zeta_t+iv_+\zeta_{\bar{z}}+2\zeta_{\bar{z}\bar{z}}-4\frac{\zeta_{\bar{z}}^2}{1+|\zeta|^2}\bar{\zeta}\right)=0 \qquad (7.82)$$

we get

$$v_{+_z}=-4i\frac{\bar{\zeta}_z\zeta_{\bar{z}}}{(1+|\zeta|^2)^2} \qquad (7.83)$$

and complex conjugate of it

$$v_{-_{\bar{z}}}=4i\frac{\bar{\zeta}_z\zeta_{\bar{z}}}{(1+|\zeta|^2)^2}. \qquad (7.84)$$

Adding (7.83) to (7.84) implies incompressibility condition

$$v_{+_z} + v_{-_{\bar{z}}} = 0 \tag{7.85}$$

and subtracting implies

$$v_{+_z} - v_{-_{\bar{z}}} = -8i\frac{\bar{\zeta}_z \zeta_{\bar{z}}}{(1+|\zeta|^2)^2} \tag{7.86}$$

which coincides with the second equation (7.77)

Under the above constraint we have the reduced system

$$i\zeta_t + iv_+\zeta_{\bar{z}} + 2\zeta_{\bar{z}\bar{z}} - 4\frac{\zeta_{\bar{z}}^2}{1+|\zeta|^2}\bar{\zeta} = 0 \tag{7.87}$$

$$i\zeta_t + \zeta_{\bar{z}}\left[iv_+ + 2\left(\ln\frac{\zeta_{\bar{z}}}{(1+|\zeta|^2)^2}\right)_{\bar{z}}\right] = 0 \tag{7.88}$$

For function

$$F \equiv v_+ - 2i\left[\ln\frac{\zeta_{\bar{z}}}{(1+|\zeta|^2)^2}\right]_{\bar{z}} \tag{7.89}$$

Eq. (7.88) becomes

$$\zeta_t + F\zeta_{\bar{z}} = 0 \tag{7.90}$$

where $F_z = 0$, due to Eq. (7.83).

7.4. Ishimori Model Reduction

Now we consider the topological magnet model (7.10) and (7.11) with incompressibility condition (7.26), which allows simplification of the equations. (Martina et al., 2003) Equation $\vec{\nabla} \cdot \vec{v} = 0$ can be solved in terms of a real function ψ, the stream function of the flow,

$$v_1 = \partial_2 \psi, \;\; v_2 = -\partial_1 \psi \,. \tag{7.91}$$

If we replace v_1 and v_2 in equations (7.10) and (7.11) respectively, we get the so called Ishimori Model (Ishimori, 1984)

$$\vec{S}_t + \partial_2 \psi \partial_1 \vec{S} + \partial_1 \psi \partial_2 \vec{S} = \vec{S} \times (\partial_1^2 \vec{S} - \partial_2^2 \vec{S}) \tag{7.92}$$

$$(\partial_1^2 + \partial_2^2)\psi \;\; = \;\; -2\vec{S} \cdot (\partial_1 \vec{S} \times \partial_2 \vec{S}) \tag{7.93}$$

where we have used

$$\partial_1 v_2 - \partial_2 v_1 = -\Delta \psi \,. \tag{7.94}$$

The Ishimori model is the first example of integrable classical spin model in 2+1 dimensions (Konopelchenko, 1987). It was shown to be gauge equivalent to the Davey-Stewartson equation, representing the 2+1 dimensional generalization of the Nonlinear Schrodinger equation (Makhankov & Pashaev, 1992), (Lepovskiy & Shirokov, 1989) , (Pashaev, 1996).

In terms of complex variables

$$v_+ \;\; = \;\; v_1 + i v_2 = -2i\psi_{\bar{z}} \tag{7.95}$$

$$v_- \;\; = \;\; v_1 - i v_2 = 2i\psi_z \tag{7.96}$$

and the stereographic projection (7.55), Eqns. (7.92) and (7.93) is written in the form, respectively

$$i\zeta_t - 2\psi_z\zeta_z + 2\psi_{\bar{z}}\zeta_{\bar{z}} + 2(\zeta_{zz} + \zeta_{\bar{z}\bar{z}}) - 4\frac{\bar{\zeta}}{1+|\zeta|^2}(\zeta_z^2 + \zeta_{\bar{z}}^2) = 0 \tag{7.97}$$

$$\psi_{z\bar{z}} = -2\frac{\zeta_z\bar{\zeta}_z - \bar{\zeta}_z\zeta_{\bar{z}}}{(1+|\zeta|^2)^2} \tag{7.98}$$

7.4.1. Anti-holomorphic Reduction of Ishimori Model

The Ishimori model appears from the topological magnet model for the incompressible flow. But according to Proposition we have seen that such flow preserves anti(holomorphicity) constraint. This is why we consider now anti(holomorphicity) constrained Ishimori model. Under constraint $\zeta_z = 0$ we have dependence $\zeta = \zeta(\bar{z}, t)$ and the model reduces to

$$i\zeta_t + 2\psi_{\bar{z}}\zeta_{\bar{z}} + 2\zeta_{\bar{z}\bar{z}} - 4\frac{\bar{\zeta}}{1+|\zeta|^2}\zeta_{\bar{z}}^2 = 0 \tag{7.99}$$

$$\psi_{z\bar{z}} = 2\frac{\bar{\zeta}_z\zeta_{\bar{z}}}{(1+|\zeta|^2)^2}\,. \tag{7.100}$$

We can rearrange the first equation as follows

$$i\zeta_t + 2\zeta_{\bar{z}}\left[\psi_{\bar{z}} + \frac{\zeta_{\bar{z}\bar{z}}}{\zeta_{\bar{z}}} - 4\frac{\bar{\zeta}\zeta_{\bar{z}}}{1+|\zeta|^2}\right] = 0 \tag{7.101}$$

$$i\zeta_t + 2\zeta_{\bar{z}}\left(\psi + \ln\frac{\zeta_{\bar{z}}}{(1+|\zeta|^2)^2}\right)_{\bar{z}} = 0 \tag{7.102}$$

so that

$$i\zeta_t + 2\zeta_{\bar{z}}\left[\psi - 2\ln(1+|\zeta|^2) + \ln\zeta_{\bar{z}}\right]_{\bar{z}} = 0\,. \tag{7.103}$$

7.4.2. Static N-Soliton Configuration

For static configurations

$$\zeta_t = 0$$

we have solution

$$\psi = 2\ln(1+|\zeta|^2) - \ln\zeta_{\bar{z}} + f(z) \tag{7.104}$$

where $f(z)$ is arbitrary holomorphic function. Due to reality of ψ

$$\psi = \bar{\psi} \Rightarrow f(z) = \ln\bar{\zeta}_z \tag{7.105}$$

then

$$\psi = 2\ln(1+|\zeta|^2) - \ln\zeta_{\bar{z}} - \ln\bar{\zeta}_z \tag{7.106}$$

Differentiating (7.106) we find that Eqn. (7.100) is satisfied automatically

$$\psi_{z\bar{z}} = [2\ln(1+|\zeta|^2)]_{z\bar{z}} = 2\frac{\bar{\zeta}_z\zeta_{\bar{z}}}{(1+|\zeta|^2)^2} \tag{7.107}$$

Then from (7.100)

$$e^{\psi} = e^{2\ln(1+|\zeta|^2)}e^{-\ln\zeta_{\bar{z}}}e^{-\ln\bar{\zeta}_z} = \frac{(1+|\zeta|^2)^2}{\zeta_{\bar{z}}\bar{\zeta}_z} \tag{7.108}$$

$$e^{-\psi} = \frac{\zeta_{\bar{z}}\bar{\zeta}_z}{(1+|\zeta|^2)^2} \tag{7.109}$$

Using Eq. (7.100) we see that function ψ (7.108) is the general solution of the Liouville equation

$$\psi_{z\bar{z}} = 2e^{-\psi} . \tag{7.110}$$

It means that any solution of the Liouville equation is a static solution of the Ishimori Model (Martina et al., 1994c). Now we consider solution of model (7.110) in the form (7.108) where function

$$\begin{aligned} \zeta &= \sin(\bar{z} - \bar{z}_1) & (7.111) \\ \zeta_{\bar{z}} &= \cos(\bar{z} - \bar{z}_1) & (7.112) \\ \bar{\zeta}_z &= \cos(z - z_1) . & (7.113) \end{aligned}$$

Then the corresponding stream function is given by

$$\begin{aligned} \psi &= 2\ln(1 + |\sin(\bar{z} - \bar{z}_1)|^2) - \ln\cos(\bar{z} - \bar{z}_1) - \ln\cos(\bar{z} - \bar{z}_1) & (7.114) \\ &= 2\ln(1 + |\sin(\bar{z} - \bar{z}_1)|^2) - \ln|\cos(\bar{z} - \bar{z}_1)|^2 & (7.115) \\ &= \ln\frac{(1 + |\sin\bar{z}|^2)^2}{|\cos\bar{z}|^2} & (7.116) \\ &= \ln\frac{[1 + (\sin x\cosh y)^2 + (\cos x\sinh y)^2]^2}{(\cos x\cosh y)^2 + (\sin x\sinh y)^2} & (7.117) \end{aligned}$$

describes periodic in x lattice of solitons .

7.4.3. Single Soliton and Soliton Lattice

Now if in Eq. (7.108) for function ζ we choose

$$\begin{aligned} \zeta &= \bar{z}\sin\bar{z} & (7.118) \\ \zeta_{\bar{z}} &= \sin\bar{z} + \bar{z}\cos\bar{z} & (7.119) \end{aligned}$$

then we find the stream function descriptive of the single soliton and the soliton lattice

$$\begin{aligned}\psi &= 2\ln(1+|\bar{z}|^2|\sin\bar{z}|^2) - \ln(\sin\bar{z}+\bar{z}\cos\bar{z}) - \ln(\sin z + z\cos z) &(7.120)\\ &= \ln\frac{[1+|\bar{z}|^2|\sin\bar{z}|^2]^2}{|\sin\bar{z}+\bar{z}\cos\bar{z}|^2} &(7.121)\\ &= \ln\frac{1+(x^2+y^2)[(\sin x\cosh y)^2+(\cos x\sinh y)^2]^2}{|\sin z|^2+|z|^2|\cos z|^2+z\cos z\sin\bar{z}+\bar{z}\cos\bar{z}\sin z}. &(7.122)\end{aligned}$$

7.4.4. Holomorphic Time Dependent Schrödinger Equation

If we choose

$$\psi = 2\ln(1+|\zeta|^2) \tag{7.123}$$

then

$$\begin{aligned}\psi_{z\bar{z}} &= 2[\frac{\bar{\zeta}_z\zeta}{1+|\zeta|^2}]_{\bar{z}} &(7.124)\\ &= 2\frac{\bar{\zeta}_z\zeta_{\bar{z}}}{(1+|\zeta|^2)^2} &(7.125)\end{aligned}$$

and Eq.(7.100) is satisfied automatically. Then from equation (7.103) for function ζ we have complex time dependent Schrödinger equation

$$i\zeta_t + 2\zeta_{\bar{z}\bar{z}} = 0 \tag{7.126}$$

Each zero of function ζ in complex plane z determines magnetic soliton of the Ishimori model. The spin vector at center of the soliton is $\vec{S} = (0,0,1)$ while at infinity $\vec{S} = (0,0,-1)$. Then a motion of zeroes of equation (7.126) determines the motion of magnetic solitons in the plane. From another side, if we consider analytic function

$$f(z,t) = \frac{\Gamma}{2\pi i} Log\,\zeta(z,t) \tag{7.127}$$

as a complex potential of an effective flow (Lavrantiev & Shabat, 1973), then every zero of function ζ corresponds to hydrodynamical vortex of the flow with intensity Γ, and to the simple pole singularity of complex velocity

$$u(\bar{z},t) = \bar{f}_{\bar{z}} = \frac{i\Gamma}{2\pi}(Log\,\bar{\zeta})_{\bar{z}}. \qquad (7.128)$$

But the last relation has meaning of the holomorphic Cole-Hopf transformation, according to which the complex velocity is subject to the holomorphic Burgers' equation (Gurkan & Pashaev, 2008)

$$iu_t + \frac{8\pi i}{\Gamma}uu_{\bar{z}} = 2u_{\bar{z}\bar{z}}. \qquad (7.129)$$

Thus, every magnetic soliton of the Ishimori model corresponds to hydrodynamical vortex of the anti-holomorphic Burgers' equation. Moreover, relation (7.127) is written in the form

$$\zeta = e^{\frac{2\pi i}{\Gamma}f} = e^{\frac{2\pi i}{\Gamma}(\phi + i\chi)} = \sqrt{\rho}\,e^{\frac{2\pi i}{\Gamma}\phi} \qquad (7.130)$$

shows that the effective flow is just the Madelung representation for the linear holomorphic Schrödinger equation (7.126), where functions ϕ and χ are the velocity potential and the stream function correspondingly. Motion of zeroes of Complex Burgers equation

$$iu_t + uu_{\bar{z}} = \nu u_{\bar{z}\bar{z}} \qquad (7.131)$$

and relations with solitons of the complex Burgers equation can be interpreted now in terms of the magnetic solitons. Particularly, to find generating function of the basic soliton solutions of this equation we consider solution in the form

$$\zeta(\bar{z},t) = e^{k\bar{z}+\omega t} \qquad (7.132)$$

where dispersion $\omega = 2ik^2$. Then

$$\zeta(\bar{z}, t) = e^{k\bar{z}+2ik^2 t} \tag{7.133}$$

Let $x \equiv k\sqrt{\frac{2t}{i}}$, then we rewrite it as the generating function for the Hermite polynomials of complex argument

$$\begin{aligned} e^{k\bar{z}+2ik^2 t} &= e^{-x^2+2(\bar{z}\sqrt{\frac{i}{8t}})x} \\ &= \sum_{n=0}^{\infty} H_n(\bar{z}\sqrt{\frac{i}{8t}})\frac{x^n}{n!} \end{aligned} \tag{7.134}$$

or

$$\begin{aligned} \zeta(\bar{z}, t) &= \sum_{n=0}^{\infty} \frac{k^n}{n!}(-2it)^{n/2} H_n(\bar{z}\sqrt{\frac{i}{8t}}) \\ &= \sum_{n=0}^{\infty} \frac{k^n}{n!}\Psi_n(\bar{z}, t) \end{aligned} \tag{7.135}$$

where at every power k^n we have a polynomial solution of order n:

$$\Psi_n(\bar{z}, t) = \left(\frac{2t}{i}\right)^{n/2} H_n(\bar{z}\sqrt{\frac{i}{8t}}) \, . \tag{7.136}$$

This polynomial has n complex roots $\bar{z}_1(t), ..., \bar{z}_n(t)$ describing positions of solitons.

7.5. N Spin Soliton System

For N soliton system in general, we can choose

$$\zeta(\bar{z}, t) = \prod_{j=1}^{N} (\bar{z} - \bar{z}_j(t)) \, . \tag{7.137}$$

Then positions of solitons are subject to the system

$$\frac{d}{dt}\bar{z}_j = \frac{4}{i}\sum_{k\neq(j)}\frac{1}{(\bar{z}_j - \bar{z}_k)}\,. \tag{7.138}$$

This system admits $2N$ integrals of motion. The first 5 integrals are of the form

$$\sum_{j=1}^{N}\bar{z}_j = I_1 - iI_2 \tag{7.139}$$

$$\sum_{j=1}^{N} z_j^2 + \bar{z}_j^2 = I_3 \tag{7.140}$$

$$\sum_{j=1}^{N}\bar{z}_j^3 + 3\sum_{j<k<l}\bar{z}_j\bar{z}_k\bar{z}_l = I_4 - iI_5 \tag{7.141}$$

This is why the dynamics of solitons in Ishimori model is integrable. The system (7.138) admits mapping to the complexified Calogero-Moser N particle problem. We differentiate it once and use the system again (Appendix D) to have Newton's equations

$$\frac{d^2}{dt^2}\bar{z}_j = \sum_{k}\frac{16}{(\bar{z}_j - \bar{z}_k)^3}\,. \tag{7.142}$$

These equations have the Hamiltonian form

$$\dot{\bar{z}}_j = \frac{\partial H}{\partial p_j} = p_j,\ \ \dot{p} = -\frac{\partial H}{\partial \bar{z}\ _j} \tag{7.143}$$

with the Hamiltonian function

$$H = \frac{1}{2}\sum_{j=1}^{n}p_j^2 + \sum_{j<k}\frac{8}{(\bar{z}_j - \bar{z}_k)^2}\,. \tag{7.144}$$

The Calogero-Moser model is finite-dimensional integrable system admitting the Lax representation, from which follows the hierarchy of constants of motion in involution.

Complexification of the classical Calogero-Moser model and holomorphic Hopf equation has been considered in connection with limit of an infinite number of particles, leading to quantum hydrodynamics and quantum Benjamin-Ono equation (Abanov & Wiegmann, 2005). From another side holomorphic version of the Burgers equation is considered in (Bonami et al., 1999) to prove existence and uniqueness of the non-linear diffusion process for the system of Brownian particles with electrostatic repulsion when the number of particles increases to infinity.

7.6. Dynamics of Topological Solitons in the Plane

In this section we study dynamics of N solitons and vortex lattices in the plane for magnetic systems under restriction of constant $\nu = -2$ (Gurkan & Pashaev, 2008). By stereographic projection formulas

$$S_1 + iS_2 = \frac{2\zeta}{1+|\zeta|^2} \qquad S_3 = \frac{1-|\zeta|^2}{1+|\zeta|^2} \tag{7.145}$$

we can see that at every zero of function $\zeta(\bar{z}_k, t) = 0$

$$(S_1 + iS_2)(\bar{z}_k, t) = 0, \;\; S_3(\bar{z}_k, t) = 1\,. \tag{7.146}$$

From another side for N degree polynomial ζ_N at infinity $|z| \to \infty$

$$(S_1 + iS_2)(\bar{z}_k, t) = 0, \;\; S_3(\bar{z}_k, t) = -1\,. \tag{7.147}$$

It shows that our zeroes correspond to the magnetic solitons located at that zeroes with the spin vector $\vec{S}$ directed up, while at infinity it is directed down (ferromagnetic type order).

If we calculate the topological charge

$$
\begin{aligned}
Q &= \frac{1}{4\pi}\int\int \vec{S}\cdot(\partial_1\vec{S}\times\partial_2\vec{S})d^2x &(7.148)\\
&= -\frac{1}{8\pi}\int\int(\Delta\psi)d^2x &(7.149)
\end{aligned}
$$

where

$$\psi = 2\ln(1+|\zeta|^2) \qquad (7.150)$$

$$Q = -\frac{1}{4\pi}\int\int(\Delta\ln(1+|\zeta|^2))d^2x\,. \qquad (7.151)$$

By Green's theorem then integral transforms

$$\int\int\left[\frac{\partial}{\partial x}\left(\underbrace{\frac{\partial}{\partial x}\ln(1+|\zeta|^2)}_{A}\right)+\frac{\partial}{\partial y}\left(\underbrace{\frac{\partial}{\partial y}\ln(1+|\zeta|^2)}_{B}\right)\right] = \oint Bdx + Ady \quad (7.152)$$

$$
\begin{aligned}
&= \oint\left[-\frac{\partial}{\partial y}\ln(1+|\zeta|^2)\right]dx+\left[\frac{\partial}{\partial x}\ln(1+|\zeta|^2)\right]dy &(7.153)\\
&= \oint -\frac{(|\zeta|^2)_y}{1+|\zeta|^2}dx+\frac{(|\zeta|^2)_x}{1+|\zeta|^2}dy &(7.154)\\
&= \oint_{R\to\infty}\frac{-(|\zeta|^2)_y dx+(|\zeta|^2)_x dy}{1+|\zeta|^2}\,. &(7.155)
\end{aligned}
$$

For N zeroes solution (7.137) asymptotically $|z|\to\infty$, $z = Re^{i\theta}$, $R\to\infty$, $|\zeta|^2\to\infty$, $\zeta\simeq\bar{z}^N$, $|\zeta|^2 = |z|^{2N}$

$$
\begin{aligned}
|\zeta|^2_x &= [(x^2+y^2)^N]_x = N(x^2+y^2)^{N-1}2x &(7.156)\\
|\zeta|^2_y &= N(x^2+y^2)^{N-1}2y &(7.157)
\end{aligned}
$$

and (7.155) is equal

$$\begin{aligned}\oint_{R\to\infty} \frac{-N(x^2+y^2)^{N-1}2ydx + N(x^2+y^2)^{N-1}2xdy}{(x^2+y^2)^N} &= 2N \oint \frac{-ydx+xdy}{x^2+y^2} \\ &= 2N\cdot 2\pi \\ &= 4\pi N\,. \end{aligned} \tag{7.158}$$

Then we find that topological charge is integer valued and equal to the number of solitons

$$Q = -N \tag{7.159}$$

In Fig. 7.1 and Fig. 7.2 we reproduce S_3 component for $N = 1$ and $N = 2$ solitons.

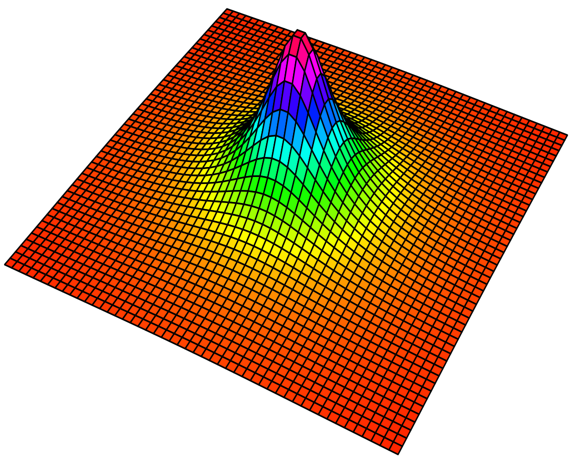


Figure 7.1. $N = 1$ Static Magnetic soliton

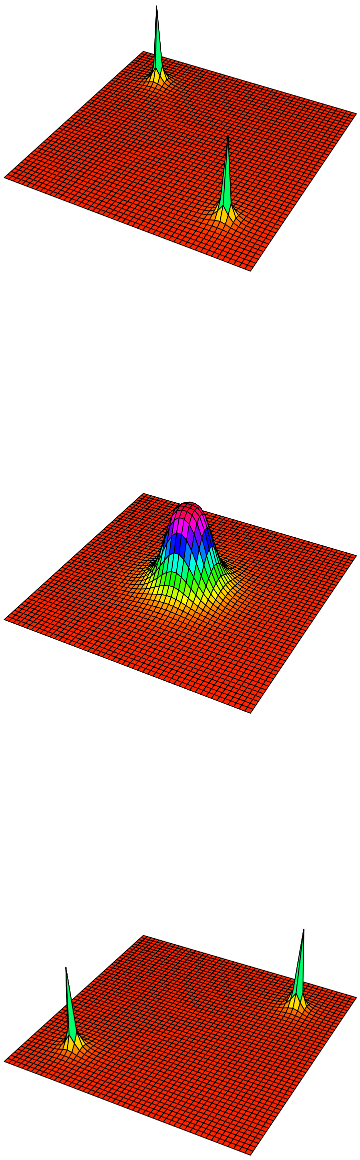


Figure 7.2. $N = 2$ Magnetic soliton Dynamics

If we consider solution

$$\zeta(\bar{z}, t) = \prod_{k=1}^{N} \sin(\bar{z} - \bar{z}_k(t)) \tag{7.160}$$

then it describes N magnetic soliton chain lattices periodic in the x direction. In Fig. 7.3 we reproduce S_3 component of these lattices for $N = 2$.

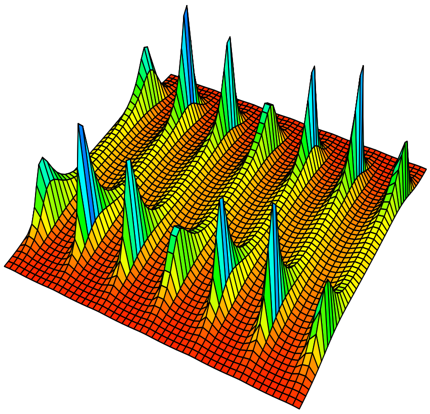


Figure 7.3. Two Magnetic soliton Lattice Dynamics

7.7. Time Dependent Schrödinger Problem in Harmonic Potential

The vorticity equation (7.100) is invariant under substitution $\psi \to \psi + U$ where U is an arbitrary harmonic function: $\Delta U = 0$. If we choose

$$\psi = 2\ln(1 + |\zeta|^2) + U(\bar{z}, t) + \bar{U}(\bar{z}, t) \tag{7.161}$$

then substituting to Eq.(7.103) we have complex Schrödinger equation with additional potential term

$$i\zeta_t + \zeta_{\bar{z}\bar{z}} + \zeta_{\bar{z}} U_{\bar{z}} = 0 \tag{7.162}$$

7.8. Bound State of Solitons

Here we choose particular form (Gurkan & Pashaev, 2008)

$$U(\bar{z}, t) = \frac{1}{2}\bar{z}^2 \tag{7.163}$$

so that

$$\psi = 2\ln(1+|\zeta|^2) + \frac{1}{2}(\bar{z}^2 + z^2)\,. \tag{7.164}$$

Then we have the time evolution subject to the equation

$$i\zeta_t + 2\zeta_{\bar{z}\bar{z}} + \bar{z}\zeta_{\bar{z}} = 0\,. \tag{7.165}$$

Looking for solution in the form

$$\zeta = \sum_n e^{int} u_n(\bar{z}) \tag{7.166}$$

we find that functions $u_n(\bar{z})$ satisfy the complex Hermite equation

$$2u''_n + \bar{z}u'_n - nu_n = 0\,. \tag{7.167}$$

It gives time dependent soliton solution in the form

$$\zeta = \sum_n e^{int} H_n(\bar{z})\,. \tag{7.168}$$

For particular value $N = 2$ we have solution

$$\zeta = H_0(\bar{z}) + e^{it}H_1(\bar{z}) + e^{2it}H_2(\bar{z}) \tag{7.169}$$

or

$$\zeta = \Re\zeta + i\Im\zeta \tag{7.170}$$

where

$$\Re\zeta = 1 + 2x\cos t + 2y\sin t + [4(x^2-y^2)-2]\cos 2t + 8xy\sin 2t \tag{7.171}$$

$$\Im\zeta = -2y\cos t + 2x\sin t - 8xy\cos 2t + [4(x^2-y^2)-2]\sin 2t\,. \tag{7.172}$$

This solution is periodic in time with period $T = 2\pi$ and it describes the bound state of two magnetic solitons. In Fig. 7.4 we demonstrate oscillation of solitons in this bound state for function

$$f = \frac{1}{1 + (\Re\zeta)^2 + (\Im\zeta)^2} \tag{7.173}$$

which characterizes projection of spin vector S_3.

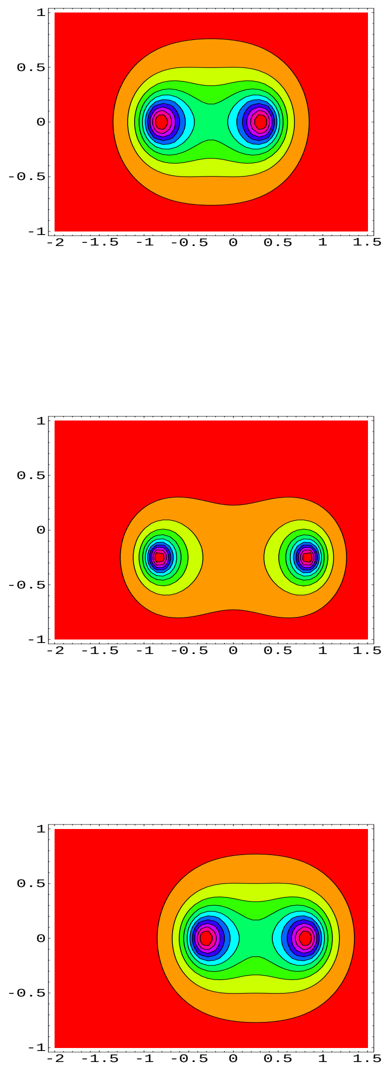


Figure 7.4. Bound State of Two Magnetic solitons

Finally we note that the holomorphic Hopf equation

$$iu_t + uu_z = 0 \tag{7.174}$$

which corresponds to the dispersionless limit of the holomorphic Burgers' equation

$$iu_t + uu_{\bar{z}} + 2u_{\bar{z}\bar{z}} = 0, \tag{7.175}$$

has been considered very recently as nonlinear bosonisation in quantum hydrodynamics for description of quantum shock waves in edge states of Fractional Quantum Hall Effect (Abanov & Wiegmann, 2005). The weak solution of this equation for point solitons with strength $\Gamma_1, ..., \Gamma_N$, so that

$$rot\ u = \sum_{k=1}^{N} \Gamma_k \delta(x - x_k(t))\delta(y - y_k(t)) \tag{7.176}$$

gives the following soliton system

$$\frac{dz_k}{dt} = 4i \sum_{l=1,(l \neq k)}^{N} \frac{\Gamma_l}{z_k - z_l}, \quad k = 1, ..., N. \tag{7.177}$$

When all the soliton strengths are equal $\Gamma_1 = ... = \Gamma_N$ then this system reduces to (8.12) when $\nu = -2$ and is integrable. However, in the general case the system is not known to be integrable. In particular, for $N = 3$ the system with constraint $\Gamma_1 = \Gamma_2 \neq \Gamma_3$ has been studied in (Calogero et al., 2005) to explain the transition from regular to irregular motion as travel on the Riemann surface.

CHAPTER 8

INTEGRABLE SOLITON DYNAMICS AND MULTI-PARTICLE PROBLEM

In the present chapter we study the mapping of the point soliton equations to the integrable multiparticle problem - the complexified Calogero-Moser problem (Gurkan & Pashaev, 2008).

8.1. Calogero-Moser Models

One dimensional problem of N-interacting particles admits the Lax representation and is integrable (Calogero et al., 1978) if in the Hamiltonian function

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + g^2 \sum_{j<k} v(q_j - q_k) \tag{8.1}$$

the pair interaction potential $v(q_j - q_k)$ has the one of the next forms

$$v(\xi) = \begin{cases} \xi^{-2}, & \text{I;} \\ a^2 \sinh^{-2}(a\xi), & \text{II;} \\ a^2 \sin^{-2}(a\xi), & \text{III;} \\ a^2 \mathcal{P}(a\xi), & \text{IV.} \end{cases} \tag{8.2}$$

where a is an arbitrary parameter, and $\mathcal{P}(\xi) = \mathcal{P}(\xi, \omega_1, \omega_2)$ is the Weierstrass function, which is a double periodic function of the complex variable ξ with periods $2\omega_1$ and $2\omega_2$ and with second order poles at the points $2(m\omega_1 + m\omega_2)$ (Perelomov, 1990). In the limit as one of the periods goes to infinity , the potential of type IV goes over into the potentials of type II or III. The potential of type I results by letting both periods go to infinity. Therefore the system of type IV is the most general one. Nevertheless the systems of type I, II and III have certain specific features that make it reasonable to treat them separately.

The Hamiltonian equations for the above potentials

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \dot{q}_j = p_j, \; j = 1, ..., N \tag{8.3}$$

are equivalent to the Lax matrix equation (Perelomov, 1990)

$$i\dot{L} = AL - LA \tag{8.4}$$

Explicit form of the Lax operators for the Case I is

$$L_{jk} = \delta_{jk} p_j + ig(1 - \delta_{jk}) \frac{1}{q_j - q_k}, \tag{8.5}$$

$$A_{jk} = g \left[\delta_{jk} \sum_{l \neq j} \frac{1}{(q_j - q_l)^2} - (1 - \delta_{jk}) \frac{1}{(q_j - q_k)^2} \right]. \tag{8.6}$$

The Lax equation (8.4) is the isospectrality condition ($\lambda_t = 0$) for the next linear problem (see Appendix C)

$$LU = \lambda U \tag{8.7}$$

$$iU_t = AU\,. \tag{8.8}$$

From this Lax representation follows that under time evolution $L(t)$ undergoes a similarity transformation

$$L(t) = U(t) L(0) U^{-1}(t)\,. \tag{8.9}$$

Due to this similarity transformation the eigenvalues of $L(t)$ are time independent and so are integrals of motion. Equivalently we can say that matrix $L(t)$ is isospectrally deformed with time. Instead of the eigenvalues it is often more convenient to take their symmetric functions as integrals of motion, for example ,

$$I_k = trL^{k+1}\,. \tag{8.10}$$

If in such a way one can find N functionally independent integrals of motion and show that they are in involution, then the system is completely integrable in the Liouville sense. It is the case for the Calogero -Moser model (8.1) of all four types I, II, II, IV. In the first part of the thesis, Chapter 5, we have considered exchange interactions in Heisenberg model depending on distance between qubits in the form (8.2), and calculated corresponding concurrence. Now we show that evolution of topological solitons is also subject to these models.

8.2. Integrable Problem for N-soliton Motion

In this section we show that the problem of N-point solitons in the plane can be reduced to the complexified version of the Calogero-Moser model (8.1) type I. The system of N point solitons is described by function

$$\Phi(\bar{z}, t) = \prod_{j=1}^{N} (\bar{z} - \bar{z}_j(t)) \tag{8.11}$$

satisfying the complex Schrodinger equation (**??**). Then positions of solitons in the complex plane, $\bar{z}_1, ..., \bar{z}_N$, are subject to the first order system

$$\frac{d}{dt}\bar{z}_j = 2\nu i \sum_{k\neq(j)}^{N} \frac{1}{(\bar{z}_j - \bar{z}_k)}\,. \tag{8.12}$$

If we differentiate once and use the system again (Appendix C), then we have the second order Newton's equations of motion

$$\frac{d^2}{dt^2}\bar{z}_j = \sum_{k\neq(j)}^{N} \frac{4\nu^2}{(\bar{z}_j - \bar{z}_k)^3}\,. \tag{8.13}$$

These equations have (complex) Hamiltonian form

$$\dot{\bar{z}}_j = \frac{\partial H}{\partial p_j} = p_j, \;\; \dot{p} = -\frac{\partial H}{\partial \bar{z}_j} \tag{8.14}$$

with the Hamiltonian function

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + 2\nu^2 \sum_{j<k} \frac{1}{(\bar{z}_j - \bar{z}_k)^2}\,. \tag{8.15}$$

The system (8.13) implies the complex conjugate one

$$\frac{d^2}{dt^2} z_j = 4\nu^2 \sum_{k} \frac{1}{(z_j - z_k)^3} \tag{8.16}$$

with Hamiltonian

$$\bar{H} = \frac{1}{2}\sum_{j=1}^{N} \bar{p}_j^2 + 2\nu^2 \sum_{j<k} \frac{1}{(z_j - z_k)^2}\,. \tag{8.17}$$

Then the real Hamiltonian for these systems is given by $H + \bar{H}$.

As easy to see, the system (8.13) is complexified version of the Calogero-Moser system discussed in the previous Section 8.1 with the Hamiltonian function (8.1) type I, where N-particle positions, $q_1, ..., q_N$ should be replaced by complex soliton positions $\bar{z}_1, ..., \bar{z}_N$, as in Eq.(8.15).

The Lax representation from Section 8.1 can be transformed to the complex case in a straightforward way. The complexified Hamiltonian equations (8.14) are equivalent to the Lax matrix equation

$$i\dot{L} = AL - LA \tag{8.18}$$

where

$$L_{jk} = \delta_{jk} p_j + ig(1-\delta_{jk}) \frac{1}{\bar{z}_j - \bar{z}_k} \tag{8.19}$$

$$A_{jk} = g\left[\delta_{jk} \sum_{l \neq j} \frac{1}{(\bar{z}_j - \bar{z}_l)^2} - (1-\delta_{jk}) \frac{1}{(\bar{z}_j - \bar{z}_k)^2}\right] \tag{8.20}$$

and the coupling constant $g = \sqrt{2}\nu$. Since matrix $L(t)$ is isospectrally deformed with time, the corresponding (complex) eigenvalues are time independent integrals of motion. If one takes their symmetric functions as integrals of motion, then they are given by

$$I_k = trL^{k+1} \tag{8.21}$$

It shows that complexified Calogero-Moser system is an integrable system and as a consequence, the N-soliton system (8.12), which has been mapped to Calogero-Moser system, is also integrable.

8.3. Integrable Problem for N-soliton Lattices

Similar to the previous case now we consider mapping of the N-soliton chain lattices to the complexified Calogero-Moser system of type II and III . For simplicity first we consider the system of two soliton chain lattices described by function

$$\Phi(\bar{z}, t) = \sin(\bar{z} - \bar{z}_1(t)) \sin(\bar{z} - \bar{z}_2(t)) \tag{8.22}$$

so that position of lattices is subject to the first order system

$$\dot{\bar{z}}_1 = 2\nu i \cot(\bar{z}_1 - \bar{z}_2) \tag{8.23}$$

$$\dot{\bar{z}}_2 = -2\nu i \cot(\bar{z}_1 - \bar{z}_2) \tag{8.24}$$

Differentiating this system once in time we get the second order equations of motion in the Newton's form

$$\ddot{\bar{z}}_1 = 2i\nu \left(-\frac{1}{\sin^2(\bar{z}_1 - \bar{z}_2)} \right) (\dot{\bar{z}}_1 - \dot{\bar{z}}_2) \tag{8.25}$$

$$= -8\nu^2 \frac{\cot(\bar{z}_1 - \bar{z}_2)}{\sin^2(\bar{z}_1 - \bar{z}_2)} \tag{8.26}$$

$$\ddot{\bar{z}}_2 = 2i\nu \left(\frac{1}{\sin^2(\bar{z}_1 - \bar{z}_2)} \right) (\dot{\bar{z}}_1 - \dot{\bar{z}}_2) \tag{8.27}$$

$$= 8\nu^2 \frac{\cot(\bar{z}_1 - \bar{z}_2)}{\sin^3(\bar{z}_1 - \bar{z}_2)} \,. \tag{8.28}$$

These equations are Hamiltonian

$$\dot{\bar{z}}_1 = \frac{\partial H}{\partial p_1} = p_1 \tag{8.29}$$

$$\dot{p}_1 = -\frac{\partial H}{\partial \bar{z}_1} = 8\nu^2 \frac{\cot(\bar{z}_1 - \bar{z}_2)}{\sin^3(\bar{z}_1 - \bar{z}_2)} \tag{8.30}$$

$$\dot{\bar{z}}_2 = \frac{\partial H}{\partial p_2} = p_2 \tag{8.31}$$

$$\dot{p}_2 = -\frac{\partial H}{\partial \bar{z}_2} = 8\nu^2 \frac{\cot(\bar{z}_2 - \bar{z}_1)}{\sin^3(\bar{z}_2 - \bar{z}_1)} \tag{8.32}$$

with the Hamiltonian function

$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{4\nu^2}{\sin^2(\bar{z}_1 - \bar{z}_2)} \,. \tag{8.33}$$

Comparing this Hamiltonian of two soliton lattices with the Calogero-Moser system, we realize that it corresponds to complexified version of the model type III.

We can generalize this result considering N soliton chain lattices periodic in the horizontal direction x. Positions of lattices are subject to the first order system

$$\dot{\bar{z}}_j = 2\nu i \sum_{k \neq j}^{N} \cot(\bar{z}_j - \bar{z}_k)\,. \tag{8.34}$$

Differentiating once we get

$$\ddot{\bar{z}}_j = -8\nu^2 \sum_{k \neq j}^{N} \frac{\cot(\bar{z}_j - \bar{z}_k)}{\sin^2(\bar{z}_j - \bar{z}_k)} \tag{8.35}$$

which is complexified Calogero-Moser system type III with Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^{N} p_j^2 + \sum_{j<k} \frac{4\nu^2}{\sin^2(\bar{z}_j - \bar{z}_k)}\,. \tag{8.36}$$

If instead of horizontal x direction, we consider N chain lattices periodic in the vertical y direction, it results in rotation of every zero of Φ (8.22) on angle $\pi/2$, which means replacement of complex function $\sin \bar{z}$ by $\sinh \bar{z}$. As a result, the corresponding Calagero-Moser system would be of type II. This consideration shows equivalence of complexified Calogero-Moser systems of type II and III.

CHAPTER 9

ABELIAN CHERN-SIMONS SOLITONS AND HOLOMORPHIC BURGERS' HIERARCHY

9.1. The Complex Galilei Group and Soliton Generation

The results shown in this chapter appeared partially in (Pashaev & Gurkan, 2007) and presented in (Pashaev & Gurkan, 2006). The complex Galilei group is generated by algebra

$$[P_0, P_z] = 0, \quad [P_0, K] = 4iP_z, \quad [P_z, K] = -i \tag{9.1}$$

where the respective energy and momentum operators are $P_0 = -i\partial_t$, $P_z = -i\partial_z$ correspondingly, and the Galilean boost is operator

$$K = z + 4it\partial_z\,. \tag{9.2}$$

The Schrödinger operator from (7.126)

$$S = i\partial_t + 2\partial_z^2 \tag{9.3}$$

corresponds to the dispersion relation $P_0 = -2P_z^2$ (comparing with previous sections for simplicity we replaced $\bar{z}$ by z) and is commuting with Galilei group operator

$$[P_0, S] = 0, \quad [P_z, S] = 0, \quad [K, S] = 0\,. \tag{9.4}$$

It is known from the theory of dynamical symmetry, that if there exists operator W such that

$$[S, W] = 0 \Rightarrow S(W\Phi) = W(S\Phi) = 0 \tag{9.5}$$

then it transforms solution Φ of the Schrödinger equation into another solution $W\Phi$. This shows that Galilei generators provide dynamical symmetries for the equation. Two of them are obvious, time translation P_0 :

$$e^{it_0 P_0}\Phi(z, t) = \Phi(z, t + t_0) \tag{9.6}$$

and the complex space translation P_z :

$$e^{it_0 P_z}\Phi(z, t) = \Phi(z + z_0, t)\,. \tag{9.7}$$

The Galilean boost creates new zero (new soliton in $\mathbb{C}$)

$$\Psi(z, t) = K\Phi(z, t) = (z + 4it\partial_z)\Phi(z, t)\,. \tag{9.8}$$

Starting from obvious solution $\Phi = 1$ we have the chain of n-soliton solutions,

$$K \cdot 1 = z = H_1(z, 2it), \tag{9.9}$$
$$K^2 \cdot 1 = z^2 + 4it = H_2(z, 2it), \tag{9.10}$$
$$K^3 \cdot 1 = z^3 + 12it = H_3(z, 2it), \tag{9.11}$$
$$\vdots$$
$$K^n \cdot 1 = H_n(z, 2it)\,, \tag{9.12}$$

in terms of the Kampe de Feriet Polynomials (Dattoli 1997)

$$H_n(z, it) = n! \sum_{k=0}^{[n/2]} \frac{(it)^k z^{n-2k}}{k!(n-2k)!} \,. \tag{9.13}$$

They satisfy the recursion relations

$$\begin{aligned} H_{n+1}(z, it) &= \left(z + 2it\frac{\partial}{\partial z}\right) H_n(z, it), & (9.14) \\ \frac{\partial}{\partial z} H_n(z, it) &= nH_{n-1}(z, it) & (9.15) \end{aligned}$$

and can be written in terms of the Hermite polynomials

$$H_n(z, 2it) = (-2it)^{n/2} H_n\left(\frac{z}{2\sqrt{-2it}}\right) \,. \tag{9.16}$$

Let $w_n^{(k)}$ is the *k-th* zero of the Hermite polynomial, $H_n(w_n^{(k)}) = 0$. Then the evolution of the corresponding soliton is given by

$$z_k(t) = 2w_n^{(k)}\sqrt{-2it} \,. \tag{9.17}$$

Under the time reflection $t \to -t$ position of the soliton rotates on 90 degrees $z_k \to z_k e^{i\pi/2}$. This transformation is also a symmetry of the soliton equations (7.138). Using formula

$$H_n(z, 2it) = \exp\left(it\frac{\partial^2}{\partial z^2}\right) z^n \tag{9.18}$$

and the superposition principle, we obtain the solution

$$\Phi(z, t) = \sum_{n=0}^{\infty} a_n H_n(z, 2it) = \sum_{n=0}^{\infty} a_n \exp\left(2it\frac{\partial^2}{\partial z^2}\right) z^n = \exp\left(2it\frac{\partial^2}{\partial z^2}\right) \sum_{n=0}^{\infty} a_n z^n \tag{9.19}$$

So if

$$\chi(z) = \sum_{n=0}^{\infty} a_n z^n \tag{9.20}$$

is an artbitrary analytic function, then

$$\Phi(z,t) = \exp\left(2it\frac{\partial^2}{\partial z^2}\right)\chi(z) \tag{9.21}$$

is a solution determined by the integrals of motion $a_0, a_1, ...$ Therefore, for a polynomial degree n describing evolution of n solitons, we have n complex integrals of motion.

The generating function of the Kampe de Feriet Polynomials

$$\sum_{n=0}^{\infty} \frac{k^n}{n!} H_n(z,it) = e^{kz+ik^2t} \tag{9.22}$$

is also solution of the plane wave type. If we exponentiate the Galilean boost

$$e^{i\lambda K} = e^{i\lambda(z+4it\partial_z)} \tag{9.23}$$

factor it by the Baker-Hausdorf formula

$$e^{A+B} = e^B e^A e^{\frac{1}{2}[A,B]}, \tag{9.24}$$

such that

$$e^{i\lambda K} = e^{i\lambda z+2i\lambda^2 t} e^{-4\lambda t\partial_z}, \tag{9.25}$$

apply it to a solution $\Phi(z,t)$, we obtain

$$e^{i\lambda K}\Phi(z,t) = e^{i\lambda z+2i\lambda^2 t}\Phi(z-4\lambda t, t) \tag{9.26}$$

the Galilean boost with velocity 4λ, where the generating function of solitons (9.22) appears as the 1-cocycle. The Galilean boost (9.8) connecting two solutions of the holomorphic Schrödinger equation (7.126) generates the auto-Bäcklund transformation :

$$v = u + \frac{i\Gamma}{2\pi}\partial_{\bar{z}} \ln(\bar{z} - \frac{8\pi t}{\Gamma}u) \tag{9.27}$$

between two solutions

$$u(\bar{z}, t) = \frac{i\Gamma}{2\pi}\frac{\bar{\Phi}_{\bar{z}}}{\bar{\Phi}}, \quad v(\bar{z}, t) = \frac{i\Gamma}{2\pi}\frac{\bar{\Psi}_{\bar{z}}}{\bar{\Psi}} \tag{9.28}$$

of anti-holomorphic Burgers' equation (7.129).

As an example, we consider double lattice solution

$$\zeta(z, t) = e^{-8it} \sin(z - z_1(t)) \sin(z + z_1(t)) \tag{9.29}$$

where $\cos 2z_1 = re^{8it}$ and r is a constant. Applying boost transformation (9.8) we obtain a solution describing collision of a soliton with the double lattice

$$\Psi(z, t) = \left(z + 4it\frac{\partial}{\partial z}\right) \zeta(z, t) \tag{9.30}$$

Generalizing we have *N*-solitons interacting with *M*-soliton lattices,

$$\Psi(z, t) = e^{iMt} \left(z + 4it\frac{\partial}{\partial z}\right)^N \prod_{k=1}^{M} \sin(z - z_k(t)) \tag{9.31}$$

where $z_1, ..., z_k$ are subject to the system

$$\dot{z_k} = 2i \sum_{l=1(\neq k)} \cot(z_k - z_l). \tag{9.32}$$

9.2. Abelian Chern-Simons Theory and Complex Burgers' Hierarchy

Now we show how the anti-holomorphic Burgers hierarchy appears in the Chern-Simons gauge field theory. The Chern-Simons functional is defined as follows

$$S(A) = \frac{\kappa}{4\pi}\int_M A \wedge dA = \frac{\kappa}{4\pi}\int_M \varepsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} \tag{9.33}$$

where M is an oriented three-dimensional manifold, A is a U(1) gauge connection, κ is the coupling constant - the statistical parameter. In the canonical approach $M = \Sigma_2 \times R$, where R we interpret as a time. Then $A_\mu = (A_0, A_i)$, $(i = 1, 2)$, where A_0 is the time component and the action takes the form

$$S = -\frac{\kappa}{4\pi}\int dt \int_\Sigma \epsilon^{ij}\left(A_i \frac{d}{dt} A_j - A_0 F_{ij}\right) \tag{9.34}$$

In the first order formalism, this implies that the Poisson bracket is

$$\{A_i(x), A_j(y)\} = \frac{4\pi}{\kappa}\epsilon_{ij}\delta(x-y) \tag{9.35}$$

and the Hamiltonian is

$$H = A_0 \epsilon^{ij} F_{ij}\,. \tag{9.36}$$

The Hamiltonian is weakly vanishing ($H \approx 0$) because of the Chern-Simons Gauss law constraint

$$\partial_1 A_2 - \partial_2 A_1 = 0 \Leftrightarrow F_{ij} = 0\,. \tag{9.37}$$

Then the evolution is determined by the Lagrange multipliers A_0 :

$$\partial_0 A_1 = \partial_1 A_0,\;\; \partial_0 A_2 = \partial_2 A_0. \tag{9.38}$$

Because of the gauge invariance

$$A_\mu \to A_\mu + \partial_\mu \lambda, \tag{9.39}$$

to fix the gauge freedom we choose the Coulomb gauge condition: $div\vec{A} = 0$. In addition, we have Chern-Simons Gauss law (9.37):

$$rot\vec{A} = 0. \tag{9.40}$$

These two equations are identical to the incompressible and irrotational hydrodynamics. Solving the first equation in terms of the velocity potential φ :

$$A_k = \partial_k \varphi, \quad (k = 1, 2)\ , \tag{9.41}$$

and the second one in terms of the stream function ψ :

$$A_1 = \partial_2 \psi \tag{9.42}$$

and

$$A_2 = -\partial_1 \psi; \tag{9.43}$$

we obtain the Cauchy- Riemann Equations:

$$\partial_1 \varphi = \partial_2 \psi, \;\; \partial_2 \varphi = -\partial_1 \psi\,. \tag{9.44}$$

Hence, these two functions are harmonically conjugate and the complex potential

$$f(z) = \varphi(x, y) + i\psi(x, y) \tag{9.45}$$

is an analytic function of $z = x + iy$:

$$\partial f / \partial \bar{z} = 0. \tag{9.46}$$

Corresponding "the complex gauge potential"

$$A = A_1 + iA_2 = \overline{f'(z)} \tag{9.47}$$

is an anti-analytic function. In analogy with hydrodynamics, the logarithmic singularities of the complex potential

$$f(z,t) = \frac{1}{2\pi i} \sum_{k=1}^{N} \Gamma_k Log(z - z_k(t)) \tag{9.48}$$

determine poles of the complex gauge field

$$A = \frac{i}{2\pi} \sum_{k=1}^{N} \frac{\Gamma_k}{\bar{z} - \bar{z}_k(t)} \tag{9.49}$$

describing point solitons in the plane. Then the corresponding "statistical" magnetic field

$$B = \partial_1 A_2 - \partial_2 A_1 = -\Delta \psi = -\Delta \Im f(z) \tag{9.50}$$

where Δ is the Laplacian, determined by the stream function

$$\psi = -\frac{1}{2\pi} \sum_{k=1}^{N} \Gamma_k Log|z - z_k(t)| \tag{9.51}$$

is equal to

$$B = \frac{1}{2\pi} \sum_{k=1}^{N} \Gamma_k \Delta Log|z - z_k(t)| = \sum_{k=1}^{N} \Gamma_k \delta(\vec{r} - \vec{r}_k(t))\,. \tag{9.52}$$

The corresponding total magnetic flux is

$$\int_{R^2}\int B d^2x = \sum_{k=1}^{N} \int\int \Gamma_k \delta(\vec{r} - \vec{r}_k(t)) d^2x = \Gamma_1 + \Gamma_2 + ... + \Gamma_N. \tag{9.53}$$

The relation (9.52) has interpretation as the Chern-Simons Gauss law

$$B = \frac{1}{\kappa}\bar{\psi}\psi = \frac{1}{\kappa}\rho \tag{9.54}$$

for point particles located at $\vec{r}_k(t)$ with density

$$\rho = \sum_{k=1}^{N} \Gamma_k \delta(\vec{r} - \vec{r}_k(t)) \tag{9.55}$$

(with masses $\Gamma_1, \Gamma_2, ..., \Gamma_N$). Then magnetic fluxes are superimposed on particles and have meaning of anyons. As a result, an integrable evolution of the complex gauge field singularities (solitons) would lead to the integrable evolution of anyons. Evolution of the anti-holomorphic complex gauge potential is determined by equation, $\partial_0 A = 2\partial_{\bar{z}} A_0$, where the function A_0, as follows, is harmonic $\Delta A_0 = 0$, and is given by

$$A_0 = \frac{1}{2}[F_0(\bar{z}, t) + \bar{F}_0(z, t)]\,. \tag{9.56}$$

Then the evolution equation is

$$\partial_0 A = \partial_{\bar{z}} F_0\,. \tag{9.57}$$

Let

$$F_0 = \sum_{n=0}^{\infty} c_n F_0^{(n)}(\bar{z}, t) \tag{9.58}$$

where

$$F_0^{(n)}(\bar{z}, t) = (\partial_{\bar{z}} + A(\bar{z}, t))^n \cdot 1 \,. \tag{9.59}$$

then for arbitrary positive integer n we have the anti-holomorphic Burgers' Hierarchy

$$\partial_{t_n} A(\bar{z}, t) = \partial_{\bar{z}}[(\partial_{\bar{z}} + A(\bar{z}, t))^n \cdot 1] \,. \tag{9.60}$$

Using the recursion operator

$$R = \partial_{\bar{z}} + \partial_{\bar{z}} A \partial_{\bar{z}}^{-1} \tag{9.61}$$

we write it in the form

$$\partial_{t_n} A = R^{n-1} \partial_{\bar{z}} A \,. \tag{9.62}$$

The above hierarchy can be linearized by anti-holomorphic Cole-Hopf transformation for the complex gauge field

$$A = \frac{\bar{\Phi}_{\bar{z}}}{\bar{\Phi}} = (\ln \bar{\Phi})_{\bar{z}} = \overline{(f(z,t))}_{\bar{z}} \tag{9.63}$$

in terms of the holomorphic Schrödinger(Heat) Hierarchy

$$\partial_{t_n} \Phi = \partial_z^n \Phi. \tag{9.64}$$

For $n = 2$ the second member of the hierarchy is just

$$i\zeta_t + 2\zeta_{zz} = 0 \tag{9.65}$$

and zeroes of this equation corresponds to magnetic solitons of the Ishimori model. The relation between Φ and complex potential f has meaning of the Madelung representation for the hierarchy

$$\Phi(z,t) = e^{f(z,t)} = e^{\varphi + i\psi} = (e^{\varphi})e^{i\psi} = \sqrt{\rho}e^{i\psi}. \tag{9.66}$$

Therefore, the hierarchy of equations for f is the Madelung form of the holomorphic Schrödinger hierarchy

$$\partial_{t_n} f = (\partial_z + \partial_z f)^n \cdot 1 = e^{-f}\partial_z^n e^f \tag{9.67}$$

or

$$\partial_{t_n}(e^f) = \partial_z^n(e^f) \tag{9.68}$$

which is the potential Burgers' hierarchy. We have the next Linear Problem for the Burgers hierarchy

$$\Phi_z = \bar{A}\Phi, \qquad \Phi_{t_n} = \partial_z^n \Phi. \tag{9.69}$$

It can be written as the Abelian zero-curvature representation for the holomorphic Burgers hierarchy,

$$\partial_{t_n} U - \partial_{\bar{z}} V_n = 0, \tag{9.70}$$

where

$$U = A, \;\; V_n = (\partial_{\bar{z}} + A)^n \cdot 1. \tag{9.71}$$

For the N-solitons of equal strength

$$\Phi(z,t) = e^f = \prod_{k=1}^{N}(z - z_k(t)) \tag{9.72}$$

positions of the solitons correspond to zeroes of $\Phi(z,t)$. As a result the soliton dynamics, leading to integrable anyon dynamics, is related to motion of zeroes subject to the soliton equations (7.138) for $n = 2$ case and for arbitrary n to equation

$$-\frac{dz_k(t_n)}{dt_n} = Res_{z=z_k}\left(\partial_z + \sum_{l=1}^{N}\frac{1}{z - z_l(t_n)}\right)^n \cdot 1, \;\; (k = 1, ..., N). \tag{9.73}$$

9.3. Galilean Group Hierarchy and Soliton Solutions

Now we consider complex Galilean Group hierarchy

$$[P_0, P_z] = 0, \quad [P_0, K_n] = i^n n P_z^{n-1}, \quad [P_z, K_n] = -i \tag{9.74}$$

where the hierarchy of the boost transformations is generated by

$$K_n = z + nt\partial_z^{n-1} \tag{9.75}$$

is commuting with the holomorphic n-Schrödinger equation

$$S_n = \partial_t - \partial_z^n. \tag{9.76}$$

As a result, application of K_n to solution Φ creates solution with additional soliton

$$\Psi(z,t) = K_n\Phi(z,t) = (z + nt\partial_z^{n-1})\Phi(z,t)\,. \tag{9.77}$$

For particular values we have

$$
\begin{aligned}
K_n \cdot 1 &= z = H_1^{(n)}(z,t), && (9.78)\\
K_n^2 \cdot 1 &= z^2 = H_2^{(n)}(z,t), && (9.79)\\
&. \\
&. \\
&., \\
K_n^{n-1} \cdot 1 &= z^{n-1} = H_{n-1}^{(n)}(z,t), && (9.80)\\
K_n^{n} \cdot 1 &= z^{n} + n!\,t = H_{n}^{(n)}(z,t), && (9.81)\\
&. \\
&. \\
&. && (9.82)\\
K_n^{m} \cdot 1 &= H_{m}^{(n)}(z,t)\,, && (9.83)
\end{aligned}
$$

where the generalized Kampe de Feriet polynomials (Dattoli, 2001) are

$$H_m^{(n)}(z,t) = m! \sum_{k=0}^{[m/n]} \frac{t^k z^{m-nk}}{k!(m-nk)!} \tag{9.84}$$

satisfy the holomorphic Schrödinger hierarchy (9.64)

$$\frac{\partial}{\partial t} H_m^{(n)}(z,t) = \partial_z^n H_m^{(n)}(z,t)\,. \tag{9.85}$$

The generating function is given by

$$\sum_{m=0}^{\infty} \frac{k^m}{m!} H_m^{(n)}(z,t) = e^{kz+k^n t}\,. \tag{9.86}$$

From operator representation

$$H_n^{(N)}(z,t) = \exp\left(t\frac{\partial^N}{\partial z^N}\right) z^n \Rightarrow \Phi(z,t) = \exp\left(t\frac{\partial^N}{\partial z^N}\right)\psi(z) \tag{9.87}$$

we have solution of (9.64) in terms of arbitrary analytic function ψ. Polynomials $H_m^{(N)}(z,t)$ are connected with the generalized Hermite polynomials (Srivastava, 1976)

$$H_m^{(N)}(z,t) = t^{[m/N]} H_m^{(N)}\left(\frac{z}{\sqrt[N]{t}}\right). \tag{9.88}$$

Then the k-th zero $w_n^{(N)k}$ of generalized Hermite polynomial $H_n^{(N)}$ determine evolution of the corresponding soliton

$$H_n^{(N)}(w_n^{(N)k}) = 0 \Rightarrow z_k(t) = w_n^{(N)k}\sqrt[N]{t}\,. \tag{9.89}$$

The zeroes are located on the circle in the plane with time dependent radius. When $t \to -t$ position of the soliton rotate on angle $z_k \to z_k e^{i\pi/N}$. The Galilean boost hierarchy (9.77) provides the Bäcklund transformation for n-th member of anti-holomorphic Burgers hierarchy (9.60)

$$v = u + \partial_z \ln[z + Nt(\partial_z + u)^{N-1} \cdot 1]\,. \tag{9.90}$$

9.4. The Negative Burgers' Hierarchy

The holomorphic Schrödinger hierarchy and corresponding Burgers hierarchy can be analytically extended to negative values of N. Introducing negative derivative (pseudo-differential) operator ∂_z^{-1}, so that,

$$\partial_z^{-m} z^n = \frac{z^{n+m}}{(n+1)...(n+m)}\,, \tag{9.91}$$

we have the hierarchy

$$\partial_{t_{-n}}\Phi = \partial_z^{-n}\Phi \tag{9.92}$$

or differentiating n times, in pure differential form $\partial_{t_{-n}}\partial_z^n\Phi = \Phi$. In terms of A defined by (9.63) we have the negative Burgers hierarchy

$$\partial_{t_{-n}}A = \partial_{\bar{z}}\left[\frac{1-\partial_{t_{-n}}(\partial_{\bar{z}}+A)^n\cdot 1}{(\partial_{\bar{z}}+A)^n\cdot 1}\right]. \tag{9.93}$$

For n = 1 we have equation $\partial_{t_{-1}}\Phi = \partial_z^{-1}\Phi$ or the Helmholtz equation $\partial_{t_{-1}}\partial_z\Phi = \Phi$. Analytical continuation of the generalized Kampe de Feriet polynomials to $n = -1$ (Dattoli, 2001) is given by

$$H_M^{(-1)}(z,t) = M!\sum_{k=0}^{\infty}\frac{t^k z^{M+k}}{k!(M+k)!}. \tag{9.94}$$

Then

$$H_M^{(-1)}(z,t) = e^{t\partial_z^{-1}}H_M^{(-1)}(z,0) \tag{9.95}$$
$$H_M^{(-1)}(z,0) = z^M. \tag{9.96}$$

Moreover higher order functions are generated by the ”negative Galilean boost”

$$H_M^{(-1)}(z,t) = (z - t\partial_z^{-2})^M H_0^{(-1)}(z,t). \tag{9.97}$$

Functions $H_M^{(-1)}(z,t)$ are related with Bessel functions (Dattoli, 2001). First, they are directly related with the Tricomi functions

$$C_M(zt) = \frac{z^{-M}}{M!}H_M^{(-1)}(z,t) \tag{9.98}$$

determined by the generating function

$$\sum_{M=-\infty}^{\infty}\lambda^M C_M(x) = e^{\lambda+x/\lambda}. \tag{9.99}$$

The last one is connected with Bessel functions according to

$$J_M(x) = \left(\frac{x}{2}\right)^M C_M(-\frac{x^2}{4})\,. \tag{9.100}$$

Then we have explicitly

$$H_M^{(-1)}(z,t) = M!\left(\frac{-z}{t}\right)^{M/2} J_M(2\sqrt{-zt})\,. \tag{9.101}$$

This provides solution of the negative (-1) flow Burgers equation

$$\partial_t A = \partial_{\bar{z}}\frac{1-\partial_t A}{A} \tag{9.102}$$

in the form

$$A = \frac{(H_M^{(-1)}(\bar{z},t))_{\bar{z}}}{H_M^{(-1)}(\bar{z},t)} = \frac{M}{2\bar{z}} + \sqrt{\frac{t}{-\bar{z}}}\frac{J_M'}{J_M} = \sqrt{\frac{t}{-\bar{z}}}\frac{J_{M-1}(2\sqrt{-\bar{z}t})}{J_M(2\sqrt{-\bar{z}t})}\,. \tag{9.103}$$

For arbitrary member of the negative hierarchy we have

$$\begin{aligned} H_M^{(-N)}(z,t) &= e^{t\partial_z^{-N}} H_M^{(-N)}(z,0) && (9.104)\\ H_M^{(-N)}(z,0) &= z^M && (9.105) \end{aligned}$$

and relation

$$W_M^{(N)}(zt^{1/N}) = \frac{z^{-M}}{M!} H_M^{(-N)}(z,t) \tag{9.106}$$

where the Wright-Bessel functions (Dattoli, 2001) $W_M^{(N)}(x)$ are given by generating function

$$\sum_{M=-\infty}^{\infty} \lambda^M W_M^{(N)}(x) = e^{\lambda + \frac{x}{\lambda^N}} \; . \tag{9.107}$$

CHAPTER 10

CONCLUSIONS

In the present thesis we studied quantum entanglement and topological soliton characteristics of spin models. By identifying spin states with qubits as a unit of quantum information, we have shown two different realization of qubit, one of them characterized by quantum states on the Bloch sphere another one related with $SU(2)$ or spin coherent states is given in terms of extended complex plane states. Then multiple qubit states are associated with spin complexes from quantum theory of magnetism. It allowed us to study quantum information characteristics as quantum entanglement in spin models. We derived entanglement characteristic in the form of concurrence for two qubit pure states and consider concurrence for the thermal states.

Starting from most general fully aniso- tropic symmetrical XYZ model with anisotropic antisymmetric DM type interaction, we constructed eigenvalues and eigenstates for two spins Hamiltonian and calculate the density matrix and concurrence characteristic of this model. As particular cases we treated explicitly pure DM, Ising, XY, XX, XXX and XXZ cases. We found that in all considered cases critical temperature for entanglement is increasing with DM coupling and in some specific cases our calculation indicates on appearance of quantum phase transitions in the model. Time evolution of two qubit states is determined and it is shown that depending on ratio of characteristic frequencies in the system periodic and quasi-periodic evolution of entanglement take place. Relation of time evolution with SWAP gate is established. Next fidelity of time evolved states are found.

We studied entanglement of two qubits with exchange interaction depending on distance $J(R)$ between spins and influence of this distance on entanglement of the system. For this we used different exchange interactions in the form of Calogero- Moser type I,II,III and Herring-Flicker potential which applicable to interaction of Hydrogen molecule. We found that entanglement decreases with the increase of distance.

For geometric quantum computations we calculated geometric(Berry) phase under the DM interaction and studied how the Berry phase changes in a two qubit XX model with DM interaction in an applied magnetic field. We showed that Berry phase in the system depend on the amount of DM interaction and also external magnetic field. We found that large DM interaction tends to wash out the Berry phase while large magnetic field produces a larger range of Berry phase.

As a topological soliton property we study classical spin models in continuum media under holomorphic reduction and constructed static N soliton configuration, soliton and soliton lattice configuration. Then holomorphic time dependent Schrödinger equation were derived for description of evolution in Ishimori model. The influence of harmonic potential and bound state soliton were studied. Relation of integrable solitons with multiparticle problem of Calogero-Moser type were established and N soliton and N soliton lattice motion were derived.

Special reduction of Abelian Chern-Simons to complex Burgers' hierarchy were derived. Galilean group of hierarchy, dynamical symmetry and Negative Burgers' hierarchy are found.

The main results presented in this thesis were published in the following papers.

- Pashaev O.K., Gurkan Z.N., 2007: Abelian Chern-Simons solitons and holomorphic Burgers' hierarchy, *Theor. Math. Phys.* , **152**, 1, 1017-1029.
- Gurkan Z. N., Pashaev O. K., 2008: Integrable soliton dynamics in anisotropic planar spin liquid model, *Chaos, Solitons and Fractals*, **38** , 238- 253.
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- Gurkan Z. N. , Pashaev O., 2010: Entanglement in two qubit magnetic models with DM antisymmetric anisotropic exchange interaction, *International Journal of Modern Physics B*, **24**, 8, 943-965.

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APPENDIX A

EIGENVALUES AND EIGENVECTORS OF XYZ MODEL

Hamiltonian for two qubit XYZ model is

$$H = \frac{1}{2}[J_x\,\sigma_1^x\sigma_2^x + J_y\;\sigma_1^y\sigma_2^y + J_z\;\sigma_1^z\sigma_2^z + (B+b)\,\sigma_1^z + (B-b)\,\sigma_2^z + \vec{D}\cdot(\vec{\sigma_1}\times\vec{\sigma_2})] \tag{A.1}$$

The Hamiltonian in matrix form

$$H = \begin{bmatrix} \frac{J_z}{2}+B & 0 & 0 & \frac{J_x-J_y}{2} \\ 0 & -\frac{J_z}{2}+b & \frac{J_x+J_y}{2}+iD & 0 \\ 0 & \frac{J_x+J_y}{2}-iD & -\frac{J_z}{2}-b & 0 \\ \frac{J_x-J_y}{2} & 0 & 0 & \frac{J_z}{2}-B \end{bmatrix} \tag{A.2}$$

Then solving

$$H|\Psi_i\rangle = E_i|\Psi_i\rangle, \quad i = 1,2,3,4. \tag{A.3}$$

we can obtain the eigenvalues E_1, E_2, E_3, E_4. The characteristic equation

$$det(H-EI) = 0 \tag{A.4}$$

or

$$\begin{vmatrix} \frac{J_z}{2}+B-E & 0 & 0 & J_- \\ 0 & -\frac{J_z}{2}+b-E & J_+ + iD & 0 \\ 0 & J_+ - iD & -\frac{J_z}{2}-b-E & 0 \\ J_- & 0 & 0 & \frac{J_z}{2}-B-E \end{vmatrix} = 0 \tag{A.5}$$

where $(J_x - J_y)/2 = J_-$ and $(J_x + J_y)/2 = J_+$.
The eigenvalues (energy levels) are:

$$E_1 = \frac{J_z}{2} - \sqrt{B^2 + J_-^2} \quad \text{(A.6)}$$
$$E_2 = \frac{J_z}{2} + \sqrt{B^2 + J_-^2} \quad \text{(A.7)}$$
$$E_3 = -\frac{J_z}{2} - \sqrt{b^2 + J_+^2 + D^2} \quad \text{(A.8)}$$
$$E_4 = -\frac{J_z}{2} + \sqrt{b^2 + J_+^2 + D^2} \quad \text{(A.9)}$$

Now we can find eigenstates corresponding to the eigenvalues which are given by the well-known Bell states.

1. For the eigenvalue $E_1 = \frac{J_z}{2} - \sqrt{B^2 + J_-^2}$:

$$(H - E_1 I)|x\rangle = 0 \quad \text{(A.10)}$$

$$\begin{bmatrix} B+\mu & 0 & 0 & J_- \\ 0 & -J_z + b + \mu & J_+ + iD & 0 \\ 0 & J_+ - iD & -J_z - b + \mu & 0 \\ J_- & 0 & 0 & -B+\mu \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \quad \text{(A.11)}$$

where $\mu = \sqrt{B^2 + J_-^2}$. Solving the system we have the corresponding eigenvector to the eigenvalue E_1

$$|x\rangle = C \begin{bmatrix} J_- \\ 0 \\ 0 \\ -(B + \sqrt{B^2 + J_-^2}) \end{bmatrix}. \quad \text{(A.12)}$$

After normalization

$$|\Psi_1\rangle = \frac{1}{\sqrt{2(B^2+J_-^2+B\sqrt{B^2+J_-^2})}}\begin{bmatrix} J_- \\ 0 \\ 0 \\ -(B+\sqrt{B^2+J_-^2}) \end{bmatrix} \tag{A.13}$$

2. For the eigenvalue $E_2 = \dfrac{J_z}{2} + \sqrt{B^2+J_-^2}$:

$$(H - E_2 I)|x\rangle = 0 \tag{A.14}$$

$$\begin{bmatrix} B-\mu & 0 & 0 & J_- \\ 0 & -J_z+b-\mu & J_+ + iD & 0 \\ 0 & J_+ - iD & -J_z-b-\mu & 0 \\ J_- & 0 & 0 & -B-\mu \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0. \tag{A.15}$$

Solving the system we have the corresponding eigenvector to the eigenvalue E_2

$$|x\rangle = C\begin{bmatrix} J_- \\ 0 \\ 0 \\ -(B-\sqrt{B^2+J_-^2}) \end{bmatrix}. \tag{A.16}$$

After normalization

$$|\Psi_2\rangle = \frac{1}{\sqrt{2(B^2+J_-^2-B\sqrt{B^2+J_-^2})}}\begin{bmatrix} J_- \\ 0 \\ 0 \\ -(B-\sqrt{B^2+J_-^2}) \end{bmatrix} \tag{A.17}$$

3. For the eigenvalue $E_3 = -\frac{J_z}{2} - \sqrt{b^2 + J_+^2 + D^2}$:

$$(H - E_3 I)|x\rangle = 0 \qquad \text{(A.18)}$$

$$\begin{bmatrix} J_z + B + \nu & 0 & 0 & J_- \\ 0 & b + \nu & J_+ + iD & 0 \\ 0 & J_+ - iD & -b + \nu & 0 \\ J_- & 0 & 0 & J_z - B + \nu \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \qquad \text{(A.19)}$$

where $\nu = \sqrt{b^2 + J_+^2 + D^2}$. Solving the system we have the corresponding eigenvector to the eigenvalue E_3

$$|x\rangle = C \begin{bmatrix} 0 \\ J_+ + iD \\ -(b + \sqrt{b^2 + J_+^2 + D^2}) \\ 0 \end{bmatrix}. \qquad \text{(A.20)}$$

After normalization

$$|\Psi_3\rangle = \frac{-i}{\sqrt{2(b^2 + J_+^2 + D^2 - b\sqrt{b^2 + J_+^2 + D^2})}} \begin{bmatrix} 0 \\ J_+ + iD \\ -(b + \sqrt{b^2 + J_+^2 + D^2}) \\ 0 \end{bmatrix}$$

4. For the eigenvalue $E_4 = -\frac{J_z}{2} + \sqrt{b^2 + J_+^2 + D^2}$:

$$(H - E_4 I)|x\rangle = 0 \qquad \text{(A.21)}$$

$$\begin{bmatrix} J_z+B-\nu & 0 & 0 & J_- \\ 0 & b-\nu & J_++iD & 0 \\ 0 & J_+-iD & -b-\nu & 0 \\ J_- & 0 & 0 & J_z-B-\nu \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0. \tag{A.22}$$

Solving the system we have the corresponding eigenvector to the eigenvalue E_4

$$|x\rangle = C \begin{bmatrix} 0 \\ J_+ + iD \\ -(b-\sqrt{b^2+J_+^2+D^2}) \\ 0 \end{bmatrix}. \tag{A.23}$$

After normalization

$$|\Psi_4\rangle = \frac{1}{\sqrt{2(b^2+J_+^2+D^2-b\sqrt{b^2+J_+^2+D^2})}} \begin{bmatrix} 0 \\ J_+ + iD \\ -(b-\sqrt{b^2+J_+^2+D^2}) \\ 0 \end{bmatrix}$$

APPENDIX B

THERMAL ENTANGLEMENT

For the Hamiltonian here we calculate $e^{-H/kT}$ and the partition function Z.

$$e^{-H/kT} = I + \left(\frac{-H}{kT}\right) + \frac{1}{2!}\left(\frac{-H}{kT}\right)^2 + ... + \frac{1}{n!}\left(\frac{-H}{kT}\right)^n + ... \tag{B.1}$$

$$e^{-H/kT} = \begin{bmatrix} A_{11} & 0 & 0 & A_{14} \\ 0 & A_{22} & A_{23} & 0 \\ 0 & A_{32} & A_{33} & 0 \\ A_{41} & 0 & 0 & A_{44} \end{bmatrix} \tag{B.2}$$

$$A_{11} = e^{\frac{-J_z}{2kT}}\left(\cosh\frac{\mu}{kT} - \frac{B}{\mu}\sinh\frac{\mu}{kT}\right) \tag{B.3}$$

$$A_{14} = e^{\frac{-J_z}{2kT}}\left(\sinh\frac{\mu}{kT}\right)\left(\frac{-J_-}{\mu}\right) \tag{B.4}$$

$$A_{22} = e^{\frac{J_z}{2kT}}\left(\cosh\frac{\nu}{kT} - \frac{b}{\nu}\sinh\frac{\nu}{kT}\right) \tag{B.5}$$

$$A_{23} = -\frac{J_+ + iD}{\nu}\sinh\frac{\nu}{kT}e^{\frac{J_z}{2kT}} \tag{B.6}$$

$$A_{32} = -\frac{J_+ - iD}{\nu}\sinh\frac{\nu}{kT}e^{\frac{J_z}{2kT}} \tag{B.7}$$

$$A_{33} = e^{\frac{J_z}{2kT}}\left(\cosh\frac{\nu}{kT} + \frac{b}{\nu}\sinh\frac{\nu}{kT}\right) \tag{B.8}$$

$$A_{41} = e^{-\frac{J_z}{2kT}}\left(\sinh\frac{\mu}{kT}\right)\left(\frac{-J_-}{\mu}\right) \tag{B.9}$$

$$A_{44} = e^{-\frac{J_z}{2kT}}\left(\cosh\frac{\mu}{kT} + \frac{B}{\mu}\sinh\frac{\mu}{kT}\right) \tag{B.10}$$

$$Z = Tr[e^{-H/kT}] = 2\left(e^{-J_z/2kT}\cosh\frac{\mu}{kT} + e^{J_z/2kT}\cosh\frac{\nu}{kT}\right) \tag{B.11}$$

where $J_\pm = \frac{J_x \pm J_y}{2}$ and $\mu = \sqrt{B^2 + J_-^2}$ and $\nu = \sqrt{b^2 + J_+^2 + D^2}$.

The concurrence C_{12} the density matrix is defined as

$$C = max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\} \tag{B.12}$$

where λ_i $(i = 1, 2, 3, 4)$ are the square roots of the eigenvalues of the operator

$$\rho_{12} = \rho(\sigma^y \otimes \sigma^y)\rho^*(\sigma^y \otimes \sigma^y) \tag{B.13}$$

and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$. In our case:

$$\begin{aligned}
\rho(\sigma^y \otimes \sigma^y) &= \frac{1}{Z}\begin{bmatrix} A_{11} & 0 & 0 & A_{14} \\ 0 & A_{22} & A_{23} & 0 \\ 0 & A_{32} & A_{33} & 0 \\ A_{41} & 0 & 0 & A_{44} \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{14} & 0 & 0 & A_{11} \\ 0 & A_{23} & A_{22} & 0 \\ 0 & A_{33} & A_{32} & 0 \\ A_{44} & 0 & 0 & A_{41} \end{bmatrix} \\
\rho^*(\sigma^y \otimes \sigma^y) &= \frac{1}{Z}\begin{bmatrix} A^*_{11} & 0 & 0 & A^*_{14} \\ 0 & A^*_{22} & A^*_{23} & 0 \\ 0 & A^*_{32} & A^*_{33} & 0 \\ A^*_{41} & 0 & 0 & A^*_{44} \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A^*_{14} & 0 & 0 & A^*_{11} \\ 0 & A^*_{23} & A^*_{22} & 0 \\ 0 & A^*_{33} & A^*_{32} & 0 \\ A^*_{44} & 0 & 0 & A^*_{41} \end{bmatrix}
\end{aligned}$$

$$\rho_{12} = \frac{1}{Z^2}\begin{bmatrix} A_{14} & 0 & 0 & A_{11} \\ 0 & A_{23} & A_{22} & 0 \\ 0 & A_{33} & A_{32} & 0 \\ A_{44} & 0 & 0 & A_{41} \end{bmatrix}\begin{bmatrix} A^*_{14} & 0 & 0 & A^*_{11} \\ 0 & A^*_{23} & A^*_{22} & 0 \\ 0 & A^*_{33} & A^*_{32} & 0 \\ A^*_{44} & 0 & 0 & A^*_{41} \end{bmatrix} \tag{B.14}$$

$$Z^2\rho_{12} = W = \begin{bmatrix} A_{14}A^*_{14} + A_{11}A^*_{44} & 0 & 0 & A_{14}A^*_{11} + A_{11}A^*_{41} \\ 0 & A_{23}A^*_{23} + A_{22}A^*_{33} & A_{23}A^*_{22} + A_{22}A^*_{32} & 0 \\ 0 & A_{33}A^*_{23} + A_{32}A^*_{33} & A_{33}A^*_{22} + A_{32}A^*_{32} & 0 \\ A_{44}A^*_{14} + A_{41}A^*_{44} & 0 & 0 & A_{44}A^*_{11} + A_{41}A^*_{41} \end{bmatrix}$$

$$W_{11} = A_{14}A_{14}^* + A_{11}A_{44}^* = e^{\frac{-J_z}{kT}} \left(\cosh^2 \frac{\mu}{kT} + \frac{J_-^2 - B^2}{\mu^2} \sinh^2 \frac{\mu}{kT}\right) \tag{B.15}$$

$$W_{14} = A_{14}A_{11}^* + A_{11}A_{41}^* = -\frac{2J_-}{\mu} \sinh \frac{\mu}{kT} e^{\frac{-J_z}{kT}} \left(\cosh \frac{\mu}{kT} - \frac{B}{\mu} \sinh \frac{\mu}{kT}\right) \tag{B.16}$$

$$W_{22} = A_{23}A_{23}^* + A_{22}A_{33}^* = e^{\frac{J_z}{kT}} \left(\cosh^2 \frac{\nu}{kT} + \frac{-b^2 + J_+^2 + D^2}{\nu^2} \sinh^2 \frac{\nu}{kT}\right) \tag{B.17}$$

$$W_{23} = A_{23}A_{22}^* + A_{22}A_{32}^* = -2\frac{J_+ + iD}{\nu} \sinh \frac{\nu}{kT} e^{\frac{J_z}{kT}} \left(\cosh \frac{\nu}{kT} - \frac{b}{\nu} \sinh \frac{\nu}{kT}\right) \tag{B.18}$$

$$W_{32} = A_{33}A_{23}^* + A_{32}A_{33}^* = -2\frac{J_+ - iD}{\nu} \sinh \frac{\nu}{kT} e^{\frac{J_z}{kT}} \left(\cosh \frac{\nu}{kT} + \frac{b}{\nu} \sinh \frac{\nu}{kT}\right) \tag{B.19}$$

$$W_{33} = A_{33}A_{22}^* + A_{32}A_{32}^* = e^{\frac{J_z}{kT}} \left(\cosh^2 \frac{\nu}{kT} + \frac{-b^2 + J_+^2 + D^2}{\nu^2} \sinh^2 \frac{\nu}{kT}\right) \tag{B.20}$$

$$W_{41} = A_{44}A_{14}^* + A_{41}A_{44}^* = -\frac{2J_-}{\mu} \sinh \frac{\mu}{kT} e^{\frac{-J_z}{kT}} \left(\cosh \frac{\mu}{kT} + \frac{B}{\mu} \sinh \frac{\mu}{kT}\right) \tag{B.21}$$

$$W_{44} = A_{44}A_{11}^* + A_{41}A_{41}^* = e^{\frac{-J_z}{kT}} \left(\cosh^2 \frac{\mu}{kT} + \frac{J_-^2 - B^2}{\mu^2} \sinh^2 \frac{\mu}{kT}\right) \tag{B.22}$$

$$|\rho_{12} - \lambda^2 I| = |\frac{W}{Z^2} - \lambda^2 I| = |W - \lambda^2 Z^2 I| = |W - \mu I| = 0 \tag{B.23}$$

where $\lambda^2 Z^2 = \mu$.

$$|W - \mu I| = \begin{vmatrix} W_{11} - \mu & 0 & 0 & W_{14} \\ 0 & W_{22} - \mu & W_{23} & 0 \\ 0 & W_{32} & W_{33} - \mu & 0 \\ W_{41} & 0 & 0 & W_{44} - \mu \end{vmatrix}$$

$$\lambda_i = \frac{\sqrt{\mu_i}}{Z} \tag{B.24}$$

$$\lambda_{1,2} = \frac{e^{J_z/2kT}}{Z} \left| \sqrt{1 + \frac{J_-^2}{\mu^2} \sinh^2 \frac{\mu}{kT}} \mp \frac{J_-}{\mu} \sinh \frac{\mu}{kT} \right| \tag{B.25}$$

$$\lambda_{3,4} = \frac{e^{-J_z/2kT}}{Z} \left| \sqrt{1 + \frac{J_+^2 + D^2}{\nu^2} \sinh^2 \frac{\nu}{kT}} \mp \frac{\sqrt{J_+^2 + D^2}}{\nu} \sinh \frac{\nu}{kT} \right| \tag{B.26}$$

APPENDIX C

LAX REPRESENTATION

Given eigenvalue problem ((Ablowitz & Segur, 1981))

$$L\Psi = \lambda\Psi \tag{C.1}$$

is called isospectral, $\partial\lambda/\partial t = 0$, if eigenfunctions evolution

$$\Psi_t \quad = \quad A\Psi \tag{C.2}$$

implies an operator equation

$$L_t = [A, L] \tag{C.3}$$

is called the Lax equation. Differentiating (C.1) according to t and using (C.2)gives:

$$\begin{aligned}
L_t\Psi + L\Psi_t &= \lambda_t\Psi + \lambda\Psi_t && \text{(C.4)}\\
L_t\Psi + LA\Psi &= \lambda_t\Psi + \lambda A\Psi && \text{(C.5)}\\
&= \lambda_t\Psi + A\lambda\Psi && \text{(C.6)}\\
&= \lambda_t\Psi + AL\Psi && \text{(C.7)}
\end{aligned}$$

Rearranging the above equations

$$
\begin{aligned}
L_t\Psi + LA\Psi_t - AL\Psi &= \lambda_t\Psi && \text{(C.8)}\\
L_t\Psi + (LA - AL)\Psi &= \lambda_t\Psi && \text{(C.9)}\\
L_t\Psi - [A, L]\Psi &= \lambda_t\Psi && \text{(C.10)}\\
(L_t - [A, L])\Psi &= \lambda_t\Psi && \text{(C.11)}
\end{aligned}
$$

we obtain

$$\lambda_t = 0 \quad \Leftrightarrow \quad L_t = [A, L] \tag{C.12}$$

APPENDIX D

N VORTEX SYSTEM

In this appendix we derived system of equations describing evolution of N vortices. Let us consider solution of complex Schrödinger equation (**??**)

$$i\Phi_t = \nu\Phi_{\bar{z}\bar{z}} \tag{D.1}$$

having N simple roots

$$\Phi(\bar{z}, t) = \prod_{k=1}^{N}(\bar{z} - \bar{z}_k(t))\,. \tag{D.2}$$

For simplicity we start with $N = 2$ case

$$\Phi(\bar{z}, t) = (\bar{z} - \bar{z}_1(t))(\bar{z} - \bar{z}_2(t))\,. \tag{D.3}$$

Substituting to the equation we have

$$-i\dot{\bar{z}}_1(\bar{z} - \bar{z}_2) - i\dot{\bar{z}}_2(\bar{z} - \bar{z}_1) = 2\nu\,. \tag{D.4}$$

This equation considered at points $\bar{z} = \bar{z}_1$ and $\bar{z} = \bar{z}_2$ gives the system

$$\dot{\bar{z}}_1 = \frac{2\nu i}{(\bar{z}_1 - \bar{z}_2)}, \quad \dot{\bar{z}}_2 = \frac{-2\nu i}{(\bar{z}_1 - \bar{z}_2)}\,. \tag{D.5}$$

For $N = 3$ case

$$\Phi(\bar{z}, t) = (\bar{z} - \bar{z}_1(t))(\bar{z} - \bar{z}_2(t))(\bar{z} - \bar{z}_3(t))\,. \tag{D.6}$$

Substituting to the equation we have

$$-i\dot{\bar{z}}_1(\bar{z}-\bar{z}_2)(\bar{z}-\bar{z}_3)-i\dot{\bar{z}}_2(\bar{z}-\bar{z}_1)(\bar{z}-\bar{z}_3)-i\dot{\bar{z}}_3(\bar{z}-\bar{z}_1)(\bar{z}-\bar{z}_2)=2\nu[3\bar{z}-(\bar{z}_1+\bar{z}_2+\bar{z}_3)]$$

This equation considered at points $\bar{z}=\bar{z}_1$, $\bar{z}=\bar{z}_2$ and $\bar{z}=\bar{z}_3$ gives the system

$$\dot{\bar{z}}_1 = 2\nu i\left[\frac{1}{(\bar{z}_1-\bar{z}_2)}+\frac{1}{(\bar{z}_1-\bar{z}_3)}\right] \quad (D.7)$$

$$\dot{\bar{z}}_2 = 2\nu i\left[\frac{1}{(\bar{z}_2-\bar{z}_1)}+\frac{1}{(\bar{z}_2-\bar{z}_3)}\right] \quad (D.8)$$

$$\dot{\bar{z}}_3 = 2\nu i\left[\frac{1}{(\bar{z}_3-\bar{z}_1)}+\frac{1}{(\bar{z}_3-\bar{z}_2)}\right]. \quad (D.9)$$

Following the same procedure, in general case of arbitrary N zeroes (D.2) we obtain the system of first order equations

$$\dot{\bar{z}}_j = 2\nu i\sum_{k\neq j}^{N}\frac{1}{(\bar{z}_j-\bar{z}_k)}. \quad (D.10)$$

Differentiating this system once more in time we get the system of Newton's equations:

$$\ddot{\bar{z}}_j = 2\nu i\sum_{k\neq j}^{n}\frac{-(\dot{\bar{z}}_j-\dot{\bar{z}}_k)}{(\bar{z}_j-\bar{z}_k)^2} = 8\nu^2\sum_{j<k}^{n}\frac{1}{(\bar{z}_j-\bar{z}_k)^3}. \quad (D.11)$$

For $N=2$ case we have two equations

$$\ddot{\bar{z}}_1 = 8\nu^2\sum_{k\neq j}^{n}\frac{1}{(\bar{z}_1-\bar{z}_2)^3}, \quad \ddot{\bar{z}}_2 = -8\nu^2\sum_{k\neq j}^{n}\frac{1}{(\bar{z}_1-\bar{z}_2)^3}. \quad (D.12)$$

For $N=3$ case we have the following equations

$$\ddot{\bar{z}}_1 = 2\nu i\left[\frac{-(\dot{\bar{z}}_1-\dot{\bar{z}}_2)}{(\bar{z}_1-\bar{z}_2)^2}+\frac{-(\dot{\bar{z}}_1-\dot{\bar{z}}_3)}{(\bar{z}_1-\bar{z}_3)^2}\right] \quad (D.13)$$

$$= 8\nu^2\left[\frac{1}{(\bar{z}_1-\bar{z}_2)^3}+\frac{1}{(\bar{z}_1-\bar{z}_3)^3}\right] \quad (D.14)$$

$$\ddot{\bar{z}}_2 = 2\nu i \left[\frac{-(\dot{\bar{z}}_2 - \dot{\bar{z}}_1)}{(\bar{z}_2 - \bar{z}_1)^2} + \frac{-(\dot{\bar{z}}_2 - \dot{\bar{z}}_3)}{(\bar{z}_2 - \bar{z}_3)^2} \right] \tag{D.15}$$

$$= 8\nu^2 \left[\frac{1}{(\bar{z}_2 - \bar{z}_1)^3} + \frac{1}{(\bar{z}_2 - \bar{z}_3)^3} \right] \tag{D.16}$$

$$\tag{D.17}$$

and

$$\ddot{\bar{z}}_3 = 2\nu i \left[\frac{-(\dot{\bar{z}}_3 - \dot{\bar{z}}_1)}{(\bar{z}_3 - \bar{z}_2)^2} + \frac{-(\dot{\bar{z}}_3 - \dot{\bar{z}}_2)}{(\bar{z}_3 - \bar{z}_2)^2} \right] \tag{D.18}$$

$$= 8\nu^2 \left[\frac{1}{(\bar{z}_3 - \bar{z}_1)^3} + \frac{1}{(\bar{z}_3 - \bar{z}_2)^3} \right] . \tag{D.19}$$