OPERATOR SPLITTING METHOD FOR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS: ANALYSES AND APPLICATIONS

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ABSTRACT

OPERATOR SPLITTING METHOD FOR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS: ANALYSES AND APPLICATIONS

This thesis presents the consistency, stability and convergence analysis of an operator splitting method, namely the iterative operator splitting method, using various approaches for parabolic partial differential equations. The idea of the method is based first on splitting the complex problems into simpler equations. Then, each sub-problem is combined with iterative schemes and efficiently solved with suitable integrators. The analyses are based on the type of the operators of the problems. When the operators are bounded, the consistency is proved in two ways: first from derived explicit local error bounds and the second using the Taylor series expansion after combining iterative schemes with midpoint rule. As for the unbounded operators, since the Taylor series expansion is no longer valid, the consistency is derived using C_0 semigroup theory. The stability is presented by constructing stability functions for each iterative schemes when the operators are bounded. For the unbounded, two stability analyses are offered: first one uses the continuous Fourier transform and the second uses semigroup theory. Lax-Richtmyer equivalence theorem and Lady Windermere's fan argument which combine the stability and consistency are proposed for the convergence. In the computational part, the method is applied to three linear parabolic PDEs and to Korteweg-de Vries equation. These three equations are capillary formation model in tumor angiogenesis, solute transport problem and heat equation. Finally, numerical results are presented to illustrate the high accuracy and efficiency of the method relative to other classical methods. These numerical results align with the obtained theoretical results.

ÖZET

PARABOLİK KISMİ DİFERANSİYEL DENKLEMLER İÇİN OPERATÖR AYIRMA METODU: ANALİZLER VE UYGULAMALAR

Bu tezde, bir operatör ayırma metodu olan tekrarlayan operatör ayırma metodunun tutarlılık, kararlılık ve yakınsama analizleri, farklı yaklaşımlarla parabolik kısmi diferansiyel denklemler için incelenmektedir. Metot karmaşık problemleri basit denklemlere ayırma ilkesine dayanır. Her alt problem tekrarlayan algoritmayla birleştirilip, elde edilen basit denklemler etkili bir şekilde uygun metotlarla çözülür. Analizler, ele aldığımız problemin operatörünün durumuna göre farkılılık göstermektedir. Operatörler sınırlı olduğunda metodun tutarlılığı iki yöntemle incelenmektedir: İlki açık yerel hata sınırları oluşturarak ve ikincisi ise tekrarlayan şemayı zamanda ortanokta metoduyla birleştirdikten sonra Taylor seriyi kullanarak. Sınırsız operatöre sahip olma durumunda, Taylor serisini kullanamayacağımızdan, tutarlılık analizinde C_0 yarıgrup özellikleri kullanılmaktadır. Sınırlı operatörler için kararlılık, her tekrarlayan şema için kararlılık fonksiyonları oluşturarak gösterilmektedir. Sınırsız operatöre sahip olma durumundaki kararlılık için ise iki yöntem sunulur: İlki sürekli Fourier dönüşümü ve ikincisi C_0 yarıgrup yöntemi. Yakınsaklık analizinde, kararlılık ve tutarlılığın birleştirilmesi ilkesine dayanan Lax-Richtmyer denklik teoremi ve Lady Windermere tezi kullanılmaktadır. Sayısal kısımda, tekrarlayan ayırma metodu üç liner parabolik kısmi diferansiyel denkleme ve bir boyutlu liner olmayan Korteweg-de Vries denklemine uygulanır. Bu üç denklem: Tümörlü hücrede kan damarları ağı oluşum probleminde kılcal damar oluşum modeli, çözününen madde taşıma problemi ve iki boyutlu ısı denklemidir. Son olarak, önerilen metodun iyi bir performansa sahip olduğu göstermek için sayısal sonuçlar sunulur. Bu sonuçlar teorik sonuçlarla uyuşur.

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CHAPTER 1

INTRODUCTION

Operator splitting is both powerful and useful method for the numerical investigation of large systems of partial differential equations. The basic idea is first based on splitting a complex problem into simpler sub-problems, whose sub-operators are chosen with respect to different physical processes. Then, each sub-equation is solved efficiently with suitable integrators. The sub-systems are connected via the initial conditions. This technique leads to a splitting error which can be estimated theoretically. The main advantage of the operator splitting technique is that the hyperbolic or the parabolic sub-problems, which are of different nature, can be solved numerically using different integrators.

The idea of splitting, which is based on Lie-Trotter splitting, dates back to the 1950s. In 1955, Peacemann and Rochford (Peaceman and Rachford, 1955) presented the splitting idea in connection with finite difference approximation to heat equation and Douglas and Rachford (Douglas and Rachford, 1956) constructed a linear implicit iterative method, which is a modification of the alternating direction of implicit method, in 1956. The first splitting scheme was suggested in 1957 by the Russian mathematicians Bagrinovskii and Gudunov (Bagrinovskii and Gudunov, 1957). Their difference scheme explicitly approximated a hyperbolic system of equations. The implicit scheme of splitting was published two year later by N. N. Yanenko, (Yanenko, 1959). In 1959, Trotter (Trotter, 1959) studied the Lie product formula and extended this formula for matrices to unbounded operator in Banach spaces. In 1968, first splitting methods were studied systematically by Marchuk and Strang, (Marchuk, 1968), (Strang, 1968). The simplest kind is sequential splitting method (or Lie-Trotter splitting), which is first order accurate in time, see (Trotter, 1959), (Marchuk, 1968), (Strang, 1968). Later, Strang-Marchuk splitting, a second order method, was constructed, (Marchuk, 1968), (Strang, 1968). In 1968, Temam (Temam, 1968) analyzed the operator splitting method extensively for non-homogenous partial differential equations. In 1963, Strang proposed a second order method, symmetrically weighted sequential splitting (SWS), which consisted of a weighted sum of splitting solutions and was obtained by a different ordering of the sub-operators, see (Strang, 1963), (Hundsdorfer and Verwer, 2003). In 1995, another splitting method, the iterative splitting was introduced in (Kelly, 1995). In 2003, the convergence analysis of iterative splitting procedure for nonlinear reactive transport problems was studied, (Kanney et al., 2003). In 2007, Faragó and Geiser (Faragó and Geiser, 2007) suggested a new scheme which was based on the combination of splitting time interval and the traditional iterative operator splitting. Then, (Faragó, 2007), (Faragó et. al., 2008a), (Geiser, 2008), (Geiser, 2008) analyzed the iterated splitting method in details. The consistency of iterative splitting was proved for bounded operators using Taylor series expansion in (Faragó, 2007). In (Faragó et. al., 2008a), the authors presented the order of the iterative schemes for bounded operators using variation of constants formula. In (Geiser, 2008), the consistency and stability of this method were studied based on the matrix representation for bounded operators.

In this thesis, we concentrate on linear and semilinear parabolic partial differential equations and a solitary water wave equation, Korteweg-de Vries (KdV) equation, in the form

$$\frac{\partial u}{\partial t} = A_1(u) + A_2(u). \tag{1.1}$$

When Problem (1.1) is linear, like an advection-diffusion-reaction problem, we split it into a linear diffusion A_1u and a linear advection-reaction A_2u. When Problem (1.1) is semilinear, we have a linear diffusion A_1u and a nonlinear reaction $A_2(u)$. When Problem (1.1) is a Korteweg-de Vries equation, we have a linear dispersion A_1u and a nonlinear term $A_2(u)$. Since iterative splitting algorithm itself makes a linearization on nonlinear terms, semilinear problems are also analyzed as linear problems.

The iterative splitting method applied to Problem (1.1) is analyzed depending on whether the operators A_1 and A_2 are bounded or unbounded. When the operators are bounded, the local error bounds are constructed for each iteration, similar to (Faragó et. al., 2008a), but we show them explicitly, see (Gücüyenen and Tanoğlu, 2011b). These bounds prove the consistency which comes from splitting. The consistency is also studied by applying midpoint rule to iterative schemes and finding the truncation error. This shows the consistency of iterative splitting combined with midpoint rule. For unbounded operators, the semigroup theory is used to show consistency and the convergence. The idea of using strongly continuous (C_0) semigroup techniques in the consistency analysis of numerical methods was applied to a homogenous abstract Cauchy problem dates back to Pazy (Pazy, 1983). Bjørhus (Bjørhus, 1998) analyzed the convergence of sequential splitting for linear non-homogenous abstract Cauchy problems. Later, Faragó and Havasi (Faragó and Havasi, 2007) studied the consistency of Strang-Markuk and symmetrically weighted sequential (SWS) splittings with similar approach. In our thesis, we prove the consistency and the convergence of iterative splitting method using C_0 semigroup approach. We reference (Geiser, 2008) for the stability of splitting methods applied to problems with bounded operators. We refer to (Holden et. al., 1999), (Regan, 2002) for stability when the operators are unbounded. Holden et. al. (Holden et. al., 1999) studied the Fourier transform of the sequential splitting applied to Korteweg-de Vries (KdV) equation with linearization on the nonlinear term. Regan (Regan, 2002) studied the symplectic integration of Hamilton PDEs and used Von Neumann analysis to achieve stability criteria. Our study explicitly derives the stability criteria using the Fourier analysis for the iterative splitting method combined with midpoint rule, see (Gücüyenen and Tanoğlu, 2011a).

Operator splitting methods have been used in many areas such as advection diffusion reaction problems (Geiser, 2008), (Hundsdorfer and Verwer, 2003), large scale air pollution models (Dimov et. al., 2001), (Dimov et. al., 2004), Navier-Stokes equations (Christov and Marinova, 2001), Hamilton-Jacobi equations (Jakobsen et. al., 2001), stochastic reaction systems (Jahnke and Altıntan, 2010), Schrödinger equations (Lubich and Jahnke, 2000), (Lubich, 2008), taxis diffusion reaction models (Gerisch and Verwer, 2002). In our thesis, the iterative splitting method is used to solve three linear parabolic partial differential equations and a one dimensional nonlinear Korteweg-de Vries (KdV) equation. These three are the one dimensional capillary formation model in tumor angiogenesis problem, two dimensional solute transport model and the two dimensional heat equation.

The mathematical model for capillary formation in tumor angiogenesis was originally presented in (Levine et. al., 2001) and described the endothelial cell movement in a capillary. (Levine et. al., 2001) combined the cell transport (chemotactic) equations and the theory of reinforced random walk (David, 1990) to develop the model. It was recently used by Othmer and Stevens (Othmer and Stevens, 1997) to model fruiting bodies. The capillary formation problem was solved by the method of lines (Serdar and Erdem, 2007) and by the shifted Legendre tau method (Saadatmandi and Dehnhan, 2008). Unlike the complicated systems (Serdar and Erdem, 2007) and (Saadatmandi and Dehnhan, 2008) had to cope with, the proposed method is based on decomposition idea, therefore, it is easier to apply.

Groundwater in many countries is the major source for drinking, irrigation, and industrial use. The contamination of this source through seepage from landfills, gas tanks, industrial waste, and agricultural chemicals is a major problem. It is vital to develop groundwater management models to keep clean and safe water under the ground.

One important segment of such management model is the numerical simulation of solute transport. Since 1970s many models have been developed, based on analytical and numerical solutions such as Laplace transform (Batu, 1979), finite difference methods (Karahan, 2006), (Karahan, 2007a), (Karahan, 2007b), finite element method (Daus and Frind, 1985), finite volume method (Verma et. al., 2000), etc. In particular, Verma et al (Verma et. al., 2000) employed overlapping control volume method which is applicable for nonorthogonal grids. The method solved two dimensional transient solute transport using an isoparametric formulation for computing the dispersion and for second order upwinding. They used an implicit approach for integrating time. We apply the iterative splitting method to the solute transport in ground water flow discussed in (Verma et. al., 2000) by splitting the equation into a diffusion part and an advection part.

Nonlinear wave equations are widely used to describe complex phenomena in various sciences such as fundamental particle physics, plasma and fluid dynamics, statistical mechanics, protein dynamics, condensed matter, biophysics, nonlinear optics, quantum field theory, see (Scott, 1999), (Drazin and Johnson, 1989), (Ablowitz and Segur, 1981), (Frody, 1990). The wide applicability of these equations is the main reason why they have attracted so much attention from many mathematicians. During the past four decades, both mathematicians and physicists have devoted considerable effort into the study of exact and numerical solutions of the nonlinear partial differential equations (PDEs) corresponding to the nonlinear problems. One of the famous nonlinear PDE is Korteweg-de Vries equation which describes the theory of water waves in shallow channels, such as a canal. It is a nonlinear equation which exhibits special solutions, known as solitons, which are stable and do not disperse with time, see (Russell, 1838). Korteweg-de Vries (Korteweg, 1895) formulated the mathematical model equation to provide explanation of the phenomena. Introduction to main ideas and techniques of the modern soliton theory is given in (Drazin, 1983), (Drazin and Johnson, 1989), (Hirota, 2004), (Munteanu and Donescu, 2004), (Pashaev, 2009). KdV equation has been solved by various analytical and numerical methods, such as an exact method (Hirota, 1972), a direct method (Hirota, 2004), a particle method (Chertock and Levy, 2002), Adomain's decomposition (Kaya, 2004), He's perturbation method (Yıldırım, 2009). Here, we use iterative operator splitting method to solve one dimensional nonlinear KdV equations for given initial and boundary conditions.

We now describe each chapter of the thesis:

- In Chapter 2, we introduce operator splitting methods, Lie-Trotter splitting, Strang-Marchuk splitting and iterative splitting by explaining their algorithms and systematic schemas on an abstract Cauchy problem.

- In Chapter 3, we investigate the consistency analysis of iterative splitting method. The analyses depend on whether the operators of the problems are bounded or unbounded. Since we study with partial differential equations, we come across spatial derivative operators which are unbounded. Here, we consider the bounded as well as the unbounded operator. Boundedness is achieved by replacing each spatial derivative with an equivalent finite difference approximation. When the operator are bounded, consistency is proved with the help of the variation of constants formula by constructing local error bounds. These error bounds also prove that more iteration numbers result in higher order accuracy, which is one of the advantages of the iterative splitting method. Also, the consistency is studied by applying midpoint rule to each sub iterative schemes and second order consistency is obtained for two iterations. When the operators are unbounded, the consistency is studied using C_0 semigroup properties. We introduce the semigroup theory briefly by presenting some necessary definitions, theorems and some key lemmas and for further details refer to (Engel and Nagel, 2000), (Pazy, 1983), (Faragó, 2005), (Bjørhus, 1998). Under the assumption of unbounded linear operators being generators of C_0 semigroups, we prove the consistency of the first and the second order iterative schemes.

- In Chapter 4, we investigate the stability of iterative splitting method. When the operators are bounded, we prove the stability of iterative splitting solutions by constructing stability functions after applying midpoint rule. Consequently, (Theorem 2.2.1, (Strikwerda, 2004)) implies stable schemes. When unbounded, we derive the stability estimates by using Fourier transform. Finally, stability estimates by using Fourier transform are derived for one dimensional advection-diffusion equation, two dimensional solute transport equation and a one dimensional nonlinear KdV equation.

- In Chapter 5, we prove the convergence of iterative splitting method by using Lax-Richtmyer Equivalence Theorem and Lady Windermere's Fan argument, which combine stability and the consistency of the method for bounded and unbounded operators, respectively.

- In Chapter 6, we investigate three parabolic partial differential equations which are one dimensional capillary formation model in tumor angiogenesis problem, two dimensional solute transport model, two dimensional heat equation and also a one dimensional Korteweg-de Vries equation. We employ various numerical methods and compare iterative splitting solutions to other classical solutions. In the last problem, we confirm the convergence results obtained in Chapter 5 using semigroup theory. Numerical experiments using Matlab confirm our theoretical results and demonstrate the effectiveness of the proposed method, which gives higher order results.

- In the conclusion we summarize the main results in the thesis.

CHAPTER 2

INTRODUCTION TO SPLITTING METHODS

For many complicated partial differential equations (PDEs), such as advection-diffusion-reaction problems

$$u_t + \nabla \cdot (au) = \nabla \cdot (D\nabla u) + f(u), \tag{2.1}$$

it is in general inefficient to find solutions by applying the same methods to different parts of the equations. For example, in chemical problems, reaction part can be very stiff, then an implicit method is recommended. On the other hand, if the advection term is discretized in space using a limiter, then explicit methods are much more suitable. In such cases, a tuned splitting approach is advocated. The basic idea behind splitting is breaking down a complex problem into simpler sub-problems, such that each sub-problem can be solved efficiently with suitable integrators.

2.1. Operator Splitting Methods

This section is devoted to operator splitting methods which are widely used for solution to the complicated PDE systems. We focus our attention on the following Cauchy problem

$$u'(t) = (A_1 + A_2)u(t), \tag{2.2}$$

$$u(0) = u_0 \tag{2.3}$$

where $t \in [0, T]$, A_1, A_2 are assumed to be linear operators in Banach space X with A_1, $A_2 : X \to X$ and $u_0 \in X$ is initial condition. When A_1 and A_2 are bounded operators, the exact solution is given by

$$u(t^{n+1}) = e^{\Delta t(A_1+A_2)}u(t^n), \tag{2.4}$$

where time step is $\Delta t = t^{n+1} - t^n$ and $u(t^n)$ is solution at $t = t^n$ time. We concentrate on the following operator splitting schemes.

2.1.1. Lie-Trotter Operator Splitting

Historically Lie-Trotter splitting is the first splitting method. It has a very simple algorithm which separates the Cauchy problem (2.2)-(2.3) into two sub-equations. The first sub-problem is solved with operator A_1 and the original initial condition. The second one is with operator A_2 and initial condition derived from the solution of the first problem. The algorithm is the following:

$$\frac{du_1(t)}{dt} = A_1u_1(t), \;\; t \in [t^n, t^{n+1}] \tag{2.5}$$

$$u_1(t^n) = u_{sp}^n \tag{2.6}$$

$$\frac{du_2(t)}{dt} = A_2u_2(t), \;\; t \in [t^n, t^{n+1}] \tag{2.7}$$

$$u_2(t^n) = u_1(t^{n+1}), \tag{2.8}$$

where split condition at $t = 0$ is given by $u_{sp}^0 = u_0$ in (2.3) and the approximated split solution at $t = t^{n+1}$ is defined as $u_{sp}^{n+1} = u_2(t^{n+1})$; here $t^{n+1} = t^n + \Delta t$, Δt is time step, and $n = 0, 1, ..., N-1$.

The chance of the original problem with sub-problems normally introduces an error, called **local splitting error.** The local splitting error of Lie-Trotter splitting method is derived as follows:

$$\rho_n = \frac{1}{\Delta t}(e^{\Delta t(A_1+A_2)} - e^{\Delta t A_2} e^{\Delta t A_1})u(t^n)$$
$$= \frac{1}{2}\Delta t[A_1, A_2]u(t^n) + \mathcal{O}(\Delta t^2). \quad (2.9)$$

We define

$$[A_1, A_2] = A_1 A_2 - A_2 A_1$$

as the commutator of A_1 and A_2. Consequently, Lie-Trotter splitting method is first order consistent if the operators A_1 and A_2 do not commute. When the operators commute, then the method is exact.

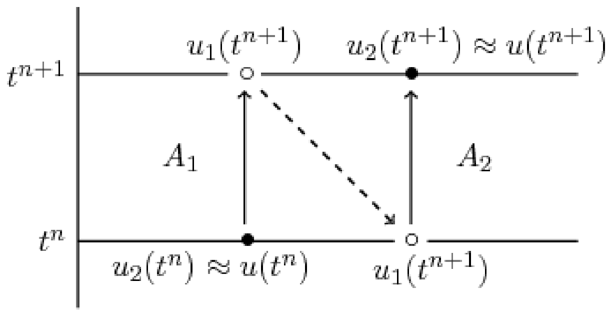


Figure 2.1. Systematic schema of Lie-Trotter splitting method.

2.1.2. Strang-Marchuk Operator Splitting

The Strang-Marchuk operator splitting method divides the split time-subinterval into two parts. Then, as in the Lie-Trotter algorithm, successively solves the problems on the first half interval with operator A_1, on the whole interval with operator A_2 and on the second half interval again with operator A_1. The first subproblem uses the original initial condition and the others use the solutions of the previous problems. The algorithm is the following:

$$\frac{du_1(t)}{dt} = A_1u_1(t), \ \ t \in [t^n, t^{n+1/2}] \tag{2.10}$$

$$u_1(t^n) = u_{sp}^n, \tag{2.11}$$

$$\frac{du_2(t)}{dt} = A_2u_2(t), \ \ t \in [t^n, t^{n+1}] \tag{2.12}$$

$$u_2(t^n) = u_1(t^{n+1/2}), \tag{2.13}$$

$$\frac{du_3(t)}{dt} = A_1u_3(t), \ \ t \in [t^{n+1/2}, t^{n+1}] \tag{2.14}$$

$$u_3(t^{n+1/2}) = u_2(t^{n+1}), \tag{2.15}$$

where split condition at $t = 0$ is given by $u_{sp}^0 = u_0$ in (2.3) and the approximated split solution at $t = t^{n+1}$ is defined as $u_{sp}^{n+1} = u_3(t^{n+1})$; here $t^{n+1} = t^n + \Delta t$, Δt is time step, and $n = 0, 1, ..., N-1$.

The local splitting error of Strang-Marchuk splitting method is derived as follows:

$$\begin{aligned} \rho_n &= \frac{1}{\Delta t}(e^{\Delta t(A_1+A_2)} - e^{\frac{\Delta t}{2}(A_1)}e^{\Delta t(A_2)}e^{\frac{\Delta t}{2}(A_1)}u(t^n) \\ &= \frac{1}{24}\Delta t^2(2[A_2,[A_2,A_1]] - [A_1,[A_1,A_2]])u(t^n) + \mathcal{O}(\Delta t^3), \end{aligned} \tag{2.16}$$

revealing a formal consistency order of two.

2.1.3. Iterative Operator Splitting

Iterative operator splitting method is similar to Lie-Trotter splitting; but each sub-problem contains both operators A_1 and A_2. For the first sub-equation, A_1 but not A_2 is included in the homogenous part. For the second sub-equation, only A_2 is included in the homogenous part. Both equations use the same initial condition. The algorithm is the

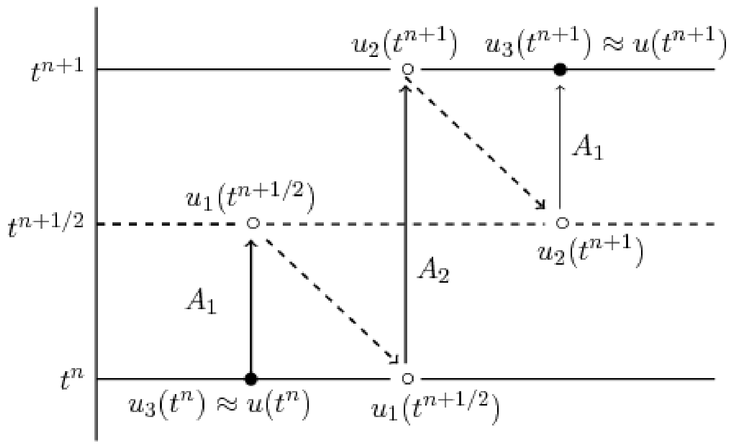


Figure 2.2. Systematic schema of Strang-Marchuk splitting method.

following:

$$\frac{du_i(t)}{dt} = A_1 u_i(t) + A_2 u_{i-1}(t), \quad t \in [t^n, t^{n+1}], \tag{2.17}$$

$$u_i(t^n) = u_{sp}^n, \tag{2.18}$$

$$\frac{du_{i+1}(t)}{dt} = A_1 u_i(t) + A_2 u_{i+1}(t), \quad t \in [t^n, t^{n+1}], \tag{2.19}$$

$$u_{i+1}(t^n) = u_{sp}^n, \tag{2.20}$$

where split condition at $t = 0$ is given by $u_{sp}^0 = u_0$ in (2.3) and the approximated split solution at $t = t^{n+1}$ is defined as $u_{sp}^{n+1} = u_{2m}(t^n)$; here $t^{n+1} = t^n + \Delta t$, Δt is time step, $n = 0, 1, ..., N-1$ and $i = 1, 3, 5, \ldots, 2m-1$. The function $u_0(t)$ is an arbitrarily initial guess.

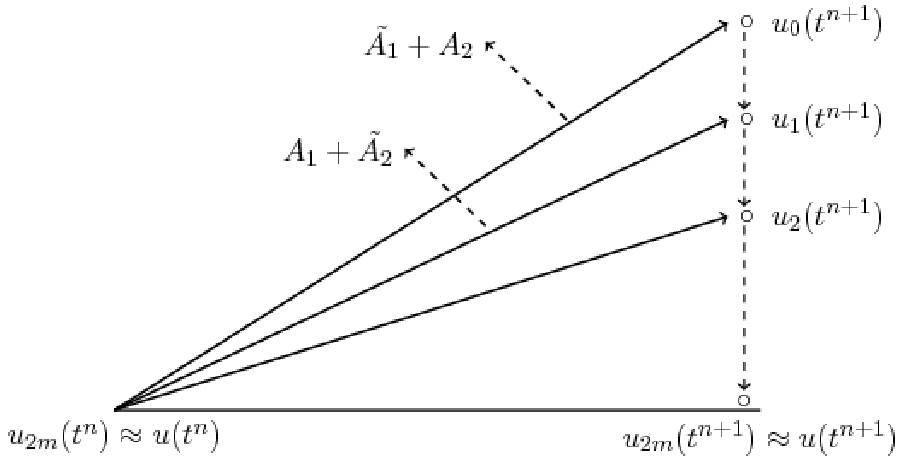


Figure 2.3. Systematic schema of iterative splitting method.

CHAPTER 3

CONSISTENCY ANALYSIS OF ITERATIVE OPERATOR SPLITTING METHOD

In this chapter, we study the consistency of iterative operator splitting method by using various techniques when applied to Cauchy problem (2.2)-(2.3). The analyses depend on whether the operators of the problems are bounded or unbounded. Boundedness is obtained using finite difference expansion on spatial derivatives, which will be explained in detailed in Section 6. When A_1 and A_2 are bounded, we first prove the consistency of iterative splitting schemes (2.17)-(2.20) by constructing local splitting error bounds. Second, we prove the consistency of iterative schemes (2.17)-(2.20) combined with midpoint rule. Lastly, when A_1 and A_2 are unbounded we prove the consistency of iterative splitting method by using C_0 semigroup theory.

3.1. Consistency Analysis for Bounded Operators

Consider the abstract Cauchy problem (2.2)-(2.3) where $u(t)$ is the exact solution. The iterative splitting method is given in (2.17)-(2.20) with numerical solution $u_i(t)$. The local error of the method after one time step $[0, \Delta t]$ is defined as $u(\Delta t) - u_i(\Delta t)$ for each iteration i. Theorem 3.1 derives explicit local splitting error bounds of iterative splitting schemes, (Gücüyenen and Tanoğlu, 2011b). Theorem 3.2 establishes the consistency of iterative splitting schemes with midpoint rule, (Gücüyenen et. al., 2011).

Theorem 3.1 *Let A_1, A_2 be bounded linear operators of Cauchy problem given in (2.2)-(2.3). The local error bounds of the iterative schemes (2.17), (2.20) are given by*

$$\|\epsilon_i\| \leq (K_2\|A_1\|)^{\frac{i-1}{2}}.(K_1\|A_2\|)^{\frac{i+1}{2}}\|\epsilon_0\|_\infty \frac{t^i}{i!}, \ i \text{ is odd} \qquad (3.1)$$

$$\|\epsilon_i\| \leq (K_1\|A_1\|)^{\frac{i}{2}}.(K_2\|A_2\|)^{\frac{i}{2}}\|\epsilon_0\|_\infty \frac{t^i}{i!}, \ i \text{ is even} \qquad (3.2)$$

where $\|\epsilon_0\|$ is the difference between one step exact solution and initial condition, $\|\epsilon_i\|$ is *the difference between one step exact solution and ith iterative solution,* $\|exp(A_1 t)\| \leq K_1$, $\|exp(A_2 t)\| \leq K_2$ *for* $t \geq 0$. $\|\cdot\|$ *is any norm defined on* $\mathbb{R}^n$.

Proof Rewriting iterative schemes, we have

$$u'_i(t) = A_1 u_i(t) + A_2 u_{i-1}(t) \tag{3.3}$$

$$u'_{i+1}(t) = A_1 u_i(t) + A_2 u_{i+1}(t) \tag{3.4}$$

with initial conditions $u_i(0) = u_0$ and $u_{i+1}(0) = u_0$, where $i = 1, 3, \ldots, 2m-1$ for $[0, t]$ time interval. The symbol ϵ_i is defined by $\epsilon_i = u(t) - u_i(t)$ and A_1, A_2 are bounded linear operators with exponential bounds $\|exp(A_1 t)\| \leq K_1$, $\|exp(A_2 t)\| \leq K_2$ for $t \geq 0$. From variation of constant formula for $i = 1$ we obtain

$$u_1(t) = e^{A_1 t} u_0 + \int_0^t e^{A_1(t-s)} A_2 u_0 \, ds, \tag{3.5}$$

and the exact solution is

$$u(t) = e^{A_1 t} u_0 + \int_0^t e^{A_1(t-s)} A_2 e^{(A_1+A_2)s} u_0 \, ds. \tag{3.6}$$

By subtracting Equation (3.5) from Equation (3.6) we obtain the following error bound

$$\begin{aligned} \|u(t) - u_1(t)\| &= \| \int_0^t e^{A_1(t-s)} A_2 (e^{(A_1+A_2)s} u_0 - u_0) \, ds \| \\ \|\epsilon_1\| &= \| \int_0^t e^{A_1(t-s)} A_2 \epsilon_0 \, ds \| \\ \|\epsilon_1\| &\leq K_1 \|A_2\| \|\epsilon_0\|_\infty t, \end{aligned} \tag{3.7}$$

for the supremum norm of $\|\epsilon_1\|$ we have

$$\|\epsilon_1\|_\infty \leq K_1 \|A_2\| \|\epsilon_0\|_\infty t. \tag{3.8}$$

For the second iteration, from the variation of constants formula, we have

$$u_2(t) = e^{A_2 t} u_0 + \int_0^t e^{A_2(t-s)} A_1 u_1 \, ds \tag{3.9}$$

and the exact solution is

$$u(t) = e^{A_2 t} u_0 + \int_0^t e^{A_2(t-s)} A_1 e^{(A_1+A_2)s} u_0 \, ds. \tag{3.10}$$

Again by subtracting Equation (3.9) from Equation (3.10) we obtain

$$\begin{aligned}
\|u(t) - u_2(t)\| &= \| \int_0^t e^{A_2(t-s)} A_1 (e^{(A_1+A_2)s} u_0 - u_1) \, ds \| \\
\|\epsilon_2\| &= \| \int_0^t e^{A_2(t-s)} A_1 \epsilon_1 \, ds \| \\
\|\epsilon_2\| &\leq K_2 \int_0^t \|A_1\| \|\epsilon_1\|_\infty \, ds \\
\|\epsilon_2\| &\leq K_2 K_1 \|A_1\| \|A_2\| \|\epsilon_0\|_\infty \frac{t^2}{2}. \qquad (3.11)
\end{aligned}$$

and for the supremum norm of $\|\epsilon_2\|_\infty$ we have

$$\|\epsilon_2\|_\infty \leq K_2 K_1 \|A_1\| \|A_2\| \|\epsilon_0\|_\infty \frac{t^2}{2}. \tag{3.12}$$

Similarly for $i = 3$, we obtain the following local error bound

$$\begin{aligned}
\|u(t) - u_3(t)\| &= \| \int_0^t e^{A_1(t-s)} A_2 (e^{(A_1+A_2)s} u_0 - u_2) \, ds \| \\
\|\epsilon_3\| &= \| \int_0^t e^{A_1(t-s)} A_2 \epsilon_2 \, ds \| \\
\|\epsilon_3\| &\leq K_1 \int_0^t \|A_2\| \|\epsilon_2\| \, ds \\
\|\epsilon_3\| &\leq K_1 K_2 K_1 \|A_2\| \|A_1\| \|A_2\| \|\epsilon_0\|_\infty \frac{t^3}{6}. \qquad (3.13)
\end{aligned}$$

Generally, for odd i, we have

$$\|u(t) - u_i(t)\| \leq (K_2\|A_1\|)^{\frac{i-1}{2}}.(K_1\|A_2\|)^{\frac{i+1}{2}}\|\epsilon_0\|_\infty \frac{t^i}{i!} \tag{3.14}$$

and for even i, we have

$$\|u(t) - u_i(t)\| \leq (K_1\|A_1\|)^{\frac{i}{2}}.(K_2\|A_2\|)^{\frac{i}{2}}\|\epsilon_0\|_\infty \frac{t^i}{i!} \tag{3.15}$$

□

by induction. Note that in (Faragó et. al., 2008a), they give the similar error bounds implicitly, but here we write these bounds in explicit forms.

Theorem 3.2 *Let A_1, A_2 be bounded linear operators of Cauchy problem given in (2.2)-(2.3). Applying iterative splitting algorithms (2.17)- (2.20) to (2.2)-(2.3) and combining with midpoint rule results in second order consistent scheme.*

Proof The Taylor series expansion of exact solution on $[0, \Delta t]$ interval is

$$\begin{aligned} u(\Delta t) &= e^{(A_1+A_2)\Delta t} u_0 \\ &= \big(I + (A_1 + A_2)\Delta t + \frac{(A_1 + A_2)^2}{2!}\Delta t^2 \\ &\quad + \frac{(A_1 + A_2)^3}{3!}\Delta t^3 + \mathcal{O}(\Delta t^4)\big)u_0. \end{aligned} \tag{3.16}$$

For $i = 1$ applying (2.17)- (2.20) to Cauchy problem (2.2)-(2.3) and combining with midpoint rule on $[0, \ \Delta t]$ interval, we get :

$$\begin{aligned} u_1(\Delta t) &= (I - \frac{\Delta t}{2}A_1)^{-1}(I + \frac{\Delta t}{2}A_1)u_1(0) \\ &\quad + (I - \frac{\Delta t}{2}A_1)^{-1}\frac{\Delta t}{2}A_2\big(u_0(0) + u_0(\Delta t)\big), \end{aligned} \tag{3.17}$$

$$\begin{aligned} u_2(\Delta t) &= (I - \frac{\Delta t}{2}A_2)^{-1}(I + \frac{\Delta t}{2}A_2)u_2(0) \\ &\quad + (I - \frac{\Delta t}{2}A_2)^{-1}\frac{\Delta t}{2}A_1\big(u_1(0) + u_1(\Delta t)\big). \end{aligned} \tag{3.18}$$

After substituting initial values $u_1(0) = u_2(0)$ with u_0, initial guess $u_0(0) = u_0(\Delta t)$ with u_0 and plugging $u_1(\Delta t)$ into $u_2(\Delta t)$, we obtain

$$\begin{aligned} u_2(\Delta t) &= \big((I - \frac{\Delta t}{2}A_2)^{-1}(I + \frac{\Delta t}{2}A_2) + (I - \frac{\Delta t}{2}A_2)^{-1}\frac{\Delta t}{2}A_1 \\ &\quad + (I - \frac{\Delta t}{2}A_2)^{-1}\frac{\Delta t}{2}A_1(I - \frac{\Delta t}{2}A_1)^{-1}(I + \frac{\Delta t}{2}A_1) \\ &\quad + (I - \frac{\Delta t}{2}A_2)^{-1}\frac{\Delta t}{2}A_1(I - \frac{\Delta t}{2}A_1)^{-1}(\Delta t A_2)\big)u_0. \end{aligned} \tag{3.19}$$

Rearranging the terms of Equation (3.19) we get

$$\begin{aligned} u_2(\Delta t) &= \big(I - \frac{\Delta t}{2}A_1)^{-1}\big(I + \frac{\Delta t}{2}A_2 + \frac{\Delta t}{2}A_1 + \frac{\Delta t}{2}A_1\big(I + \frac{\Delta t}{2}A_1 \\ &\quad + \frac{\Delta t^2}{4}A_1^2 + \frac{\Delta t^3}{8}A_1^3 + \mathcal{O}(\Delta t^4)\big)(I + \frac{\Delta t}{2}A_1) + \frac{\Delta t}{2}A_1\big(I \\ &\quad + \frac{\Delta t}{2}A_1 + \frac{\Delta t^2}{4}A_1^2 + \frac{\Delta t^3}{8}A_1^3 + \mathcal{O}(\Delta t^4)\big)\Delta t A_2\big)u_0, \\ &= \big(I + \frac{\Delta t}{2}A_2 + \frac{\Delta t^2}{4}A_2^2 + \frac{\Delta t^3}{8}A_2^3 + \mathcal{O}(\Delta t^4)\big)\big(I + \Delta t(\frac{A_2 + A_1}{2} + \frac{A_1}{2}) \\ &\quad + \Delta t^2(\frac{A_1^2}{4} + \frac{A_1^2}{4} + \frac{A_1 A_2}{2}) + \Delta t^3(\frac{A_1^3}{8} + \frac{A_1^3}{8} + \frac{A_1^2 A_2}{4})\big)u_0, \\ &= \big(I + \Delta t(A_1 + A_2) + \Delta t^2(\frac{A_1^2}{2} + \frac{A_1 A_2}{2} + \frac{A_2^2}{2} + \frac{A_2 A_1}{2}) \\ &\quad + \Delta t^3(\frac{A_1^3}{4} + \frac{A_1^2 A_2}{4} + \frac{A_2 A_1^2}{4} + \frac{A_2 A_1 A_2}{4} + \frac{A_2^3}{8} + \frac{A_2^2 A_1}{4} + \frac{A_2^3}{8})\big)u_0 \\ &\quad + \mathcal{O}(\Delta t^4). \end{aligned} \tag{3.20}$$

Subtracting Equation (3.16) from Equation (3.20), we obtain

$$u_2(\Delta t) - u(\Delta t) = \frac{A_1^3 - 3A_1^2 A_2 - 6A_1 A_2^2 - 3A_2 A_1 A_2 - 3A_2^2 A_1 + A_2^3}{12}\Delta t^3 + \mathcal{O}(\Delta t^4).$$

Taking the norm of both sides yields

$$\|u_2(\Delta t) - u(\Delta t)\| \approx C\Delta t^3, \tag{3.21}$$

where

$$C = \begin{cases} \frac{17}{12}\|A_1\|^3, & \|A_2\| \le \|A_1\|; \\ \frac{17}{12}\|A_2\|^3, & \|A_1\| \le \|A_2\|. \end{cases}$$

Hence $\|u_2(\Delta t) - u(\Delta t)\| \to 0$ as $\Delta t \to 0$ with suitable matrix norm defined on $(\mathbb{R}^n, \|\cdot\|)$ and by definition yields the second order consistency.

Remark 3.1 *Since we use two iterations and apply midpoint rule to each scheme, the second order consistency is obtained. If we use more iterative schemes and higher order integrator, then we get higher order accuracy.*

Remark 3.2 *Since the operators A_1 and A_2 are obtained with the expansion of spatial derivative terms in PDEs, they include spatial discretization step. In order to avoid to reduce the order, the balance between time discretization step and space discretization step is important.*

□

3.2. Consistency Analysis for Unbounded Operators via C_0 Semigroup

So far we have considered the bounded operators consistency. Now we examine the consistency for unbounded operators. Theorem 3.6 proves the first order consistency for first iterative scheme and Theorem 3.7 proves second order consistency for the second iterative scheme. $\|\cdot\|$ is any norm defined in X Banach space and $\|\cdot\|_{L(X)}$ is corresponding induced operator norm.

3.2.1. Semigroup Theory

Consider the abstract homogenous Cauchy problem in a Banach space X

$$u'(t) = Au(t), \quad t \in [0,T] \tag{3.22}$$

$$u(0) = u_0 \tag{3.23}$$

where $u \in X$ and $A : X \to X$ is a linear operator. If A is a bounded operator then the solution is $u(t) = e^{At}u(0)$. On the other hand, when A is an unbounded linear operator in Banach space then $u(t)$ can not be expressed as $e^{At}u(0)$. Hence we ascertain some reasonable conditions on the operator A so that operator A generates a C_0 semigroup. This implies the existence of a unique solution of Cauchy problem (3.22)-(3.23) for each initial point $u_0 \in X$. Before explaining these conditions, we shall give some necessary definitions, lemmas and theorems about this semigroup theory.

Definition 3.1 *A family $\{S(t)\}_{t\geq 0}$ of bounded linear operators on a Banach space X, such as $S : \mathbb{R}_+ \to L(X)$, is called **strongly continuous semigroup** or C_0 **semigroup** if the following conditions are satisfied:*

(i) $S(0)u = u$ *($u \in X$)*

(ii) $S(t+s)u = S(t)S(s)u = S(s)S(t)u$ *(for $\forall\, t, s \geq 0$, $u \in X$).*

(iii) $\lim_{t\to 0^+} S(t)u \to u$ *for each $u \in X$ with respect to the norm on X.*

The first two axioms are algebraic and state that S is a representation of semigroup, the last is topological, and states that the map S is continuous in the strong operator topology.

Lemma 3.1 *For every strongly continuous semigroup $\{S(t)\}_{t\geq 0}$, there exists constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that*

$$\|S(t)\|_{L(X)} \leq Me^{\omega t} \tag{3.24}$$

for all $t \geq 0$.

Proof See (Chapter I, Proposition 1.4, (Engel and Nagel, 2006)). □

The real problem is to determine which operators A generates semigroups, which is answered after recording some further general facts.

Definition 3.2 *The* ***generator*** $A : D(A) \subseteq X \to X$ *of a strongly continuous semigroup* $\{S(t)\}_{t\geq 0}$ *on a Banach space* X *is the operator*

$$Au = \lim_{t\to 0^+} \frac{S(t)u - u}{t} \quad (u \in D(A)) \tag{3.25}$$

and defined for every u *in its* ***domain***

$$D(A) := \{u \in X : \lim_{t\to 0^+} \frac{S(t)u - u}{t} \; exists \; in \; X\}. \tag{3.26}$$

Lemma 3.2 *The generator* $(A, D(A))$ *of a strongly continuous semigroup* $\{S(t)\}_{t\geq 0}$ *has the following properties.*

(i) $A : D(A) \subseteq X \to X$ *is a linear operator.*

(ii) *If* $u \in D(A)$ *then* $S(t)u \in D(A)$ *and* $\frac{d}{dt}S(t)u = S(t)Au = AS(t)u$ *for all* $t \geq 0$.

(iii) *For every* $t \geq 0$ *and* $u \in X$, $\int_0^t S(s)u\,dx \in D(A)$.

(iv) *For every* $t \geq 0$

$$\begin{aligned} S(t)u - u &= A\int_0^t S(s)u\,dx \;\; \text{if } u \in X, \\ &= \int_0^t S(s)Au\,dx \;\; \text{if } u \in D(A). \end{aligned} \tag{3.27}$$

Proof See (Chapter II, Sec.1, Lemma 1.3, (Engel and Nagel, 2000)), (Section 7.4, Theorem 1, (Lawrence, 1998)). □

Lemma 3.3 *The generator of a strongly continuous semigroup is a closed and densely defined linear operator that determines the semigroup uniquely.*

Proof See (Chapter II, Sec.1, Theorem 1.4, (Engel and Nagel, 2000)). □

In order to retrieve the semigroup $\{S(t)\}_{t\geq 0}$ from its generator $(A, D(A))$, we need the resolvent operator which is given in the following definitions.

Definition 3.3 **(i)** *The **resolvent set of** A (i.e. $\rho(\mathbf{A})$) is the set of complex numbers λ such that the operator*

$$\lambda I - A : D(A) \to X$$

is one-to-one and onto.

(ii) *The **resolvent operator of** A (i.e. R_λ) is the operator defined by*

$$R_\lambda = (\lambda I - A)^{-1}. \tag{3.28}$$

According to the Closed Graph Theorem (see Appendix A), R_λ is a bounded linear operator.

Lemma 3.4 *If $\lambda, \mu \in \rho(A)$ then we have*

$$R_\lambda - R_\mu = (\lambda - \mu)R_\lambda R_\mu \;\; (\textit{resolvent identity}) \tag{3.29}$$

and $R_\lambda R_\mu = R_\mu R_\lambda$.

Proof See (Section 7.4, Theorem 3, (Lawrence, 1998)), (Engel and Nagel, 2000). □

It is seen that resolvent operator is the Laplace transform of semigroup.

Lemma 3.5 *Let $\{S(t)\}_{t\geq 0}$ be a strongly continuous semigroup on the Banach space X and take constants $\omega \in R$, $M \geq 1$ such that*

$$\|S(t)\|_{L(X)} \leq Me^{\omega t} \tag{3.30}$$

for $t \geq 0$. For the generator $(A, D(A))$ of $\{S(t)\}_{t\geq 0}$ the following properties hold.

(i) *If $\lambda \in \mathbb{C}$ such that $R_\lambda u = \displaystyle\int_0^\infty e^{-\lambda s} S(s) u ds$ exists for all $u \in X$, then $\lambda \in \rho(A)$.*

(ii) *If $Re\lambda \geq \omega$, then $\lambda \in \rho(A)$ and the resolvent is given by the integral expression in (i).*

(iii) $\|R_\lambda\|_{L(X)} \leq \dfrac{M}{Re\lambda - \omega}$ *for all $Re\lambda \geq \omega$.*

Proof See (Chapter II, Sec.1, Theorem 1.10, (Engel and Nagel, 2000)). □

To state the various relations between the objects that we have seen so far, we illustrate the following triangle.

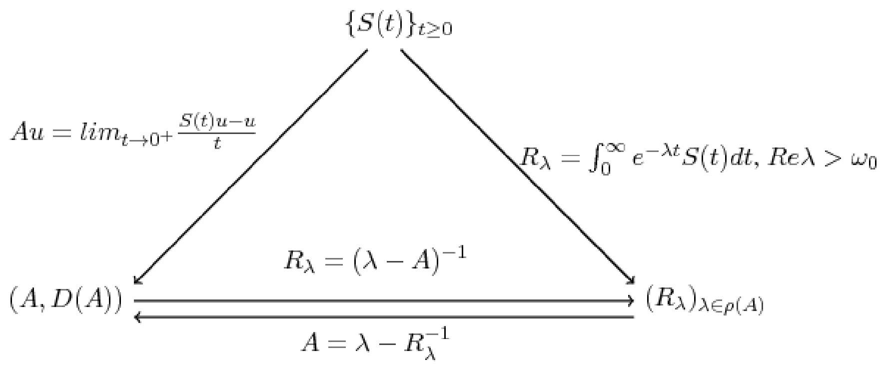


Figure 3.1. The relations between a semigroup, its generator and its resolvent.

Theorem 3.3 (Hille-Yosida) *Let A be closed, densely-defined linear operator on X. Then A is the generator of a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ if and only if*

$$(\omega, \infty) \subset \rho(A), \ \ and \ \|R_\lambda\|_{L(X)} \leq \frac{M}{Re\lambda - \omega} \ for \ Re\lambda > \omega. \tag{3.31}$$

Proof See (Section 7.4, Theorem 4, (Lawrence, 1998)), (Chapter II, Sec.3, (Engel and Nagel, 2000)). □

Theorem 3.4 *The abstract Cauchy problem (3.22)-(3.23) is well posed if and only if A is the (infinitesimal) generator of a C_0 semigroup $\{S(t)\}_{t\geq 0}$. In this case, the solution of*

(3.22)-(3.23) is given by

$$u(t) = S(t)u_0,\ t \geq 0. \tag{3.32}$$

Proof See (Chapter II, Theorem 1.2, (Goldstein, 1985)). $\square$

In the following example, we demonstrate that a certain second order partial differential equation can be realized within the semigroup framework.

Example 3.1 (Section 7.4.3, Applications a, (Lawrence, 1998)) *Consider the second order parabolic initial boundary value problem*

$$\begin{aligned} u_t &= Au,\ \textit{in}\ \Omega, \\ u &= g,\ \textit{on}\ \Omega \times \{t = 0\}, \\ u &= 0,\ \textit{on}\ \partial\Omega \times [0, T]. \end{aligned} \tag{3.33}$$

We assume that $A : D(A) \subset X \to X$ has the divergence structure, (see Appendix B), satisfies the usual strong ellipticity condition, has smooth coefficients, which do not depend on t. We also assume that bounded open set Ω has a smooth boundary. Recall the energy estimate

$$\beta\|u\|^2_{H^1_0(\omega)} \leq B[u, u] + \gamma\|u\|_{L^2(\omega)}, \tag{3.34}$$

for constants $\beta > 0$, $\gamma \geq 0$, where $B[u, u]$ is the bilinear form associated with A. We want to reinterpret (3.33) as the flow determined by a semigroup on $X = L^2(\Omega)$ Banach space induced with L^2 norm. For this purpose, we let

$$D(A) = H^1_0(\Omega) \cap H^2(\Omega). \tag{3.35}$$

Clearly A is an unbounded linear operator on X. We must verify the hypotheses the variant of Theorem 3.3 (Hille-Yosida).

1. *First, we shall show $D(A) = H^1_0(\Omega) \cap H^2(\Omega)$ is dense in $L^2(\Omega)$. Which means: let* *any point* $\{u_n\}_{n=0}^{\infty} \in L^2(\Omega)$, *then* u_n *or the limit point of* u_n *must be in* $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$. *Hence* $D(A)$ *is dense in* X. *We know subspace* C_0^{∞} *is dense in* $L^2(\Omega)$ *and in* $H_0^1(\Omega) \cap H^2(\Omega)$, *(see Lemma 4.2, Lemma 6.4, (Prokert, 2005)). Also we know* $H_0^1(\Omega) \cap H^2(\Omega)$, *is subspace of* $L^2(\Omega)$. *This implies* $H_0^1(\Omega) \cap H^2(\Omega)$ *is dense in* $L^2(\Omega)$.

2. *We shall prove that the operator* A *is closed.*

Let $\{u_n\}_{n=0}^{\infty} \in D(A) = H_0^1(\Omega) \cap H^2(\Omega)$ *such that*

$$u_n \to u, \;\; Au_n \to f \;\; in \;\; L^2(\Omega), \tag{3.36}$$

then $u \in D(A)$ *and* $Au = f$. *According to the regularity in (Theorem 4, Section 6.3.2, (Lawrence, 1998)), we have*

$$\|u_n - u_l\|_{H^2(\Omega)} \leq C(\|Au_n - Au_l\|_{L^2(\Omega)} + \|u_n - u_l\|_{L^2(\Omega)})$$

for all n, l. *Thus Equation (3.37) implies* $\{u_n\}_{n=1}^{\infty}$ *is a Cauchy sequence in* $H^2(\Omega)$ *so we have*

$$u_n \to u \;\; in \;\; H^2(\Omega). \tag{3.37}$$

Therefore $u \in D(A)$, $Au_n \to Au$ *in* $L^2(\Omega)$, *and consequently* $f = Au$.

3. *Next, we shall check the resolvent condition, with* γ *replacing* ω. $\lambda \in \mathbb{C}$ *belongs to the resolvent set of* A, $\rho(A)$, *if*

$$\begin{aligned} (\lambda I - A)u &= f, \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{3.38}$$

has a unique weak solution $u \in H_0^1(\Omega)$ *for each* $f \in L^2(\Omega)$.

By (Theorem 3, Section 6.2.2, (Lawrence, 1998)), see (Appendix B), it is shown (3.38) has a unique weak solution. Owing to the elliptic regularity theory, in fact $u \in H_0^1(\Omega) \cap H^2(\Omega)$ *and thus* $u \in D(A)$. *Also* $(\lambda I - A) : D(A) \to X$ *is one-to-one* *and onto, provided $\lambda \geq \gamma$. Hence $\rho(A) \supset [\gamma, \infty)$.*

4. *Finally, we need to show the resolvent operator is bounded. The weak form of Equation (3.38) is*

$$(\lambda u, v) + B[u, v] = (f, v), \tag{3.39}$$

for each $v \in H_0^1(\Omega)$, where $B[u, v]$ is the bilinear form associated with A and $(\, , \,)$ is the inner product in $L^2(\Omega)$. Recall the energy estimate given in (Section 6.2.2, (Lawrence, 1998)). Set $v = u$ then for $\lambda > \gamma$:

$$(\lambda - \gamma)\|u\|^2_{L^2(\Omega)} \leq \|f\|^2_{L^2(\Omega)}\|u\|_{L^2(\Omega)}. \tag{3.40}$$

Hence, we have

$$\|R_\lambda f\|^2_{L^2(\Omega)} \leq \frac{1}{(\lambda - \gamma)}\|f\|_{L^2(\Omega)} \tag{3.41}$$

since $u = R_\lambda f$. This bound is valid for all $f \in L^2(\Omega)$ and $\|R_\lambda\|_{L^2(\Omega)} \leq \frac{1}{\lambda - \gamma}$ $(\lambda > \gamma)$.

Hence all the criterias of Theorem 3.3 (Hille-Yosida) are satisfied and A generates a C_0 semigroup $\{S(t)\}_{t \geq 0}$.

In the remainder of this subsection, we assume that A is a closed, densely-defined linear operator on X Banach space, which satisfies the resolvent condition (3.31). Hence A generates a C_0 semigroup $\{S(t)\}_{t \geq 0}$. Then Cauchy problem (3.22)-(3.23) has a unique solution

$$u(t) = S(t)u_0, \;\; t > 0, \tag{3.42}$$

for any $u_0 \in D(A)$. Or it can be written recursively at $t = t^n$ with time step h as

$$u(t^n) = S(h)u(t^{n-1}) = E^h u(t^{n-1}) \tag{3.43}$$

where $E^h = S(h)$ is the exact solution operator.

Consider the abstract nonhomogeneous Cauchy problem in a Banach space X

$$u'(t) = Au(t) + f, \quad t \in [0, T] \tag{3.44}$$

$$u(0) = u_0; \tag{3.45}$$

if $u_0 \in D(A)$ and $f \in C'([0, T]; X)$, then the Cauchy problem has a unique solution on $[0, T]$, (see (Kato, 1980)), given by

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)\, ds. \tag{3.46}$$

If F is

$$F(t, \tau) = \int_\tau^t S(t-s)f(s)\, ds, \quad t \geq \tau, \tag{3.47}$$

Equation (3.46) reduces to

$$u(t) = S(t)u_0 + F(t, 0). \tag{3.48}$$

Now, consider the Cauchy problem (2.2)-(2.3). Let A_1 and A_2 be (infinitesimal) generators of C_0 semigroups $S_1(t)$, $S_2(t)$ on a Banach space X satisfying

$$A_1 + A_2 = A, \quad D(A_1^k) = D(A_2^k) = D(A^k), \quad k = 1, 2, 3. \tag{3.49}$$

Then

$$D_k = D(A_1^k) \cap D(A_2^k) \cap D(A^k), \quad k = 1, 2, 3 \text{ are dense in } X$$
$$\text{and } \; A_i^k \mid_{D_k}, i = 0, 1, 2, \; k = 1, 2, 3, \text{closed operators } (A_0 = A). \tag{3.50}$$

Remark 3.3 *If we assume* $D(A_1^k) = D(A_2^k) = D(A^k), \ \ k = 1, 2, 3,$ *and the resolvent set* $\rho(A_i), \ \ i = 0, 1, 2$ *are not empty, as it is assumed for* $k = 1, 2$ *in (Bjørhus, 1998), the (3.50) are satisfied. (See (Hille and Phillips, 1957) and (Engel and Nagel, 2000), appendix B, B. 14)*

Rewriting the Cauchy problem (2.2)-(2.3) yields

$$u'(t) \ = \ A_1 u(t) + A_2 u(t), \ \ u(0) = u_0 \ \ t \in [0, T]. \tag{3.51}$$

The solution of (3.51) is given by

$$u(t) = S_1(t)u_0 + F_0(t, 0) \tag{3.52}$$

where

$$F_0(t, \tau) = \int_{\tau}^{t} S_1(t - s)A_2 S(s)u_0 \, ds. \tag{3.53}$$

Or the solution of nonhomogeneous problem (3.51) is given by

$$u(t) = S_2(t)u_0 + F_1(t, \tau) \tag{3.54}$$

where

$$F_1(t, \tau) = \int_{\tau}^{t} S_2(t - s)A_1 S(s)u_0 \, ds. \tag{3.55}$$

Next we apply and analyze the iterative splitting sechemes (2.17)-(2.20) to have approximate solutions of the Cauchy problem (2.2)-(2.3). We examine first and second order iterative schemes.

3.2.2. Solution to First Iterative Scheme

The first iterative scheme is given

$$u_1'(t) \;=\; A_1 u_1(t) + A_2 u_0(t), \quad u_1(0) = u_0 \tag{3.56}$$

for $[0, t]$ interval and $u(t)$ is approximated with $u_1(t)$. Applying variation of constants formula to Equation (3.56) we have

$$u_1(t) = S_1(t)u_0 + \int_0^t S_1(t-s)A_2 u_0 \, ds \tag{3.57}$$

where $u_1(t)$ is the approximate solution to Cauchy problem. If

$$F_2(t, \tau) = \int_\tau^t S_1(t-s)A_2 u_0 \, ds, \quad 0 < \tau < t \tag{3.58}$$

then Equation (3.57) can be rewritten as

$$u_1(t) = S_1(t)u_0 + F_2(t, 0). \tag{3.59}$$

Denoting the first iterative solution $u_1(t^n)$ by U^n at $t = t^n$ time, then recursive relation is

$$U^{n+1} = S_1(h)U^n + \int_{t^n}^{t^{n+1}} S_1(t^{n+1} - s)A_2 U^n \, ds \tag{3.60}$$

where $U^0 \approx u_0$.

3.2.3. Solution to Second Iterative Scheme

The second iterative scheme is given

$$u_2'(t) \quad = \quad A_1 u_1(t) + A_2 u_2(t), \quad u_2(0) = u_0 \tag{3.61}$$

for $[0, t]$ interval and $u(t)$ is approximated with $u_2(t)$. Applying variation of constants formula to Equation (3.61) we have

$$u_2(t) = S_2(t)u_0 + \int_0^t S_2(t-s)A_1 u_1(s)\, ds \tag{3.62}$$

where $u_2(t)$ is the approximate solution. If

$$F_3(t,\tau) = \int_\tau^t S_2(t-s)A_1 u_1(s)\, ds, \quad 0 < \tau < t \tag{3.63}$$

then Equation (3.62) can be written as

$$u_2(t) = S_2(t)u_0 + F_3(t,0). \tag{3.64}$$

Denoting the second iterative solution $u_2(t^n)$ by U^n at $t = t^n$ time, then recursive relation is

$$U^{n+1} = S_2(h)U^n + \int_{t^n}^{t^{n+1}} S_2(t^{n+1} - s)A_1 u_1^n(s)\, ds \tag{3.65}$$

where $U^0 \approx u_0$. Substituting Equation (3.57) into Equation (3.62) yields

$$\begin{aligned} u_2(t) \quad &= \quad S_2(t)u_0 + \int_0^t S_2(t-s)A_1 S_1(s)u_0 ds \\ &+ \quad \int_0^t S_2(t-s)A_1 \int_0^s S_1(s-\tau)A_2 u_0\, d\tau\, ds, \end{aligned} \tag{3.66}$$

and recursively

$$U^{n+1} = S_2(h)U^n + \int_{t^n}^{t^{n+1}} S_2(t^{n+1}-s)A_1 S_1(s)U^n ds$$
$$+ \int_{t^n}^{t^{n+1}} S_2(t^{n+1}-s)A_1 \int_0^s S_1(s-\tau)A_2 U^n \, d\tau \, ds. \quad (3.67)$$

3.2.4. Consistency of Iterative Operator Splitting Method

Recall the following consistency definitions and results for iterative splitting method.

Definition 3.4 *Define $T_h : X \times [0, T-h] \to X$ by*

$$T_h(u_0, t) = S(h)u(t) - S_{iter}(h)u(t), \quad (3.68)$$

*where $u(t)$ is given in Equation (3.42), $S_{iter}(h)u(t)$ is in Equations (3.59) and (3.64). For each u_0, t, the difference $T_h(u_0, t)$ is called the **local truncation error** of the iterative splitting method.*

Definition 3.5 *The iterative splitting method is said to be **consistent** on $[0, T]$ if*

$$\lim_{h\to 0} \sup_{0\le t_n \le T-h} \frac{\|T_h(u_0, t_n)\|}{h} = 0 \quad (3.69)$$

whenever $u_0 \in D_k$ where D_k is a dense subspace of X.

Definition 3.6 *If the consistency relation (3.69) holds, we have*

$$sup_{0\le t_n\le T-h} h^{-1} \|T_h(u_0, t_n)\| = \mathcal{O}(h^p) \quad p > 0, \quad (3.70)$$

*and the method is said to be **consistent of order p**.*

Theorem 3.5 *For any C_0 semigroups $\{S(t)\}_{t\ge 0}$ of bounded linear operators with corresponding (infinitesimal) generator A, we have the Taylor series expansion*

$$S(t)u = \sum_{j=0}^{n-1} \frac{t^j}{j!} A^j u + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} S(s) A^n u \, ds, \quad \textit{for all } u \in D(A^n). \quad (3.71)$$

Proof See (Section 11.8, (Hille and Phillips, 1957)). □

For $n = 3, 2$ and 1 we get the relations,

$$S(h)u = u + hAu + \frac{h^2}{2} A^2 u + \frac{1}{2} \int_0^h (h-s)^2 S(s) A^3 u ds, \quad (3.72)$$

$$S(h)u = u + hAu + \int_0^h (h-s) S(s) A^2 u ds, \quad (3.73)$$

$$S(h)u = u + \int_0^h S(s) A u ds, \quad (3.74)$$

respectively.

Lemma 3.6 *Let A (resp. B) be a closed linear operator from $D(A) \subset X$ (resp. $D(B) \subset X$) into X. If $D(A) \subset D(B)$, then there exists a constant **C** such that*

$$\|Bu\| \leq C(\|Au\| + \|u\|) \quad \textit{for all } u \in D(A). \quad (3.75)$$

This implies that there exists a constant C that is for $u \in D_k$, $k = 1, 2, 3$

$$\|A_i^k u\| \leq C(\|A_j^k u\| + (\|u\|), \quad i, j = 0, 1, 2, \quad (3.76)$$

where D_k is given in Equation (3.50). (Note that in our case $A_0 = A$.)

Proof See (Chapter II.6, Theorem 2, (Yosida, 1980).) □

Lemma 3.7 *Let A be (infinitesimal) generator of a C_0 semigroup $\{S(t)\}_{t \geq 0}$. Let $T > 0$ and $n \in N$ arbitrary. If $f, Af \in C'([0,T]; X)$, then u in Equation (3.46) satisfies $u(t) \in$*

$D(A^n)$ for $0 \leq t \leq T$ whenever $u_0 \in D(A^n)$, and we have

$$sup_{0\leq t\leq T}\|A^k u(t)\| \leq C_k(T), \quad k = 0, 1, 2, \ldots, n, \tag{3.77}$$

where $C_k(T)$ depends on the choices of T, A, f and u_0.

Proof See (Bjørhus, 1998). □

3.2.4.1. Consistency of First Iterative Scheme

To show the consistency of first iterative scheme, we must argue that the local truncation error

$$T_h = S(h)u(t) - S_1(h)u(t) - F_2(t+h, t) \tag{3.78}$$

which appears inside the norm in Equation (3.70), is $\mathcal{O}(h^2)$. With the aid of Equation (3.52), T_h can be rewritten as

$$\begin{aligned} T_h &= S_1(h)u(t) + F_0(t+h, t) - S_1(h)u(t) - F_2(t+h, t) \\ &= F_0(t+h, t) - F_2(t+h, t). \end{aligned} \tag{3.79}$$

By means of the integral representations of F_0 and F_2 in Equations (3.53) and (3.58), respectively, we obtain

$$\begin{aligned} F_0(h, 0) - F_2(h, 0) &= \int_0^h S_1(h-s)A_2S(s)u_0ds - \int_0^h S_1(h-s)A_2u_0ds \\ &= \int_0^h S_1(h-s)(A_2S(s)u_0 - A_2u_0)ds. \end{aligned} \tag{3.80}$$

This expression motives the following proposition.

Proposition 3.1 *Let A (resp. A_1, A_2) be an (infinitesimal) generator of a C_0 semigroup $\{S(s)\}_{s\geq 0}$ (resp. $S_1(s)$, $S_2(s)$) that satisfies (3.49) with $T > 0$, then*

$$\|A_2S(h)u - A_2u\| \leq hC(T)(\|Au\| + \|u\|), \quad 0 \leq h \leq T, \tag{3.81}$$

whenever $u \in D_k$, where $C(T)$ is constant independent of h.

Proof Let (3.49) be satisfied. For $u \in D_k$, by using a fundamental property of semigroups in (Theorem $2.4b, p5$, (Pazy, 1983)) and Estimate (3.74), we obtain

$$\begin{aligned} A_2S(h)u - A_2u &= A_2u + A_2\int_0^h S(s)Auds - A_2u\,, \\ &= A_2\int_0^h S(s)Auds. \end{aligned} \tag{3.82}$$

Taking the norm of (3.82) yields

$$\|A_2S(h)u - A_2u\| = \|A_2\int_0^h S(s)Auds\|. \tag{3.83}$$

Starting with the right hand side of (3.83) and using Lemma 3.6 yield

$$\|A_2\int_0^h S(s)Auds\| \leq C_1(\|A\int_0^h S(s)Auds\| + \|\int_0^h S(s)Auds\|) \tag{3.84}$$

where

$$A\int_0^h S(s)Auds = S(h)Au - Au = \int_0^h S(s)A^2uds. \tag{3.85}$$

Substituting Equations (3.84), (3.85) into (3.83) and taking the norm, we obtain

$$\|A_2S(h)u - A_2u\| \leq C_t(\|\int_0^h S(s)A^2uds\| + \|\int_0^h S(s)Auds\|. \tag{3.86}$$

For each terms on the right hand side of Equation (3.86), by using Lemma 3.6 and Equation (3.24), we obtain

$$\|\int_0^h S(s)A^2uds\| \leq hMe^{|\omega_1|h}\|A^2u\| \leq hC_1(T)\|A^2u\|, \quad 0 \leq h \leq T, \quad (3.87)$$

$$\|\int_0^h S(s)Auds\| \leq hMe^{|\omega_1|h}\|Au\| \leq hC_2(T)\|Au\|, \quad 0 \leq h \leq T. \quad (3.88)$$

Consequently, we have

$$\|A_2S(h)u - A_2u\| \leq hC_3(T)(\|A^2u\| + \|Au\|), \quad (3.89)$$

where $C_3(T) = max\{C_1(T), C_2(T)\}$ is independent of h. □

Later, applying Proposition 3.1 and Lemma 3.7 to truncation error Estimate (3.80) yields

$$\begin{aligned}
\|\int_0^h S_1(h-s)(A_2S(s)u_0 - A_2u_0)ds\| &\leq \int_0^h \|S_1(h-s)(A_2S(s)u_0 - A_2u_0)\|ds \\
&\leq \int_0^h \|S_1(h-s)\|\|A_2S(s)u_0 - A_2u_0\|ds \\
&\leq M_1e^{\omega_1 h}\int_0^h \|A_1S(s)u_0 - A_1u_0\|ds \\
&\leq M_1e^{\omega_1 h}\int_0^h sC_3(T)(\|A^2u_0\| + \|Au_0\|)ds \\
&\leq M_1e^{\omega_1 h}C_3(T)\frac{h^2}{2}(\|A^2u_0\| + \|Au_0\|) \\
&\leq h^2\hat{C}_3(T)(\|A^2u_0\| + \|Au_0\|). \quad (3.90)
\end{aligned}$$

Thus we have proved the following Theorem 3.6.

Theorem 3.6 *Let A (resp. A_1, A_2) be an (infinitesimal) generator of a C_0 semigroup $\{S(t)\}_{t\geq 0}$ (resp. $S_1(t)$, $S_2(t)$) that satisfies the Equation (3.49) with $T > 0$. Then the first* *iterative solution (3.59) for any $u_0 \in D_k$ meets the uniform bound*

$$\|u(t) - u_1(t)\| \leq t^2\, C(T), \tag{3.91}$$

where $C(T)$ is a constant independent of t.

3.2.4.2. Consistency of Second Iterative Scheme

To show consistency of second iterative scheme, we must argue that the local truncation error

$$T_h = S(h)u(t) - S_2(h)u(t) - F_3(t+h,t), \tag{3.92}$$

which appears inside the norm in Equation (3.70), is $\mathcal{O}(h^3)$. With the aid of Identity (3.54), T_h can be rewritten as

$$\begin{aligned} T_h &= S_2(h)u(t) + F_1(t+h,t) - S_2(h)u(t) - F_3(t+h,t) \\ &= F_1(t+h,t) - F_3(t+h,t). \end{aligned} \tag{3.93}$$

By means of the integral representations of F_1, F_3 in Equations (3.55) and (3.63), respectively, we obtain

$$\begin{aligned} F_1(h,0) - F_3(h,0) &= \int_0^h S_2(h-s)A_1u(s)ds - \int_0^h S_2(h-s)A_1u_1(s)ds \\ &= \int_0^h S_2(h-s)(A_1u(s) - A_1u_1(s))ds. \end{aligned} \tag{3.94}$$

Rearranging Equation (3.94) and using Estimate (3.74), we obtain

$$\begin{aligned}
A_1u(s) - A_1u_1(s) &= A_1(S_1(s)u_0 + \int_0^s S_1(s-\tau)A_2u(\tau)d\tau)) \\
&\quad -A_1(S_1(s)u_0 + \int_0^s S_1(s-\tau)A_2u_0d\tau) \\
&= A_1\int_0^s S_1(s-\tau)A_2S(\tau)u_0d\tau - A_1\int_0^s S_1(s-\tau)A_2u_0d\tau \\
&= \int_0^s S_1(s-\tau)A_1A_2(u_0 + \int_0^\tau S(\tau-\sigma)Au_0d\sigma)d\tau \\
&\quad -\int_0^s S_1(s-\tau)A_1A_2u_0d\tau \\
&= \int_0^s S_1(s-\tau)A_1A_2\int_0^\tau S(\tau-\sigma)Au_0d\sigma d\tau. \qquad (3.95)
\end{aligned}$$

Therefore

$$\begin{aligned}
\|A_1u(s) - A_1u_1(s)\| &\leq \|\int_0^s S_1(s-\tau)A_1A_2\int_0^\tau S(\tau-\sigma)Au_0d\sigma d\tau\|, \\
&\leq \int_0^s \|S_1(s-\tau)\|\|A_1A_2\int_0^\tau S(\tau-\sigma)Au_0d\sigma\|d\tau, \\
&\leq M_1e^{\omega_1h}\int_0^s \|A_1A_2\int_0^\tau S(\tau-\sigma)Au_0d\sigma\|d\tau. \qquad (3.96)
\end{aligned}$$

Applying Lemma 3.7, we get

$$
\begin{aligned}
\|A_1A_2\int\limits_0^\tau S(\tau-\sigma)Au_0d\sigma\| &\leq \|A_2^2\int\limits_0^\tau S(\tau-\sigma)Au_0d\sigma\| + \|A_2\int\limits_0^\tau S(\tau-\sigma)Au_0d\sigma\| \\
&\leq \|A^2\int\limits_0^\tau S(\tau-\sigma)Au_0d\sigma\| + \|\int\limits_0^\tau S(\tau-\sigma)Au_0d\sigma\| \\
&\quad +\|A\int\limits_0^\tau S(\tau-\sigma)Au_0d\sigma\| + \|\int\limits_0^\tau S(\tau-\sigma)Au_0d\sigma\| \\
&\leq \|\int\limits_0^\tau S(\tau-\sigma)A^3u_0d\sigma\| + \|\int\limits_0^\tau S(\tau-\sigma)A^2u_0d\sigma\| \\
&\quad +2\|\int\limits_0^\tau S(\tau-\sigma)Au_0d\sigma\|. \qquad (3.97)
\end{aligned}
$$

Rearranging Equation (3.97), we get

$$
\begin{aligned}
\|A_1A_2\int\limits_0^\tau S(\tau-\sigma)Au_0d\sigma\| &\leq \tau Me^{\omega h}\|A^3u_0\| + \tau Me^{\omega h}\|A^2u_0\| + \tau Me^{\omega h}\|Au_0\| \\
&\leq \tau(C_1(T)\|A^3u_0\| + C_2(T)\|A^2u_0\| + C_3(T)\|Au_0\|) \\
&\leq \tau C_4(T)(\|A^3u_0\| + \|A^2u_0\| + \|Au_0\|), \qquad (3.98)
\end{aligned}
$$

where $C_4(T) = max\{C_1(T), C_2(T), C_3(T)\}$.
Going back to Equation (3.96) we have

$$
\begin{aligned}
\|A_1u(s) - A_1u_1(s)\| &\leq M_1e^{\omega_1 h}\int\limits_0^s \|A_1A_2\int\limits_0^\tau S(\tau-\sigma)Au_0d\sigma\|d\tau \\
&\leq M_1e^{\omega_1 h}\int\limits_0^s \tau C_4(T)(\|A^3u_0\| + \|\|\|A^2u_0\| + \|Au_0\|)d\tau \\
&\leq s^2C_5(T)(\|A^3u_0\| + \|A^2u_0\| + \|Au_0\|). \qquad (3.99)
\end{aligned}
$$

Taking the norm of both sides of Equation (3.94) and then using the bound obtained in Equation (3.99), we get

$$\begin{aligned}\|F_1(h,0)-F_3(h,0)\| &\leq \int_0^h \|S_2(h-s)\|\|(A_1u(s)-A_1u_1(s))\|ds \\ &\leq M_2e^{\omega_2 h}\int_0^h s^2C_5(T)(\|A^3u_0\|+\|A^2u_0\|+\|Au_0\|)ds \\ &\leq h^3\hat{C}(T)(\|A^3u_0\|+\|A^2u_0\|+\|Au_0\|). \qquad (3.100)\end{aligned}$$

Thus we have proved the following Theorem 3.7.

Theorem 3.7 *Let A (resp. A_1, A_2) be an (infinitesimal) generator of a C_0 semigroup $\{S(t)\}_{t\geq 0}$ (resp. $S_1(t)$, $S_2(t)$) that satisfies the Equation (3.49) with $T>0$. Then the second iterative solution (3.64) for any $u_0 \in D_k$ meets the uniform bound*

$$\|u(t)-u_2(t)\| \leq t^3\,\hat{C}(T) \qquad (3.101)$$

where $\hat{C}(T)$ is a constant independent of t.

CHAPTER 4

STABILITY ANALYSIS OF ITERATIVE OPERATOR SPLITTING METHOD

In this chapter, we study the stability analyses of iterative operator splitting method by using various techniques when applied to the Cauchy problem (2.2)-(2.3). The analyses depend on whether the operators of the problems are bounded or unbounded. Boundedness is obtained using finite difference expansion on spatial derivatives, which will be explained in detailed in Section 6. Then we prove the stability of solutions by constructing stability functions, see (Gücüyenen and Tanoğlu, 2011b). When the operators are unbounded, Fourier transform and C_0 semigroup approaches are studied. First, we examine the stability analyses of iterative splitting solutions using Fourier transform for three special problems. These are one dimensional advection-diffusion equation, two dimensional solute transport equation and one dimensional nonlinear KdV equation, see (Gücüyenen and Tanoğlu, 2011a). Next, C_0 semigroup theory properties are used to prove the stability of the first and the second iterative splitting solutions.

4.1. Stability Analysis for Bounded Operators

Theorem 4.1 derives a stability function bound for each iterative splitting scheme (2.17)-(2.20) when applied to Cauchy problem (2.2)-(2.3), see (Gücüyenen and Tanoğlu, 2011b).

Theorem 4.1 *Let A_1, A_2 be bounded linear operators with Cauchy problem given in (2.2)- (2.3) . The midpoint rule applied to the iterative schemes (2.17)- (2.20) with $Z_1 = \Delta t A_1$ and $Z_2 = \Delta t A_2$ are stable if and only if there exists functions $R_i(Z_1, Z_2)$ such that*

$$\forall\ Z_1, Z_2 \in \mathbb{R}^{n\times n},\ \|R_i(Z_1, Z_2)\| \leq 1 + K\Delta t, \tag{4.1}$$

where K is a positive real constant (independent of Δt, Δx). $\|\cdot\|$ is any norm defined on $\mathbb{R}^n$.

Proof The midpoint rule applied to the iterative schemes (2.17)- (2.20) gives

$$u_i^{n+1} = (I - \frac{\Delta t}{2}A_1)^{-1}(I + \frac{\Delta t}{2}A_1)u_i^n + (I - \frac{\Delta t}{2}A_1)^{-1}\frac{\Delta t}{2}A_2(u_{i-1}^n + u_{i-1}^{n+1}) \quad (4.2)$$

$$u_{i+1}^{n+1} = (I - \frac{\Delta t}{2}A_2)^{-1}(I + \frac{\Delta t}{2}A_2)u_{i+1}^n + (I - \frac{\Delta t}{2}A_2)^{-1}\frac{\Delta t}{2}A_1(u_i^n + u_i^{n+1}). \quad (4.3)$$

For $i = 1$ we have that

$$u_1^{n+1} = R(Z_1)u_1^n + \Delta t(I - \frac{Z_1}{2})^{-1}\frac{A_2}{2}(u_0^n + u_0^{n+1}) \quad (4.4)$$

$$u_2^{n+1} = R(Z_2)u_2^n + \Delta t(I - \frac{Z_2}{2})^{-1}\frac{A_1}{2}(u_1^n + u_1^{n+1}) \quad (4.5)$$

where

$$R(Z) = (I - \frac{Z}{2})^{-1}(I + \frac{Z}{2}) \quad (4.6)$$

and $Z_1 = \Delta t A_1$, $Z_2 = \Delta t A_2$.

After setting $u_1^n = u_2^n = u^n$ and initializing with $u_0^{n+1} = u_0^n = u^n$, then for the first Equation (4.4), we have the following stability equation

$$u_1^{n+1} = (R(Z_1) + \Delta t(I - \frac{Z_1}{2})^{-1}A_2)u^n = \tilde{R}_1(Z_1, \Delta t)u^n \quad (4.7)$$

where $\tilde{R}_1(Z_1, \Delta t)$ is the stability function for the first iterative scheme.

In (Chapter II, Sect. 1.3, (Hundsdorfer and Verwer, 2003)), Hundsdorfer and Verwer show that the stability region for $R(Z) = (I - \frac{Z}{2})^{-1}(I + \frac{Z}{2})$ is exactly the left half of the complex plane and $\|R(Z)\| \leq 1$ for this region. Then by taking the norm of $\tilde{R}_1(Z_1, \Delta t)$ we obtain

$$\|\tilde{R}_1(Z_1, \Delta t)\| \leq 1 + \Delta t K_1$$

where $K_1 = \|(I - \frac{Z_1}{2})^{-1}A_2\|$, which is independent of Δt and Δx. Setting $u_1^n = u_2^n = u^n$ and substituting $u_1^{n+1} = \tilde{R}_1 u^n$ into Equation (4.5), we have the following stability equation

$$u_2^{n+1} = (R(Z_2) + \Delta t(I - \frac{Z_2}{2})^{-1}\frac{A_1}{2} + \Delta t(I - \frac{Z_2}{2})^{-1}\frac{A_1}{2}\tilde{R}_1(Z_1, \Delta t))u^n. \tag{4.8}$$

Taking the norm of both sides and using above mentioned result in (Hundsdorfer and Verwer, 2003) we obtain a stability function for the second iterative scheme such as

$$\|\tilde{R}_2(Z_1, Z_2, \Delta t)\| \leq 1 + \Delta t\|(I - \frac{Z_2}{2})^{-1}\|\|A_1\| + \frac{\Delta t^2}{2}\|(I - \frac{Z_2}{2})^{-1}\|\|A_1\|K_1. \tag{4.9}$$

Since $\Delta t^2 < (T+1)\Delta t$ for $\Delta t \in [0, T]$, we obtain

$$\|\tilde{R}_2(Z_1, Z_2, \Delta t)\| \leq 1 + \Delta t K_2$$

where $K_2 = \|(I - \frac{Z_2}{2})^{-1}\|\|A_1\|(1 + K_1\frac{(T+1)}{2})$ which is independent of Δt and Δx.

As shown in (Chapter II, Theorem 2.2.1, (Strikwerda, 2004)), a one step finite difference scheme (with constant coefficients) is stable in a stability region Λ if and only if there is a constant K (independent of θ, Δt and Δx) such that

$$|g(\theta, \Delta t, \Delta x)| \leq 1 + K\Delta t$$

with $(\Delta t, \Delta x) \in \Lambda$, which completes the proof. It follows similarly for every i. □

4.2. Stability Analysis for Unbounded Operators

We study the stability of iterative splitting solutions in two ways: first by using Fourier transform and the other using C_0 semigroup techniques given in Section 3.2.

4.2.1. Stability via Fourier transform

Consider Cauchy problem (2.2)-(2.3). Suppose that we have a linear map obtained from the application of iterative splitting schemes (2.17)-(2.20) with the midpoint rule to Equation (2.2) over one time step, that is

$$\begin{pmatrix} La_{11} & La_{21} \\ La_{21} & La_{22} \end{pmatrix} \begin{pmatrix} u_i^{n+1} \\ u_{i+1}^{n+1} \end{pmatrix} = \begin{pmatrix} Lb_{11} & Lb_{21} \\ Lb_{21} & Lb_{22} \end{pmatrix} \begin{pmatrix} u_i^{n} \\ u_{i+1}^{n} \end{pmatrix} + \frac{\Delta t}{2} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \quad (4.10)$$

equivalently

$$\tilde{L_a} \begin{pmatrix} u_i^{n+1} \\ u_{i+1}^{n+1} \end{pmatrix} = \tilde{L_b} \begin{pmatrix} u_i^{n} \\ u_{i+1}^{n} \end{pmatrix} + \frac{\Delta t}{2} \bar{\varphi} \quad (4.11)$$

where $\tilde{L_a}$, $\tilde{L_b}$ are matrices of the linear operators; φ_1 and φ_2 are constants from previous step. u_1^n, u_2^n are the approximations of u_1, u_2 in function space at time $t^n = t^{n-1} + \Delta t$.

To apply the stability theory, $\tilde{L_a}$ and $\tilde{L_b}$ must be manipulated into matrices of scalars, which can be done by taking Fourier transforms of (4.10). We restrict this discussion to linear operators that are spatial derivatives of order at least one or the identity multiplied by complex scalar. Given this restriction, applying the continuous Fourier transform

$$\hat{u}(w) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-iwx} u(x) dx \quad (4.12)$$

to (4.10) results in

$$\begin{aligned} \begin{pmatrix} \hat{u}_i^{n+1}(w) \\ \hat{u}_{i+1}^{n+1}(w) \end{pmatrix} &= \begin{pmatrix} z_{11}(w) & z_{21}(w) \\ z_{21}(w) & z_{22}(w) \end{pmatrix} \begin{pmatrix} \hat{u}_i^{n}(w) \\ \hat{u}_{i+1}^{n}(w) \end{pmatrix} + \frac{\Delta t}{2} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \\ &= \tilde{A} \begin{pmatrix} \hat{u}_i^{n}(w) \\ \hat{u}_{i+1}^{n}(w) \end{pmatrix} + \frac{\Delta t}{2} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \end{aligned} \quad (4.13)$$

where $\tilde{A}$ is the matrix of scalars $z_{ij}(w)$ involving the frequency $w \in \mathbb{R}$ and Δt time step;

the non-homogenous term is neglected since it is constant.

The stability for the linear map is related to the eigenvalues of the matrix $\tilde{A}$. The eigenvalues of $\tilde{A}$ are solutions to $\lambda^2 - Tr(\tilde{A})\lambda + det(\tilde{A}) = 0$. For the stability of the linear maps, when the roots λ_1 and λ_2 of the equation are given by

$$\lambda_{1,2} = \frac{Tr(\tilde{A})}{2} \pm i\sqrt{det(\tilde{A}) - \left(\frac{Tr(\tilde{A})}{2}\right)^2}$$

with $\mid Tr(\tilde{A}) \mid < 2\sqrt{det(\tilde{A})}$, the eigenvalues must satisfy $|\lambda_{1,2}| \leq 1$.

Rewriting iterative operator splitting algorithms (2.17)-(2.20) yields

$$u_i'(t) = A_1 u_i(t) + A_2 u_{i-1}(t) \tag{4.14}$$

$$u_{i+1}'(t) = A_1 u_i(t) + A_2 u_{i+1}(t) \tag{4.15}$$

where A_1 and A_2 are unbounded linear operators. The direct application of midpoint to iterative splitting schemes (4.14)-(4.15) yields the following components

$$u_i^{n+1} = u_i^n + \Delta t(\frac{A_1 u_i^n + A_2 u_{i-1}^n + A_1 u_i^{n+1} + A_2 u_{i-1}^{n+1}}{2}) \tag{4.16}$$

$$u_{i+1}^{n+1} = u_{i+1}^n + \Delta t(\frac{A_1 u_i^n + A_2 u_{i+1}^n + A_1 u_i^{n+1} + A_2 u_{i+1}^{n+1}}{2}). \tag{4.17}$$

Regrouping (4.16), (4.17)

$$(1 - \frac{\Delta t}{2}A_1)u_i^{n+1} = (1 + \frac{\Delta t}{2}A_1)u_i^n + \frac{\Delta t}{2}A_2(u_{i-1}^{n+1} + u_{i-1}^n) \tag{4.18}$$

$$-\frac{\Delta t}{2}A_1 u_i^{n+1} + (1 - \frac{\Delta t}{2}A_2)u_{i+1}^{n+1} = (1 + \frac{\Delta t}{2}A_2)u_{i+1}^n + \frac{\Delta t}{2}A_1 u_i^n, \tag{4.19}$$

which in matrix form reads

$$\begin{pmatrix} 1-\frac{\Delta t}{2}A_1 & 0 \\ -\frac{\Delta t}{2}A_1 & 1-\frac{\Delta t}{2}A_2 \end{pmatrix}\begin{pmatrix} u_i^{n+1} \\ u_{i+1}^{n+1} \end{pmatrix} = \begin{pmatrix} 1+\frac{\Delta t}{2}A_1 & 0 \\ \frac{\Delta t}{2}A_1 & 1+\frac{\Delta t}{2}A_2 \end{pmatrix}\begin{pmatrix} u_i^{n} \\ u_{i+1}^{n} \end{pmatrix} + \frac{\Delta t}{2}\begin{pmatrix} A_2(u_{i-1}^n + u_{i-1}^{n+1}) \\ 0 \end{pmatrix}. \tag{4.20}$$

Equivalently

$$\tilde{L_a}\begin{pmatrix} u_i^{n+1} \\ u_{i+1}^{n+1} \end{pmatrix} = \tilde{L_b}\begin{pmatrix} u_i^{n} \\ u_{i+1}^{n} \end{pmatrix} + \frac{\Delta t}{2}\begin{pmatrix} A_2(u_{i-1}^n + u_{i-1}^{n+1}) \\ 0 \end{pmatrix} \tag{4.21}$$

where

$$\tilde{L_a} = \begin{pmatrix} 1-\frac{\Delta t}{2}A_1 & 0 \\ -\frac{\Delta t}{2}A_1 & 1-\frac{\Delta t}{2}A_2 \end{pmatrix}, \quad \tilde{L_b} = \begin{pmatrix} 1+\frac{\Delta t}{2}A_1 & 0 \\ \frac{\Delta t}{2}A_1 & 1+\frac{\Delta t}{2}A_2 \end{pmatrix}.$$

Each element of the matrices $\tilde{L_a}$, $\tilde{L_b}$ are polynomials in the linear operators A_1 and A_2; initial conditions are $u_1^0 = u_1(x,0)$ and $u_2^0 = u_2(x,0)$ and u_1^1, u_2^1 are the approximations of u_1, u_2 in function space at time $t^1 = t^0 + \Delta t$ for $i = 1$. For the non-homogenous part, all components are known from the previous steps. The following three problems are analyzed using Fourier transform.

4.2.1.1. First Example: One Dimensional Advection-Diffusion Equation

Consider the one dimensional advection-diffusion equation

$$\partial_t u + v\partial_x u - \partial_x D \partial_x u = 0 \tag{4.22}$$

where v is the advection parameter and D is the diffusion parameter. Rearranging Equation (4.22) in an abstract Cauchy form yields

$$\partial_t u = A_1 u + A_2 u \tag{4.23}$$

where $A_1 = -v\partial_x$, $A_2 = D\partial_{xx}$. Implementing iterative splitting schemes (2.17)-(2.20) to Equation (4.23) we obtain

$$u_i' = A_1 u_i + A_2 u_{i-1} \tag{4.24}$$

$$u_{i+1}' = A_1 u_i + A_2 u_{i+1}. \tag{4.25}$$

Applying midpoint rule to Equations (4.24), (4.25) on $[t^n, t^{n+1}]$ interval with Δt time step results in

$$\tilde{L_a}\begin{pmatrix} u_i^{n+1} \\ u_{i+1}^{n+1} \end{pmatrix} = \tilde{L_b}\begin{pmatrix} u_i^{n} \\ u_{i+1}^{n} \end{pmatrix} + \frac{\Delta t}{2}\begin{pmatrix} L_2(u_{i-1}^n + u_{i-1}^{n+1}) \\ 0 \end{pmatrix} \tag{4.26}$$

where

$$\tilde{L_a} = \begin{pmatrix} 1-\frac{\Delta t}{2}A_1 & 0 \\ -\frac{\Delta t}{2}A_1 & 1-\frac{\Delta t}{2}A_2 \end{pmatrix}, \quad \tilde{L_b} = \begin{pmatrix} 1+\frac{\Delta t}{2}A_1 & 0 \\ \frac{\Delta t}{2}A_1 & 1+\frac{\Delta t}{2}A_2 \end{pmatrix}.$$

We begin the iterations by assigning $i = 1, 3, \dots, 2m-1$. Non-homogenous part is negligible since the terms are constant. Applying a continuous Fourier transform (4.12) to (4.26) yields

$$\begin{aligned}\begin{pmatrix} \tilde{u}_i^{n+1} \\ \tilde{u}_{i+1}^{n+1} \end{pmatrix} &= \begin{pmatrix} \frac{1-v\frac{\Delta t}{2}wi}{1+v\frac{\Delta t}{2}wi} & 0 \\ \frac{-\frac{\Delta t}{2}vwi(1-\frac{\Delta t}{2}vwi)}{(1+\frac{\Delta t}{2}vwi)(1+\frac{\Delta t}{2}Dw^2)} + \frac{-\frac{\Delta t}{2}vwi}{1+\frac{\Delta t}{2}Dw^2} & \frac{1-\frac{\Delta t}{2}Dw^2}{1+\frac{\Delta t}{2}Dw^2} \end{pmatrix}\begin{pmatrix} \tilde{u}_i^{n} \\ \tilde{u}_{i+1}^{n} \end{pmatrix} + \frac{\Delta t}{2}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \\ &= \tilde{A}\begin{pmatrix} \tilde{u}_i^{n} \\ \tilde{u}_{i+1}^{n} \end{pmatrix} + \frac{\Delta t}{2}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \end{aligned} \tag{4.27}$$

The eigenvalues of $\tilde{A}$ are

$$\lambda_1 = \frac{1-v\frac{\Delta t}{2}wi}{1+v\frac{\Delta t}{2}wi} \quad \text{and} \quad \lambda_2 = \frac{1-\frac{\Delta t}{2}Dw^2}{1+\frac{\Delta t}{2}Dw^2}.$$

Stability requires that $|\lambda_i| \leq 1, i = 1, 2$

$$\left|\frac{1-v\frac{\Delta t}{2}wi}{1+v\frac{\Delta t}{2}wi}\right| \leq 1 \text{ and } \left|\frac{1-\frac{\Delta t}{2}Dw^2}{1+\frac{\Delta t}{2}Dw^2}\right| \leq 1.$$

The first inequality is true for any choice of w and v, and the second is valid whenever $D \geq 0$. Hence, when iterative splitting combined with midpoint rule is applied to one dimensional advection-diffusion equation, stable solutions are obtained whenever $D \geq 0$.

4.2.1.2. Second Example: Two Dimensional Solute Transport Equation

It is also possible to extend our stability analysis to higher dimensional PDEs. A Fourier transform in N dimensions of the function $u(x)$ where $x \in \mathbb{R}^N$ is

$$\hat{u}(\overrightarrow{\omega}) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{R^N} e^{-i\overrightarrow{\omega}x} u(x)dx \tag{4.28}$$

where $\overrightarrow{\omega} \in \mathbb{R}^N$.

Consider the two dimensional solute transport equation (Verma et. al., 2000)

$$\partial_t u + v u_x - D_{xx} u_{xx} - D_{yy} u_{yy} = 0, \tag{4.29}$$

where the uniform pore velocity v is 0.1 m/day; the longitudinal and transverse dispersivities are $D_{xx} = 1$ and $D_{yy} = 0.1$, respectively. Rearranging Equation (4.29) in an abstract Cauchy form yields

$$\partial_t u = A_1 u + A_2 u \tag{4.30}$$

where $A_1 = -v\partial_x$ and $A_2 = D_{xx}\partial_{xx} + D_{yy}\partial_{yy}$. We start with application of iterative splitting schemes (2.17)- (2.20) with the midpoint rule to Equation (4.30) over single time step Δt. Later taking the continuous Fourier transform results in

$$\begin{aligned}
\begin{pmatrix} \tilde{u}_i^{n+1} \\ \tilde{u}_{i+1}^{n+1} \end{pmatrix} &= \begin{pmatrix} \frac{1-\frac{\Delta t}{2}(v\omega_1)i}{1+\frac{\Delta t}{2}(v\omega_1)i} & 0 \\ \frac{-\frac{\Delta t}{2}(v\omega_1)(1-\frac{\Delta t}{2}(v\omega_1)i)i}{(1+\frac{\Delta t}{2}(v\omega_1)i)(1-\frac{\Delta t}{2}(D_{xx}\omega_1^2+D_{yy}\omega_2^2))} & \frac{1-\frac{\Delta t}{2}(D_{xx}\omega_1^2+D_{yy}\omega_2^2)}{1+\frac{\Delta t}{2}(D_{xx}\omega_1^2+D_{yy}\omega_2^2)} \end{pmatrix} \begin{pmatrix} \tilde{u}_i^{n} \\ \tilde{u}_{i+1}^{n} \end{pmatrix} \\
&\quad + \frac{\Delta t}{2} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \\
&= \tilde{A} \begin{pmatrix} \tilde{u}_i^{n} \\ \tilde{u}_{i+1}^{n} \end{pmatrix} + \frac{\Delta t}{2} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}.
\end{aligned} \tag{4.31}$$

The eigenvalues of $\tilde{A}$ are

$$\lambda_1 = \frac{1-\frac{\Delta t}{2}(v\omega_1)i}{1+\frac{\Delta t}{2}(v\omega_1)i} \quad \text{and} \quad \lambda_2 = \frac{1-\frac{\Delta t}{2}(D_{xx}\omega_1^2+D_{yy}\omega_2^2)}{1+\frac{\Delta t}{2}(D_{xx}\omega_1^2+D_{yy}\omega_2^2)}.$$

For stability we know that $|\lambda_i| \leq 1$, $i = 1, 2$

$$\left|\frac{1-\frac{\Delta t}{2}(v\omega_1)i}{1+\frac{\Delta t}{2}(v\omega_1)i}\right| \leq 1 \text{ and } \left|\frac{1-\frac{\Delta t}{2}(D_{xx}\omega_1^2+D_{yy}\omega_2^2)}{1+\frac{\Delta t}{2}(D_{xx}\omega_1^2+D_{yy}\omega_2^2)}\right| \leq 1.$$

First inequality is true for any choice of v and $\overrightarrow{\omega} = (\omega_1, \omega_2)$, and the second is valid whenever $D_{xx}\omega_1^2 + D_{yy}\omega_2^2 \geq 0$. Hence, when iterative splitting combined with midpoint rule is applied to two dimensional solute transport model, stable solutions are obtained whenever $D_{xx}\omega_1^2 + D_{yy}\omega_2^2 \geq 0$.

4.2.1.3. Third Example: One Dimensional Korteweg-de Vries Equation

Consider one dimensional nonlinear Korteweg de-Vries equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{4.32}$$

We begin by splitting Equation (4.32) into linear and nonlinear parts such as

$$u_t = -u_{xxx} \text{ and } u_t = -6uu_x. \tag{4.33}$$

We first apply iterative splitting schemes and obtain

$$u_i' = -(u_i)_{xxx} + 6u_{i-1}(u_{i-1})_x, \tag{4.34}$$

$$u_{i+1}' = -(u_i)_{xxx} + 6u_i(u_{i+1})_x \tag{4.35}$$

where $i = 1, 3, \ldots, 2m-1$. Then rearranging Equations (4.34)-(4.35) with a linearization about steady state $6u_{i-1} = k_1$, $6u_i = k_2$ yields

$$u_i' = A_1 u_i + k_1 A_2 u_{i-1} \tag{4.36}$$

$$u_{i+1}' = A_1 u_i + k_2 A_2 u_{i+1} \tag{4.37}$$

where $A_1 = -\frac{\partial^3}{\partial x^3}$, $A_2 = -\frac{\partial}{\partial x}$ and $i = 1, 3, \ldots, 2m-1$. Next after applying midpoint rule to (4.36), (4.37), we obtain

$$\begin{pmatrix} u_i^{n+1} \\ u_{i+1}^{n+1} \end{pmatrix} = \begin{pmatrix} u_i^n \\ u_{i+1}^n \end{pmatrix} + \Delta t \begin{pmatrix} A_1 \frac{u_i^n + u_i^{n+1}}{2} + k_1 A_1 \frac{u_{i-1}^n + u_{i-1}^{n+1}}{2} \\ A_1 \frac{u_i^n + u_i^{n+1}}{2} + k_2 A_2 \frac{u_{i+1}^n + u_{i+1}^{n+1}}{2} \end{pmatrix} \tag{4.38}$$

where Δt is the time step on $[t^n, t^{n+1}]$ interval. Now, it is time to take Fourier transform of Equation (4.38):

$$\begin{aligned}
\begin{pmatrix} \tilde{u}_i^{n+1} \\ \tilde{u}_{i+1}^{n+1} \end{pmatrix} &= \begin{pmatrix} \frac{1+\frac{\Delta t}{2}w^3 i}{1-\frac{\Delta t}{2}w^3 i} & 0 \\ \frac{\frac{\Delta t}{2}w^3 i(1+\frac{\Delta t}{2}w^3 i)}{(1-\frac{\Delta t}{2}w^3 i)(1+\frac{\Delta t}{2}k_2 wi)} + \frac{\frac{\Delta t}{2}w^3 i}{1+\frac{\Delta t}{2}k_2 wi} & \frac{1-\frac{\Delta t}{2}k_2 wi}{1+\frac{\Delta t}{2}k_2 wi} \end{pmatrix} \begin{pmatrix} \tilde{u}_i^{n} \\ \tilde{u}_{i+1}^{n} \end{pmatrix} + \frac{\Delta t}{2} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \\
&= \tilde{A} \begin{pmatrix} \tilde{u}_i^{n} \\ \tilde{u}_{i+1}^{n} \end{pmatrix} + \frac{\Delta t}{2} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \qquad (4.39)
\end{aligned}$$

The eigenvalues of $\tilde{A}$ are

$$\lambda_1 = \frac{1+\frac{\Delta t}{2}w^3 i}{1-\frac{\Delta t}{2}w^3 i}, \ \lambda_2 = \frac{1-\frac{\Delta t}{2}k_2 wi}{1+\frac{\Delta t}{2}k_2 wi}.$$

For stability we know that $|\lambda_i| \leq 1$, $i = 1, 2$

$$\left| \frac{1+\frac{\Delta t}{2}w^3 i}{1-\frac{\Delta t}{2}w^3 i} \right| \leq 1 \qquad \text{and} \qquad \left| \frac{1-\frac{\Delta t}{2}k_2 wi}{1+\frac{\Delta t}{2}k_2 wi} \right| \leq 1.$$

However, one can easily deduce that $|\lambda_1| = |\lambda_2| = 1$ holds for any choices of Δt, w and k_2. Hence, when iterative splitting combined with midpoint rule is applied to one dimensional Korteweg-de Vries equation, unconditionally stable solutions are obtained. This approach is advantageous since the nonlinear problem can be analyzed as a linear problem.

4.2.2. Stability via C_0 Semigroup

In Section 3.2, the solutions of first and the second iterative schemes obtained using C_0 semigroup techniques are presented. Here, we prove the stability of each scheme by using these solutions.

4.2.2.1. Stability of First Iterative Scheme

The first iterative scheme and its solution based on C_0 semigroup is given in Subsection 3.2.2. Let $u_1(t)$ and $\tilde{u}_1(t)$ be solutions of Equation (3.56) with initial conditions u_0 and $\tilde{u}_0$, respectively. Then for $[0, t]$ time interval, we obtain the following equations

$$u_1'(t) = A_1 u_1(t) + A_2 u_0 \tag{4.40}$$

$$\tilde{u}_1'(t) = A_1 \tilde{u}_1(t) + A_2 \tilde{u}_0, \tag{4.41}$$

and substraction yields

$$(u_1(t) - \tilde{u}_1(t))' = A_1(u_1(t) - \tilde{u}_1(t)) + A_2(u_0 - \tilde{u}_0). \tag{4.42}$$

It has solution

$$u_1(t) - \tilde{u}_1(t) = S_1(t)(u_0 - \tilde{u}_0) + \int_0^t S_1(t-s) A_2(u_0 - \tilde{u}_0) ds \tag{4.43}$$

and taking the norm yields

$$\begin{aligned}
\|u_1(t) - \tilde{u}_1(t)\| &\leq \|S_1(t)\| \|u_0 - \tilde{u}_0\| + \int_0^t \|S_1(t-s)\| \|A_2(u_0 - \tilde{u}_0)\| ds \\
&\leq M_1 e^{\omega_1 t} \|u_0 - \tilde{u}_0\| + \int_0^t M_1 e^{\omega_1 (t-s)} C_1(T) ds \\
&\leq M_1 e^{\omega_1 t} \|u_0 - \tilde{u}_0\| + \frac{M_1}{\omega_1}(1 - e^{\omega_1 t}) C_1(T) \\
&\leq M_1 e^{\omega_1 t} \|u_0 - \tilde{u}_0\|
\end{aligned} \tag{4.44}$$

where $\|S_1(t)\|_{L(X)} \leq M_1 e^{\omega_1 t}$ by Equation (3.24) and $sup_{0 \leq t \leq T} \|A_2(u_0 - \tilde{u}_0)\| \leq C_1(T)$ by Lemma 3.7. The first iterative splitting propagator is written

$$\Phi^t_{A_2, A_1} = M_1 e^{\omega_1 t}. \tag{4.45}$$

The method is said to be **stable** on $[0, T]$ if there are constants K_T, h' such that

$$\|(\Phi^t_{A_2, A_1})^j\| \leq K_T \tag{4.46}$$

for all $j = 0, 1, 2, \ldots$ and $0 < t < h'$ satisfying $jt \leq T$. Here,

$$\|(\Phi^t_{A_2, A_1})^j\| \leq M_1^j e^{\omega_1 t j}, \tag{4.47}$$

where the right-hand side is unbounded as $j \to \infty$, $tj \leq T$ if and only is $M_1 > 1$. Thus, for stability requirement we assume $\|S_1(t)\|_{L(X)} \leq e^{\omega_1 t}$ and the propagator operator becomes equal to

$$\Phi^t_{A_2, A_1} = e^{\omega_1 t}. \tag{4.48}$$

4.2.2.2. Stability of Second Iterative Scheme

The second iterative scheme and its solution based on C_0 semigroup is given in Subsection 3.2.3. Let $u_2(t)$ and $\tilde{u}_2(t)$ be solutions of Equation (3.61) with initial conditions u_1 and $\tilde{u}_1$, respectively. (Note that the first iterative solution is taken as an initial condition of the second iterative scheme). Then for $[0,t]$ time interval, we obtain following equations

$$u_2(t) = A_1u_1 + A_2u_2(t) \tag{4.49}$$

$$\tilde{u_2}(t) = A_1\tilde{u}_1 + A_2\tilde{u}_2(t), \tag{4.50}$$

and substraction yields

$$(u_2(t) - \tilde{u}_2(t))' = A_1(u_1 - \tilde{u}_1) + A_2(u_2(t) - \tilde{u}_2(t)). \tag{4.51}$$

It has solution

$$u_2(t) - \tilde{u}_2(t) = S_2(t)(u_1 - \tilde{u}_1) + \int_0^t S_2(t-s)A_1(u_1 - \tilde{u}_1)ds \tag{4.52}$$

and taking the norm yields

$$\begin{aligned}\|u_2(t) - \tilde{u}_2(t)\| &\leq \|S_2(t)\|\|u_1 - \tilde{u}_1\| + \int_0^t \|S_2(t-s)\|\|A_1(u_1 - \tilde{u}_1)\|ds \\ &\leq e^{\omega_2 t}\|u_1 - \tilde{u}_1\| + \int_0^t e^{\omega_2(t-s)}C_1(T)ds \\ &\leq e^{\omega_2 t}\|u_1 - \tilde{u}_1\| + \frac{(1 - e^{\omega_2 t})}{\omega_2}C_1(T) \\ &\leq e^{\omega_2 t}\|u_1 - \tilde{u}_1\| \end{aligned} \tag{4.53}$$

where $\|S_2(t)\|_{L(X)} \leq e^{\omega_2 t}$ is assumed for stability requirement and $sup_{0 \leq t \leq T}\|A_1(u_1 - \tilde{u}_1)\| \leq C_1(T)$ by Lemma 3.7. Then the second iterative splitting propagator is written

$$\Phi^t_{A_1,A_2} = e^{\omega_2 t} \tag{4.54}$$

and hence

$$\|\Phi^t_{A_1,A_2}\| \leq K_T \tag{4.55}$$

which shows the stability of the second iterative scheme.

CHAPTER 5

CONVERGENCE ANALYSIS OF ITERATIVE OPERATOR SPLITTING METHOD

In Chapter 3 and Chapter 4, we have studied consistency and the stability. It remains to asses the convergence of the iterative splitting method when applied to the Cauchy problem (2.2)-(2.3). For the bounded case, we utilize Lax-Richtmyer Equivalence Theorem, see (Strikwerda, 2004), whereas we adapt the Lady Windermere's Fan argument, see (Lubich and Jahnke, 2000), (Lubich, 2008), when the operators are unbounded.

5.1. Convergence Analysis for Bounded Operators

The following theorem provides a simple characterization of convergent schemes.

Theorem 5.1 (The Lax-Richtmyer Equivalence Theorem) *A consistent finite difference scheme for partial differential equation for which the initial value problem is well-posed is convergent if and only if it is stable.*

Proof See (Chapter X, Theorem 10.5.1, (Strikwerda, 2004)). □

Hence, consistency shown in Theorem 3.2 and stability shown in Theorem 4.1 together imply the convergence of iterative splitting schemes by Theorem 5.1.

5.2. Convergence Analysis for Unbounded Operators

Here, we are concerned with deducing an estimate for global error $U^N - u(t^N)$ of iterative splitting method (2.17)-(2.20) when applied to Cauchy problem (2.2)-(2.3). U^N is numerical solution and $u(t^N)$ is the exact solution at $t^N = T$. To this purpose we follow a standard approach based on a Lady Windermere's Fan argument.

Generally, when the local error is equal to $d_n = (\Phi(h_{n-1}) - E(h_{n-1}))u(t^{n-1})$ with $t^0 < t^1 < \ldots < t^N = T$, non-uniform meshes $h_{n-1} = t^n - t^{n-1}$, $1 \leq n \leq N$ where $\Phi(h_{n-1})$ is a numerical solution operator and $E(h_{n-1})$ is an exact solution operator, in order to relate global and local error, we employ the telescoping series identity

$$U^N - u(t^N) = \prod_{j=0}^{N-1} \Phi(h_j)(u_0 - u(t^0)) + \sum_{n=1}^{N} \prod_{j=n}^{N-1} \Phi(h_j) d_n. \tag{5.1}$$

The validity of Equation (5.1) is verified by a short calculation

$$\begin{aligned}
& \prod_{j=0}^{N-1} \Phi(h_j)(u_0 - u(t^0)) + \sum_{n=1}^{N} \prod_{j=n}^{N-1} \Phi(h_j) d_n \\
= & \prod_{j=0}^{N-1} \Phi(h_j)(u_0 - u(t^0)) + \sum_{n=1}^{N} \prod_{j=n}^{N-1} \Phi(h_j)(\Phi(h_{n-1}) - E(h_{n-1}))u(t^{n-1}) \\
= & \prod_{j=0}^{N-1} \Phi(h_j) u_0 - \prod_{j=0}^{N-1} \Phi(h_j) u(t^0) \\
& + \sum_{n=1}^{N} \prod_{j=n-1}^{N-1} \Phi(h_j) u(t^{n-1}) - \sum_{n=1}^{N} \prod_{j=n}^{N-1} \Phi(h_j) u(t^n) \\
= & U^N - \prod_{j=0}^{N-1} \Phi(h_j) u(t^0) + \sum_{n=0}^{N-1} \prod_{j=n}^{N-1} \Phi(h_j) u(t^n) - \sum_{n=1}^{N} \prod_{j=n}^{N-1} \Phi(h_j) u(t^n) \\
= & U^N - \prod_{j=0}^{N-1} \Phi(h_j) u(t^0) + \prod_{j=0}^{N-1} \Phi(h_j) u(t^0) - u(t^N) \\
= & U^N - u(t^N).
\end{aligned}$$

In the present study, the local errors are equal to

$d_n = U^n - u(t^n) = (\Phi^h_{A_2,A_1} - E^h)u(t^{n-1})$, where $U^n = u_1(t^n)$ is the first iterative scheme

or

$d_n = U^n - u(t^n) = (\Phi^h_{A_1,A_2} - E^h)u(t^{n-1})$, where $U^n = u_2(t^n)$ is the second iterative scheme,

the numerical solution operators $\Phi^h_{A_2,A_1}$, $\Phi^h_{A_1,A_2}$ and the exact solution operator E^h are given in Equations (4.48), (4.54), (3.43), respectively, and uniform mesh h is used.

After rearranging the above Lady Windermere's fan argument, then Equation (5.1) turns into

$$U^N - u(t^N) = \sum_{j=0}^{N-1} \Phi^{(N-j-1)h}(\Phi^h u(t^j) - u(t^{j+1})) = \sum_{j=0}^{N-1} \Phi^{(N-j-1)h}(\Phi^h - E^h)u(t^j) \tag{5.2}$$

where $t^j = jh$, initial condition is $u_0 = u(t^0)$ and numerical solution operators are $\Phi^h = \Phi^h_{A_2,A_1}$ or $\Phi^h = \Phi^h_{A_1,A_2}$, for first and the second iterative solutions, respectively. In order to deal with Lady Windermere's fan, we need to show $u(t^j)$ is bounded at every time t^j, $j = 1, \ldots, N$ with

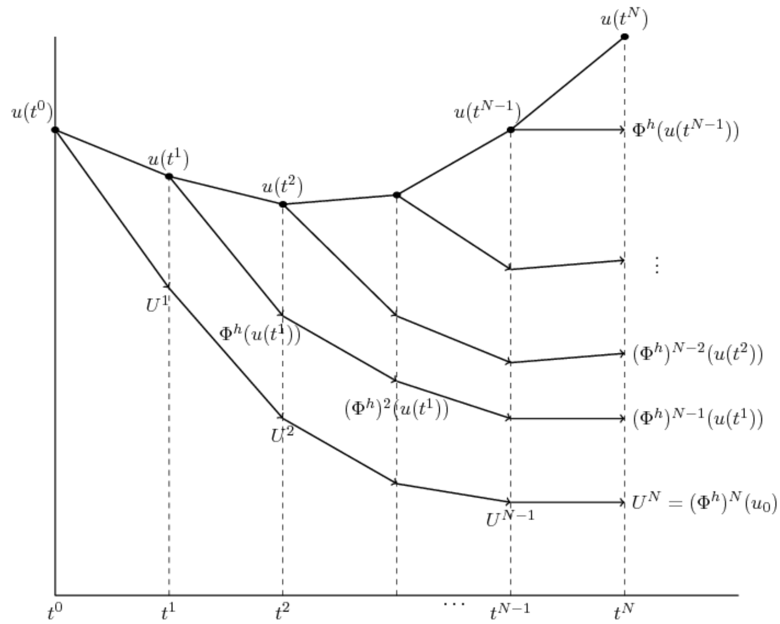


Figure 5.1. Lady Windermere's fan for the convergence analysis of iterative splitting solution.

respect to the used norm. The exact solution is given in Equation (3.42), then we obtain

$$\begin{aligned} u(t^1) &= S(h)u_0, \\ u(t^2) &= S(h)u(t^1) = (S(h))^2 u_0 \\ &\vdots \\ u(t^N) &= (S(h))^N u_0. \end{aligned} \tag{5.3}$$

Taking the norm of both sides yields

$$\|u(t^N)\| = \|S(h)^N u_0\| \leq \|S(h)\|^N \|u_0\| \leq D\|u_0\| \tag{5.4}$$

where $D = e^{\omega T}$ and $T = N/h$ with $\|S(h)\| \leq e^{\omega h}$. Hence it is bounded for every N.

5.2.1. Convergence of First Iterative Scheme

For the first iterative splitting scheme, the local error is given in Theorem 3.6 and the propagator operator is in Equation (4.48). Then by using rearranged Lady Windermere's fan (5.2) and after taking the norm we get

$$\begin{aligned}\|U^N - u(t^N)\| &\leq \sum_{j=0}^{N-1} \|\Phi_{A_2,A_1}^{(N-j-1)h}\| \|(\Phi_{A_2,A_1}^h - E^h)u(t^j)\| \\ &\leq \sum_{j=0}^{N-1} e^{\omega_1 (N-j-1)h} C(T) h^2 \\ &\leq N e^{\omega_1 T} C(T) h^2 \\ &\leq T e^{\omega_1 T} C(T) h \end{aligned} \tag{5.5}$$

since $e^{\omega_1 (N-j-1)h} \leq e^{\omega_1 T}$ and $Nh = T$. Hence we have proved Theorem 5.2.

Theorem 5.2 *Let the Cauchy problem (2.2)-(2.3) satisfies the Equation (3.49) with $T > 0$. The exact solution is bounded (5.4). Then the first order iterative splitting scheme (3.59) for any $u_0 \in D_k$ has the first order global error for $0 \leq t_N = Nh \leq T$ i. e.*

$$\|U^N - u(t^N)\| \leq h\, K(T) \tag{5.6}$$

where $K(T)$ is constant independent of h.

5.2.2. Convergence of Second Iterative Scheme

For the second iterative splitting scheme, the local error is given in Theorem 3.7 and the propagator operator is in Equation (4.54). Then by using rearranged Lady Windermere's fan (5.2) and after taking the norm we get

$$\begin{aligned}\|U^N - u(t^N)\| &\leq \sum_{j=0}^{N-1} \|\Phi_{A_1,A_2}^{(N-j-1)h}\| \|(\Phi_{A_1,A_2}^h - E^h)u(t^j)\| \\ &\leq \sum_{j=0}^{N-1} e^{\omega_2 (N-j-1)h} \hat{C}(T) h^2 \\ &\leq N e^{\omega_2 T} \hat{C}(T) h^3 \\ &\leq T e^{\omega_2 T} \hat{C}(T) h^2 \end{aligned} \tag{5.7}$$

since $e^{\omega_2 (N-j-1)h} \leq e^{\omega_2 T}$ and $Nh = T$. Hence we have proved Theorem 5.3.

Theorem 5.3 *Let the Cauchy problem (2.2)-(2.3) satisfies the Equation (3.49) with $T > 0$. The exact solution is bounded (5.4). Then the second order iterative splitting scheme (3.64) for any $u_0 \in D_k$ has the second order global error for $0 \leq t^N = Nh \leq T$ i. e.*

$$\|U^N - u(t^N)\| \leq h^2\, \hat{K}(T) \tag{5.8}$$

where $\hat{K}(T)$ is constant independent of h.

CHAPTER 6

APPLICATIONS AND NUMERICAL EXPERIMENTS

In this chapter, we shall perform numerical experiments in three parabolic problems and a one dimensional nonlinear Korteweg-de Vries equation. These three equations are one dimensional capillary formation model in tumor angiogenesis problem, two dimensional solute transport model and two dimensional heat equation. These examples set precedents for the theoretical results derived before. All simulations are run by programs written in Matlab programming language.

6.1. Capillary Formation Model in Tumor Angiogenesis Problem

In this model, Levine et. al. (Levine et. al., 2001) introduces the following initial boundary value problem which describes the endothelial cell movement in capillary that is

$$\frac{\partial u}{\partial t} = D\frac{\partial}{\partial x}(u\frac{\partial}{\partial x}(\ln\frac{u}{f(x)})), \quad x \in (0,1), \quad t \in (0,T] \tag{6.1}$$

where T is total time. Initial condition is given

$$u(x,0) = 1, \quad x \in (0,1), \tag{6.2}$$

and the boundary conditions are given

$$Du\frac{\partial}{\partial x}(\ln\frac{u}{f(x)}) \mid_{(0,t)} \quad = \quad 0, \quad t \in [0,T], \tag{6.3}$$

$$Du\frac{\partial}{\partial x}(\ln\frac{u}{f(x)}) \mid_{(1,t)} \quad = \quad 0, \quad t \in [0,T], \tag{6.4}$$

where $f(x)$ is the so-called transition probability function which has the effect of biasing the random walk of endothelial cells that is

$$f(x) = (\frac{a + a_1 x^k (1-x)^k}{b + a_1 x^k (1-x)^k})^{\alpha_1} (\frac{c + 1 - a_2 x^k (1-x)^k}{d + 1 - a_2 x^k (1-x)^k})^{\alpha_2}.$$

In this initial boundary value problem (6.1)-(6.4), $u(x,t)$ is the concentration of Endothelial Cells, D is the cell diffusion constant and a, b, c, d, a_1, a_2, k, α_1, α_2 are some arbitrary constants, see (Serdar and Erdem, 2007), (Saadatmandi and Dehnhan, 2008). The Equation (6.1) can be written as

$$D\frac{\partial}{\partial x}(u\frac{\partial}{\partial x}(\ln \frac{u}{f(x)})) = D\frac{\partial}{\partial x}(u(\frac{u'}{u} - \frac{f'(x)}{f(x)})) \tag{6.5}$$

and by setting $F(x) = \frac{f'(x)}{f(x)}$, we get the following simple equation

$$u_t = D(u_{xx} - (uF(x))_x). \tag{6.6}$$

The boundary conditions (6.3), (6.4) become

$$D(\frac{\partial u}{\partial x} - uF) \mid_{(0,t)} = 0 \text{ for } t > 0, \tag{6.7}$$

$$D(\frac{\partial u}{\partial x} - uF) \mid_{(1,t)} = 0 \text{ for } t > 0. \tag{6.8}$$

We split the equation

$$u_t = D(u_{xx} - u_x F - F_x u) \tag{6.9}$$

into two parts: diffusion

$$u_t = Du_{xx} \tag{6.10}$$

and advection-reaction part

$$u_t = -Du_x F - DF_x u. \tag{6.11}$$

Next we combine these equations by using the iterative splitting algorithm and obtain

$$u_i = D(u_i)_{xx} - D((u_{i-1})_x F - F_x u_{i-1}), \tag{6.12}$$

$$u_{i+1} = D(u_i)_{xx} - D((u_{i+1})_x F - F_x u_{i+1}) \tag{6.13}$$

where $i = 1, 3, \ldots, 2m-1$.

To solve these iterative schemes, finite difference discretization is used in space. Then initial condition becomes

$$u_m = 1, \quad 0 \leq m \leq N, \tag{6.14}$$

and the boundary conditions (6.7), (6.8) turn into

$$D(\frac{\partial u_0}{\partial x} - u_0 F_0) = 0, \text{ for } t > 0, \tag{6.15}$$

$$D(\frac{\partial u_N}{\partial x} - u_N F_N) = 0, \text{ for } t > 0, \tag{6.16}$$

where m defines the spatial discretization step and N is the spatial discretization number. The derivatives in Equations (6.15), (6.16) are approximated by using backward and forward difference formulas. The central difference approximation for each derivatives u_{xx} and u_x are taken into account at each grid point (x_m, t) that is

$$\frac{\partial^2 u}{\partial x^2} \mid_{(x_m,t)} \approx \frac{1}{\Delta x^2}(u_{m+1}(t) - 2u_m(t) + u_{m-1}(t)) \tag{6.17}$$

and

$$\frac{\partial u}{\partial x} \mid_{(x_m,t)} \approx \frac{1}{2\Delta x}(u_{m+1}(t) - u_{m-1}(t)) \tag{6.18}$$

where Δx is the spatial stepping and $m = 1, \ldots, N + 1$.

Assembling the unknowns of (6.17) and embedding the boundary conditions (6.15), (6.16) yield the following system of equations:

$$u_{xx} = AU \tag{6.19}$$

where

$$A = \frac{1}{\Delta x^2} \begin{pmatrix} -2 + (1 - hF_0) & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 + (1 + hF_N) \end{pmatrix}_{(N+1)\times(N+1)}$$

and after assembling the unknowns of (6.18) we obtain the following system

$$u_x = BU \tag{6.20}$$

where

$$B = \frac{1}{2\Delta x} \begin{pmatrix} -(1 - hF_0) & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & (1 + hF_N) \end{pmatrix}_{(N+1)\times(N+1)}$$

and U is $(N + 1) \times 1$ dimensional vector.

We fix the functions $F(x)$ and $F'(x)$ at each discretization points $m = 0, 1, \ldots, N$ and have

$$F(\bar{x}) = \begin{pmatrix} F(x_0) & 0 & \dots & 0 \\ 0 & F(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & F(x_N) \end{pmatrix}_{(N+1)\times(N+1)},$$

$$F'(\bar{x}) = \begin{pmatrix} F'(x_0) & 0 & \dots & 0 \\ 0 & F'(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & F'(x_N) \end{pmatrix}_{(N+1)\times(N+1)}.$$

After redefining Equations (6.12), (6.13), we obtain the following bounded linear systems

$$U'_i = A_1 U_i + A_2 U_{i-1} \tag{6.21}$$

$$U'_{i+1} = A_1 U_i + A_2 U_{i+1} \tag{6.22}$$

where $A_1 = DA$, $A_2 = -DF(\bar{x})B - DF'(\bar{x})$.

Finally, midpoint method on each subinterval $[t^n, t^{n+1}]$ is applied to Equations (6.21), (6.22) and the algorithms are obtained

$$U_i^{n+1} = (I - \frac{\Delta t}{2}A_1)^{-1}((I + \frac{\Delta t}{2}A_1)U_i^n + \frac{\Delta t}{2}A_2(U_{i-1}^n + U_{i-1}^{n+1})) \tag{6.23}$$

$$U_{i+1}^{n+1} = (I - \frac{\Delta t}{2}A_2)^{-1}((I + \frac{\Delta t}{2}A_2)U_{i+1}^n + \frac{\Delta t}{2}A_1(U_i^n + U_i^{n+1})) \tag{6.24}$$

where Δt is time discretzation step. We start with $i = 1$, initial guess is $U_0(t) = (0, \dots)^T$, initial conditions are $U_1(t) = (1, 1, \dots)^T$ and $U_2(t) = (1, 1, \dots)^T$.

For the purpose of comparative analysis, the same numerical parameters used in (Serdar and Erdem, 2007), (Saadatmandi and Dehnhan, 2008) are employed, which are $D = 0.00025$, $a = 1$, $b = 2$, $c = 10$, $d = 0.1$, $\alpha_1 = \alpha_2 = 1$, $a_1 = 28 \times 10^7$, $a_2 = 0.22 \times 10^9$ and $k = 16$.

In Figure 6.1, the concentration of Endothelial Cells, $u(x, t)$, is plotted at different values of T with computational domain, $N = 100$, $M = 1000$ (in x-direction, t-direction). It is seen that graphs, in Figure 6.1, show similar trends as the ones obtained by method of lines in (Serdar and Erdem, 2007) and Tau method in (Saadatmandi and Dehnhan, 2008).

In Figure 6.2 and Figure 6.3, the numerical solutions taken by iterative splitting, Lie-Trotter splitting, Strang splitting methods, and the exact solution at $T = 30$, $T = 750$ times are simulated, with $\Delta t = 3$ at $T = 30$ and $\Delta t = 10$ at $T = 750$, respectively. It is seen that iterative splitting approximates exact solution more accurately than the other classical methods, especially with large time steps.

In Figure 6.4 and Figure 6.5, the relative errors of iterative splitting, Lie-Trotter splitting and Strang splitting methods are simulated for various final times T. The below lines in the figures represents the highest order numerical convergence.

In Table 6.2, Table 6.1 and Table 6.3, we compare the errors of various splitting methods and non-splitting method (expanded with backward time-central space) at times $T = 150$, $T = 300$ and $T = 750$. The numerical rates of convergence in tables are calculated by

$$\rho(\Delta t_1, \Delta t_2) = \frac{\log(Err(\Delta t_1)/Err(\Delta t_2))}{\log(\Delta t_1/\Delta t_2)} \tag{6.25}$$

where $Err(\Delta t_1)$ and $Err(\Delta t_2)$ are l^2 errors taken with time steps Δt_1 and Δt_2, respectively. Note that in this case, $\Delta t_2 = \Delta t_1/10$. It is seen that iterative splitting method gives the smallest error and provides very accurate numerical solution for mathematical model in comparison with other classical splitting methods and non-splitting method.

Furthermore, in Table 6.4, Table 6.5 and Table 6.6, we simulate the l^2 errors and convergence rates of the same three splitting methods at time $T = 3$, $T = 50$ and $T = 150$ for decreasing time steps. The iterative splitting (with two iterations) and Strang splitting methods have second order convergence, while Lie-Trotter has first order.

The Matlab package *expm* is used to calculate an exact solution.

	Error l^1	Error l^2	Error l^∞
Iterative splitting	1.9941e-007	2.2098e-008	3.3364e-009
Lie-Trotter splitting	0.0081	0.0013	3.7108e-004
Strang splitting	2.9793e-005	4.5777e-006	1.0991e-006
Non-splitting	4.7212e-004	5.2296e-005	7.4543e-006

Table 6.1. The errors of different splitting methods and non-splitting (backward time-central space) for $\Delta x = 0.01$, $\Delta t = 0.3$ at $T = 300$.

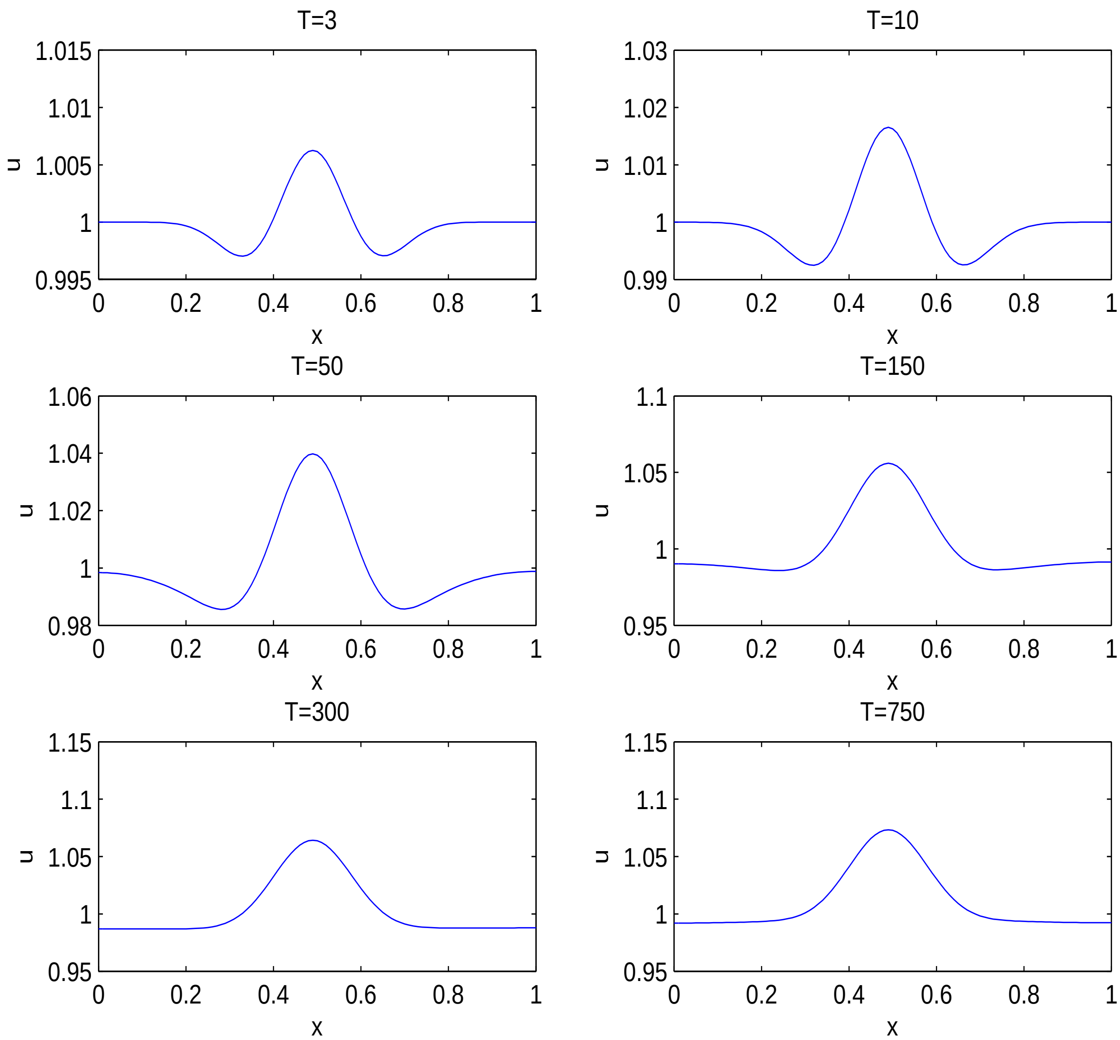


Figure 6.1. Numerical solutions of Problem (6.1)-(6.4) by using iterative operator splitting methods at $T = 3$, $T = 10$, $T = 50$, $T = 150$, $T = 300$, $T = 750$.

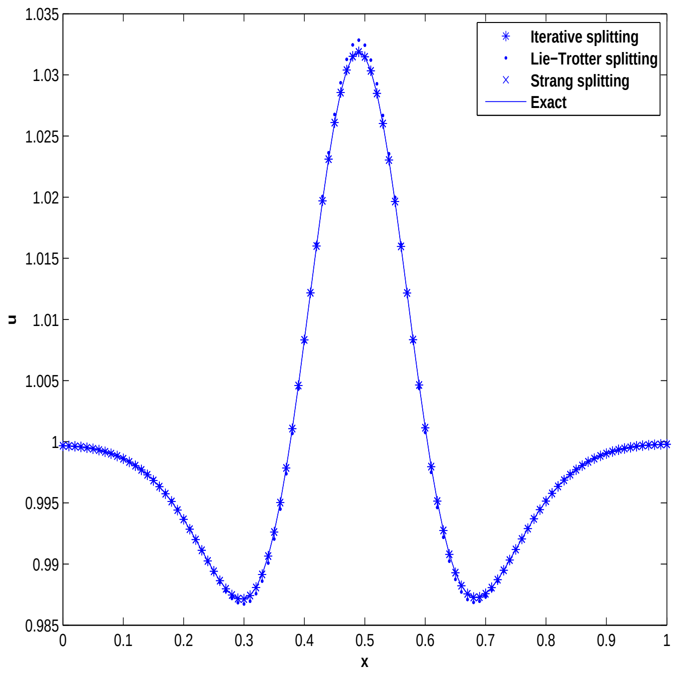


Figure 6.2. The comparison of solutions of Problem (6.1)-(6.4) for $\Delta x = 0.01$, $\Delta t = 3$ at $T = 30$.

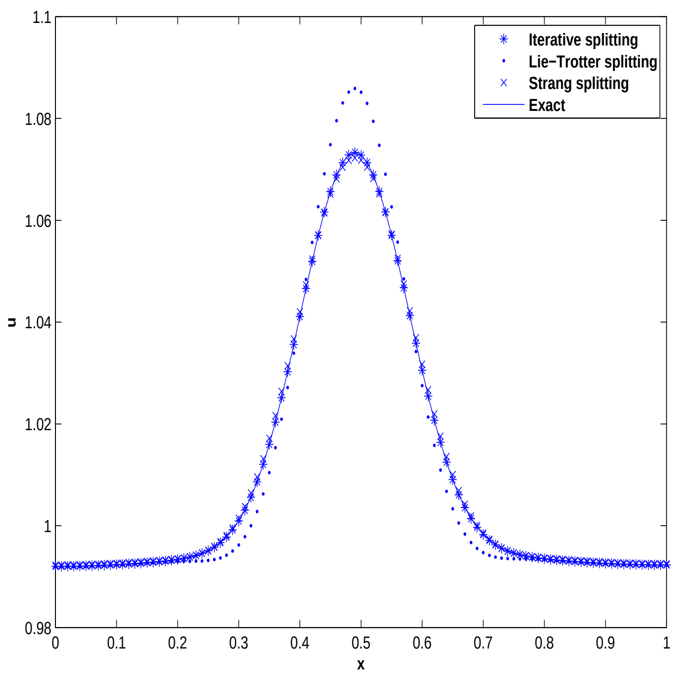


Figure 6.3. The comparison of solutions of Problem (6.1)-(6.4) for $\Delta x = 0.01$, $\Delta t = 10$ at $T = 750$.

	Error l^1	Error l^2	Error l^∞
Iterative splitting	4.7701e-007	5.3524e-008	9.4143e-009
Lie-Trotter splitting	0.0082	0.0013	3.5973e-004
Strang splitting	2.8190e-005	4.4547e-006	1.1447e-006
Non-splitting	9.6361e-004	1.0662e-004	1.6967e-005

Table 6.2. The errors of different splitting methods and non-splitting method (backward time-central space) for $\Delta x = 0.01$, $\Delta t = 0.3$ at $T = 150$.

	Error l^1	Error l^2	Error l^∞
Iterative splitting	8.3485e-009	9.2748e-010	1.4392e-010
Lie-Trotter splitting	0.0082	0.0013	3.7690e-004
Strang splitting	3.9090e-005	5.1606e-006	1.1341e-006
Non-splitting	1.8739e-005	2.1709e-006	4.2305e-007

Table 6.3. The errors of different splitting methods and non-splitting method (backward time-central space) for $\Delta x = 0.01$, $\Delta t = 0.3$ at $T = 750$.

Time steps	Iterative Error	Order	Lie-Trotter Error	Order	Strang Error	Order
1	1.62282e-005		9.6425e-004		1.1943e-005	
0.1	1.6226e-007	2.0015	9.4902e-005	1.0069	1.1939e-007	2.0001
0.01	1.6285e-009	1.9984	9.4760e-006	1.0007	1.1940e-009	2

Table 6.4. The l^2 errors and convergence rates of different splitting methods with $\Delta x = 0.01$ for decreasing time steps at $T = 3$.

Time steps	Iterative Error	Order	Lie-Trotter Error	Order	Strang Error	Order
1	5.0177e-006		0.0039		4.7238e-005	
0.1	5.0176e-008	2	3.8925e-004	1.0008	4.7238e-007	2
0.01	5.0180e-010	2	3.8921e-005	1	4.7249e-009	1.9999

Table 6.5. The l^2 errors and convergence rates of different splitting methods with $\Delta x = 0.01$ for decreasing time steps at $T = 50$.

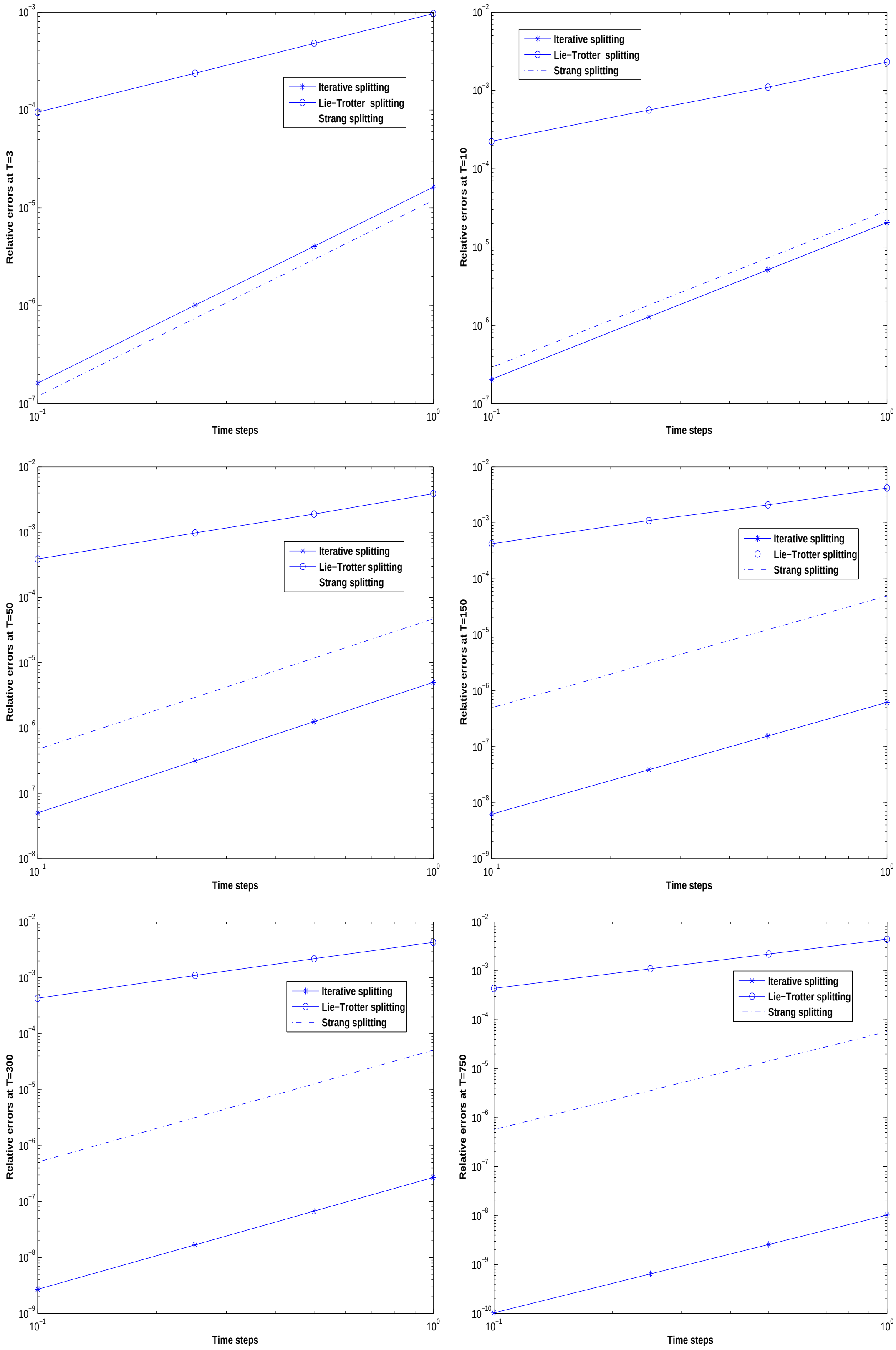


Figure 6.4. l^2 errors versus scaled step sizes for Problem (6.1)-(6.4) at $T = 3$, $T = 10$, $T = 50$, $T = 150$, $T = 300$, $T = 750$.

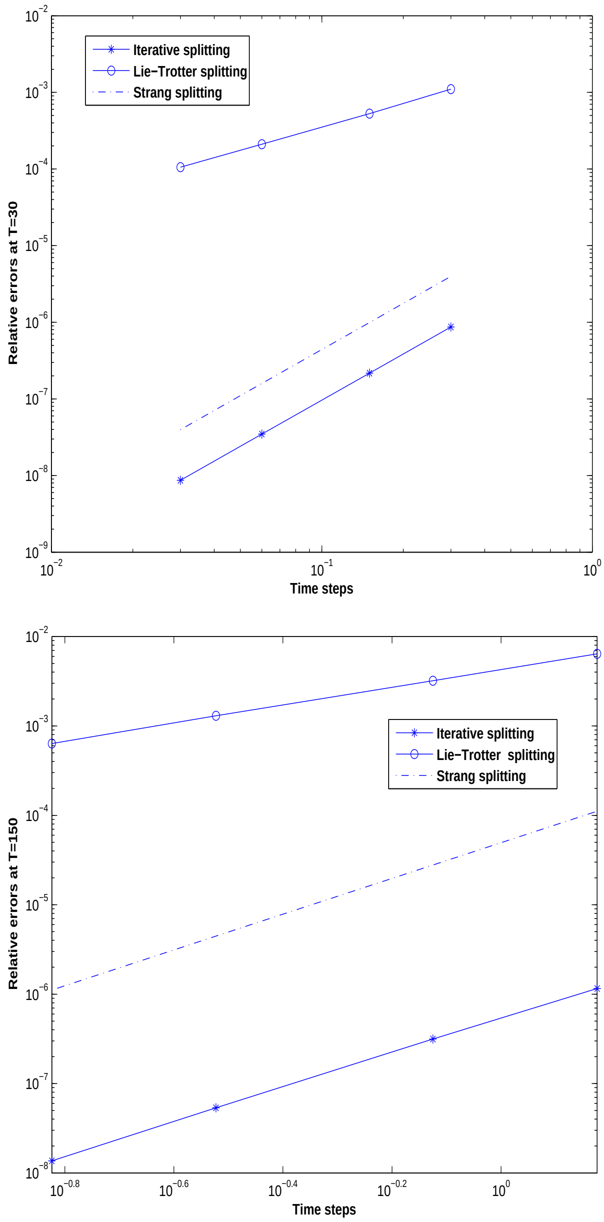


Figure 6.5. (a) l^2 errors versus scaled step sizes for Problem (6.1)-(6.4) at $T = 30$.
(b) l^2 errors versus scaled step sizes for the Problem (6.1)-(6.4) at $T = 150$.

Time steps	Iterative Error	Order	Lie-Trotter Error	Order	Strang Error	Order
1	6.2150e-007		0.0042		4.9497e-005	
0.1	6.2153e-009	2	4.2363e-004	0.9963	4.9496e-007	2
0.01	6.1947e-011	2.0014	4.2363e-005	1	4.9575e-009	1.9993

Table 6.6. The l^2 errors and convergence rates of different splitting methods with $\Delta x = 0.01$ for decreasing time steps at $T = 150$.

6.2. Solute Transport Problem in Ground Water Flow

The transient two dimensional solute transport model in ground water flow, (see (Verma et. al., 2000)), is given with the following partial differential equation

$$R_d \frac{\partial u}{\partial t} = \nabla \cdot (D_h \cdot \nabla u) - \nabla \cdot (Vu) - \lambda R_d u \tag{6.26}$$

where u=solute concentration; V=pore-water velocity vector; R_d=retardation factor; λ=first-order decay coefficient; D_h=hydrodynamic dispersion tensor, D_{xx}, D_{yy}, $D_{xy}(= D_{yx})$ are generally functions of velocity and the molecular diffusion. In this study, the case of homogenous and isotropic medium under $2D$ ground-water flow with $2D$ dispersion is considered.

Equation (6.26), using the iterative splitting method, is solved to simulate unsteady $2D$ solute transport between two impervious boundaries. A finite-length strip solute source, whose concentration is a given function of time, is located asymmetrically along the y-axis at $x = 0$ in a unidirectional seepage velocity field, as shown in Figure 6.6. The rectangular domain is 75 m in the x-direction and 50 m in the y-direction. The uniform pore velocity v is 0.1 m/day. The longitudinal, transverse, and cross dispersivities, $D_{xx} = 1$, $D_{yy} = 0.1$, and $D_{xy} = 0$, respectively. The retardation factor $R_d = 1$, and decay coefficient $\lambda = 0$. The initial condition is

$$u(x, y, 0) = 0. \tag{6.27}$$

The boundary condition at $x = 0$, $t > 0$, is given by

$$u(0, y, t) = 0, \quad 0 < y < 5 \text{ m} \tag{6.28}$$

$$u(0, y, t) = 1, \quad 5 < y < 15 \text{ m} \tag{6.29}$$

$$u(0, y, t) = 0, \quad 15 < y < 50 \text{ m}. \tag{6.30}$$

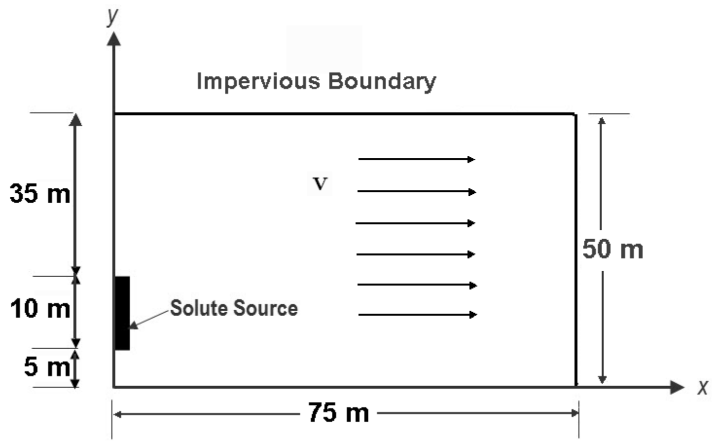


Figure 6.6. Schema of solute transport problem.

We start with splitting transient solute transport Equation (6.26) into two parts: diffusion

$$u_t = D_{xx}u_{xx} + D_{yy}u_{yy} \tag{6.31}$$

and convection part

$$u_t = -vu_x. \tag{6.32}$$

Next we combine these equations with iterative algorithms (2.17), (2.19) and obtain

$$u_i = D_{xx}(u_i)_{xx} + D_{yy}(u_i)_{yy} - vu_{i-1} \tag{6.33}$$

$$u_{i+1} = D_{xx}(u_i)_{xx} + D_{yy}(u_i)_{yy} - vu_{i+1} \tag{6.34}$$

where $i = 1, 3, \dots, 2m-1$.

The second order central difference expansion is used for each derivative terms u_{xx}, u_{yy} and u_x such as

$$\frac{\partial^2 u}{\partial x^2} |_{(x_m, y_k, t)} = \frac{1}{\Delta x^2}(u_{m+1,k}(t) - 2u_{m,k}(t) + u_{m-1,k}(t)) \tag{6.35}$$

and

$$\frac{\partial^2 u}{\partial y^2} |_{(x_m, y_k, t)} = \frac{1}{\Delta y^2}(u_{m,k+1}(t) - 2u_{m,k}(t) + u_{m,k-1}(t)) \tag{6.36}$$

and advection term at each grid point (x_m, y_k, t) becomes

$$\frac{\partial u}{\partial x} \mid_{(x_m,y_k,t)} = \frac{1}{2\Delta x}(u_{m+1,k}(t) - u_{m-1,k}(t)) \tag{6.37}$$

where Δx and Δy are the spatial discretization steps and $m = 1, \ldots, N+1$, $k = 1, \ldots, M+1$.

After assembling the unknowns in (6.35), (6.36), (6.37) for each m, k and embedding the boundary conditions, we have the following systems of equations

$$u_{xx} = AU, \;\; u_{yy} = \tilde{A}U, \text{ and } \; u_x = BU. \tag{6.38}$$

Redefining Equations (6.33), (6.34) yields

$$U_i' = A_1 U_i + A_2 U_{i-1} + f \tag{6.39}$$

$$U_{i+1}' = A_1 U_i + A_2 U_{i+1} + g \tag{6.40}$$

where $A_1 = D_{xx}A + D_{yy}\tilde{A}$, $A_2 = -vB$ are $(M-1)(N+1) \times (M-1)(N+1)$ dimensional matrixes and f, g are vectors which come from boundary conditions. Finally, applying midpoint rule to each iteration on each $[t^n, t^{n+1}]$ interval, we obtain

$$\begin{aligned} U_i^{n+1} &= (I - \frac{\Delta t}{2}A_1)^{-1}((I + \frac{\Delta t}{2}A_1)U_i^n + \frac{\Delta t}{2}A_2(U_{i-1}^n + U_{i-1}^{n+1}) \\ &+ \frac{\Delta t}{2}(f^n + g^n + f^{n+1} + g^{n+1})) \end{aligned} \tag{6.41}$$

$$\begin{aligned} U_{i+1}^{n+1} &= (I - \frac{\Delta t}{2}A_2)^{-1}((I + \frac{\Delta t}{2}A_2)U_{i+1}^n + \frac{\Delta t}{2}A_1(U_i^n + U_i^{n+1}) \\ &+ \frac{\Delta t}{2}(f^n + g^n + f^{n+1} + g^{n+1})) \end{aligned} \tag{6.42}$$

where Δt is time discretization step.

The computational domain is represented by $N = 60$, $M = 40$ (in x-direction and y-direction). Here, for the purpose of comparative analysis, the same discretization numbers and numerical parameters used in (Verma et. al., 2000) are employed.

Figure 6.7 (a) and Figure 6.8 (a) present the longitudinal concentration distributions taken with iterative splitting, Lie-Trotter splitting, Strang splitting methods and exact solution at $T =$ 100 days as a function of x at $y = 10$ m and $y = 16.25$ m for computational time step $\Delta t = 1$ day and $\Delta t = 10$ days, respectively.

Figure 6.7 (b) and Figure 6.8 (b) present the lateral concentration distributions taken with iterative splitting, Lie-Trotter splitting, Strang splitting methods and exact solution at $x = 5$ m and $x = 20$ m for $\Delta t = 1$ day and $\Delta t = 10$ days, respectively.

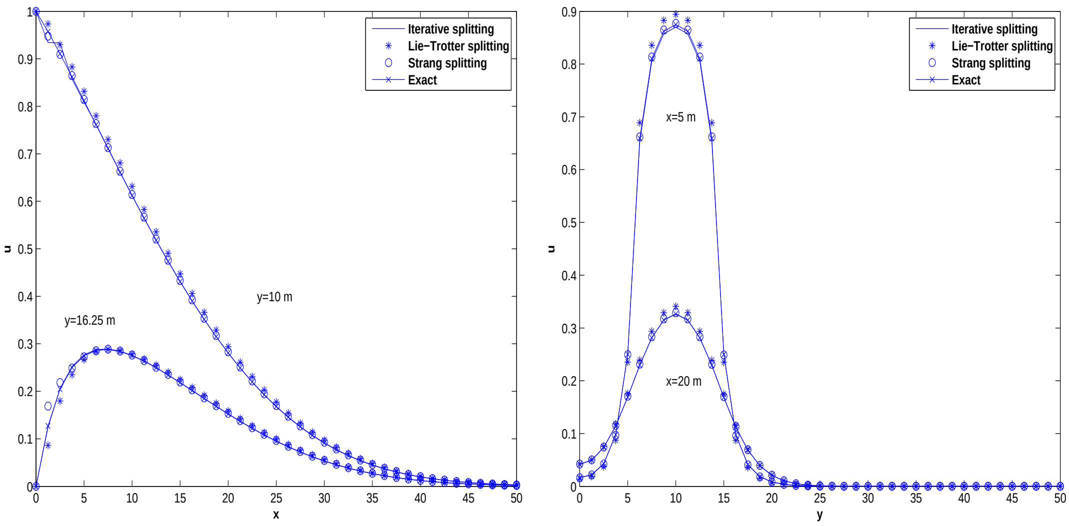
In Figure 6.9 (a) and Figure 6.9 (b), the relative errors of iterative splitting, Lie-Trotter splitting and Strang splitting methods are simulated for $x = 5$ m, $x = 20$ m at $T = 100$ days. The below lines in the figures represents the highest order numerical convergence. In Figure 6.10 (a) and Figure 6.10 (b), the relative errors of iterative splitting, Lie-Trotter splitting and Strang splitting methods are simulated for $y = 10$ m, $y = 16.25$ m at $T = 100$ days.

In Table 6.7 and Table 6.8, we present the l^2 errors of the lateral and longitudinal concentration distributions taken with various splitting methods at $T = 100$ days.

The Matlab package *expm* is used to calculate an exact solution. The pointwise errors are estimated using Richardson extrapolation, (see (Burg and Erwin, 2009)), such as

$$e(x_m, y_k, T) = \frac{1}{3}|4u^{\Delta t_{j2}}(x_m, y_k, T) - u^{\Delta t_{j1}}(x_m, y_k, T)| \tag{6.43}$$

where $u^{\Delta t_{j1}}(x_m, y_k, T)$ and $u^{\Delta t_{j2}}(x_m, y_k, T)$ are the approximate solutions obtained with Δt_{j1} and Δt_{j2}, respectively, at T days. Note that in this case, $\Delta t_{j2} = \Delta t_{j1}/2$.



(a) (b)

Figure 6.7. (a) Longitudinal concentration profile for $\Delta t = 10$ days.
(b) Transverse concentration profile for $\Delta t = 10$ days.

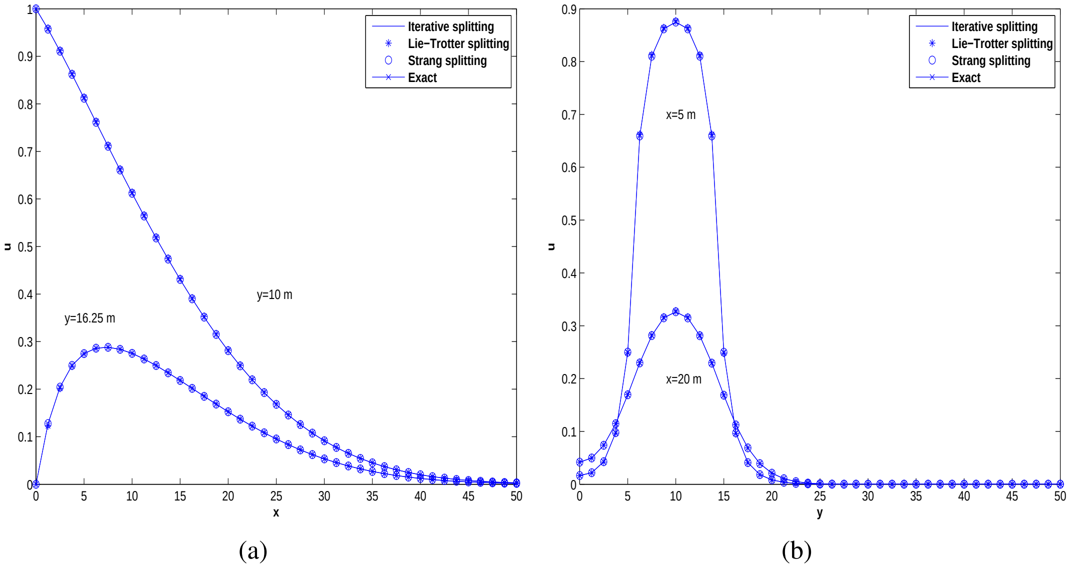


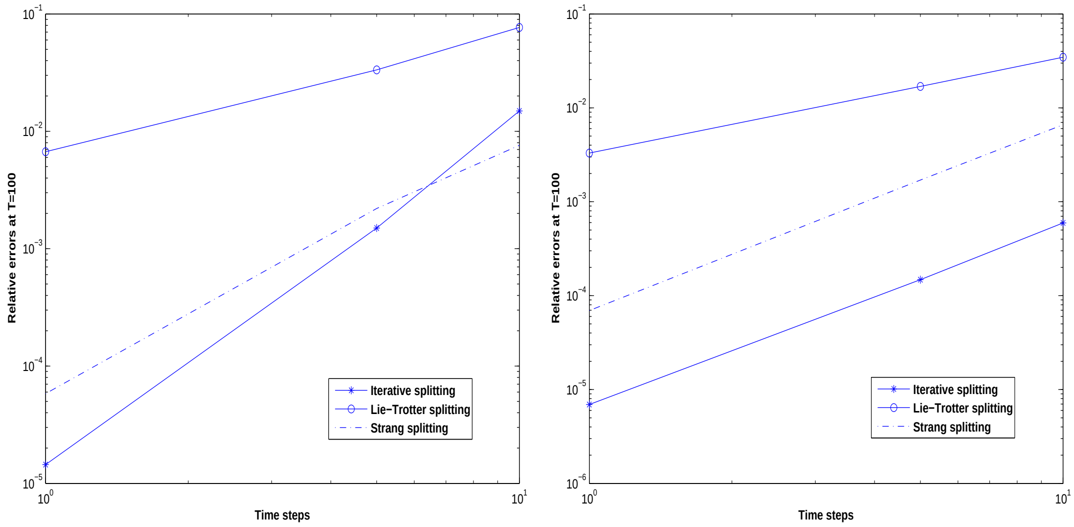
Figure 6.8. (a) Longitudinal concentration profile for $\Delta t = 1$ day.
(b) Transverse concentration profile for $\Delta t = 1$ day.

	at $y = 10$	at $y = 16.25$
Iterative Splitting	2.3716e-005	4.5885e-005
Lie-Trotter Splitting	0.0025	0.0018
Strang Splitting	1.6546e-004	5.4205e-004

Table 6.7. The l^2 errors of longitudinal concentration distributions for $\Delta t = 1$ at $T = 100$.

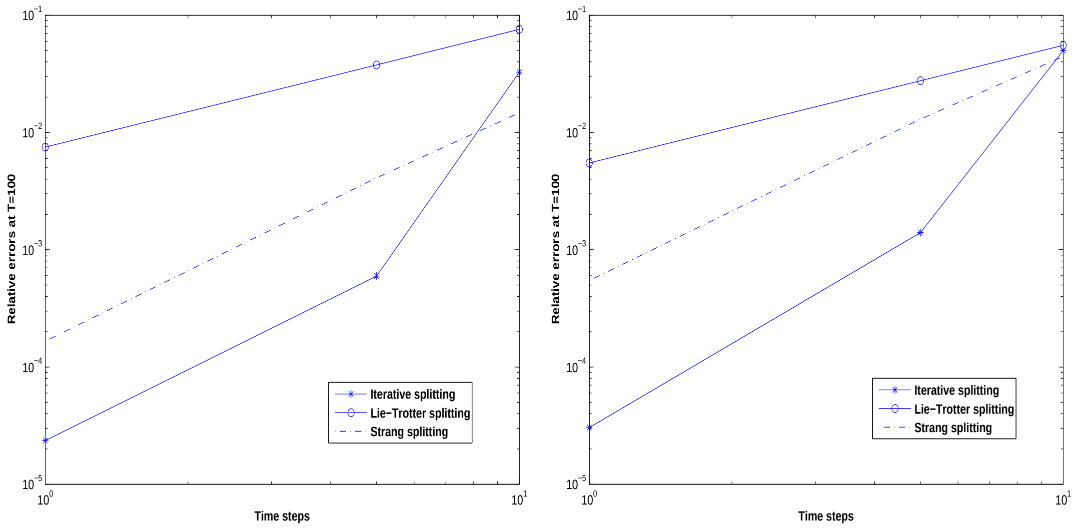
	at $x = 20$	at $x = 5$
Iterative Splitting	6.2517e-006	3.3810e-005
Lie-Trotter Splitting	0.0011	0.0022
Strang Splitting	6.9421e-005	6.3913e-005

Table 6.8. The l^2 errors of transverse concentration distributions for $\Delta t = 1$ at $T = 100$.



(a) (b)

Figure 6.9. (a) l^2 errors versus scaled step sizes for transverse concentration profile for $x = 5$ m at $T = 100$ days.
(b) l^2 errors versus scaled step sizes for transverse concentration profile for $x = 20$ m at $T = 100$ days.



(a) (b)

Figure 6.10. (a) l^2 errors versus scaled step sizes for longitudinal concentration profile for $y = 10$ m at $T = 100$ days.
(b) l^2 errors versus scaled step sizes for longitudinal concentration profile for $y = 16.25$ m at $T = 100$ days.

6.3. Korteweg-de Vries Equation

We consider the nonlinear Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{6.44}$$

which admits soliton solutions and models the propagation of solitary wave on water surface. In 1834, its phenomena is first discovered by Russell (Russell, 1838) and Korteweg-de Vries (Korteweg, 1895) formulates the mathematical model equation to provide explanation of the phenomena. The term uu_x describes the sharpening of waves and u_{xxx} is the dispersion term.

For purpose of illustration of iterative splitting method in solving KdV Equation (6.44), consider the following three initial boundary value problems.

Example 6.1

$$u_t + 6uu_x + u_{xxx} = 0, \;\; u(x,0) = \frac{1}{2}sech^2(\frac{1}{2}x), \;\; x \in (l_1, l_2) \tag{6.45}$$

$$u(x,t)\mid_{l_1} = 0, \;\; u(x,t)\mid_{l_2} = 0, \;\; t \in (0,T], \tag{6.46}$$

where the analytic solution is

$$u_{analy}(x,t) = \frac{1}{2}sech^2(\frac{1}{2}(x-t)). \tag{6.47}$$

The Equation (6.45) is splitted into linear and nonlinear parts:

$$u_t = -u_{xxx},$$

$$u_t = -6uu_x. \tag{6.48}$$

Applying iterative schemes yields the following algorithms

$$u_i' = -(u_i)_{xxx} - 6u_{i-1}(u_{i-1})_x, \tag{6.49}$$

$$u_{i+1}' = -(u_i)_{xxx} - 6u_i(u_{i+1})_x \tag{6.50}$$

where $i = 1, 3, \ldots, 2m-1$. In order to solve them, applying spatial discretization for initial condition yields

$$u_m = \frac{1}{2}sech^2(\frac{1}{2}x_m), \quad 1 \le m \le N+1, \tag{6.51}$$

and boundary conditions (6.46) become

$$u_1 \;=\; 0, \quad 0 \leq t, \tag{6.52}$$

$$u_{N+1} \;=\; 0, \quad 0 \leq t. \tag{6.53}$$

We derive the second order discretization for u_{xxx} and central difference expansion for u_x such as

$$\frac{\partial^3 u}{\partial x^3}\mid_{(x_m,t)}= \frac{1}{2\Delta x^3}(u_{m+2}(t) - 2u_{m+1}(t) + 2u_{m-1}(t) - u_{m-2}(t)) \tag{6.54}$$

and

$$\frac{\partial u}{\partial x}\mid_{(x_m,t)}= \frac{1}{2\Delta x}(u_{m+1}(t) - u_{m-1}(t)) \tag{6.55}$$

where Δx is the spatial stepping and $m = 1, \ldots, N+1$.

After assembling the unknowns of (6.54), (6.55) for each m, we have the following systems of equations

$$u_{xxx} = A_1 U, \quad u_x = A_2 U. \tag{6.56}$$

Redefining Equations (6.49), (6.50) we have

$$U_i' \;=\; -A_1 U_i - 6\tilde{U}_{i-1} A_2 U_{i-1} \tag{6.57}$$

$$U_{i+1}' \;=\; -A_1 U_i - 6\tilde{U}_i A_2 U_{i+1} \tag{6.58}$$

where the nonlinear term $U \simeq \tilde{U}$ are fixed at each space discretization points $m = 1, \ldots, N+1$. Applying midpoint rule to Equations (6.57), (6.58) yields

$$U_i^{n+1} = (I + \tfrac{\Delta t}{2} A_1)^{-1}((I - \tfrac{\Delta t}{2} A_1)U_i^n - \tfrac{\Delta t}{2}(6\tilde{U}_{i-1}^n A_2 U_{i-1}^n + 6\tilde{U}_{i-1}^{n+1} A_2 U_{i-1}^{n+1})) \tag{6.59}$$

$$U_{i+1}^{n+1} = (I + \tfrac{\Delta t}{2} 6\tilde{U}_i^{n+1} A_2)^{-1}((I - \tfrac{\Delta t}{2} 6\tilde{U}_i^n A_2)U_{i+1}^n - \tfrac{\Delta t}{2} A_1(U_i^n + U_{i+1}^n)) \tag{6.60}$$

where Δt is time discretization step.

Example 6.2 *As a second example we present the following two-soliton problem*

$$u_t + 6uu_x + u_{xxx} \;=\; 0, \quad u(x,0) = 6sech^2(x), \quad x \in (l_1, l_2) \tag{6.61}$$

$$u(x,t)\mid_{l_1} \;=\; 0, \quad u(x,t)\mid_{l_2}= 0, \quad t \in (0, T] \tag{6.62}$$

where the analytic solution, (see (Chertock and Levy, 2002), (Drazin and Johnson, 1989)), is

$$u_{analy}(x,t) = 12\frac{3 + 4cosh(2x - 8t) + cosh(4x - 64t)}{(3cosh(x - 28t) + cosh(3x - 36t))^2}. \tag{6.63}$$

Example 6.3 *As a third example we present a double-soliton problem such that*

$$u_t + 6uu_x + u_{xxx} = 0, \quad u(x,0) = \frac{1}{2}sech^2(\frac{1}{2}x) + 6sech^2(x), \quad x \in (l_1, l_2) \tag{6.64}$$

$$u(x,t)\mid_{l_1} = 0, \quad u(x,t)\mid_{l_2} = 0, \quad t \in (0,T]. \tag{6.65}$$

The iterative splitting method has been successfully applied to finding the numerical solution of KdV equations with different initial conditions.

In Figure 6.12 (a), one soliton iterative splitting and exact solutions are presented for $\Delta x = 0.3$, $\Delta t = 0.05$ at $T = 5$ and in Figure 6.12 (b), iterative splitting solutions are presented at different times. The results show that iterative splitting and exact solutions behave similarly. In Figure 6.11, three dimensional one soliton iterative splitting and exact solutions presented with same values.

In Figure 6.13, two soliton iterative splitting solutions are plotted at different times. Finally, we computed the double soliton collision by taking the initial condition as a sum of two solitons and the results are presented in Figure 6.14.

In Table 6.9, the errors of KdV Equation (6.45)-(6.46) taken with various methods are given for $\Delta x = 0.3$, $\Delta t = 0.05$ at $T = 5$.

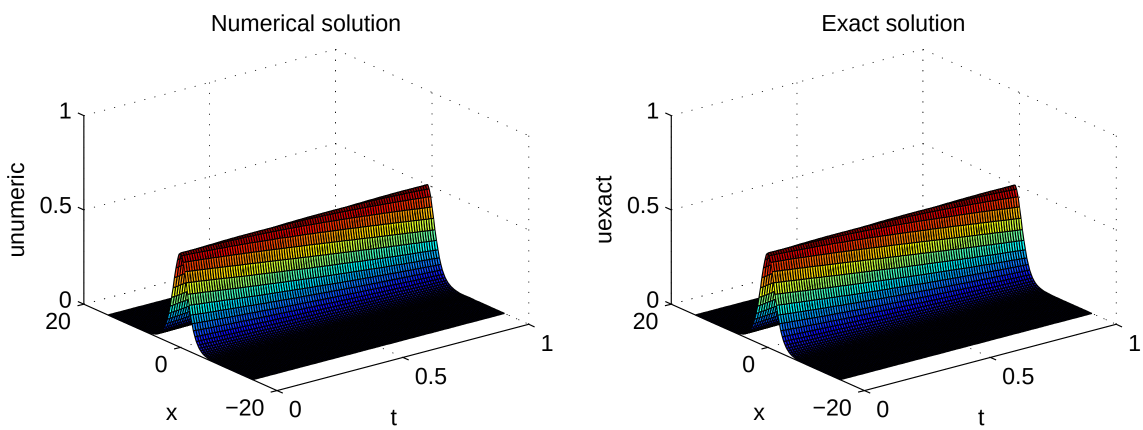
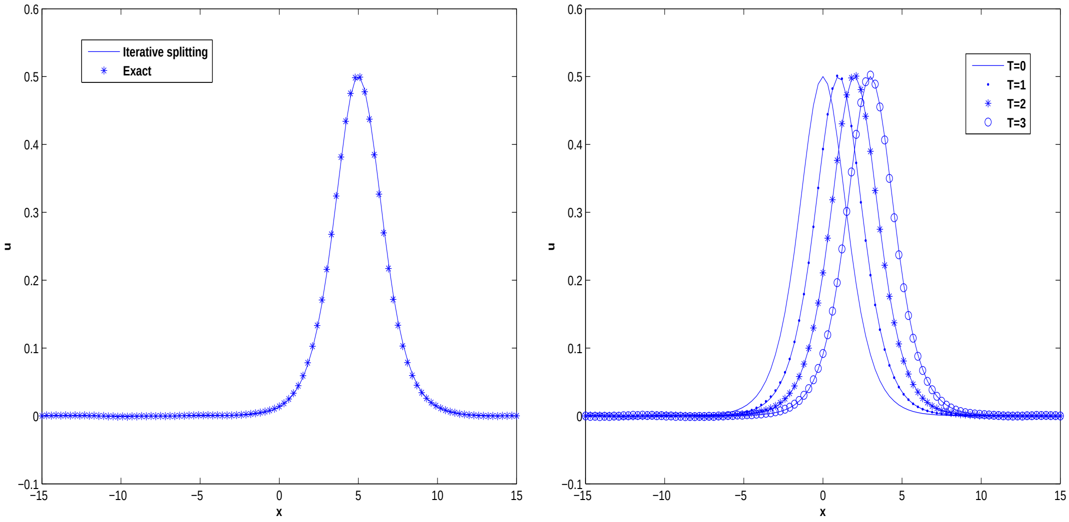


Figure 6.11. Comparison of iterative splitting and exact solutions of KdV Equation (6.45)-(6.46) on $-15 \leq x \leq 15$ interval at $T = 5$.



(a) (b)

Figure 6.12. (a) Comparison of numerical and exact solutions of KdV Equation (6.45)-(6.46) on $-15 \leq x \leq 15$ interval at $T = 5$.
(b) The solutions of KdV Equation (6.45)-(6.46) on $-15 \leq x \leq 15$ interval at different times. The points represent the location of iterative splitting solutions. The solid lines represent exact solution (6.47).

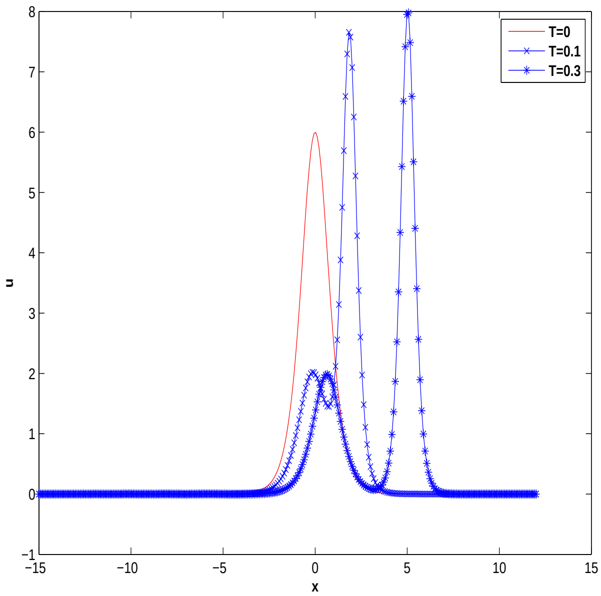
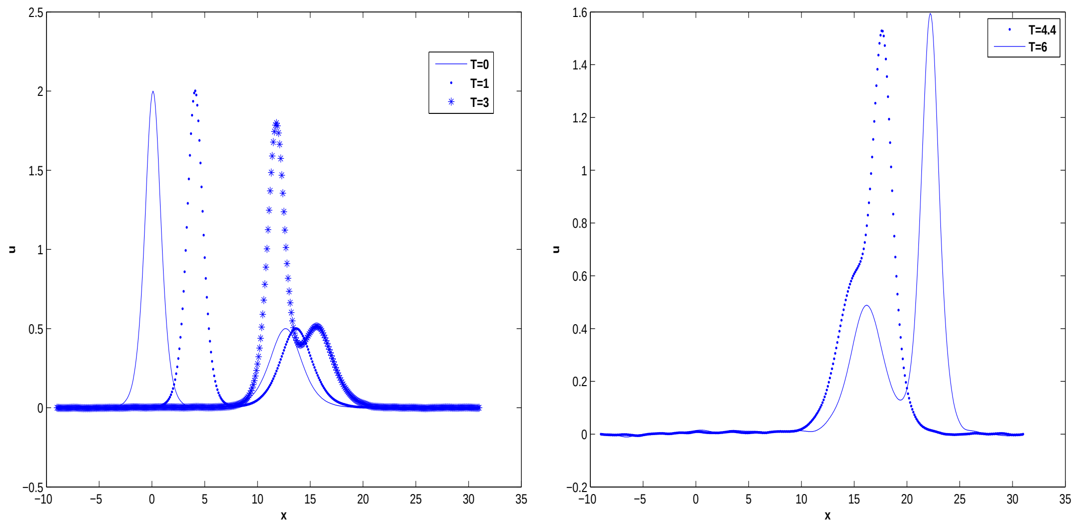


Figure 6.13. The solutions of KdV Equation (6.61)-(6.62) on $-15 \leq x \leq 12$ interval at different times. The points represent the location of iterative splitting solutions. The solid lines represent exact solution (6.63).

	Error l^2	Error l^∞
Iterative Splitting	0.0301	0.0098
Lie-Trotter Splitting	0.1434	0.0503
Non splitting(BTCS)	0.3076	0.1103

Table 6.9. The errors of KdV Equation (6.45)-(6.46) on $[-15, 15]$ interval for $\Delta x = 0.3$, $\Delta t = 0.05$ at $T = 5$.



(a)

(b)

Figure 6.14. (a) The iterative splitting solutions of KdV Equation (6.64)-(6.65) on $-9 \leq x \leq 31$ interval at different times.
(b) The figure goes on.

6.4. Heat Equation

Consider the two dimensional heat equation which is given

$$u_t = \epsilon^2 u_{xx} + u_{yy} + f \text{ in } \Omega \tag{6.66}$$

$$u = e^{x+\epsilon y^2} \text{ on } \Omega \times \{t = 0\}, \tag{6.67}$$

$$u = e^{-t} e^{x+\epsilon y^2} \text{ on } \partial\Omega \times [0, 1] \tag{6.68}$$

where $f(x,y,t) = -(1+\epsilon^2+4\epsilon^2y^2+2\epsilon)e^{-t}e^{x+\epsilon y^2}$, $\Omega = (-1,1)\times(-1,1)$ and the exact solution is

$$u(x,y,t) = e^{-t}e^{x+\epsilon y^2}. \tag{6.69}$$

We want to study (6.66)-(6.68) within the semigroup framework given in Section 3.2 to confirm the second order convergence of the second iterative scheme. We employ splitting to Equation (6.66) which yields

$$u_t = A_1u + f/2 \tag{6.70}$$

and

$$u_t = A_2u + f/2 \tag{6.71}$$

where the operators $A_1 = \epsilon^2\partial_{xx}$, $A_2 = \partial_{yy}$ and $A = A_1 + A_2$. The operators have the divergence structure and smooth coefficients, also satisfy ellipticity condition (see Appendix B). Note that a problem with nonzero boundary values can be transformed into zero boundary conditions, then problem (6.66) is examined similar to Example 3.1. Hence we let $X = L^2(\Omega)$, $D(A_1) = H_0^1(\Omega)\cap H^2(\Omega)$, $D(A_2) = H_0^1(\Omega)\cap H^2(\Omega)$. Clearly the A_1 and A_2 are unbounded linear operators on X and satisfy the hypotheses of Theorem 3.3 (Hille-Yosida). Equation (3.49) is also satisfied such as $D(A_1) = D(A_2) = D(A)$. Hence, Theorem 5.3 is satisfied such that the second iterative scheme gives the second order convergence. We illustrate the results in Table 6.10.

Time steps	Error l^2	Order
$1/5$	0.0031	
$1/50$	3.0902e-005	2.0014
1/500	1.4892e-007	2.3170

Table 6.10. The l^2 errors and convergence rates of second iterative scheme for decreasing time steps at $T = 1$.

CHAPTER 7

CONCLUSIONS

In this thesis, we investigated the consistency, stability and convergence analyses of an operator splitting method, namely the iterative operator splitting method using various approaches for parabolic partial differential equations. The analyses depended on whether the operators were bounded or unbounded. We used finite difference approximation in each spatial derivative term to obtain a bounded system. In the bounded case, we showed that the accuracy of the iterative splitting method increases with the number of iterative schemes. For unbounded operators, we used Fourier transform method and C_0 semigroup theory. Fourier transform analysis achieved the linear stability criteria and also enabled a nonlinear problem, KdV equation, to be analysed as a linear problem. C_0 semigroup theory allowed the consistency and the stability estimates for unbounded operators. Then, we investigated the convergence issues. For that purpose, Lax-Richtmyer equivalence theorem and Lady Windermere's fan argument were used.

Finally, we studied three parabolic partial differential equations and a one dimensional KdV equation. These three equations were capillary formation model in tumor angiogenesis problem, solute transport model and heat equation. For the first three problems, we exhibited the numerical errors and orders in tables and figures to show the effectiveness of the iterative splitting method. The error results revealed that the proposed method gave smaller error for chosen problems. In the last model, we confirmed the convergence result obtained using semigroup framework with an order table.

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APPENDIX A

A REMINDER OF SOME FUNCTIONAL ANALYSIS

Definition A.1 (Normed space) *A **normed space** is a vector space with a normed defined on it. Here a **norm** on a (real or complex) vector space X is a real valued function on X whose value at an $x \in X$ is denoted by*

$$\|x\|$$

and for any arbitrary x and y vectors in X, has the properties

(N1) $\|x\| \geq 0$,

(N2) $\|x\| = 0 \Leftrightarrow x = 0$,

(N3) $\|\alpha x\| = |\alpha|\|x\|$,

(N4) $\|x + y\| \leq \|x\| + \|y\|$.

Definition A.2 (Banach space) *A **Banach space** is a complete normed vector space $(X, \|\cdot\|)$, i. e. every Cauchy sequence in (X, d) converges.*

Definition A.3 (Completeness) *A normed vector space X is said to be **complete** if every Cauchy sequence converges to a limit in X.*

Definition A.4 (Cauchy sequence) *A sequence $\{x_n\}_{n=1}^{\infty}$ in a normed vector space X is called a **Cauchy sequence** if for any $\epsilon >$, there exists an integer N (depends on ϵ) such that*

$$\|x_n - x_m\| < \epsilon \tag{A.1}$$

for all $n > N, m > N$.

Definition A.5 (Equivalence of norms) *Two norms $\|\cdot\|_1$ $\|\cdot\|_2$ on a real vector space X are equivalent, if*

$$\exists\, \epsilon > 0,\, \forall x \in X\, :\, \frac{1}{c}\|x\|_1 \leq \|x\|_2 \leq c\|x\|_1.$$

Lemma A.1 *Let X be finite dimensional vector space. Then any two norms on X are equivalent.*

Lemma A.2 *Every finite dimensional vector space is $(X, \|\cdot\|)$ is complete.*

Here, the vector spaces with norms defined on them used in this thesis are given

- $l^p = \{(x_n)_{n\in\mathbb{N}} \subset \mathbb{R} : \sum_{n\in\mathbb{N}} |x_n|^p < \infty\}, p \geq 1$ with norm

$$\|(x_n)_{n\in\mathbb{N}}\|_p = \big(\sum_{i=1}^{n} |x_i|^p\big)^{1/p},$$

- $l^\infty = \{(x_n)_{n\in\mathbb{N}} \subset \mathbb{R} : sup\|x_n\|_{n\in\mathbb{N}} < \infty\}$ with norm

$$\|(x_n)_{n\in\mathbb{N}}\|_\infty = sup_{n\in\mathbb{N}}|x_n|,$$

where $x_n \in \mathbb{R}^n$.

The space of all bounded linear operators on X is denoted by $L(X)$ and becomes a Banach space for the norm

$$\|A\| = sup\{Ax : x \leq 1\}, A \in L(X). \tag{A.2}$$

Definition A.6 (Densely defined operator) *A operator $A : Y \to X$ is called a* ***densely defined operator*** *if its domain is a dense set in X.*

Definition A.7 (Dense set) *A set $D(A)$ is called* ***dense*** *in X if every point x in X either belongs to Y or is a limit point of Y.*

Definition A.8 *A linear operator $A : Y \to X$ is* ***closed*** *if for any sequences $\{x_n\}_{n=1}^{\infty} \in Y$ such that $x_n \to x$ and $Ax_n \to f$ in X, it follows that $x \in Y$ and $Ax = f$.*

Theorem A.1 (Closed graph theorem) *A closed operator on a closed domain is necessarily bounded.*

Definition A.9 (Square integrable function space) *$L^2(\Omega)$ is the vector spaces of square integrable functions defined on Ω such that*

$$u \in L^2(\Omega) \text{ if and only if } \int_\Omega u^2 < \infty. \tag{A.3}$$

The norm defined on $L^2(\Omega)$ is

$$\|u\|_{L^2(\Omega)} = (\int_\Omega u^2)^{1/2}. \tag{A.4}$$

Definition A.10 (Sobolev space) *For a positive index k, the Sobolev space $H^k(\Omega)$ is the set of function $u : \Omega \to \mathbb{R}$ such that u and all derivatives up to and including k are Sobolev integrable:*

$$u \in H^k(\Omega) \Leftrightarrow \int_\Omega u^2 < \infty, \int_\Omega (\frac{du}{dx})^2 < \infty, \ldots, \int_\Omega (\frac{d^k u}{dx^k})^2 < \infty. \tag{A.5}$$

The norm defined on $H^k(\Omega)$ is

$$\|u\|_{H^k(\Omega)} = \big(\int_\Omega u^2 + \int_\Omega (\frac{du}{dx})^2 + \ldots + \int_\Omega (\frac{d^k u}{dx^k})^2\big)^{1/2}. \tag{A.6}$$

APPENDIX B

A REMINDER OF SECOND ORDER ELLIPTIC EQUATIONS

Definition B.1 (Elliptic equations) *Consider the following boundary value problem*

$$Au = f \text{ in } \Omega \tag{B.1}$$

$$u = 0 \text{ on } \partial\Omega, \tag{B.2}$$

where Ω *is an open, bounded subset of* $\mathbb{R}^n$, $u : \bar{\Omega} \to \mathbb{R}$ *is the unknown,* $u = u(x)$, $f : \Omega \to \mathbb{R}$ *and* A *denotes a second order partial differentail operator having either the form*

$$Au = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u, \tag{B.3}$$

$$Au = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u \tag{B.4}$$

for given coefficient functions a^{ij}, b^i, c $(i, j = 1, \ldots, n)$.

The PDE $Au = f$ is in divergence form if A is given by (B.3) and is in nondivergence form provided A is given by (B.4).

Definition B.2 (Ellipticity condition) *We say that a partial differential operator* A *is (uniformly)* ***elliptic*** *if there exists a constant* $\theta > 0$ *such that*

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \tag{B.5}$$

for a. e. $u \in \Omega$ *and all* $\xi \in \mathbb{R}^n$.

Definition B.3 (Bilinear form) **(i)** *The bilinear form* $B[\,,\,]$ *associated with the divergence form elliptic operator* A *is defined by (B.3) is*

$$B[u, v] = \int_\Omega \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv \, dx, \tag{B.6}$$

for u, $v \in H_0^1(\Omega)$.

(ii) *We say that* $u \in H_0^1(\Omega)$ *is a weak solution of the boundary value problem (B.1)-(B.2) if*

$$B[u,v] = (f,v) \tag{B.7}$$

for all $v \in H_0^1(\Omega)$, *where* $(\ ,\ )$ *denotes the inner product in* $L^2(\Omega)$.

Theorem B.1 (Energy estimates) *There exists constants* $\alpha,\ \beta > 0$ *and* $\gamma \geq 0$ *such that*

(i)

$$|B[u,v]| \leq \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}, \tag{B.8}$$

(ii)

$$\beta \|u\|^2_{H_0^1(\Omega)} \leq B[u,u] + \gamma \|u\|^2_{L^2(\Omega)}, \tag{B.9}$$

for all $u,\ v \in H_0^1(\Omega)$.

Theorem B.2 (First existence theorem for weak solutions) *There is a number* $\gamma \geq 0$ *such that for each*

$$\mu \geq \gamma \tag{B.10}$$

and each function

$$f \in L^2(\Omega), \tag{B.11}$$

there is a weak solution $u \in H_0^1(\Omega)$ *of the boundary value problem*

$$Au + \mu u = f \ \ in \ \ \Omega \tag{B.12}$$

$$u = 0 \ \ on \ \ \partial\Omega. \tag{B.13}$$

Theorem B.3 (Boundary H^2 regularity) *Assume*

$$a^{ij} \in C^1(\bar{\Omega}), b^i, c \in L^\infty(\Omega),\ (i,j = 1,\ldots,n), \tag{B.14}$$

$$f \in L^2(\Omega). \tag{B.15}$$

Suppose that $u \in H_0^1(\Omega)$ *is a weak solution of the elliptic boundary value problem*

$$Au = f \ \ in \ \ \Omega \tag{B.16}$$

$$u = 0 \ \ on \ \ \partial\Omega. \tag{B.17}$$

Assume finally

$$\partial\Omega \text{ is } C^2. \tag{B.18}$$

Then

$$u \in H^2(\Omega), \tag{B.19}$$

and we have the estimate

$$\|u\|_{H^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}), \tag{B.20}$$

the constant C depending on Ω and the coefficients of A.

See (Lawrence, 1998) for the proof of theorems given in this section.

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