STABILIZED FINITE ELEMENT METHODS FOR TIME DEPENDENT CONVECTION-DIFFUSION EQUATIONS

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ABSTRACT

STABILIZED FINITE ELEMENT METHODS FOR TIME DEPENDENT CONVECTION-DIFFUSION EQUATIONS

In this thesis, enriched finite element methods are presented for both steady and unsteady convection diffusion equations. For the unsteady case, we follow the method of lines approach that consists of first discretizing in space and then use some time integrator to solve the resulting system of ordinary differential equation. Discretization in time is performed by the generalized Euler finite difference scheme, while for the space discretization the streamline upwind Petrov-Galerkin (SUPG), the Residual free bubble (RFB), the more recent multiscale (MS) and specific combination of RFB with MS (MIX) methods are considered. To apply the RFB and the MS methods, the steady local problem, which is as complicated as the original steady equation, should be solved in each element. That requirement makes these methods quite expensive especially for two dimensional problems. In order to overcome that drawback the pseudo approximation techniques, which employ only a few nodes in each element, are used. Next, for the unsteady problem a proper adaptation recipe, including these approximations combined with the generalized Euler time discretization, is described. For piecewise linear finite element discretization on triangular grid, the SUPG method is used. Then we derive an efficient stability parameter by examining the relation of the RFB and the SUPG methods. Stability and convergence analysis of the SUPG method applied to the unsteady problem is obtained by extending the Burman's analysis techniques for the pure convection problem. We also suggest a novel operator splitting strategy for the transport equations with nonlinear reaction term. As a result two subproblems are obtained. One of which we may apply using the SUPG stabilization while the other equation can be solved analytically. Lastly, numerical experiments are presented to illustrate the good performance of the method.

ÖZET

ZAMANA BAĞLI KONVEKSIYON DİFÜZYON DENKLEMLERİ İÇİN KARARLI SONLU ELEMANLAR YÖNTEMLERİ

Bu tezde hem durağan hemde durağan olmayan konveksiyon difüzyon denklemleri için zenginleştirilmiş sonlu elemanlar yöntemleri verildi. Durağan olmayan problemler için "method of lines" tekniği ele alınıp denklemin önce uzaysal kısmı ayrıklaştırılıp sona zamansal ayrıklaştırması ortaya çıkan adi differansiyel denklem sistemine uygulandı. Zamandaki ayrıklaştırma için genelleştirilmiş Euler sonlu fark şeması kullanılırken uzaysal ayrıklaştırma için "streamline upwind Petrov-Galerkin" (SUPG), "residual free bubble" (RFB) ve daha güncel olan "multiscale" (MS) ile RFB ve MS in özel bir kombinasyonu olan MIX motodları incelendi. Özellikle iki boyutlu problemlerde RFB ve MS algoritmaları için her bir eleman içinde orjinal durağan differansiyel denklem kadar karmaşık bir denlem çözme gerekliliği bu algoritmaları oldukça kullanışsız yapmaktadır. Fakat "pseudo" yaklaşım tekniği sayesinde eleman içinde sadece bir kaç nokta kullanarak bu denklemlerin etkili ve pratik yaklaşık çözümleri elde edilebildi. Daha sonra bu metodların ve genelleştirilmiş Euler şemasının uygun bir kombinasyon formülü verilerek durağan olmayan denklemler için bir adaptasyon sağlanmış oldu. Üçgensel ağ üzerinde parçalı sürekli doğrusal baz fonksiyonları için SUPG metodu incelendi. Bu sayede, etkili bir SUPG stabilizasyon parametresi RFB metodu kullanılarak elde edildi. SUPG için stabilite ve yakınsama analizleri ayrıca incelenip, Burman'ın durağan olmayan salt konveksiyon denklemi için önerdiği analiz tekniği burada konveksiyon difüzyon denklemi için genelleştirildi. Ayrıca yeni bir operatör ayırma stratejisi linear olmayan reaksiyon terimi içeren taşınım denklemi için önerildi. Bunun sonucu olarak bir tanesi SUPG methodu kullanılarak yaklaşık olarak çözülebilen diğeri ise analitik olarak çözülebilen iki alt probleme ulaşıldı. Son olarak metodumuzun etkinliği sayısal deneylerle gösterildi.

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CHAPTER 1

INTRODUCTION

1.1. Introduction

It is known that the standard Galerkin finite element method (SGFEM) based on low order piecewise polynomials is unsuitable for the solution of singularly perturbed problems. When the advection term dominates the diffusion one or small time steps are employed, numerical solutions obtained by SGFEM suffer from nonphysical oscillations unless appropriately designed mesh is used (Ross, Stynes and Tobiska, 2008), (Harari, 2004). Although a number of studies focuses on the steady problems, little attention is given to the unsteady cases. The present study contributes to filling in that gap. For the unsteady problems, the method of lines approach, which is based on separating spatial and temporal discretization, is applied. Consequently the steady case of the convection diffusion equation is considered in detail.

Many stabilization techniques for steady equations have been proposed to cure the drawback of the SGFEM in the convection dominated case. One of the frequently used method is the streamline-upwind Petrov-Galerkin (SUPG) introduced by Brooks and Hughes (Brooks and Hughes, 1982) and analyzed by Johnson et al. (Johnson, Nävert and Pitkäranta, 1984). This method corresponds to adding a consistent term providing an additional diffusion in the streamline direction to improve the numerical stability of the Galerkin method without compromising accuracy. A wide variety of applications of this method to many worthwhile problems can be found in the literature (Hughes, Franca and Balestra, 1986), (Brezzi and Douglas, 1988), (Franca and Frey, 1992), (Franca, Frey and Hughes, 1992), (Harari and Hughes, 1994) and (Franca and Valentin, 2000). However, a common drawback of this method is that the amount of additional diffusion should be carefully selected by user through a stabilization parameter δ, which is usually seen as a drawback of the method. In this work we derive δ by examining the relation between of the SUPG and a more recent strategy known as the residual-free bubble (RFB) method introduced in (Brezzi and Russo, 1994) (see also (Brezzi, Franca and Russo, 1998), (Franca, Neslitürk and Stynes, 1998), (Brezzi, Hughes, Marini, Russo and Süli, 1999), (Brezzi, Marini and Russo, 2000) and (Sangalli, 2000)).

To capture the small scales, the RFB method is based on the enrichment of the finite element spaces. This strategy maintains the Galerkin method by enhances the polynomial spaces with the so called the residual-free bubble functions which satisfy a differential equation inside each element and vanish on its boundary. However, the vanishing boundary condition along inter-element boundaries in higher dimensions may lead to inaccuracies in the numerical solution since the approximate solution is still interpolated by polynomials along element's edges. In fact, the RFB method in advection-dominated problems is less accurate at the outflow boundaries when compared to other stabilized methods (Franca, Madureira and Valentin, 2005). Thus an improvement in the RFB method is required. An alternative strategy based on enriching the test space with bubble functions and the trial space with the so called the multiscale (MS) functions, which satisfy the same differential equation as the RFBs but do not vanish on the element edges, was proposed by Franca et. al. in (Franca, Madureira and Valentin, 2005), (Franca, Madureira, Tobiska and Valentin, 2005) and (Franca, Ramalho and Valentin, 2005). However, contrary to the residual-free bubbles, internal layers are not well captured by the latter algorithm if the mesh is not aligned with the convection field. Therefore, Franca and his co-workers combine these two approaches and report that employing the MS functions in elements connected to the outflow boundaries and the RFB functions in the rest of the domain increases the accuracy of the numerical approximations considerably (Franca, Ramalho and Valentin, 2005); this approach is renamed as MIX method. The common point of these two approaches is that either they employ the exact solutions of the equations defining the enriching functions (those are the bubble functions or the multiscale functions) or their approximations using a very fine mesh inside each element; both approaches make the numerical method less practical. Regarding the RFBs, the implementation of the method requires the solution of a local boundary value problem which may not be easier to solve than the original problem. Therefore, owing to the simplicity of element geometry, researchers have been proposed several numerical methods to compute an inexpensive approximate solution to the local problem on a specially chosen sub-grid consisting of a few nodes. Nevertheless the approximate counterpart of the RFB functions retain the crucial features of the exact RFBs from the convergence point of view (Brezzi, Hauke, Marini and Sangalli, 2003) and (Brezzi, Marini and Russo, 2005).

In this work, we extend the idea above to the MS functions and propose a stable, fully discrete, yet inexpensive numerical method for convection-diffusion problems on rectangular grids. As we simply enrich the test space by bubble functions, to enrich the trial space, we employ the MS functions in elements connected to the outflow boundaries and the RFB functions in the rest of the domain. However, the numerical method proposed suggests to use suitable approximate counterparts of enriching functions. The significant feature of it is that they retain the stabilizing feature of the exact ones. This can be achieved by using a specially chosen sub-grid with a single internal node in the interior of each element in the approximation of the RFBs, which are also known as pseudo residual-free bubbles (PRFBs) and the associated method is denoted by PRFB (Neslitürk, 2010). Regarding the MS functions, they only differ along element's edges from the bubbles, therefore the same strategy in the element's interior is used. Along element's edges, we apply the same method reduced to 1D, which uses a single additional node per edge to approximate the restriction of the MS function on the element's edges. The resulting algorithm , which is renamed as PMS, numerically performs well and the results are comparable with previous ones found in the literature.

Since enriched methods are originally developed for the steady problem, their adaptation to the unsteady case has some difficulties. Following papers can be given as examples of this issue (Franca, Ramalho and Valentin, 2006), (Asensio, Ayuso and Sangalli, 2007) and (Frutos and Novo, 2008). One of our goals is to find a way that combines the methods RFB, MS and MIX with generalized Euler time integration (θ method) for the unsteady problem on a rectangular grid. In order to construct practical algorithm we have enriched the bilinear trial function space with MS or/and RFB functions while the bilinear test function space has been used without making an enrichment. Then enriched part of the solution of the steady problem has been directly employed for the unsteady problem. Since the enriching basis multiscale and bubble functions are obtained from steady equation, their shapes do not change at different time levels, which makes the method quite cheap with utilizing the pseudo approximate forms of these functions instead of their exact counterparts in the full discrete algorithm. The efficiency of the proposed algorithms on the unsteady problem is investigated by the numerical experiments.

For the case of continuous piecewise linear elements, the RFB and the SUPG methods have an identical structure (Brezzi, Marini and Russo, 1998). After choosing the SUPG parameter δ by the relation of these methods, the combination of the SUPG method in space and θ method in time for the unsteady convection diffusion equation is studied. According to the coercivity estimate in (Bochev, Gunzburger and Shadid, 2004) such implicit algorithm can be considered as well posed regardless of the time step size. Another important recent papers about SUPG type stabilization with θ method are (Burman, 2010) for pure advection equation, (Burman and Smith, 2011) for advection diffusion equation and (Lube and Weiss, 1995), (Frutos, Garcia-Archilla and Novo, 2010), (John and Novo, 2011) for advection diffusion reaction equation. Using the fact that the approximation space consists of piecewise linear polynomials, we extend stability and convergence analysis for the pure convection equation given in (Burman, 2010) to the convection diffusion equation. For the time integration both two A-stable cases backward Euler ($\theta = 0$) and Crank-Nicolson ($\theta = 1/2$) are considered and similar results in (Burman, 2010) are obtained i.e. uniform stability of the general formulation is proved under a regularity condition on the data and then quasi-optimal convergence is shown under sufficiently smoothness condition of the exact solution. After proving the robustness of the SUPG/θ method (for $\theta \in [0, 1/2]$), we apply it to more general problems. The last chapter provides an illustrating example.

Finally a chapter is devoted to mathematical models describing the transport phenomena which is time dependent convection diffusion reaction equations. This kind of equation with linear or nonlinear reaction term is one for which approximate solution procedures persistently exhibit significant limitations for certain problems of physical interest. The most interesting cases appear when convection is dominated. In such situations users are usually forced to choose either nonphysical oscillations or excessive diffusion. Here we investigate another alternative: an operator splitting method widely used to simulate models coming from environmental processes (Zlatev, 1995), (Geiser, 2008), (Levine, Pamuk, Sleeman and Hamilton, 2010), (Ewing, 2002) and (Frolkovič and Geiser, 2000). In essence we split the transport equation into two unsteady subproblems. The main advantage of splitting is that each subproblem can be discretized separately by the convenient method independently from each other. In our splitting strategy the first part becomes a first order nonlinear differential equation without space derivatives and the second one becomes an unsteady linear convection diffusion equation. The first problem can be solved exactly by using simple analytical techniques or numerically by appropriate time integrator. However the second one is problematic when convection is dominated. In this regime, the SUPG method for space discretization and θ-method for time discretization are employed. Numerical results that illustrate the good performance of this method for both one and two dimensional test problems are reported.

CHAPTER 2

MOTIVATION WITH ONE DIMENSIONAL PROBLEMS

2.1. Steady Convection Diffusion Equation

In this section we study some stabilization techniques for the following linear steady convection-diffusion equation.

$$\begin{aligned} \mathcal{L}u := -\epsilon\, u^{''} + \beta u^{'} &= f \quad \text{in } \Omega = (0,1) \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

where we assume the diffusion term ϵ is positive constant, the convection (advection) term $\beta > 0$ and the right hand side function f are piecewise constant with respect to the standard partition $\mathfrak{T}_h$ of Ω. Then the weak formulation of (2.1) can be written as:
Find $u \in V := H_0^1(\Omega)$ such that

$$a(u,v) = \epsilon(u',v') + (\beta u', v) \;=\; (f,v) \quad \forall v \in V \tag{2.2}$$

where $a(.,.)$ defines continuous and coercive bilinear form on

$$H_0^1(\Omega) := \{v \in L_2(\Omega) : \; v' \in L_2(\Omega) \;\text{ and }\; supp(v) \subset \Omega\}$$

and the notation $(.,.)_{\mathfrak{D}}$ is used standard inner product on $L^2(\mathfrak{D})$ .To simplify the notation we drop subscript $\mathfrak{D}$ from $(.,.)_{\mathfrak{D}}$ in the case $\mathfrak{D} = \Omega$. Under the conditions described above, existence and uniqueness of the solution of (2.2) are guaranteed by Lax-Milgram theorem (Raviart and Thomas, 1992).

In the following section we mention about the (standard) Galerkin and SUPG approximations to the solution of the problem (2.1).

2.1.1. The Galerkin and The SUPG Methods

We first consider the Galerkin approximation of the problem (2.2). Let $V_L \subset V$ be a finite dimensional space of Lagrangian finite elements according to

$$V_L := \{v_L \in H_0^1(\Omega) : v_L|_K \in P_1(K) \ \ \forall K \in \mathfrak{T}_h\}$$

where $\mathfrak{T}_h$ is the partitions of Ω into M elements with the interval size $h_i = x_i - x_{i-1}$ for $i = 1, 2, ..., M$.
It is known that

$$V_L = span\{\varphi_i\}_{i=0}^{M} \quad \text{where} \qquad \varphi_i(x) = \begin{cases} \left(1 + \dfrac{x - x_i}{h_i}\right)_+, & x \leq x_i \\ \left(1 - \dfrac{x - x_i}{h_{i+1}}\right)_+, & x \geq x_i \end{cases} \tag{2.3}$$

then the Galerkin finite element problem reads: Find $u_L \in V_L \subset H_0^1(\Omega)$ such that

$$a(u_L, v_L) = (f, v_L) \ \ \forall v_L \in V_L \tag{2.4}$$

It is well known that the Galerkin method fails to provide a satisfactory approximation for the convection diffusion equation when the convection term (β) dominates the diffusion one (ϵ) (more concrete description of regimes will be given in the next sections). In this case the Galerkin method produce non-physical oscillations that pollute the whole computational domain. Because of this undesirable feature of the Galerkin method several approaches have been proposed to cure this problem with in the framework of finite element methods. Now we consider the most favorite one the SUPG method. In order to construct the SUPG method to the problem (2.2) we introduce the space of test functions W_L defined by

$$W_L := \{w_L : w_L = v_L + \delta\beta v_L' \ and \ v_L \in V_L\}$$

where δ is stabilizing parameter and it is piecewise constant with respect to $\mathfrak{T}_h$ such that

$$\delta|_K = \delta_K \ \ \forall K \in \mathfrak{T}_h.$$

Then the approximation u_h obtained by the SUPG method can be written as: Find $u_L \in V_L$ such that

$$a(u_L, v_L) + \sum_K \delta_K(\beta u_L', \beta v_L')_K = (f, v_L) + \sum_K \delta_K(f, \beta v_L') \tag{2.5}$$

Here the stabilizing parameter δ must be sellected in a suitable way. According to thumb-rule arguments and a lot of numerical tests, several recipes have been proposed for the choice of δ. Nevertheless the need for a suitable convincing argument to guide the choice of δ is still considered as a major drawback of the method by several users. In recent times, the SUPG method has been related to the process of addition and elimination of suitable bubble functions (Brezzi, Baiocchi and Franca, 1992), (Baiocchi, Brezzi and Franca, 1993) than aroused considerable interest, although the problem of the optimal choice of δ was simply translated into the problem of the optimal choice of the bubble space.

2.1.2. The RFB Method and The SUPG Stabilization Parameter

One way to recover intrinsically the value of δ_K is to use the RFB approach (Brooks and Hughes, 1982), (Franca and Russo, 1996) and (Neslitürk, 2006) that will be recalled here. The idea is to enlarge the finite element space V_L in the following way: For each element K, we define the space of bubbles in K as

$$B_K := H_0^1(K) \tag{2.6}$$

and enlarging space

$$V_B := \oplus_K B_K. \tag{2.7}$$

Then we solve the weak problem (2.2) on $V_h = V_L \oplus V_B$. Now the Petrov-Galerkin formulation on V_h is: Find $u_h \in V_h$ such that

$$a(u_h, v_h) \quad = \quad (f, v_h) \quad \forall v_h \in V_h. \tag{2.8}$$

Since $v_h \in V_h$, it can be split into a linear part $v_L \in V_L$ and into a bubble part $v_B \in V_B$ in a unique way:

$$v_h \quad = \quad v_L + v_B \tag{2.9}$$

and the bubble part itself can be uniquely split element by element :

$$v_B = \sum_K v_{B,K}, \quad v_{B,K} \in B_K. \tag{2.10}$$

Then the variational problem (2.8) in V_h is equivalent to the following set of problems: Find $u_L \in V_L$ such that

$$a(u_L + u_B, v_L) = (f, v_L), \quad \forall v_L \in V_L \tag{2.11}$$

and

$$a(u_L + u_{B,K}, v_{B,K})_K = (f, v_{B,K})_K, \quad \forall K \in \mathfrak{T}_h \text{ and } \forall v_{B,K} \in B_K. \tag{2.12}$$

Let's consider (2.11), it can be written as

$$a(u_L, v_L) + \sum_K a(u_{B,K}, v_L)_K = (f, v_L). \tag{2.13}$$

The term $\sum_K a(u_{B,K}, v_L)_K$ represents the effect of the bubble part $u_{B,K}$ onto the linear part u_L. Observing that, for suitable u and v

$$a(u, v)_K = (\mathcal{L}u, v)_K = (u, \mathcal{L}_K^* v)_K \tag{2.14}$$

where $\mathcal{L}_K^*$ is the formal adjoint of $\mathcal{L}$ on K. Then the bubble part of (2.13) is represented by

$$\sum_K a(u_{B,K}, v_L)_K = \sum_K (u_{B,K}, \mathcal{L}_K^* v_L)_K. \tag{2.15}$$

Now consider (2.12) to determine $u_{B,K}$ in terms of u_L.

$$a(u_{B,K}, v_{B,K})_K = (f - \mathcal{L}u_L, v_B)_K = (f - \beta\, u_L^{'}, v_B)_K \tag{2.16}$$

or using the differential form

$$\begin{aligned} \mathcal{L}u_{B,K} &= f - \beta\, u_L^{'} \quad \text{in } K = (x_k, x_{k+1}) \\ u_{B,K} &= 0 \quad \text{in } \partial K. \end{aligned} \tag{2.17}$$

For each $u_L \in V_L$ the problem (2.17) has always a unique weak solution $u_{B,K} \in B_K$ that can be represented by

$$u_{B,K} = M_K(f - \beta\, u_L^{'}) \tag{2.18}$$

where M_K is the inverse of the operator defined by (2.17), it is also a bounded linear operator from $H^{-1}(K)$ to $B_K = H_0^1(K)$. Substituting (2.18) into (2.13), we obtain:

$$a(u_L, v_L) + \sum_K a(M_K(f - \beta\, u_L^{'}), v_L)_K = (f, v_L) \tag{2.19}$$

or using (2.14),

$$a(u_L, v_L) + \sum_K (M_K(f - \beta\, u_L^{'}), \mathcal{L}_K^* v_L)_K = (f, v_L) \quad \forall v_L \in V_L. \tag{2.20}$$

Since $(f - \beta u_L')$ is constant on $K \in \mathfrak{T}_h$,

$$a(u_L, v_L) + \sum_K ((f - \beta\, u_L^{'})(-\beta\, u_L^{'})(M_K(1), 1) = (f, v_L)\; \forall v_L \in V_L. \tag{2.21}$$

Then

$$a(u_L, v_L) + \sum_K \hat{\delta}_K((\beta\, u_L^{'} - f), (\beta\, v_L^{'}))_K = (f, v_L)\; \forall v_L \in V_L \tag{2.22}$$

where

$$\hat{\delta}_K = \frac{1}{h_K} \int_K b_K \, dx \tag{2.23}$$

and $b_K \in H_0^1(K)$ is the solution of the boundary value problem (local bubble problem)

$$\begin{aligned} -\epsilon\, b_K^{''} + \beta b_K^{'} &= 1 \quad \text{in } K \\ b_K &= 0 \quad \text{on } \partial K. \end{aligned} \tag{2.24}$$

Then (2.21) implies

$$a(u_L, v_L) + \sum_K a(u_L, \beta\hat{\delta}_K v_L^{'})_K = (f, v_L) + \sum_K (f, \beta\hat{\delta}_K v_L^{'})_K \quad \forall v_L \in V_L. \qquad (2.25)$$

It can be seen that the SUPG scheme (2.5) and (2.25) have an identical structure. We only need to obtain the approximate value of $\hat{\delta}_K$. For this purpose we use the pseudo bubble approximation (Brezzi, Marini and Russo, 1998).

The problem of finding the optimum value for $\hat{\delta}_K$ would be solved if we know explicitly in each interval K and for any given value of ϵ and $\beta|_K$, the exact solution of problem (2.24) (or at least its integral on K). However in general this can not be computable in an easy way. Now we will present a strategy to solve this problem, at least in a reasonably good approximate way. The idea is to look for a solution of (2.24) having the shape of a triangle with vertex in a point P_K internal to K.

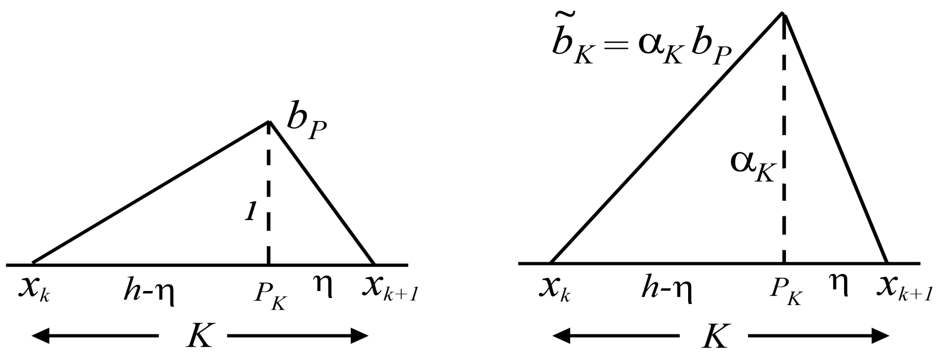

Figure 2.1. Basic Pseudo Bubble is presented in left side and in the right side approximate solution of local bubble problem is presented.

The height of the triangle will be determined by solving the problem (2.24) in the following way:

$$a(\tilde{b}_K, b_P)_K = (1, b_P)_K. \qquad (2.26)$$

We look for α_K such that

$$a(\alpha_K b_P, b_P)_K = (1, b_P)_K.$$

then

$$\alpha_K = \frac{\int_K b_P \, dx}{\epsilon ||b_P^{'}||_K} = \frac{h_K/2}{\epsilon \, h_K/\eta(h_K - \eta)} = \frac{\eta(h_K - \eta)}{2\,\epsilon}. \tag{2.27}$$

On the other hand, to determine the optimum position of the point P_K in K, we need to minimize the following integral

$$J(P_K) = \int_{x_k}^{x_{k+1}} \left| -\epsilon \tilde{b}_K^{''} + \beta \tilde{b}_K^{'} - 1 \right| dx. \tag{2.28}$$

Note that $\tilde{b}_K$ being piecewise linear on K the term $\tilde{b}_K^{''}$ will have only distributional meaning, so that the integral appearing in (2.28) has to be intended in the sense of measures. It is obvious that

$$\begin{aligned} J(P_K) &\leq \epsilon \int_{x_k}^{x_{k+1}} |\tilde{b}_K^{''}| \, dx + \int_{x_k}^{x_{k+1}} \left| \beta \tilde{b}_K^{'} - 1 \right| dx \\ &= \epsilon \int_{x_k}^{x_{k+1}} |\tilde{b}_K^{''}| \, dx + \int_{x_k}^{P_K} \left| \beta \tilde{b}_K^{'} - 1 \right| dx + \int_{P_K}^{x_{k+1}} \left| \beta \tilde{b}_K^{'} - 1 \right| dx. \end{aligned} \tag{2.29}$$

Firstly let us consider the integral

$$\epsilon \int_{x_k}^{x_{k+1}} |\tilde{b}_K^{''}| \, dx = \epsilon \alpha_K \int_{x_k}^{x_{k+1}} |b_P^{''}| \, dx. \tag{2.30}$$

Assume ∂b_P and $\partial^2 b_P$ are the first and second order derivatives of b_P in a weak sense such that they satisfies:

$$(\partial b_P, \phi)_K = -(b_P, \frac{d\phi}{dx})_K \quad \forall \phi \in C_0^{\infty}(K) \tag{2.31}$$

$$(\partial^2 b_P, \phi)_K = -(\partial b_P, \frac{d\phi}{dx})_K \quad \forall \phi \in C_0^{\infty}(K) \tag{2.32}$$

where $C_0^{\infty}(K)$ is the space of continuously infinitely many differentiable functions which has compact support included in K It is easy to see that

$$\partial b_P(x) = \begin{cases} 1/h - \eta, & x \in (x_k, P_K) \\ -1/\eta, & x \in (P_K, x_{k+1}) \end{cases} \tag{2.33}$$

$$\partial^2 b_P(x) = (-\frac{1}{\eta} - \frac{1}{h_K - \eta})\delta_P(x) \tag{2.34}$$

where δ_P denotes the dirac-delta notation such that

$$\delta_P(x) = \begin{cases} \infty, & x = P_K \\ 0, & x \neq P_K \end{cases} \tag{2.35}$$

From functional analysis, we know that

$$\int_{x_k}^{x_{k+1}} \delta_P(x)dx = 1$$

and using (2.27) , we can easily compute the integral

$$\epsilon\alpha_k \int_{x_k}^{x_{k+1}} |b''_P|dx = \epsilon\frac{\eta(h_K - \eta)}{2\epsilon}\frac{h_K}{\eta(h_K - \eta)} = \frac{h_K}{2}. \tag{2.36}$$

Now minimizing (2.29) amounts to minimize

$$\tilde{J}(P_K) = \int_{x_k}^{P_K} |J_1|dx + \int_{P_K}^{x_{k+1}} |J_2|dx \tag{2.37}$$

where

$$J_1 = \beta\tilde{b}|'_{(x_k,P_K)} - 1 \ \text{ and } \ J_2 = \beta\tilde{b}|'_{(P_K,x_{k+1})} - 1.$$

On the other hand

$$\int_{x_k}^{P_K} J_1 dx + \int_{P_K}^{x_{k+1}} J_2 dx = (\frac{\beta\alpha_K}{h_K - \eta} - 1)(h_K - \eta) + (-\frac{\beta\alpha_K}{\eta} - 1)\eta = -h_K < 0$$

implies

$$\int_{P_K}^{x_{k+1}} J_2 dx < 0.$$

If we take

$$\int_{x_k}^{P_K} J_1 dx \leq 0 \tag{2.38}$$

then

$$\int_{x_k}^{P_K} |J_1| dx + \int_{P_K}^{x_{k+1}} |J_2| dx = (1 - \frac{\beta\alpha_K}{h_K - \eta})(h_K - \eta) + (1 + \frac{\beta\alpha_K}{\eta})\eta = h_K. \tag{2.39}$$

It is the minimum value of $\tilde{J}(P_K)$ since

$$h_K = |\int_{x_k}^{P_K} J_1 dx + \int_{P_K}^{x_{k+1}} J_2 dx| \leq \int_{x_k}^{P_K} |J_1| dx + \int_{P_K}^{x_{k+1}} |J_2| dx = \tilde{J}(P_K). \tag{2.40}$$

Consequently from this analysis the upper bound of η can be found as in the following way

$$\begin{aligned} \int_{x_k}^{P_K} J_1 dx \leq 0 \quad &\Longleftrightarrow \quad (\frac{\beta\alpha_K}{h_K - \eta} - 1)(h_K - \eta) = \frac{\beta\eta(h_K - \eta)}{2\epsilon} - (h_K - \eta) \leq 0 \\ &\Longleftrightarrow \quad \eta \leq \frac{2\epsilon}{\beta}. \end{aligned} \tag{2.41}$$

The criteria (2.41) describes the position of the point P_K i.e. for the convection dominated regime $\eta = 2\epsilon/\beta$ and for the other case $\eta = h_K/2$. More preciously, in order to satisfy the continuity of the transition of the regimes we use the relation

$$\epsilon = \frac{\beta h_K}{4}. \tag{2.42}$$

Then the position of P_K in the interval $[x_k, x_{k+1}]$ can be given by the following rule

$$P_K = \begin{cases} x_{k+1} - \dfrac{2\epsilon}{\beta}, & \epsilon \leq \dfrac{\beta h_K}{4} \\ x_{k+1} - \dfrac{h_K}{2}, & \epsilon \geq \dfrac{\beta h_K}{4} \end{cases} \tag{2.43}$$

Hence the SUPG parameter $\hat{\delta}_K$ can be computed such that

$$\hat{\delta}_K = \frac{1}{h_K}\int_K b_K dx \approx \frac{1}{h_K}\int_{x_k}^{x_{k+1}} \tilde{b}_K dx = \frac{1}{h_K}\frac{\alpha_P h_K}{2} = \frac{\eta(h_K - \eta)}{4\epsilon}.$$

which implies

$$\hat{\delta}_K = \begin{cases} \dfrac{h_K}{2\beta} - \dfrac{\epsilon}{\beta^2}, & \epsilon \leq \dfrac{\beta h_K}{4} \\ \dfrac{h_K^2}{16\epsilon}, & \epsilon \geq \dfrac{\beta h_K}{4}. \end{cases} \tag{2.44}$$

2.1.3. Numerical Experiments

In this section we test the methods considered in previous sections for the steady convection-diffusion equation to assess the performance of the stabilization method. These tests show the effect of the stabilization coefficient computed with the pseudo residual free bubble.

In our calculations we take a uniform partition of Ω into subintervals of length $h = 1/M$.

Firstly we will test our methods with following steady problem:

$$-\epsilon u'' + \beta u' = 1 \text{ in } \Omega := (0,1) \tag{2.45}$$
$$u(0) = u(1) = 0$$

In the Figs. 2.2 and 2.3 exact solutions u (red curves) and the Galerkin approximations u_L (blue doted curves) are compared. This figures shows that the Galerkin method gives

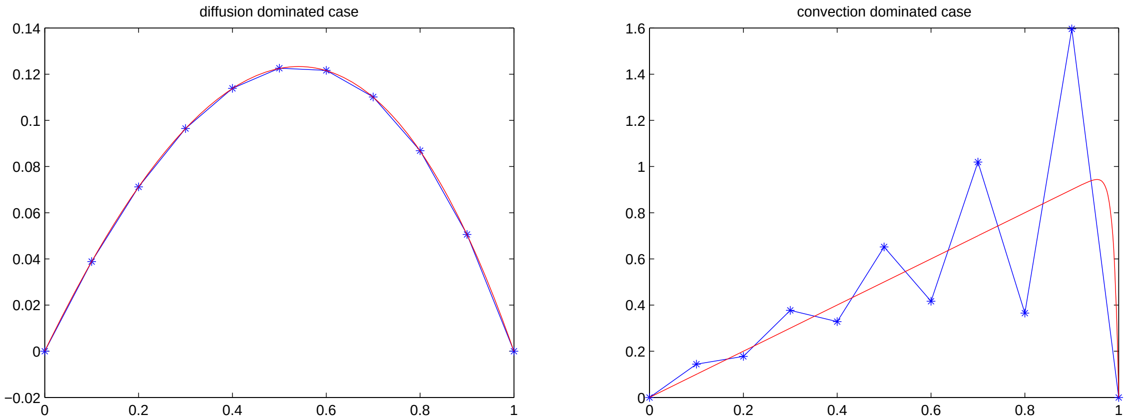

Figure 2.2. The Galerkin approximation with $\epsilon = 1,\ \beta = 1\ and\ M = 10$ (left) and $\epsilon = 10^{-3},\ \beta = 1\ and\ M = 10$ (right)

satisfactory results in diffusion dominated case. However in the convection dominated case it fails to yield stable solutions unless the discretization step size is small enough. Stabilized methods have been introduced in order to overcome this undesirable feature. We shall deal here with the RFB and the SUPG methods.

In the Fig. 2.4 left side we present the refine -grid numerical solution u_h (green curve) , the coarse-grid numerical solution u_L (blue curve) and the exact solution u (red curve). Since we have obtained the SUPG parameter from bubble stabilization, coarse grid numerical solution u_L is identically same as the SUPG solution. According to these figures stabilized methods give accurate results.

2.2. Time Dependent Convection Diffusion Equation

In this section we will consider the numerical solution of the given unsteady problem:

$$\begin{aligned} \mathcal{L}_t u := u_t + \mathcal{L}u &= f \ \text{ in } \ \Omega_t := (0,1) \times (0,T] \\ u(0,t) = u(1,t) &= 0 \ \text{ in } \ [0,T] \\ u(x,0) &= u_0(x) \ \text{ in } \ \Omega = (0,1) \end{aligned} \tag{2.46}$$

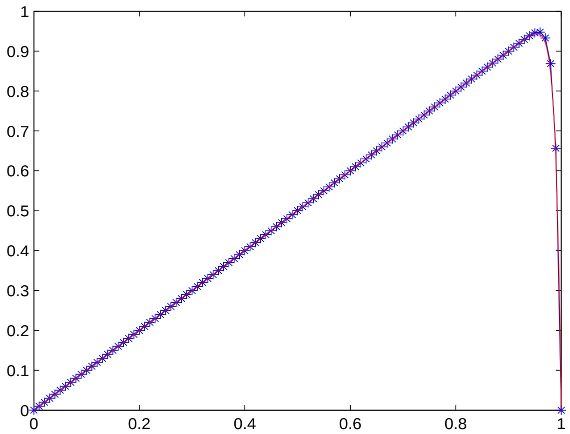

Figure 2.3. The Galerkin approximation with $\epsilon = 10^{-3},\ \beta = 1\ and\ M = 50$.

where the elliptic operator $\mathcal{L}$, which depends on space variable, is defined in previous chapter. We assume $f \in L_2(\Omega)$ for each $t \in (0, T]$ and $u_0 \in L_2(\Omega)$. Then the weak formulation of (2.46) reads:
Find $u \in \mathcal{L}_2(0, T; V) \cap C^0([0, T]; \mathcal{L}_2(\Omega))$ such that

$$\begin{aligned} \frac{d}{dt}(u(t), v) + a(u(t), v) &= (f(t), v) \ \ \forall v \in V \\ u(0) &= u_0 \end{aligned} \tag{2.47}$$

It can be shown that, under the conditions given above the weak problem (2.47) is well-posed. Following sections have been devoted to semi-discretization and full discretization of (2.46) by the SUPG method in space and by the θ method in time, respectively.

2.2.1. Semi-Discrete Approximation by The SUPG Method

Firstly the standard Galerkin approximation for the problem (2.47) will be considered. Let $V_L \subset V$ be a finite dimensional space of Lagrangian finite elements as in (2.3). Then the semi-discrete Galerkin finite element problem reads:

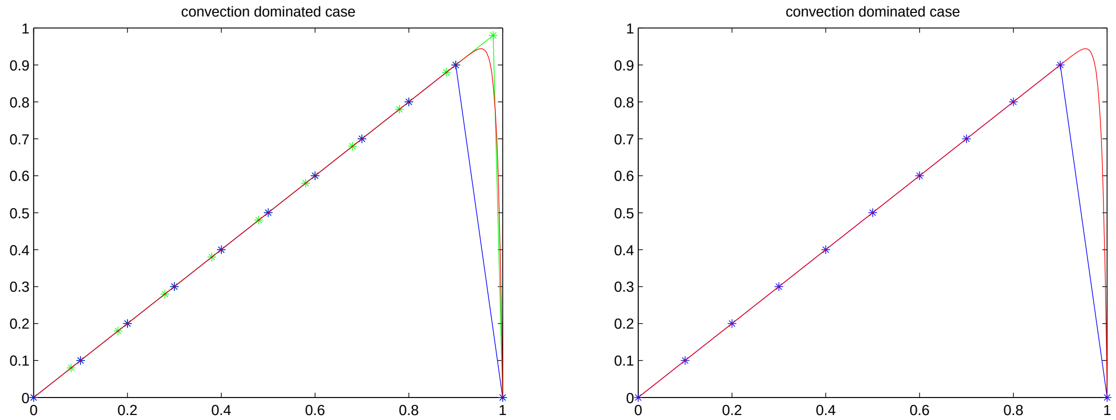

Figure 2.4. The RFB method with $\epsilon = 10^{-3}$, $\beta = 1$ *and* $M = 10$ (left) and The SUPG method with $\epsilon = 10^{-3}$, $\beta = 1$ *and* $M = 10$ (right)

For each $t \in (0, T]$ find $u_L(t) \in V_L$ such that

$$\frac{d}{dt}(u_L(t), v_L) + a(u_L(t), v_L) = (f(t), v_L) \quad \forall v_L \in V_L \tag{2.48}$$

$u_L(0)$ is chosen as a suitable approximation of u_0. SGFEM is also unsuitable for the solution of transient advection-diffusion problems as well as the steady form so stabilizing term must be added to the semi discrete algorithm (2.48). Then the semi-discrete problem related to the SUPG method reads as follows:
For all $t \in (0, T]$ find $u_L(t) \in V_L$ such that

$$\begin{aligned}\frac{d}{dt}(u_L(t), v_L) \;&+\; \sum_K \delta_K \frac{d}{dt}(u_L(t), \beta v_L')_K + a(u_L(t), v_L) + \sum_K \delta_K(\beta u_L'(t), \beta v_L')_K \\ &= \; (f, v_L) + \sum_K \delta_K(f, \beta v_L') \quad \forall v_L \in V_L. \end{aligned} \tag{2.49}$$

Here the stability parameter δ_K is determined by (2.44). We introduce the following bilinear forms in order to simplify (2.49).

$$
\begin{aligned}
Z(\frac{\partial}{\partial t}u_L, v_L) &= \frac{d}{dt}(u_L(t), v_L) + \sum_K \delta_K \frac{d}{dt}(u_L(t), \beta v_L')_K \\
G(u_L, v_L) &= a(u_L(t), v_L) + \sum_K \delta_K (\beta u_L'(t), \beta v_L')_K \\
F(f, v_L) &= (f, v_L) + \sum_K \delta_K (f, \beta v_L').
\end{aligned}
$$

Then (2.49) reduces to

$$
Z(\frac{\partial}{\partial t}u_L, v_L) + G(u_L, v_L) = F(f, v_L) \tag{2.50}
$$

On the other hand numerical solution $u_L(x,t)$ can be written in the following form:

$$
u_L(x,t) = \sum_{i=0}^{M} \alpha_i(t)\varphi_i(x). \tag{2.51}
$$

where $\alpha_i(t)$'s are the coefficients of φ's at the time level $t \in (0, T]$. Also using the boundary conditions $\alpha_0(t) = \alpha_M(t) = 0$ for all $t \in (0, T]$. Combining (2.51) and (2.49), we get the system of ordinary differential equation (ODE) such that

$$
Z\dot{\alpha}(t) + G\alpha(t) = F(t) \tag{2.52}
$$

where $\alpha(t) = (\alpha_0(t), \alpha_1(t), ..., \alpha_M(t))^t$ is unknown vector. The matrices Z and G are generated in the usual manner from the bilinear forms $Z(.,.)$ and $G(.,.)$ respectively and F is a vector whose components are L_2 products of the source term and the model shape functions φ_i. On the other hand initial condition $\alpha(0) = (\alpha_0(0), \alpha_1(0), ..., \alpha_M(0))^t$ can be taken as

$$
\alpha_i(0) = u_0(ih) \quad \text{for} \;\; i = 0, ..., M. \tag{2.53}
$$

2.2.2. Full Discretization by θ Method

The system of ODE (2.52) can be solved by suitable ODE solvers. In this work we use θ-method which is also known as the generalized trapezoidal Euler rule.

In order to obtain full discretization, the time interval $(0, T)$ is subdivided into N equal subintervals $[t^j, t^{j+1}]$ with the length $\tau = t^{j+1} - t^j$ for $j = 0, ..., N-1$. Using the notations u_L^n for linear approximation to $u_L(t^n)$ (or $u_L(x, t^n)$). Then fully discrete problem is linear system of algebraic equations:

$$(Z + (1-\theta)\tau G)\alpha^n = F^{n-\theta} + (Z - \theta\tau G)\alpha^{n-1} \text{ for } n = 1, ..., N. \tag{2.54}$$

where $F^{n-\theta} = (1-\theta)F(t_n) + \theta F(t_{n-1})$, the initial condition is defined as

$$\alpha(0) = \alpha^0 = (u_0(0),\ u_0(h),\ ...,\ u_0(Mh))^t \text{ for } n = 0, ..., N-1 \tag{2.55}$$

and the unknown vector is

$$\alpha(t^n) = \alpha(n\tau) \approx \alpha^n = (\alpha_0^n,\ \alpha_1^n,\ ...,\ \alpha_M^n). \tag{2.56}$$

Applying the boundary conditions, we obtain

$$\alpha_0^n = \alpha_M^n = 0 \quad for\ all\ n = 0, 1, ..., N. \tag{2.57}$$

For $\theta = 1$ the scheme (2.54) is the explicit Euler method, $\theta = 1/2$ gives Crank-Nicolson method and $\theta = 0$ gives the implicit Euler rule. Having found the unknown vectors α^n approximate solution of (2.47) can be written as follows

$$u(x, t^n) \approx u_L(x, t^n) = \sum_{i=0}^{M} \alpha_i^n \varphi_i(x). \tag{2.58}$$

2.2.3. Numerical Experiments

In this section we present some numerical experiments for the time dependent convection diffusion equation to assess the performance of the stabilization method introduced this chapter. These tests indicate the effect of the stabilization term τ. In the calculations we take uniform partition of Ω and of $(0, T)$ into M and N subintervals, respectively. We also fix the following parameters in (2.46): $\epsilon = 10^{-3}$, $\beta = 1$, $T = 3/5$, $N = 100$, $f = 0$ and $u_0(x) = \sin(\pi x)$.

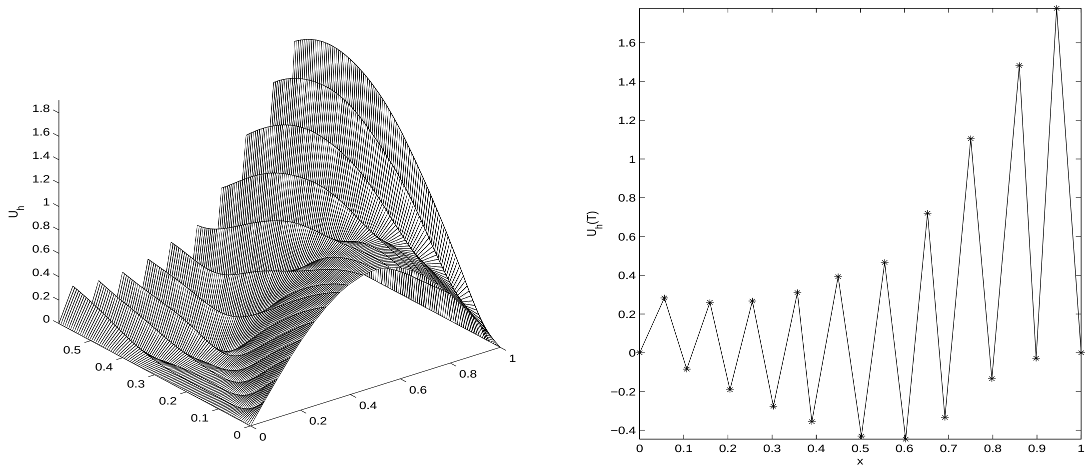

Figure 2.5. Numerical simulation (left) and final time result (right) of the method Galerkin disc. in space and $\theta = 1/2$ disc. in time with $M = 20$.

In the Figs. 2.5, 2.6, 2.7, 2.8 and 2.9 approximate solutions u_L of the problem (2.46) are presented in all computational domain (left) and in final time T restriction. Similarly to the steady case, the Galerkin method again fails to satisfy accurate solution unless discretization step size is small enough (see Figs. 2.5 and 2.6). Contrary to the Galerkin method in space, the SUPG method gives oscillation free approximation with Crank Nicolson method ($\theta = 1/2$) as in Fig. 2.7. We have also applied the Galerkin and SUPG methods in Figs. 2.8 and 2.9, respectively with the forward Euler ($\theta = 1$) method in order to verify the accuracy of the method.

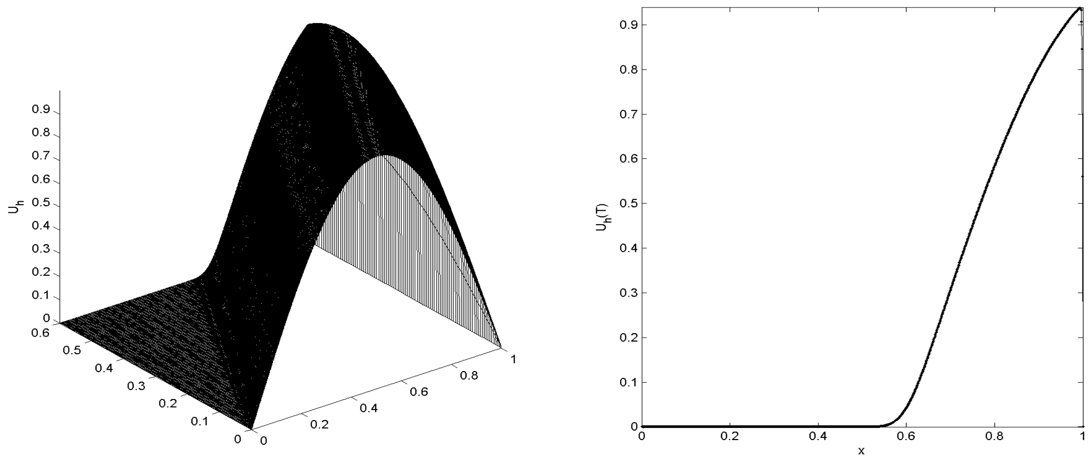

Figure 2.6. Approximate solution (left) and its final time result (right) obtained by the Galerkin in space and $\theta = 1/2$ in time with $M = 1000$.

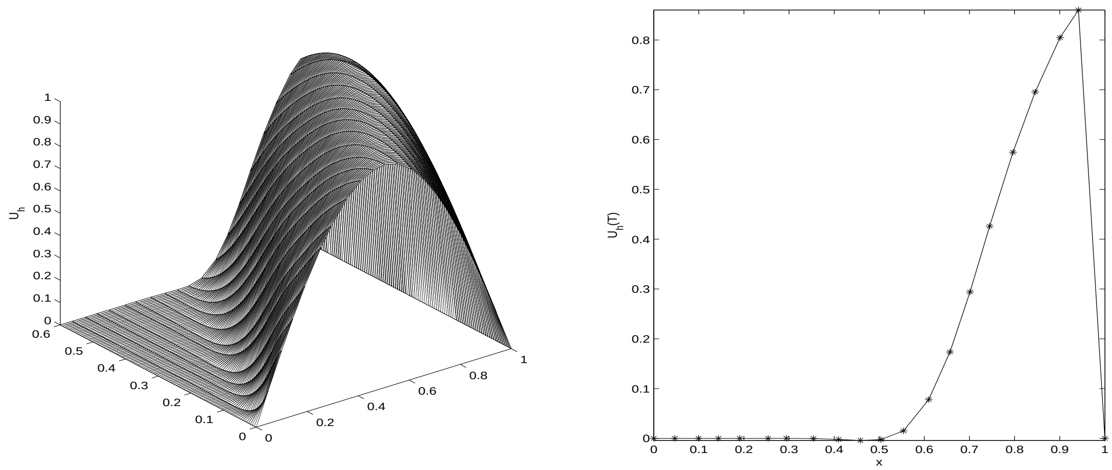

Figure 2.7. Approximate solution (left) and its final time result (right) obtained by the SUPG in space and $\theta = 1/2$ disc. in time with $M = 20$.

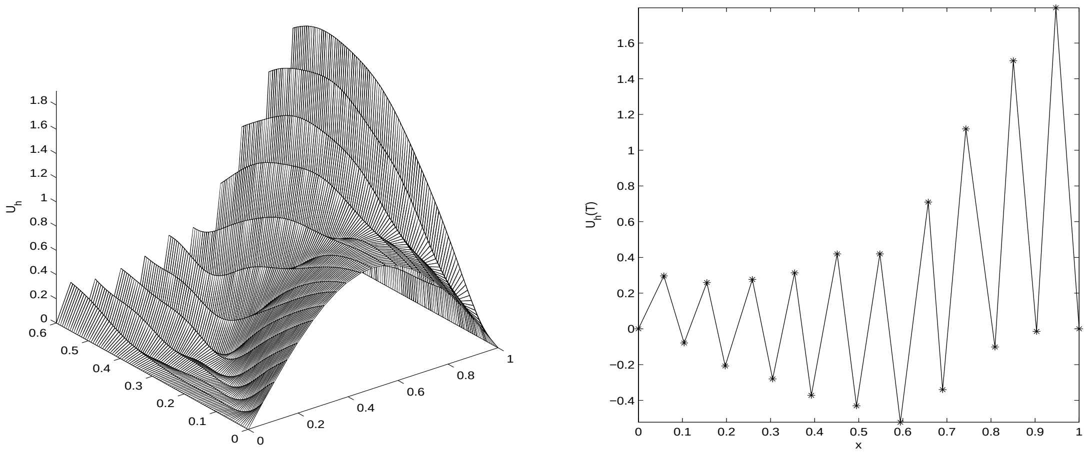

Figure 2.8. Approximate solution (left) and its final time result (right) obtained by the Galerkin disc. in space and $\theta = 1$ disc. in time with $M = 20$.

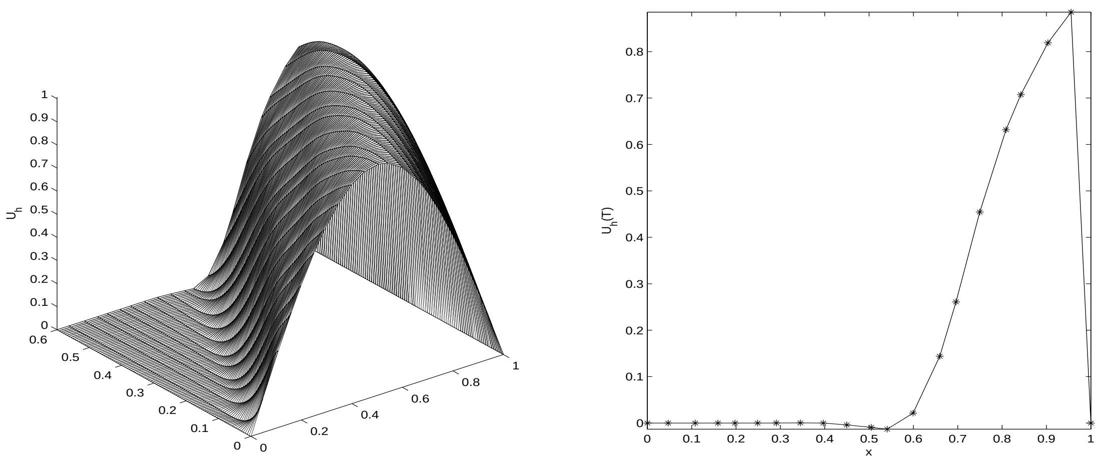

Figure 2.9. Approximate solution (left) and its final time result (right) obtained by the SUPG disc. in space and $\theta = 1$ disc. in time with $M = 20$.

CHAPTER 3

PSEUDO RESIDUAL FREE BUBBLES FOR STABILIZATION OF THE STEADY EQUATION ON TRIANGULAR GRID

Previous chapter contains the results that are representative of the body of discretization dealing with singular perturbed boundary value problems in one space variable. We now move to two space dimensions, where one encounters technical problems that are much more varied and challenging.

In this chapter we discuss the linear singularly perturbed boundary value problem

$$\begin{aligned} \mathcal{L}u := -\epsilon\triangle u + \beta.\nabla u &= f(x,y) \text{ in } \Omega := (0,1)^2 \\ u(x,y) &= 0 \text{ on } \partial\Omega \end{aligned} \tag{3.1}$$

As usual the parameter ϵ positive constant and let $\mathfrak{T}_h = \{K\}$ be a regular decomposition of Ω into triangles, h_K be the diameter of the element K and $h = \max_{K\in\mathfrak{T}} h_K$. We assume also that the convection field β and the source term f are piecewise constants with respect to decomposition $\mathfrak{T}_h$.

Then the classical variational formulation of the problem (3.1): Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) := \epsilon(\nabla u, \nabla v) + (\beta.\nabla u, v) = (f,v) \text{ forall } v \in H_0^1(\Omega) \tag{3.2}$$

where $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defines a continuous and coercive bilinear form. Hence the weak problem (3.2) is well-posed by Lax-Milgram theorem.

3.1. Standard Galerkin Approximation

The Galerkin approximation of the problem (3.1) consists of taking a finite dimensional subspace V_h of $H_0^1(\Omega)$ and solving the variational problem (3.2) on V_h. In this chapter, we choose the finite element space as space of continuous, piecewise linear functions. More formally, this space can be represented by

$$V_L = \{v \in H_0^1(\Omega) : v|_K \text{ linear for all } K \in \mathfrak{T}_h\} \tag{3.3}$$

Then the standard Galerkin approximation of the problem (3.2) on V_L reads:
Find $u_L \in V_L$ such that

$$a(u_L, v_L) = (f, v_L) \text{ forall } v_L \in V_L. \tag{3.4}$$

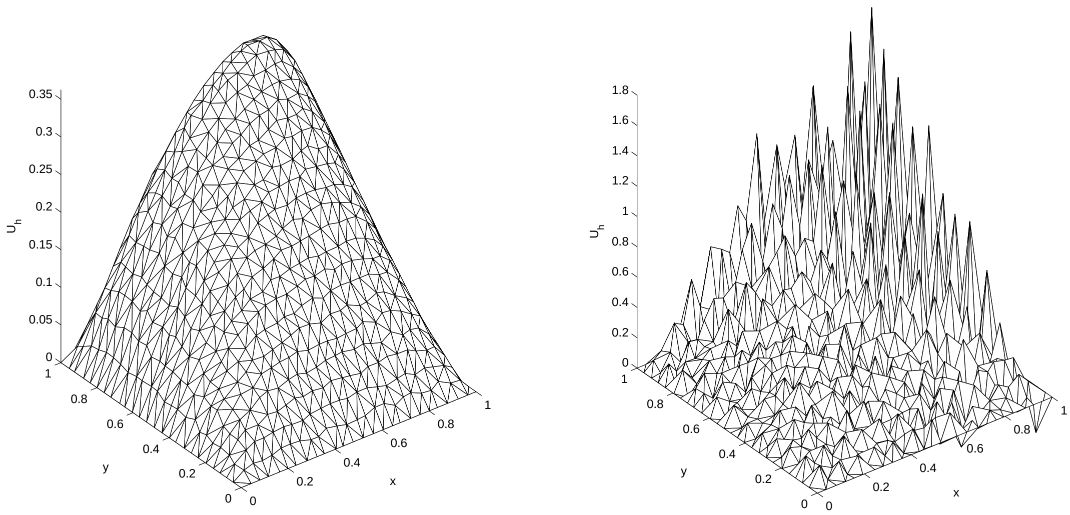

Figure 3.1. The Galerkin Approximations for diffusion dominated (left) and convection dominated (right) problems.

It is well known that the exact solution of the problem (3.1) will have a boundary layer at the outflow when $\epsilon << \beta h$. In this case, the Galerkin finite element approximation (3.4) will typically fail showing strong oscillations near the boundary layer (see Fig.

3.1-right) and stabilization is necessary.

3.2. Relation Between The RFB and The SUPG methods

The most classical and extensively used technique to stabilize (3.4) is to use the SUPG method as follows: Find $u_L \in V_L$ such that

$$\begin{aligned} a(u_L, v_L) + \sum_{K \in \mathfrak{T}_h} \tau_K(\beta.\nabla u_L, \beta.\nabla v_L)_K &= (f, v_L) \\ &+ \sum_{K \in \mathfrak{T}_h} \tau_K(f, \beta.\nabla v_L)_K \ \ \forall\, v_L \in V_L. \end{aligned} \tag{3.5}$$

Since $\beta.\nabla u_L$ and $\beta.\nabla v_L$ are constants in each element for the space of continuous piecewise linear elements, (3.5) implies

$$a(u_L, v_L) + \sum_{K \in \mathfrak{T}_h} \tau_K(\beta.\nabla u_L - f)(\beta.\nabla v_L)|K| = (f, v_L) \ \text{ forall } v_L \in V_L. \tag{3.6}$$

A way to recover the value of τ_K is to use the RFB approach (Brezzi and Russo, 1994) the idea consists of enlarging the finite element space V_L. For this method, in each element K, the space of bubbles $B_K = H_0^1(K)$ and the enlarging space V_B as $V_B = \bigoplus_{K \in \mathfrak{T}_h} B_K$ are constructed. Then we define

$$V_h = V_L \bigoplus V_B. \tag{3.7}$$

Thus any $v_h \in V_h$ can be split into a linear part $v_L \in V_L$ and a bubble part $v_B \in V_B$ in a unique way:

$$v_h = v_L + v_B \in V_h = V_L \bigoplus V_B \tag{3.8}$$

and the bubble part itself can be uniquely split element by element

$$v_B = \sum_K v_{B,K} \ , \ v_{B,K} \in B_K. \tag{3.9}$$

The variational problem (3.2) in V_h can be given as follows:
Find $u_h = u_L + u_B \in V_L \bigoplus V_B$ for all $v_L \in V_L$ and $v_{B,K} \in B_K$ such that

$$a(u_L + u_B, v_L) = (f, v_L) \tag{3.10}$$

$$a(u_L + u_{B,K}, v_{B,K})_K = (f, v_{B,K})_K \tag{3.11}$$

Solving (3.11) for $u_{B,K}$ we obtain the following expression for triangular element

$$u_{B,K} \ = \ (f - \mathcal{L}u_L)M_K(1) \tag{3.12}$$

$$b_K : \ = \ M_K(1) \tag{3.13}$$

where M_K is the inverse operator of $\mathcal{L}$. M_K on $L_2(K)$ is defined such a way:

$$g := M_K(f) \iff \mathcal{L}g = f \text{ in } K \text{ and } g = 0 \text{ on } \partial K. \tag{3.14}$$

Then substituting $u_{B,K}$ into the first equation, it can be shown that the effect of the bubbles which are chosen to be residual-free inside each K, can be identified with an additional term that has an identical structure with the mesh-dependent term in the SUPG method. Consequently the resulting scheme on V_L reads: Find $u_L \in V_L$ such that

$$a(u_L, v_L) + \sum_{K \in \mathfrak{T}_h} \hat{\tau}_K(\beta.\nabla u_L - f, \beta.\nabla v_L)_K = (f, v_L) \text{ forall } v_L \in V_L. \tag{3.15}$$

where

$$\hat{\tau}_K = \frac{1}{|K|}\int_K b_k dk. \tag{3.16}$$

A priori error estimates for RFB method were proved for linear elements in (Brezzi, Hughes, Marini, Russo and Süli, 1999) such that
If the solution belongs to $H^s(\Omega)$ for some s with $1 < s \leq 2$ then

$$\epsilon|u - u_L^R|^2_{H^1(\Omega)} + \sum_{K\in\mathfrak{T}_h} h_K||\beta.\nabla(u - u_L^R)||^2_{L^2(\Omega)} \leq C \sum_{K\in\mathfrak{T}_h} (\epsilon h_K^{2s-2} + h_K^{2s-1})|u|^2_{H^s(\Omega)} \tag{3.17}$$

where u_L^R is the linear component of the RFB solution. See (Brezzi, Marini and Russo, 2000) and (Franca, Neslitürk and Stynes, 1998) for additional results.

3.3. The Pseudo Residual-Free Bubbles (PRFB)

To find the optimum value of $\hat{\tau}_K$, we need to solve (3.13) explicitly or at least find its integral on K. Although it couldn't be evaluated in most cases, the reasonably good approximate way, which is called pseudo residual free bubble (PRFB) approximation (Neslitürk, 2006), (Brezzi, Marini and Russo, 2005), can be given. Now we are going to summarize this strategy for linear triangular elements.

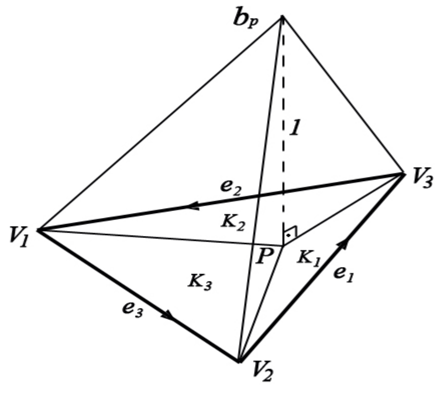

Figure 3.2. Bases of the Pseudo-bubbles for triangular element

The idea is to look for a solution of (3.13) having the shape of pyramid with vertex in a point P internal to K as in the Fig. 3.3.
The notation $\hat{b}_K = \alpha_K b_P$ is used for the approximate solution of (3.13) i.e.

$$b_K \approx \hat{b}_K = \alpha_K b_P.$$

The height of the pyramides α can be determined by solving the local bubble problem (3.13) in the Galerkin sense as follows

$$a(\alpha_K b_P, b_P)_K = (1, b_P)_K. \tag{3.18}$$

Then

$$\alpha_K = \frac{(1, b_P)_K}{\epsilon(\nabla b_P, \nabla b_P)_K} = \frac{|K|/3}{\epsilon \sum_i \frac{|e_i|^2}{4|K_i|}} = \frac{4|K|^2}{3\epsilon \sum_i \frac{|e_i|^2}{w_i}}. \tag{3.19}$$

where $|e_i|$ will denote the length of the edge e_i, n_i the outward unit normal to e_i and $w_i = |e_i|n_i$. Now we only need to determine the exact position P. Without loss of generality, the components of the convection field $\beta = (\beta_1, \beta_2)$ are nonnegative. According to the pseudo bubble strategy, the exact location of P is to be chosen on the line $\mathcal{I}$ (see Fig. 3.3) in order to minimize the L_1-norm of the residual equation.

$$J(P) := \int_K |-\epsilon\triangle\hat{b}_K + \beta\nabla\hat{b}_K - 1|dk \tag{3.20}$$

Note that the single basis function of the bubble space B_K being piecewise linear on K, the term $\triangle\hat{b}_K$ will have only in distributional meaning, so that the integral appearing in (3.20) has to be intended in the sense of measures.

If the problem is advection dominated in K the projection of the max point of $\hat{b}_K$ on xy-plane will be very close to outflow part of the boundary ∂K. Hence we choose the stability point P in K such a strategy:

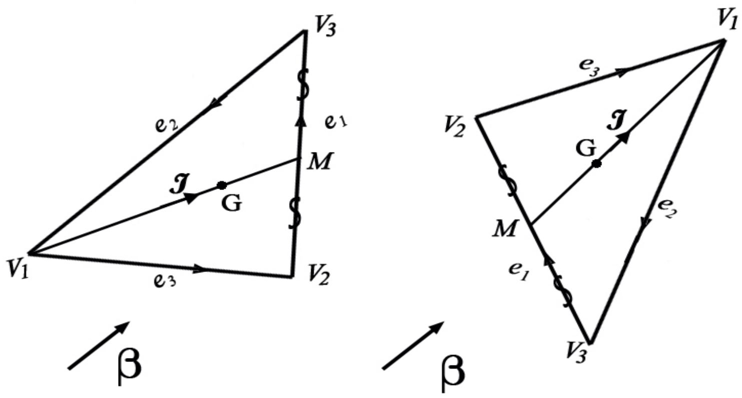

Figure 3.3. Case of 2 inflow boundary edges (left) and two outflow boundary edges (right).

For the inflow boundary is made of two edges:

$$P \in [G, M) \subset [V_1, M] \text{ and } P \rightarrow M \text{ whenever } \epsilon \rightarrow 0.$$

For the inflow boundary is made of single edge:

$$P \in [G, V_1) \subset [V_1, M] \text{ and } P \rightarrow V_1 \text{ whenever } \epsilon \rightarrow 0.$$

The set of points on the median (V_1, M) can be described as a function depending on a single parameter σ : $P = (1 - \sigma)V_1 + \sigma M$ where $0 < \sigma < 1$ for details see (Neslitürk, 2006).

Hence the minimization of (3.20) gives the exact position of the subgrid point P as following rules

Case 1: The inflow boundary is made of two edges of K

$$\sigma = \begin{cases} \sigma_1^* := 1 + \frac{\epsilon |e_1|^2}{\epsilon |e_2 - e_3|^2 + \frac{4}{3}|K|(\beta, w_2)} & , \epsilon \leq \epsilon_1^* := \frac{-4|K|(\beta, w_2)/3}{3|e_1|^2 + |e_2 - e_3|^2}, \\ \frac{2}{3} & , \text{otherwise.} \end{cases} \tag{3.21}$$

Case 2: The inflow boundary is made of one edge of K

$$\sigma = \begin{cases} \sigma_2^* := \frac{\epsilon(|e_2|^2+|e_3|^2)}{\epsilon|e_2-e_3|^2/2-|K|(\beta,w_1)/3} & ,\epsilon \le \epsilon_2^* := \frac{-2|K|(\beta,w_1)/3}{|e_1|^2+|e_2|^2+|e_3|^2}, \\ \frac{2}{3} & ,\text{otherwise.} \end{cases} \tag{3.22}$$

Case 3: One edge of the triangle is parallel to β

From the error estimates point of view in (Brezzi, Marini and Russo, 2005) P can be taken as the definition of point either in case1 or case2 if convection field is parallel to any one edge of the triangle. The solution will has small changing but it is not important.

Under these definitions, the approximate solution of the local bubble problem (3.13) can be explicitly described in order to obtain the numerical solution of (3.1).

Another possible way of choosing the position of P was suggested in the context of Standard Galerkin method on such augmented grid (SGAG) strategy (see (Brezzi, Marini and Russo, 2005)).

3.4. Numerical Results

In this section we examine the numerical methods presented here for two different test problems. The nontrivial boundary condition described in Fig. 3.4 is used for both two problems. The basic mesh is made 2*25*25 nonuniform triangles for linear elements.

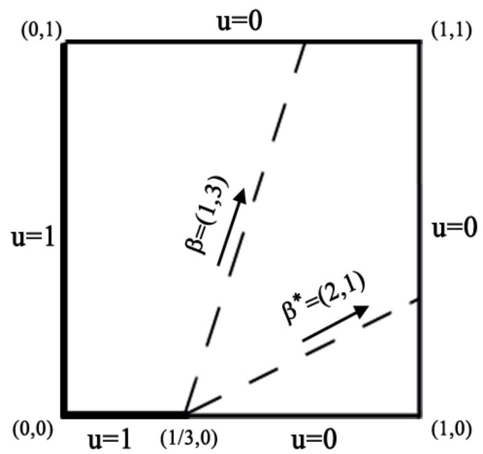

Figure 3.4. Problem description on square domain

1^{st} **Test Problem:**

Test 1.a) $-0.01\triangle u + (1,3).\nabla u = 0$ in $\Omega := (0,1)^2$

Test 1.b) $-0.0001\triangle u + (1,3).\nabla u = 0$ in $\Omega := (0,1)^2$

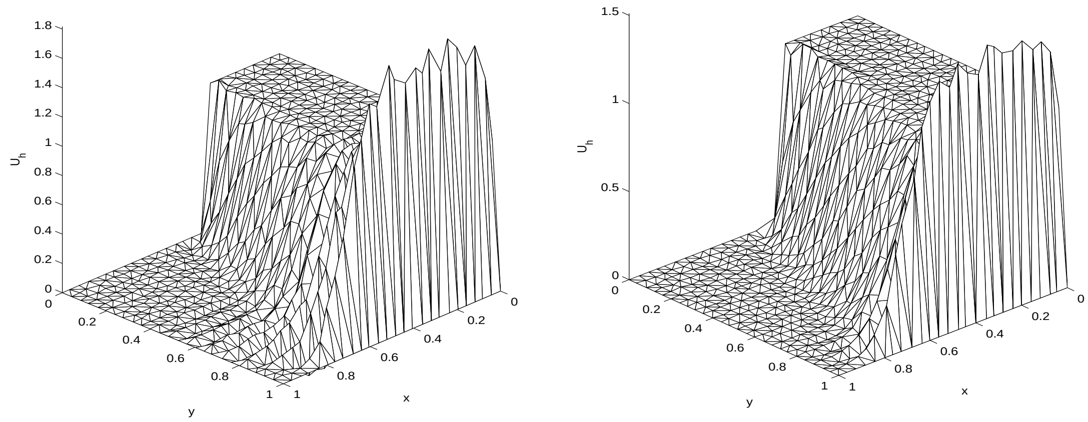

Figure 3.5. Galerkin (left) and PRFB solution (right) of Test(1.a) on triangular elements

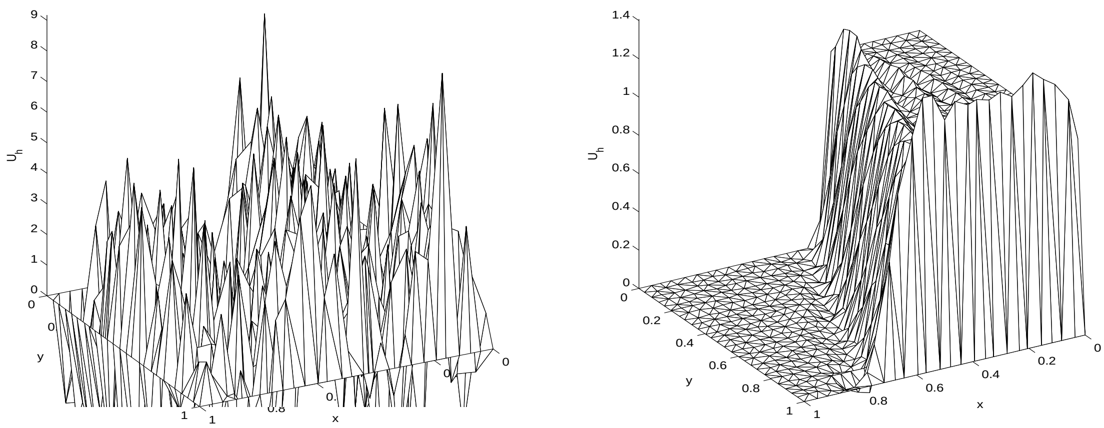

Figure 3.6. Galerkin (left) and PRFB solution (right) of Test 1.b on triangular elements

2^{nd} **Test Problem:**

Test 2.a) $-0.01\triangle u + (2,1).\nabla u = 0$ in $\Omega := (0,1)^2$

Test 2.b) $-0.0001\triangle u + (2,1).\nabla u = 0$ in $\Omega := (0,1)^2$

In the Figs. 3.5, 3.6, 3.7 and 3.8 we compare the standard Galerkin (left side) and the PRFB method (right side) for all test problems considered here. As a result, PRFB method is able to produce more stable results than the Galerkin method that validate the accuracy of the PRFB algorithm.

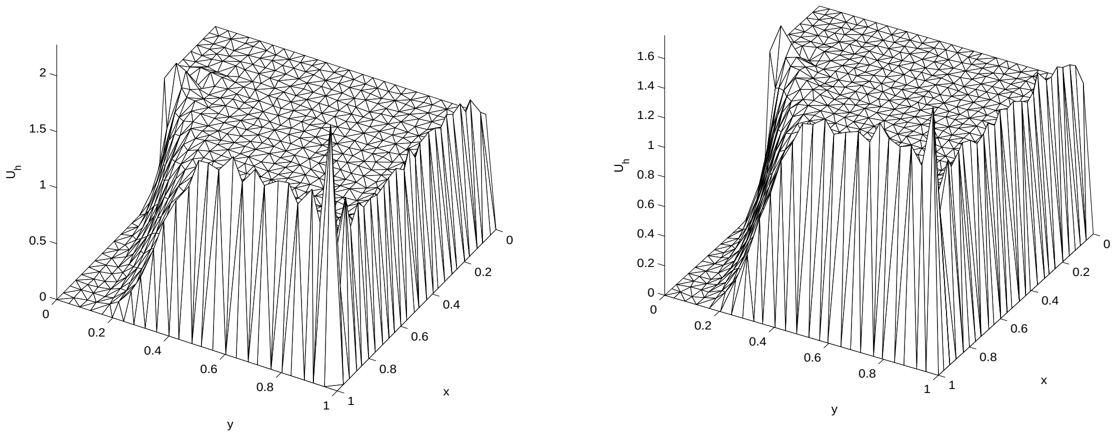

Figure 3.7. Galerkin (left) and PRFB solution (right) of Test 2.a on triangular elements

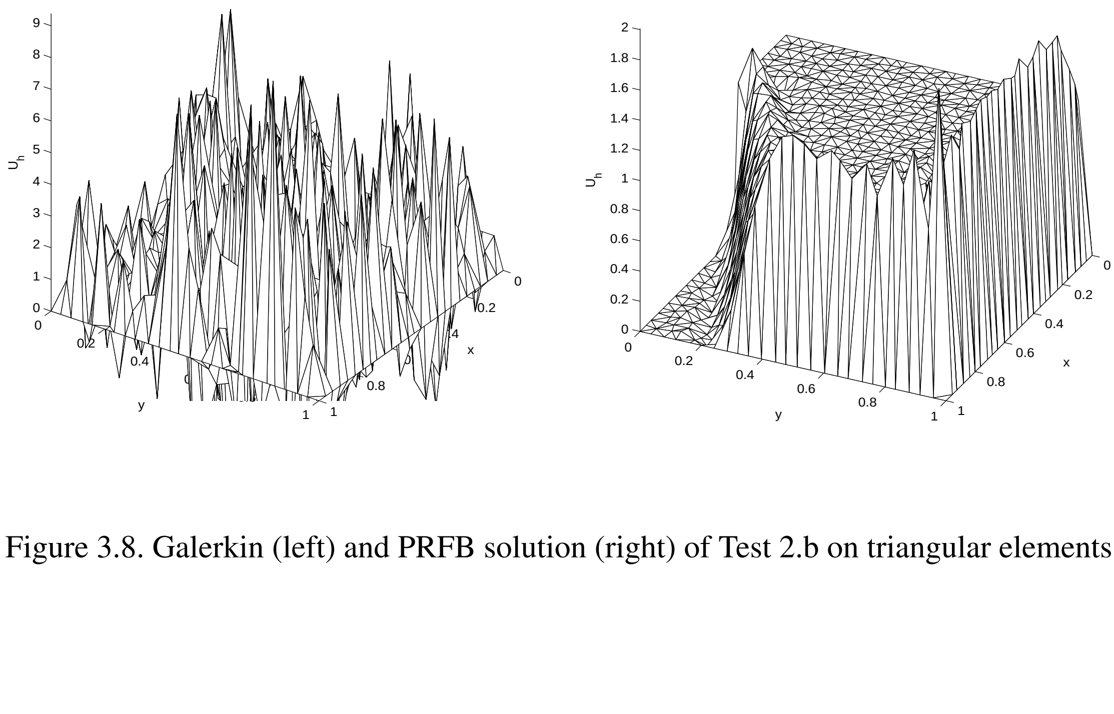

Figure 3.8. Galerkin (left) and PRFB solution (right) of Test 2.b on triangular elements

CHAPTER 4

PSEUDO MULTISCALE FUNCTIONS FOR THE STABILIZATION OF THE STEADY EQUATION ON RECTANGULAR GRIDS

In this chapter, we extend the idea of the pseudo bubble techniques to the multiscale functions and propose a stable, fully discrete, yet inexpensive numerical method for convection-diffusion problems on rectangular grids.

4.1. Problem Description

Let Ω be a bounded open domain in $\mathbb{R}^2$ with polygonal boundary $\partial\Omega$. We consider the following linear advection-diffusion problem:

$$\begin{aligned} \mathcal{L}u := -\epsilon\triangle u + \beta.\nabla u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{4.1}$$

where $\epsilon > 0$ is a constant diffusion coefficient. Let $\mathfrak{T}_h$ be a standard partition of domain into rectangles K and h refers to the level of refinement of the discretization that is defined by

$$h := \max_{K\in\mathfrak{T}_h} h_K$$

where h_K is the diameter of K. As usual we assume that the source term f and the convection field $\beta = (\beta_1, \beta_2)$ are piecewise constants with respect to the decomposition $\mathfrak{T}_h$. Without loss of generality we take components of β are positive in each element K. The outflow boundary is a part of $\partial\Omega$ defined by

$$\partial\Omega^{out} := \{(x, y) \in \partial\Omega |\ \beta.n(x, y) > 0\}$$

where n is the outflow normal to boundary. We denote by $\mathfrak{T}_h^{out}$ the set of elements in $\mathfrak{T}_h$ that has at least one boundary contained by $\partial\Omega^{out}$.

Then associated weak formulation is to find $u \in H_0^1(\Omega)$ such that

$$a(u, v) := \epsilon(\nabla u, \nabla v) + (\beta.\nabla u, v) = (f, v) \;\; \forall v \in H_0^1(\Omega) \tag{4.2}$$

Due to the coercivity of the bounded bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ and the Lax-Milgram theorem, the weak problem (4.2) is well-posed.

To introduce the Galerkin approach, we introduce the finite dimensional subspace $V_L(\Omega)$ of $H_0^1(\Omega)$ as

$$V_L(\Omega) = \{v \in H_0^1(\Omega) : v|_K \text{ is a bilinear polynomial } \; \forall K \in \mathfrak{T}_h\}$$

Then the classical Galerkin approach of (4.2) reads: Find $u_L \in V_L \subset H_0^1(\Omega)$ such that

$$a(u_L, v_L) = (f, v_L) \;\; \forall v_L \in V_L \tag{4.3}$$

Here u_L is the bilinear approximation of u and it can be represented by the linear combination of standard nodal basis functions ψ_i with the coefficients u_i.

It is well known that the Galerkin method inherits the stability of the continuous problem and it yields to spurious oscillations when the advection coefficient is larger than the diffusive one ($\epsilon << |\beta|h$). Since we are interested in finding a finite element discretization for (4.2) that is stable and coarse mesh accurate for all ϵ and β, we consider a Petrov-Galerkin type stabilization so that we respectively enrich the trial and test spaces as

$$U_h(\Omega) : \;\; = \;\; V_L(\Omega) \oplus E_h(\Omega) \tag{4.4}$$

$$W_h(\Omega) : \;\; = \;\; V_L(\Omega) \oplus B(\Omega). \tag{4.5}$$

Here $B(\Omega)$ is the bubble space defined by

$$B(\Omega) := \{v \in H_0^1(\Omega) : v|_K \in H_0^1(K)\ \forall K \in \mathfrak{T}_h\}$$

and we later define the enriching space $E_h(\Omega)$. Now the Pertrov-Galerkin problem (4.3) reads: Find $u_h \in U_h(\Omega)$ such that

$$a(u_h, w_h) = (f, w_h)\ \ \forall w_h \in W_h(\Omega). \tag{4.6}$$

Since typical member u_h of $U_h(\Omega)$ can be split into a bilinear part $u_L \in V_L(\Omega)$ and a enriching part $u_e \in E_h(\Omega)$, solving (4.6) is equivalent to find $u_L \in V_L(\Omega)$ such that

$$a(u_L + u_e, v_L) \ = \ (f, v_L)\ \ \forall v_L \in V_L(\Omega) \tag{4.7}$$

where for all $K \in \mathfrak{T}_K$, u_e is the weak solution of the following residual equation:

$$\begin{aligned} \mathcal{L}u_e &= f - \mathcal{L}u_L \ \text{ in } \ K \\ u_e &= \mu \ \text{ on } \ \partial K. \end{aligned} \tag{4.8}$$

In order to evaluate u_e uniquely, we need to choose a boundary condition. It is known that for RFB method we set $\mu = \mu_b := 0$. In this case enriching space $E_h(\Omega)$ can be represented direct sum of the two dimensional bubble spaces (Franca and Tobiska, 2002) such that

$$E_h(\Omega) = B_h(\Omega) := \bigoplus_{K \in \mathfrak{T}_h} B_h(K)$$

where $B_h(K) = span\{b_0^K, b_2^K\}$ and b_0^K , b_1^K are the solutions of the following problems

$$\left\{ \begin{aligned} \mathcal{L}b_0^K &= 1 \quad \text{in } K \\ b_0^K &= 0 \quad \text{on } \partial K \end{aligned} \right. \quad \text{and} \quad \left\{ \begin{aligned} \mathcal{L}b_1^K &= \beta_1(x - x_K) + \beta_2(y - y_K) \quad &\text{in } K \\ b_1^K &= 0 \quad &\text{on } \partial K \end{aligned} \right. \tag{4.9}$$

where (x_K, y_K) is the barycenter of K. Then enriching part of the solution can be written as

$$u_e = u_b := \sum_{K \in \mathfrak{T}_h} (\alpha_0^K b_0^K + \alpha_1^K b_1^K) \tag{4.10}$$

where

$$\alpha_0^K = (f - \beta.\nabla u_L)(x_K, y_K) \text{ and } \alpha_1^K = -\frac{\partial^2 u_L}{\partial x \partial y}.$$

For the MS method we choose $\mu = \mu_m$ that satisfies the following ordinary differential equation:

$$\begin{aligned} \mathcal{L}_{\partial K} \mu_m &= -\mathcal{L}_{\partial K} u_L \text{ on } \partial K \\ \mu_m &= 0 \text{ at the vertices of } K \end{aligned} \tag{4.11}$$

where

$$\mathcal{L}_{\partial K} v := -\epsilon \frac{\partial^2 v}{\partial s^2} + \mathbf{P}(\beta, s) \frac{\partial v}{\partial s}$$

with s is a variable that parameterizes ∂K by arc-length and $\mathbf{P}(\beta, s)$ is the usual projection of the convection field onto ∂K. In order to make the method more practical the boundary condition in the original MS algorithm in (Franca, Ramalho and Valentin, 2005) was modified as in (4.11) but numerical tests indicate that this difference is negligible.

Let I_0 be the set of indexes of internal nodal points with respect to the discretization of Ω then MS space can be given by

$$E_h(\Omega) = M_h(\Omega) := span\{\phi_i, \phi_f\}_{i \in I_0}$$

where ϕ_i and ϕ_f are enriching basis functions, defined by the following auxiliary problems

$$\begin{cases} \mathcal{L}\phi_i = -\mathcal{L}\psi_i & \text{in } K \\ \phi_i = \nu_i & \text{on } \partial K \end{cases} \text{ where } \begin{cases} \mathcal{L}_{\partial K}\nu_i = -\mathcal{L}_{\partial K}\psi_i & \text{on } \partial K \\ \nu_i = 0 & \text{at the vertices of } K \end{cases} \tag{4.12}$$

and

$$\phi_f = \sum_{K \in \mathfrak{T}_h} f|_K \, b_0^K \tag{4.13}$$

Then enriching part of the solution can be written in terms of the global MS basses functions

$$u_e = u_m := \sum_{i \in I_0} u_i \phi_i + \phi_f \tag{4.14}$$

It is reported in (Franca, Ramalho and Valentin, 2005) that the MS method capture layers near the outflow boundary better than the RFB method does. On the other hand the RFB method produces more accurate results than the MS one in some parts of the domain close to the internal layer if mesh is not aligned with the advection field. Motivated by this observation, the RFB-MS (MIX) algorithm was proposed by Franca et. al. in (Franca, Ramalho and Valentin, 2005). This algorithm is based on the idea that the MS functions are used in the elements connected to the outflow boundaries and the RFB functions are used in the rest of the domain. According to MIX method we set the boundary condition of the residual equation (4.8) $\mu = \mu_{bm}$ such that

$$\mu_{bm} = \begin{cases} \mu_m & , K \in \mathfrak{T}_h^{out} \\ \mu_b & , \text{otherwise} \end{cases}$$

In this case enriching part of the solution can be written as:

$$u_e = u_{bm} := \sum_{K \in \mathfrak{T}_h / \mathfrak{T}_h^{out}} u_b|_K + \sum_{K \in \mathfrak{T}_h^{out}} u_m|_K. \tag{4.15}$$

4.2. Computing the Enriching Functions

In order to obtain the enriching part of the solution u_e, equations (4.9) and (4.12) should be solved. This task is as difficult as solving the original problem that makes the method impractical. So in this study we are just interested in obtaining cheap and efficient approximation of enriching bubble and MS functions.

We first consider the bubble equation in (4.9). Although $B_h(K)$ spanned by two bubble basis functions b_0^K and b_1^K in each element, considering only the dominant one b_0^K is sufficient to obtain stable results. We also use the PRFB approximation for b_0^K as illustrated in Fig. 4.1 -left (Neslitürk, 2010). Therefore the resulting method is called PRFB.

The height of the pyramid can be determined by the formulation (3.19) and exact location P is to be chosen on the line, whose end points are V_1 and V_3, in order to minimize the integral (3.20). Without loss of generality, we assume components of the convection field are nonnegative. Then the stability point P in K is choosen by the strategy

$$P \in [G, V_3) \subset [V_1, V_3] \text{ and } P \to V_3 \text{ whenever } \epsilon \to 0.$$

where G is the barycenter of the rectangle. Hence the minimization of (3.20) gives the exact position of the subgrid point P as following rule. Let σ be a parameter that describes $P = (1-\sigma)V_1 + \sigma V_3$ then

$$\sigma = \begin{cases} \sigma^* := 1 - \frac{3\epsilon h_K^2}{|K|(\beta, w_2+w_3)} & , \epsilon \leq \epsilon^* := \frac{|K|(\beta, w_2+w_3)}{6h_K^2}, \\ \frac{1}{2} & , \text{otherwise.} \end{cases} \tag{4.16}$$

where $w_i = |z_i| n_i$ with $|z_i|$ and n_i denotes the length of z_i and the outward unit normal to z_i, respectively.

We secondly consider the pseudo multiscale (PMS) approximation for the MS basis functions ϕ_i. Without loss of generality, we may consider typical element K whose lower-left vertex is located at the origin and (h_x, h_y) is the i^{th} inner node in the discretization as in Fig. 4.1 -right. We only display the computation of ϕ_i in K, as the other enriched functions can be found in a similar way.

The problem (4.12) can be written in terms of the enriched basis function $\lambda_i := \phi_i + \psi_i$:

$$\begin{cases} \mathcal{L}\lambda_i = 0 & \text{in } K \\ \lambda_i = \theta_i & \text{on } \partial K \end{cases} \quad \text{where} \quad \begin{cases} \mathcal{L}_{\partial K}\theta_i = 0 & \text{on } \partial K \\ \theta_i = \psi_i & \text{at the vertices of } K. \end{cases} \tag{4.17}$$

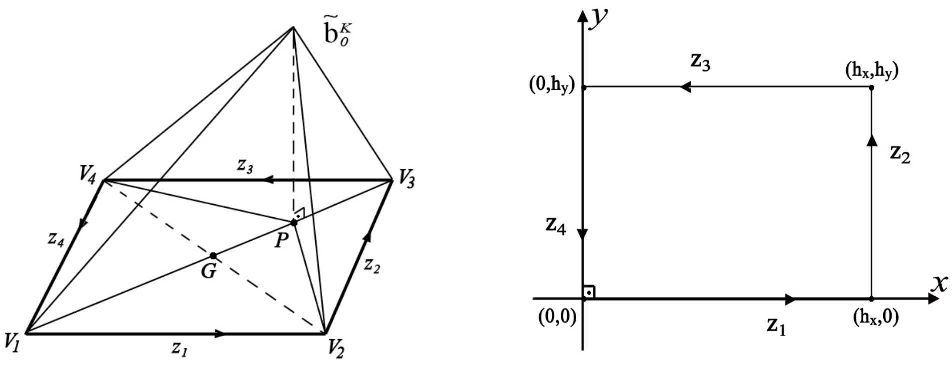

Figure 4.1. Approximation of the bubble bases function b_0^K (left) and typical element K (right)

Since λ_i and its restriction to ∂K satisfies (4.17), its solution can be written as

$$\lambda_i(x,y) = \lambda_i^x(x)\lambda_i^y(y) \text{ where } \lambda_i^x(x) := \theta_i|_{z_1} \text{ and } \lambda_i^y(y) := \theta_i|_{z_4} \tag{4.18}$$

and the single variable functions λ_i^x and λ_i^y satisfy

$$\begin{aligned} \mathcal{L}_{\partial K}|_{z_1}\lambda_i^x = -\epsilon\frac{d^2\lambda_i^x}{dx^2} + \beta_1\frac{d\lambda_i^x}{dx} &= 0 \text{ in } (0,h_x) \\ \lambda_i^x(0) = 0 \text{ and } \lambda_i^x(h_x) &= 1 \end{aligned} \tag{4.19}$$

$$\begin{aligned} \mathcal{L}_{\partial K}|_{z_4}\lambda_i^y = -\epsilon\frac{d^2\lambda_i^y}{dy^2} + \beta_2\frac{d\lambda_i^y}{dy} &= 0 \text{ in } (0,h_y) \\ \lambda_i^y(0) = 0 \text{ and } \lambda_i^y(h_y) &= 1. \end{aligned} \tag{4.20}$$

Enriched functions λ_i^x and λ_i^y can be brought to the form of bubble functions before we suggest a suitable subgrid. That is, they should vanish at the boundary of domain where the equations are posed, without upsetting the nature of the differential operator. To bring that end, let us define two auxiliary functions α_i^x and α_i^y by

$$\alpha_i^x := \lambda_i^x - \psi_i^x \quad \text{and} \quad \alpha_i^y := \lambda_i^y - \psi_i^y. \tag{4.21}$$

where $\psi_i^x(x) := \psi_i(x,0)$ and $\psi_i^y(y) := \psi_i(0,y)$.
Then from (4.19)-(4.21), following equations are satisfied

$$\begin{cases} -\epsilon\dfrac{d^2\alpha_i^x}{dx^2} + \beta_1\dfrac{d\alpha_i^x}{dx} = \dfrac{\beta_1}{h_x} \text{ in } (0,h_x) \\ \alpha_i^x(0) = \alpha_i^x(h_x) = 0 \end{cases} \text{ and } \begin{cases} -\epsilon\dfrac{d^2\alpha_i^y}{dy^2} + \beta_2\dfrac{d\alpha_i^y}{dy} = \dfrac{\beta_2}{h_y} \text{ in } (0,h_y) \\ \alpha_i^y(0) = \alpha_i^y(h_y) = 0. \end{cases} \tag{4.22}$$

We now can add specially chosen internal nodes into the domains $(0,h_x)$ and $(0,h_y)$, on which we approximate α_i^x and α_i^y so that the resulting approximations, say $\tilde{\alpha}_i^x$ and $\tilde{\alpha}_i^y$ retain stabilizing features of α_i^x and α_i^y. The location of the additional node is crucial for the stabilization and its choice depending on different configurations can be found in the literature (Brezzi, Hauke, Marini and Sangalli, 2003) and (Neslitürk, 2006). Here we use the formulation (2.43) for the position of this additional critical node. A straightforward application of the asserted approach to the problem (4.22) results in

$$\tilde{\alpha}_i^x(x) = \begin{cases} \dfrac{\beta_1(h_x-P)x}{2h_x\epsilon}, & x \le P \\ \\ \dfrac{\beta_1 P(h_x-x)}{2h_x\epsilon}, & x > P \end{cases} \quad \text{where } P = \begin{cases} h_x - 2\epsilon/\beta_1, & \epsilon \le \beta_1 h_x/4 \\ h_x/2, & \text{otherwise.} \end{cases} \tag{4.23}$$

$\tilde{\alpha}_i^y(y)$ is similarly obtained by replacing x by y, h_x by h_y and β_1 by β_2 in (4.23). Thus approximate enriching basis function $\tilde{\phi}_i$ can be written as

$$\begin{aligned} \tilde{\phi}_i(x,y) &= \tilde{\lambda}_i(x,y) - \psi_i(x,y) = \tilde{\lambda}_i^x(x)\tilde{\lambda}_i^y(y) - \psi_i(x,y) \\ &= \Big(\tilde{\alpha}_i^x(x) - \psi_i^x(x)\Big)\Big(\tilde{\alpha}_i^y(y) - \psi_i^y(y)\Big) - \psi_i(x,y) \\ &= \tilde{\alpha}_i^x(x)\tilde{\alpha}_i^y(y) - \tilde{\alpha}_i^x(x)\psi_i^y(y) - \tilde{\alpha}_i^y(y)\psi_i^x(x). \end{aligned} \tag{4.24}$$

A comparison of ϕ_i and its approximate counterpart $\tilde{\phi}_i$ are displayed on a patch of four rectangular elements in the Figs. 4.2-4.3 for decreasing values of ϵ. It is remarkable that although a few additional nodes are used in each element, the results are very comparable with the exact solution. Therefore, it is quite reasonable to employ the approximate enriching functions $\tilde{\phi}_i$ in place of ϕ_i. Hence PMS is used for the resulting algorithm.

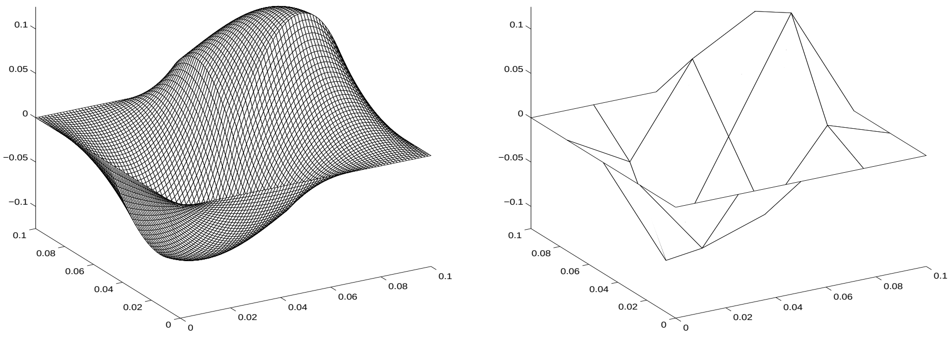

Figure 4.2. ϕ_i(left) and $\tilde{\phi}_i$(right) for $\epsilon = 0.1$, $\beta = (1, 2)$ and $h_x = h_y = 0.05$

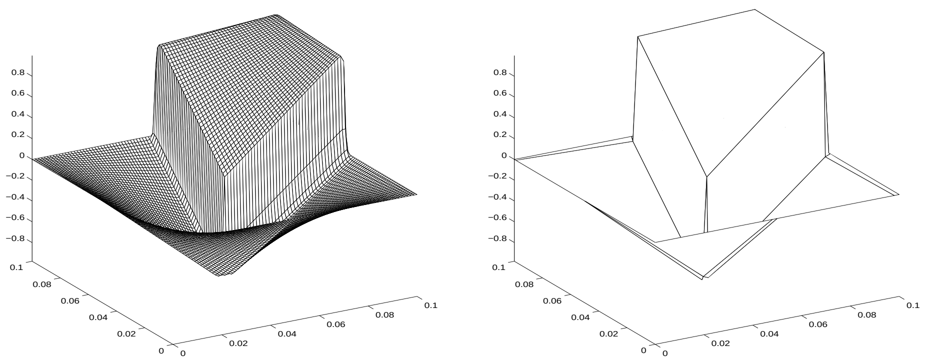

Figure 4.3. ϕ_i(left) and $\tilde{\phi}_i$(right) for $\epsilon = 0.001$, $\beta = (1, 2)$ and $h_x = h_y = 0.05$

4.3. Numerical Results

Experiment 1: We examine the numerical method presented here on a benchmark problem posed on the unit square, subject to the nontrivial boundary conditions as depicted in Fig. 4.4. The basic mesh is made up of 20×20 rectangles, whose edges are parallel to the coordinate axes. The only exception is in Fig. 4.5-right, in which we use 20×10 rectangular elements. We test the method for high Peclet numbers, that is $\epsilon = 10^{-6}$, and three different convection fields: $\beta = (1, 2)$, $\beta = (1, 1)$ and $\beta = (2, 1)$. Since the basis functions in both the MS and the PMS method are comparable (see Figs. 4.2 and 4.3), they produce almost the same results. Therefore we only display the numerical results obtained by the PMS method due to its little cost.

Case 1. $f = 0$**:** On non-aligned meshes, we observe that the PMS method produces

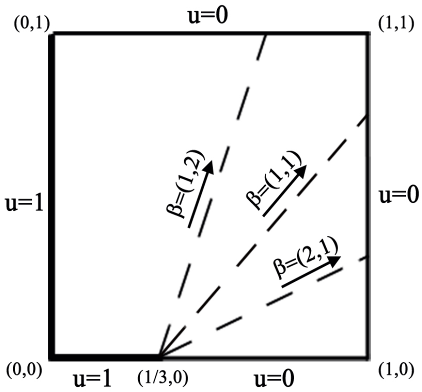

Figure 4.4. Problem description on square domain

accurate results at the outflow boundaries, yet it does not well capture internal layers (Fig. 4.5-left), in which case, numerical solution presents oscillations in some parts of the domain close to the internal layer. We now apply the MIX strategy, yet, the approximate counterparts of the MS and RFB functions are used. The resulting numerical method will be denoted by PMIX. In order to display the performance of our method, we compare it with the PRFB method on the uniform mesh in Figs. 4.6,4.7 and 4.8 and nonuniform mesh in Fig. 4.9. It is obvious that the proposed algorithms improves over the PRFB method.

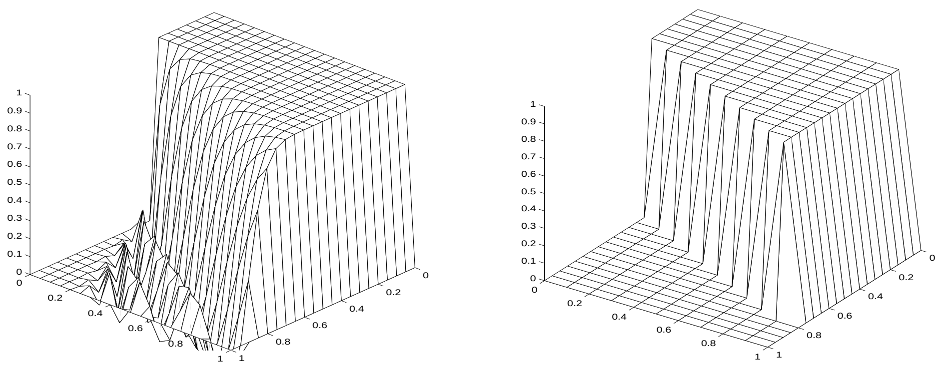

Figure 4.5. PMS approximations on nonaligned (left) and aligned (right) uniform rectangular mesh with $\beta = (1, 2)$.

Case 2. $f = 1$: In this part, we report some results for convection diffusion problem with

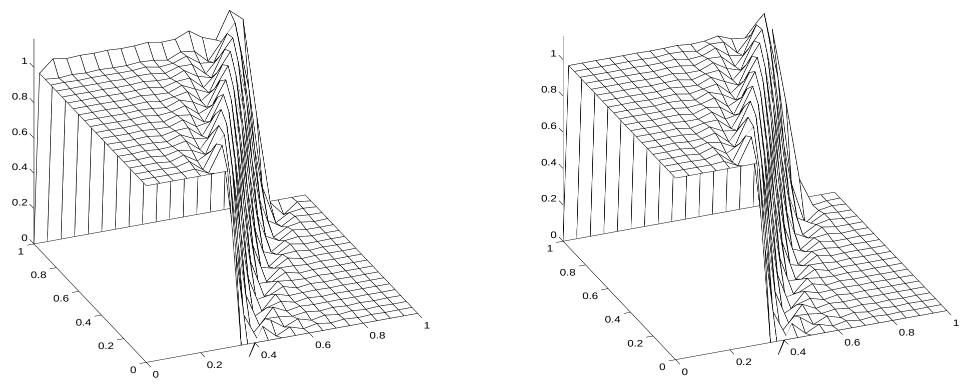

Figure 4.6. PRFB (left) and PMIX (right) approximations with $\beta = (1, 2)$.

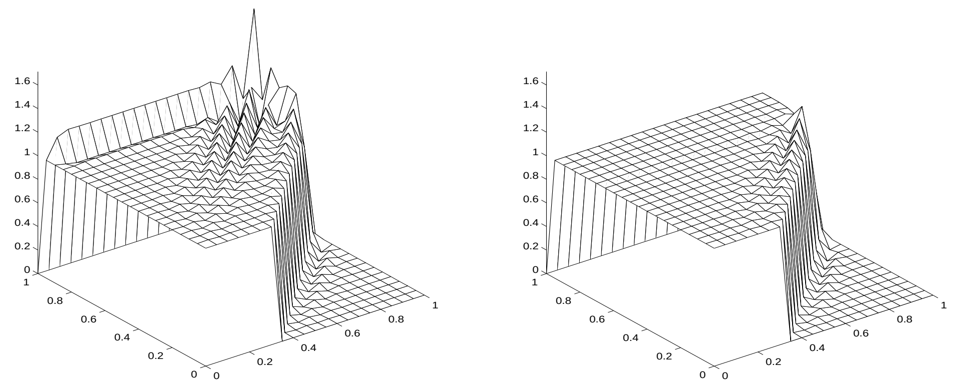

Figure 4.7. PRFB (left) and PMIX (right) approximations with $\beta = (1, 1)$.

nonzero source term. For this case ϕ_f satisfies

$$
\begin{aligned}
\mathfrak{L}\phi_f &= 1 \text{ in } K \\
\phi_f &= 0 \text{ on } \partial K
\end{aligned}
\tag{4.25}
$$

By using (4.13), we see $\phi_f|_K = b_0^K$. Since b_0^K is a bubble function, the PRFB method on rectangular elements can be applied to (4.25). Thus, we employ the approximation of ϕ_f instead of ϕ_f in the MIX algorithm. In Figs. 4.10 and 4.11 we again compare the PRFB and PMIX methods for $f = 1$ and the convection fields $\beta = (1, 1)$ and $\beta = (2, 1)$. As predicted, the PMIX method performs better than the PRFB in each cases.

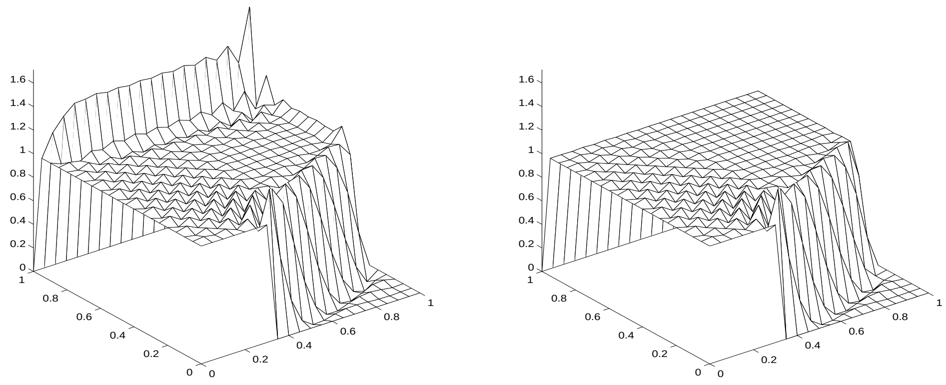

Figure 4.8. PRFB (left) and PMIX (right) approximations with $\beta = (2, 1)$.

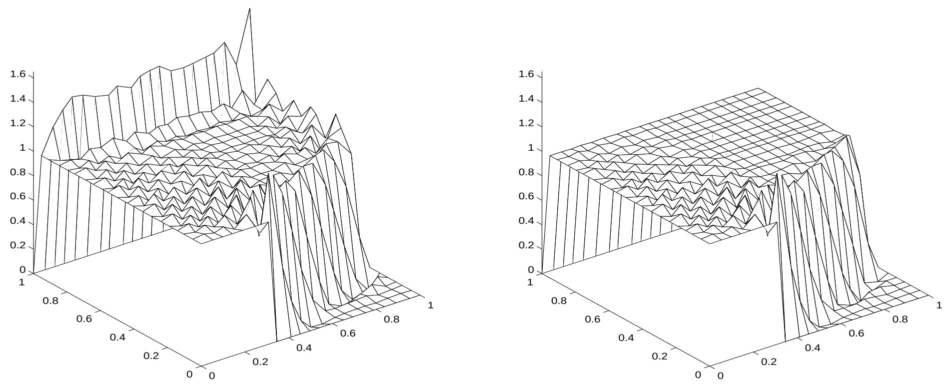

Figure 4.9. PRFB (left) and PMIX (right) approximations on nonuniform mesh with $\beta = (2, 1)$.

Experiment 2: We now test the proposed algorithm for non-constant flow field on L-shape domain with boundary conditions depicted in Fig. 4.12(left). We take $\beta = (y, 1 - x), \epsilon = 0.005$ and $f = 0$. Exact solution of the problem exhibits boundary layers near the outflow boundaries which are upper side, right-upper side and below the corner of the domain (Brezzi, Marini and Russo, 1998). We discretize the domain into 300 uniform rectangular elements (Fig. 4.12(right)). In computations, we use the average value of the flow field over the whole element, that is,

$$\bar{\beta}_i|_K = \frac{1}{|K|} \iint\limits_K \beta_i \, \mathrm{d}x \, \mathrm{d}y \quad \text{for} \quad i = 1, 2.$$

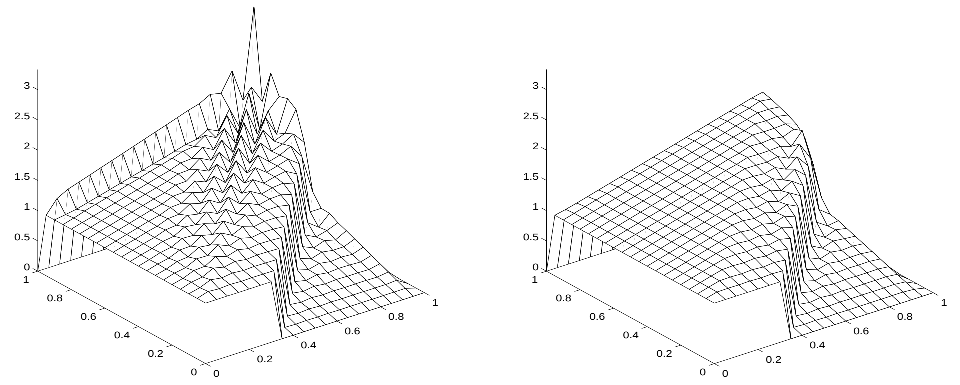

Figure 4.10. PRFB (left) and PMIX (right) approximations with $\beta = (1, 1)$ and $f = 1$.

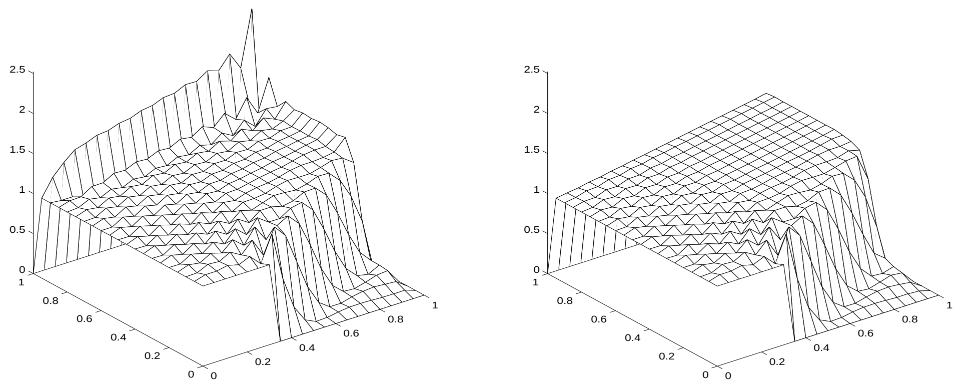

Figure 4.11. PRFB (left) and PMIX (right) approximations with $\beta = (2, 1)$ and $f = 1$.

We apply the same approach used in Experiment 1 and observe that PMIX method is still able to produce better approximations than the PRFB for more complicated problem configurations (Fig. 4.13).

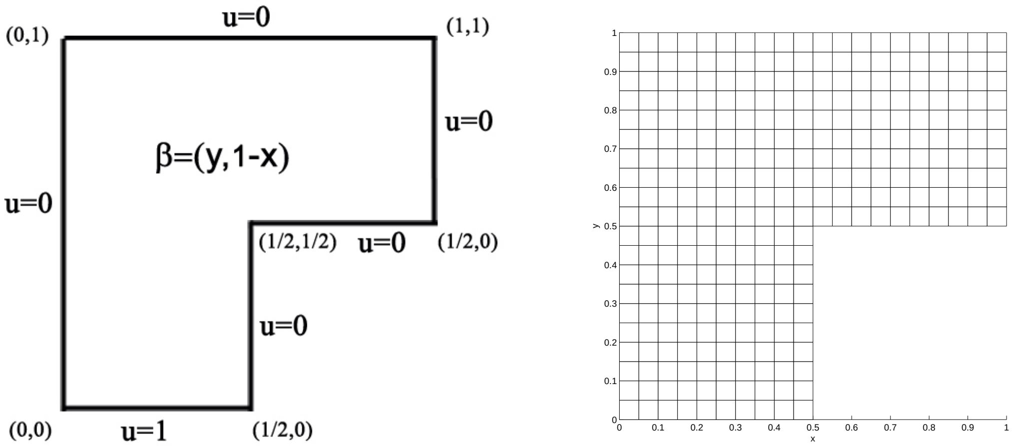

Figure 4.12. Problem description on L-shape domain and the mesh employed.

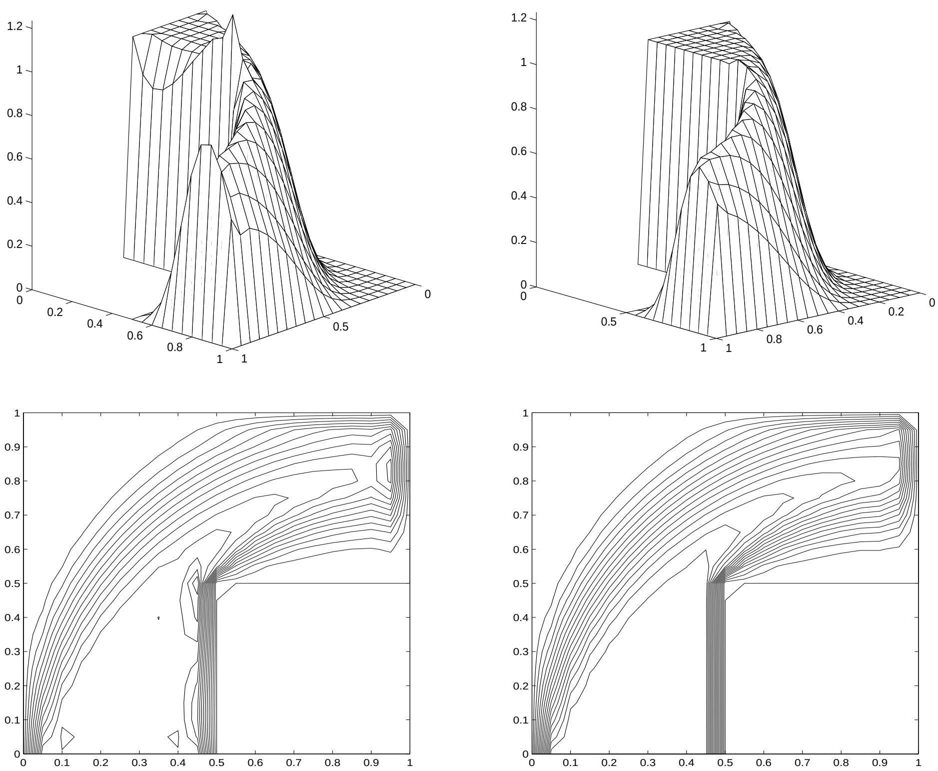

Figure 4.13. PRFB (left) and PRFB-PMS (right) approximations on L-shape domain..

CHAPTER 5

BUBBLE AND MULTISCALE STABILIZATION FOR UNSTEADY EQUATIONS ON RECTANGULAR GRIDS

Here we consider the approximation technics based on enriched methods which consist of the Galerkin method with enhanced approximation spaces then applied the simple time integration to the resulting semidiscrete formulation. In this chapter we also discuss how the stabilized finite element methods RFB, MS and MIX designed for steady problem could be properly combined with generalized Euler time integration for the numerical solution of the unsteady advection diffusion equation.

5.1. Problem Description

In this part we consider the following initial boundary value problem:

$$\mathcal{L}_t u := u_t + \mathcal{L}u = f \text{ in } \Omega_t := \Omega \times (0,\mathrm{T}) \tag{5.1}$$
$$u = 0 \text{ on } \partial\Omega \times [0,T] \ , \ \ u(.,0) = u_0(.) \text{ in } \Omega$$

where $\mathcal{L}$ is taken as previous section, $u_0 \in L_2(\Omega)$ is an initial datum and right hand side function $f(.,t)$ (or $f(t)$) is chosen from $L_2(\Omega)$ for each $t \in [0,T]$.

Then the weak formulation associated with (5.1) reads: For all $t \in (0,T]$ find $u(t) \in H_0^1(\Omega)$, $u(0) = u(.,0)$ satisfying

$$\frac{d}{dt}(u(t),v) + a(u(t),v) = (f(t),v) \quad \forall v \in H_0^1(\Omega). \tag{5.2}$$

We first discretize the space variable in order to construct an approximation to (5.1). This process leads to a system of ODE whose solution $u_h(t)$ is an approximation of the exact solution u at $t \in [0,T]$.

Then semi discrete Petrov-Galerkin approximation of (5.1) can be given by: For all $t \in (0,T]$, find $u_h(t) \in U_h$ such that

$$\frac{d}{dt}(u_h(t), v_L) + a(u_h(t), v_L) = (f(t), v_L) \quad \forall v_L \in V_L(\Omega). \tag{5.3}$$

In order to simplify algorithm we only enrich the trial space as in (4.4) for each $t \in (0,T]$. However, test space will be taken as $W_h = V_L$ instead of (4.5) and enriched part of the solution of steady equation will be adapted for unsteady case by the following recipe.

- RFB method: $u_e(t) = u_b(t) := \sum_{K \in \mathfrak{T}_h} (f(t) - \mathcal{L}u_L(t))|_{(x_K, y_K)} b_0^K$.
- MS method: $u_e(t) = u_m(t) := \sum_{i \in I_0} u_i(t)\phi_i + \phi_f(t)$.
- MIX method: $u_e(t) = u_{bm}(t) := \sum_{K \in \mathfrak{T}_h / \mathfrak{T}_h^{out}} u_b(t)|_K + \sum_{K \in \mathfrak{T}_h^{out}} u_e(t)|_K$.

Here

$$\phi_f(t) = \sum_{K \in \mathfrak{T}_h} f(t)|_K b_0^K$$

and $u_i(t)$ is the unknown coefficients of the $u_L(t)$, more formally,

$$u_L(t) = \sum_{i \in I_0} u_i(t)\psi_i(x,y).$$

In order to obtain full discretization of (5.1) we take a uniform subintervals for the time variable with

$$t^n = n\tau, \quad n = 0, .., N$$

where τ is the time-step and $N = T/\tau$.
Next we replace the time derivative with θ method in (5.3) therefore we obtain the following problem:
Given u_h^0 as some suitable approximation of $u(0)$, for $n \geq 1$, find $u_h^n \in U_h$ s.t.

$$\left(\frac{u_h^n - u_h^{n-1}}{\tau}, v_L\right) + a(u_h^{n-\theta}, v_L) = (f^{n-\theta}, v_L) \; \forall v_L \in V_L \tag{5.4}$$

where $u_h^{n-\theta} = \theta u_h^{n-1} + (1-\theta)u_h^n$. For $\theta = 0$, $\theta = 1/2$, and $\theta = 1$ we have the Backward

Euler, Crank-Nicolson and Forward Euler Method, respectively.
Since $u_h^n = u_L^n + u_e^n$, full discrete algorithm (5.4) can be given by:
For all $t \in (0, T]$, find $u_L(t) \in V_L$ such that

$$\left(\frac{u_L^n - u_L^{n-1}}{\tau}, v_L\right) + \left(\frac{u_e^n - u_e^{n-1}}{\tau}, v_L\right) + a(u_L^{n-\theta}, v_L) + a(u_e^{n-\theta}, v_L) = (f^{n-\theta}, v_L) \ \ \forall v_L \in V_L. \quad (5.5)$$

Here u_e^n denotes the discrete version of the enriched function $u_e(t)$, more precisely,

- RFB method: $u_e^n = u_b^n := \sum_{K \in \mathfrak{T}_h} (f^n - \mathcal{L}u_L^n)|_{(x_K, y_K)} b_0^K$
- MS method: $u_e^n = u_m^n := \sum_{i \in I_0} u_i^n \phi_i + \phi_f^n$
- MIX method: $u_e^n = u_{bm}^n := \sum_{K \in \mathfrak{T}_h / \mathfrak{T}_h^{out}} u_b^n|_K + \sum_{K \in \mathfrak{T}_h^{out}} u_e^n|_K$

Since the enriching basis functions ϕ_i and b_0^K are obtained from the steady equation, their shape do not change at different time levels. Therefore, we employ the approximate forms $\tilde{\phi}_i$ and $\tilde{b}_0^K$ instead of their exact forms in the full discrete formulation (5.5).

5.2. Numerical Experiments and Conclusion

We illustrate the accuracy of the method by different two test problems on $\Omega = (0, 1)^2$ with uniform 20*20 rectangular discretization. For all numerical simulations, $\epsilon = 10^{-6}$ and $\tau = 0.0025$ are used.

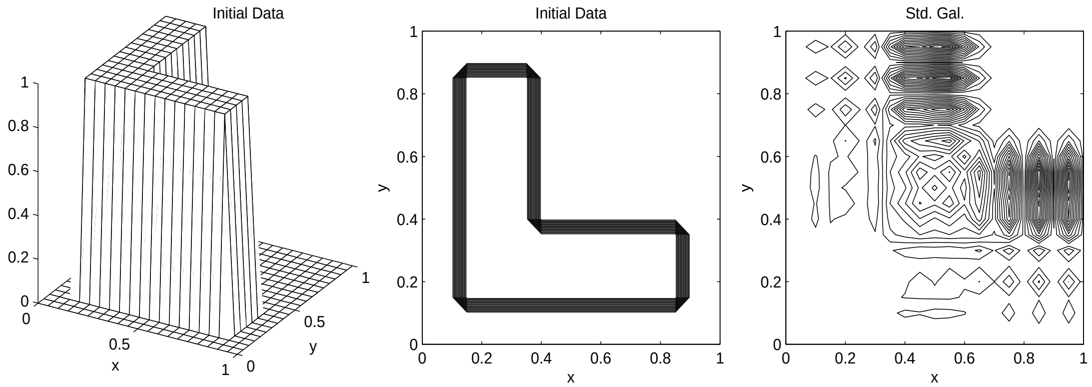

Figure 5.1. Initial condition (left), its counter plot (center) and Galerkin solution at $T = 1/4$ (right).

For the first test problem, we chose $f = 0$, $\beta = (1, 1)$, $\theta = 0$ with L-shape discrete type initial condition (see Fig. 5.1-left and center). The instable result for standard Glaerkin solution at $T = 1/8$ has been shown in Fig. 5.1 (right). We also compared the enriching finite element methods RFB, MS and MIX algorithms in Figs. 5.2 and 5.3 at the final time $T = 1/8$ and $T = 1/4$, respectively. It is known that the MS method captures the outflow layer and the RFB methods capture the internal layer well for the steady problems. In these figures, similar results can be seen for unsteady problems that verify the accuracy of the MIX algorithm.

In the second test problem, we examine the enriching methods with the problem parameters $f = 1$, $\beta = (1, 1/2)$, $T = 1/2$ and homogeneous initial condition. We again compared the RFB, MS and MIX method in Figs. 5.4 and 5.5 for backward Euler ($\theta = 0$) and Crank-Nicholson ($\theta = 0.5$) time discretization, respectively. Both cases MIX algorithm shows more stable behaviors. As a result, numerical simulations indicate that MIX algorithm combined with the implicit Euler time integration can be considered a reliable and accurate method for unsteady convection diffusion problems.

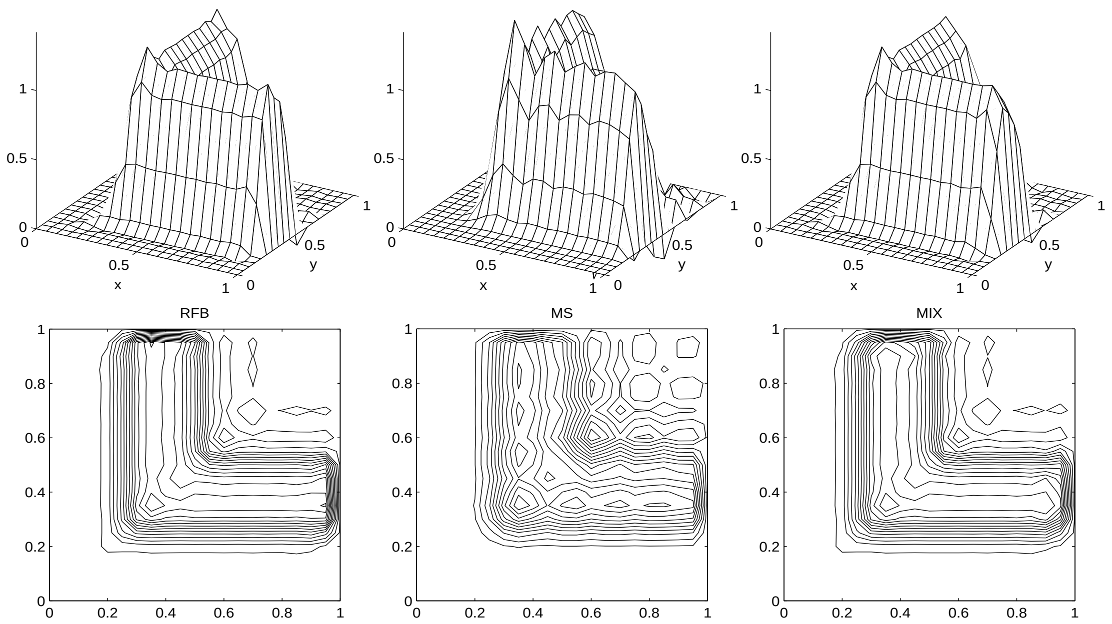

Figure 5.2. Results for RFB (left), MS (center) and MIX (right) methods at $T = 1/8$.

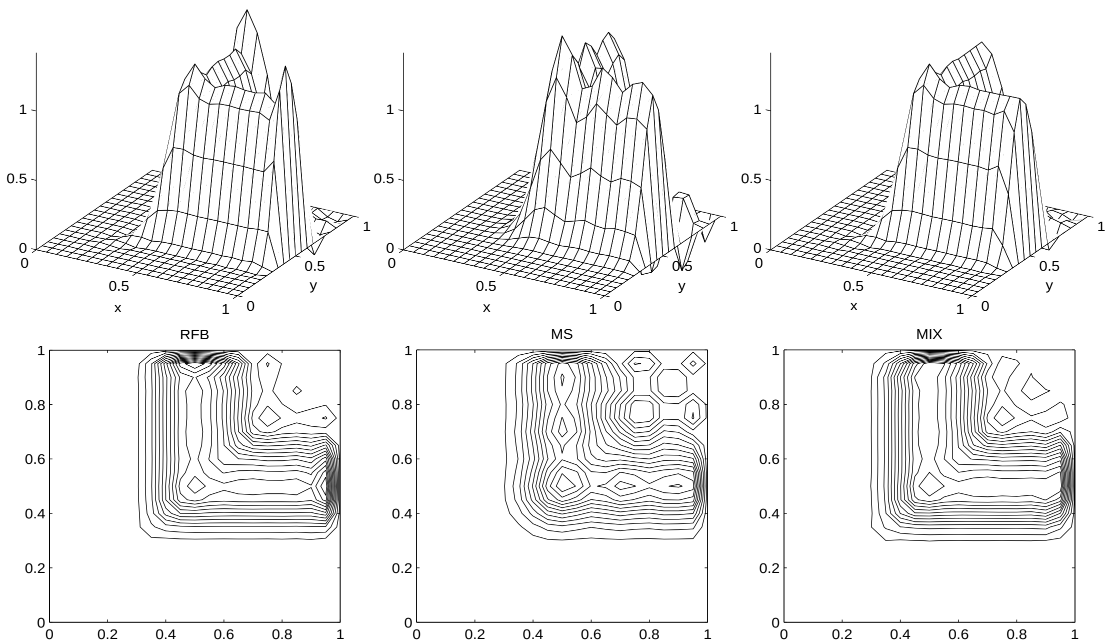

Figure 5.3. Results for RFB (left), MS (center) and MIX (right) methods at $T = 1/4$.

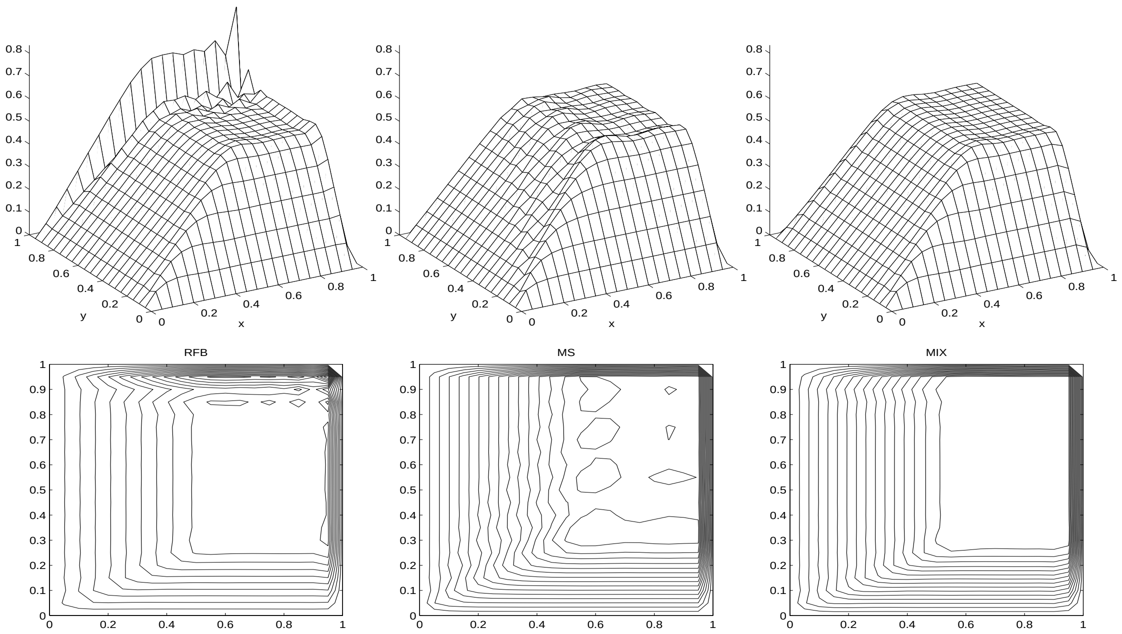

Figure 5.4. Results obtained by RFB (left), MS (center) and MIX (right) algorithms for $\theta = 0$.

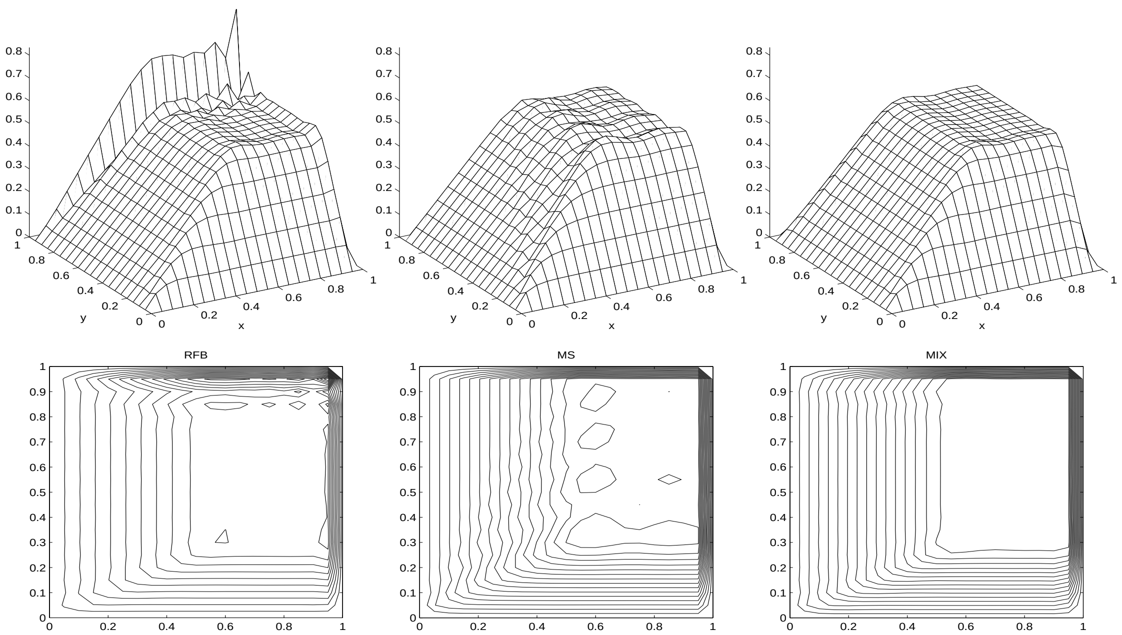

Figure 5.5. Results obtained by RFB (left), MS (center) and MIX (right) algorithms at $\theta = 1/2$.

CHAPTER 6

STABILITY AND CONVERGENCE ANALYSIS OF THE SUPG/θ METHOD FOR THE UNSTEADY PROBLEM

In this chapter we consider the time/space discretization of the transient advection-diffusion equation and apply Burman's analysis in (Burman, 2010) to this equation under the restriction that approximation space consists of piecewise linear polynomials. Discretization in space is performed by the streamline upwind Petrov-Galerkin (SUPG) method and in time we use the generalized Euler rule (θ-method). Two A-stable cases backward Euler ($\theta = 0$) and Crank-Nicolson ($\theta = 1/2$) are considered.

6.1. Problem Setting

Let Ω be a bounded open domain in $\mathbb{R}^d (d = 1, 2, 3)$ with Lipschitz continuous boundary $\partial\Omega$. We consider the following initial boundary value problem:

$$\begin{aligned} \mathcal{L}_t u := u_t + \mathcal{L}u &= f \text{ in } \Omega_\mathrm{t} := \Omega \times (0, \mathrm{T}) \\ u &= 0 \text{ on } \partial\Omega \times [0, \mathrm{T}] \\ u(., 0) &= u_0(.) \text{ in } \Omega \end{aligned} \tag{6.1}$$

where $\mathcal{L}$ is the second order differential operator defined by

$$\mathcal{L}v := -\epsilon \triangle v + \beta.\nabla v.$$

Here $\epsilon > 0$ is a constant diffusion coefficient, $\beta \in [L^\infty(\Omega)]^d$ is solenoidal convection field ($\nabla.\beta = 0$) which is constant in time, f is given source function assuming that it is function of bounded variation in time i.e. $f \in BV^0(0, T; L^2(\Omega))$, and $u_0 \in L^2(\Omega)$ is an initial datum, with $u_0|_{\partial\Omega} = 0$. We also assume that variation of field is bounded such that

$$\exists C_{\beta_*}, C_{\beta^*} \in \mathbb{R}^+ \ \text{ with } \ C_{\beta_*} \sup_{x\in\Omega} |\beta| \leq \overline{\beta} \leq C_{\beta^*} \sup_{x\in\Omega} |\beta| \tag{6.2}$$

where $\overline{\beta}$ denote the average of $|\beta|$ over Ω.

As usual we use the notation $H^m(D)$ for the Hilbertian Sobolev space of order m on an open set $D \subseteq \Omega$. Then the norm and semi norm of $H^m(D)$ are denoted by $||.||_{m,D}$ and $|.|_{m,D}$,respectively. To simplify the notation, we drop the subscript m in the case $m = 0$ and D in the case $D = \Omega$. In particular we denote the inner product by $(.,.)$ and the norm by $||.||$ in $L^2(\Omega)$.

Then the weak formulation of (6.1): For all $t \in (0, T]$ find $u(t) \in V^0 := H_0^1(\Omega)$, $u(0) = u(.,0)$ satisfying

$$(\partial_t u, v) + \epsilon(\nabla u, \nabla v) + (\beta\nabla u, v) = (f, v) \quad \forall v \in V^0. \tag{6.3}$$

Under the conditions described above, existence and uniqueness of the solution of (6.3) are guaranteed (Raviart and Thomas, 1992).

6.2. Semi-Discrete Approximation by SUPG Method

We first discretize the space variable in order to construct an approximation to (6.1). This process leads to a system of ordinary differential equations whose solution $u_h(t)$ is an approximation of the exact solution for all t in $[0, T]$. Let $\mathfrak{T}_h$ be a standard partition of Ω into triangles K, we introduce mesh diameter

$$h := \max_{K\in\mathfrak{T}_h} h_K, \ \text{ with } \ h_K := \max_{F\in\mathfrak{F}(K)} h_F$$

where h_F the diameter of the face F and $\mathfrak{F}(K)$ denotes the set of faces such that $F \in \partial K$. We choose a finite dimensional subspace V_h^0 of $H_0^1(\Omega)$ that is space of continuous piecewise affine polynomials and vanish on $\partial\Omega$:

$$V_h^0 := \{v_h \in C(\overline{\Omega}) : v_h|_K \in P_1(K) \;\; \forall\, K \in \mathfrak{T}_h \;\; \text{and} \;\; v_h|_{\partial\Omega} = 0\}.$$

Then the classical Galerkin approach can be given by solving the variational problem (6.3) on discrete space V_h^0 instead of V^0. Due to the instable results for the Galerkin method on coarse mesh, we consider the most popular stabilized method SUPG. In order to obtain SUPG approximation we replace the the trial space V^0 by V_h^0 and the test space V^0 by W_h defined by

$$W_h := \{w_h(v_h) = v_h + \delta\beta.\nabla v_h : \; v_h \in V_h^0\}.$$

We assume that stabilization parameter δ is constant in space and time. We choose it as

$$\delta = \begin{cases} \dfrac{\mu h}{\overline{\beta}}, & Pe := \dfrac{\overline{\beta} h}{\epsilon} > 1 \\ 0, & Pe \leq 1. \end{cases}$$

Here $\mu \in \mathbb{R}^+$ so that

$$\mu < \frac{1}{2} \min\left\{ \frac{\overline{\beta}}{c_i ||\beta||_\infty}, \frac{1}{c_i^2} \right\}$$

where c_i satisfies the following inverse inequality for all $K \in \mathfrak{T}_h$ and $v_h \in V_h^0$,

$$||\nabla v_h||_K \leq \frac{c_i}{h_K} ||v_h||_K. \tag{6.4}$$

It is important to note that we only consider the convection dominated case ($Pe > 1$), analysis for other case is similar to the analysis for standard parabolic problems (Thomee, 2006).

Then the SUPG approximation of (6.1) can be given by: For all $t \in (0, T]$, find $u_h(t) \in V_h^0$ such that

$$(\partial_t u_h, w_h) + \sum_{K \in \mathfrak{T}_h} (-\epsilon \triangle u_h, w_h)_K + (\beta \nabla u_h, w_h) = F(w_h) \; \forall w_h \in W_h,$$

with $F(w) := (f, w)$ or equivalently;

$$(\partial_t u_h, w_h(v_h)) + a(u_h, v_h) = F(w_h(v_h)) \quad \forall v_h \in V_h^0, \tag{6.5}$$

where

$$a(u, v) := \epsilon(\nabla u, \nabla v) - \sum_{K \in \mathfrak{T}_h} (\epsilon \triangle u, \delta\beta\nabla v)_K + (\beta\nabla u, v + \delta\beta\nabla v) \tag{6.6}$$

and $u_h(0) \in V_h^0$ is a suitable approximation of $u(0)$, to be specified. Since we restrict ourselves to the piecewise affine polynomials, the summation term in (6.6) will be 0. The formulation (6.5) is consistent in the sense that the time derivative is included in the stabilization term i.e.

$$(\partial_t(u_h - u), w_h(v_h)) + a(u_h - u, v_h) = 0 \quad \text{forall} \;\; v_h \in V_h^0.$$

Before considering the full discrete formulation, we state the Young's lemma (Thomee, 2006).

Lemma 6.1 (Young's Inequality) *Let $a, b \in \mathbb{R}$. Then $\forall \kappa \in \mathbb{R}^+$*

$$a.b \leq \frac{\kappa}{2}a^2 + \frac{1}{2\kappa}b^2, \tag{6.7}$$

and

$$(a + b)^2 \leq (1 + \kappa)a^2 + (1 + \frac{1}{\kappa})b^2. \tag{6.8}$$

6.3. Full Discretization by θ Method

In order to obtain full discretization of (6.1) we take a uniform subintervals for the time variable with

$$t^n = n\tau, \quad n = 0, .., N$$

Here τ is the time-step and $N = T/\tau$. Next we replace the time derivative with suitable difference quotients in (6.5) therefore we construct a sequence $u_h^n(x)$ which is the approximation of $u(t^n, x)$. For simplicity we restrict ourselves to the one step scheme θ method (Quarteroni and Valli, 1996). Applying this method to the equation (6.5), we obtain the following problem:
Given u_h^0 as some suitable approximation of $u(0)$, for $n \geq 1$, find $u_h^n \in V_h^0$ s.t.

$$(\partial_\tau u_h^n, w_h(v_h)) + a(u_h^{n-\theta}, v_h) = F^{n-\theta}(w_h(v_h)) \; \forall v_h \in V_h^0. \tag{6.9}$$

Here

$$\partial_\tau u_h^n = \frac{u_h^n - u_h^{n-1}}{\tau}, \qquad u_h^{n-\theta} = \theta u_h^{n-1} + (1-\theta) u_h^n$$

and

$$F^{n-\theta}(w_h) := (f(t^{n-\theta}), w_h) = (f(\theta t^{n-1} + (1-\theta)t^n), w_h).$$

For $\theta = 0$, $\theta = 1/2$, and $\theta = 1$ we have the Backward Euler, Crank-Nicolson and Forward Euler Method, respectively. We will now state a property valid for all cases.

Lemma 6.2 *Let $s(.,.)$ be a symmetric positive semi-definite bilinear form. Then there holds,*

$$2\tau s(u_h^{n-\theta}, \partial_\tau u_h^n) = ||u_h^n||_s^2 - ||u_h^{n-1}||_s^2 + (1-2\theta)||u_h^n - u_h^{n-1}||_s^2, \tag{6.10}$$

and for $\theta \in [0, 1/2]$

$$||u_h^n||_s^2 \leq 2\tau \sum_{m=1}^{n} s(u_h^{m-\theta}, \partial_\tau u_h^m) + ||u_h^0||_s^2 \quad \forall n \geq 1. \tag{6.11}$$

Here notation $||.||_s$ used for semi-norm defined by $s(.,.)$.

Proof (6.10) can be obtained by the following expression

$$\begin{aligned} 2\tau s(u_h^{n-\theta}, \partial_\tau u_h^n) &= s(u_h^n + u_h^{n-1} + (1-2\theta)(u_h^n - u_h^{n-1}), u_h^n - u_h^{n-1}) \\ &= s(u_h^n, u_h^n) - s(u_h^{n-1}, u_h^{n-1}) + (1-2\theta) s(u_h^n - u_h^{n-1}, u_h^n - u_h^{n-1}). \end{aligned}$$

Since $(1-2\theta)$ is nonnegative for $\theta \in [0, 1/2]$, (6.11) is an open result of (6.10). □

Now we introduce the following semi-norm on V_h^0

$$|||v_h^n|||^2 := \delta||\partial_\tau v_h^n + \beta.\nabla v_h^{n-\theta}||^2 + \epsilon||\nabla v_h^{n-\theta}||^2$$

and the norm on V_h^0

$$||v_h^n||_\delta^2 \;:=\; ||v_h||^2 + \delta^2||\beta.\nabla v_h||^2 + \epsilon\delta||\nabla v_h||^2.$$

In the rest of this work,the symbol $\lesssim$ referring an inequality up to a multiplicative constant, that is independent of the discretization parameters h, τ and problem parameters β, ϵ and T, on the other hand it depends on the domain, mesh geometry or θ. Under these definitions some useful inequalities can be given by a lemma.

Lemma 6.3 *For any $v_h \in V_h^0$, following norm equivalence holds*

$$||w_h(v_h)|| \leq \sqrt{2}||v_h||_\delta \lesssim ||v_h|| \leq ||v_h||_\delta. \tag{6.12}$$

and the linear form $F^{n-\theta}(w_h(v_h))$ satisfies the upper bound

$$|F^{n-\theta}(w_h(v_h))| \leq \sqrt{2}||f(t^{n-\theta})||.||v_h||_\delta. \tag{6.13}$$

Proof The first inequality in (6.12) can be obtained from triangular and Young's inequalities, second one follows from (6.2) and inverse inequality (6.4) and we get easily third one by the definition of $||.||_\delta$.

Other part of the proof can be done by the definition of $F^{n-\theta}$, Cauchy-Schwarz inequality and (6.12) such that

$$|F^{n-\theta}(w_h(v_h))| \;=\; |(f(t^{n-\theta}), w_h(v_h))| \leq ||f(t^{n-\theta})||.||w_h(v_h)|| \leq \sqrt{2}||f(t^{n-\theta})||.||v_h||_\delta.$$

□

6.4. Stability Estimates

Before giving the main stability results that holds for the formulation (6.9), we state the discrete Gronwall lemma.

Lemma 6.4 (Discrete Gronwall Lemma) *Let* $\{k, B\} \cup \{a_m, b_m, c_m, d_m\}_{m=0}^{\infty} \subset \mathbb{R}_0^+$ *and* $kd_m < 1$ *for all* $m \geq 0$*, such that*

$$a_n + k\sum_{m=0}^{n} b_m \leq k\sum_{m=0}^{n}(d_m a_m + c_m) + B \quad \text{for} \;\; n \geq 0.$$

Then

$$a_n + k\sum_{m=0}^{n} b_m \leq \exp\left(k\sum_{m=0}^{n}\frac{d_m}{1-kd_m}\right)\left[k\sum_{m=0}^{n} c_m + B\right] \quad \text{for} \;\; n \geq 0.$$

For the proof of this lemma see (Heywood and Rannacher, 1990).

Theorem 6.1 *Let* $\{u_h^n\}_{n=0}^{N}$ *be the solution of the scheme (6.9) with* $\theta \in [0, \frac{1}{2}]$ *then at every time level* $1 \leq n \leq N$ *there holds*

$$\begin{aligned} ||u_h^n||_\delta^2 + \tau\sum_{m=1}^{n}|||u_h^m|||^2 \;\; &\lesssim \;\; t^n\tau\sum_{m=1}^{n}||f(t^{m-\theta})||^2 + t^n\tau\sum_{m=1}^{n}\delta^2||\partial_\tau f(t^{m-\theta})||^2 \\ &\quad +\delta^2 \sup_{t\in(0,t^n]}||f(t)||^2 + ||u_h^0||_\delta^2 \end{aligned} \tag{6.14}$$

or if $f \in C^0(0,T;L^2(\Omega))$ only, then

$$||u_h^n||_\delta^2 + \tau \sum_{m=1}^{n} |||u_h^m|||^2 \lesssim \tau \sum_{m=1}^{n} t^n \left(1 + \frac{\delta}{\tau}\right)^2 ||f(t^{m-\theta})||^2 + ||u_h^0||_\delta^2 \tag{6.15}$$

Proof First take $v_h = u_h^{m-\theta}$ in (6.9) to obtain

$$\begin{aligned}(\partial_\tau u_h^m, u_h^{m-\theta}) &+ (\partial_\tau u_h^m, \delta\beta.\nabla u_h^{m-\theta}) + \epsilon||\nabla u_h^{m-\theta}||^2 + \delta||\beta.\nabla u_h^{m-\theta}||^2 \\ &= F^{m-\theta}(w_h(u_h^{m-\theta})).\end{aligned} \tag{6.16}$$

Now take $v_h = \delta\partial_\tau u_h^m$ in (6.9) to obtain

$$\begin{aligned}\delta||\partial_\tau u_h^m||^2 &+ (\partial_\tau u_h^m, \delta^2\beta.\nabla\partial_\tau u_h^m) + \epsilon(\nabla u_h^{m-\theta}, \delta\nabla\partial_\tau u_h^m) + (\beta.\nabla u_h^{m-\theta}, \delta\partial_\tau u_h^m) \\ &+ (\beta.\nabla u_h^{m-\theta}, \delta^2\beta.\nabla\partial_\tau u_h^m) = F^{m-\theta}(w_h(\delta\partial_\tau u_h^m))\end{aligned} \tag{6.17}$$

Consider the summation

$$\begin{aligned}(\partial_\tau u_h^m, \delta\beta.\nabla u_h^{m-\theta}) &+ \delta||\beta.\nabla u_h^{m-\theta}||^2 + \delta||\partial_\tau u_h^m||^2 + (\beta.\nabla u_h^{m-\theta}, \delta\partial_\tau u_h^m) \\ &= \delta||\partial_\tau u_h^m + \beta.\nabla u_h^{m-\theta}||^2.\end{aligned} \tag{6.18}$$

This trick in (Burman, 2010) shows that the combination of the functions $u_h^{m-\theta}$ and $\delta\partial_\tau u_h^m$ positively contributes in the estimate from the term

$$(\partial_\tau u_h^m, \delta\beta.\nabla u_h^{m-\theta}).$$

Then if we take $v_h = u_h^{m-\theta} + \delta\partial_\tau u_h^m$ in (6.9) with (6.18) and product both side by τ, we obtain

$$\begin{aligned}\tau(\partial_\tau u_h^m, u_h^{m-\theta}) &+ \tau|||u_h^m|||^2 + \tau\epsilon(\nabla u_h^{m-\theta}, \delta\nabla\partial_\tau u_h^m) + \tau(\beta.\nabla u_h^{m-\theta}, \delta^2\beta\nabla\partial_\tau u_h^m) \\ &= \tau F^{m-\theta}(w_h(u_h^{m-\theta} + \delta\partial_\tau u_h^m)).\end{aligned} \tag{6.19}$$

Now sum over $m = 1, ..., n$ and use the inequality (6.11),

$$\begin{aligned}\frac{1}{2}[||u_h^n||^2 - ||u_h^0||^2] &- \frac{\delta^2}{2}[||\beta.\nabla u_h^n||^2 - ||\beta.\nabla u_h^0||^2] + \frac{\epsilon\delta}{2}[||\nabla u_h^n||^2 - ||\nabla u_h^0||^2] \\ &- \sum_{m=1}^{n} \tau |||u_h^m|||^2 \leq \tau \sum_{m=1}^{n} F^{m-\theta}(w_h(u_h^{m-\theta} + \delta\partial_\tau u_h^m)). \quad (6.20)\end{aligned}$$

Using the definition of $||.||_\delta$, we obtain

$$||u_h^n||_\delta^2 - ||u_h^0||_\delta^2 + 2\tau \sum_{m=1}^{n} |||u_h^m|||^2 \leq 2\tau |\sum_{m=1}^{n} F^{m-\theta}(w_h(u_h^{m-\theta} + \delta\partial_\tau u_h^m))|. \quad (6.21)$$

It follows that

$$||u_h^n||_\delta^2 + 2\tau \sum_{m=1}^{n} |||u_h^n|||^2 \lesssim 2\tau |\sum_{m=1}^{n} F^{m-\theta}(w_h(u_h^{m-\theta} + \delta\partial_\tau u_h^m))| + ||u_h^0||_\delta^2. \quad (6.22)$$

Observe that by using the definition of $F^{m-\theta}(.)$ and that of $w_h(.)$ and by a summation by parts we may write

$$\begin{aligned}\tau & \sum_{m=1}^{n} F^{m-\theta}(w_h(u_h^{m-\theta} + \delta\partial_\tau u_h^m)) \\ &= \tau \sum_{m=1}^{n} \Big((f(t^{m-\theta}), u_h^{m-\theta} + \delta\beta.\nabla u_h^{m-\theta}) + (\delta f(t^{m-\theta}), \partial_\tau u_h^m + \delta\beta.\nabla\partial_\tau u_h^m) \Big) \\ &= \tau \sum_{m=1}^{n} \underbrace{(f(t^{m-\theta}), u_h^{m-\theta} + \delta\beta.\nabla u_h^{m-\theta})}_{I} + \underbrace{(\delta f(t^{n-\theta}), u_h^n + \delta\beta\nabla u_h^n)}_{II} \\ &- \underbrace{(\delta f(t^{1-\theta}), u_h^0 + \delta\beta\nabla u_h^0)}_{III} - \tau \sum_{m=2}^{n} \underbrace{(\delta\partial_\tau f(t^{m-\theta}), u_h^{m-1} + \delta\beta\nabla u_h^{m-1})}_{IV}. \quad (6.23)\end{aligned}$$

Using the Cauchy-Schwarz and Young's inequality with $\kappa_I = st^n/2, \kappa_{II} = 1/4, \kappa_{III} = 1/2$ and $\kappa_{IV} = st^n/2$ for the I, II, III and IV terms in (6.23) respectively where $s > 0$ will be specified later.

$$
\begin{aligned}
\tau \quad & \sum_{m=1}^{n} |F^{m-\theta}(w_h(u_h^{m-\theta} + \delta\partial_\tau u_h^m))| \\
\leq \quad & \tau \sum_{m=1}^{n} \left(\frac{st^n}{4} ||f(t^{m-\theta})||^2 + \frac{1}{st^n} ||u_h^{m-\theta} + \delta\beta\nabla u_h^{m-\theta}||^2 \right) \\
+ \quad & \frac{1}{8} ||u_h^n + \delta\beta\nabla u_h^n||^2 + 2\delta^2 ||f(t^{n-\theta})||^2 + \frac{1}{4} ||u_h^0 + \delta\beta\nabla u_h^0||^2 + \delta^2 ||f(t^{1-\theta})||^2 \\
+ \quad & \tau \sum_{m=2}^{n} \left(\frac{st^n}{4} \delta^2 ||\partial_\tau f(t^{m-\theta})||^2 + \frac{1}{st^n} ||u_h^{m-1} + \delta\beta\nabla u_h^{m-1}||^2 \right). \qquad (6.24)
\end{aligned}
$$

Now using (6.24) and (6.12) in (6.22), we obtain

$$
\begin{aligned}
||u_h^n||_\delta^2 \quad - \quad & ||u_h^0||_\delta^2 + 2\tau \sum_{m=1}^{n} |||u_h^m|||^2 \\
\leq \quad & \tau \sum_{m=1}^{n} \left(\frac{st^n}{2} ||f(t^{m-\theta})||^2 + \frac{4}{st^n} (||u_h^{m-\theta}||_\delta^2 + ||u_h^{m-1}||_\delta^2) \right) \\
+ \quad & \frac{1}{2} ||u_h^n||_\delta^2 + 4\delta^2 ||f(t^{n-\theta})||^2 + ||u_h^0||_\delta^2 + 2\delta^2 ||f(t^{1-\theta})||^2 \\
+ \quad & \frac{\tau t^n s}{2} \sum_{m=2}^{n} \delta^2 ||\partial_\tau f(t^{m-\theta})||^2. \qquad (6.25)
\end{aligned}
$$

Using triangular inequality for δ-norm, it can be written that

$$
\begin{aligned}
||u_h^{m-\theta}||_\delta^2 \quad \leq \quad & (1-\theta)^2 ||u_h^m||_\delta^2 + \theta^2 ||u_h^{m-1}||_\delta^2 + 2\theta(1-\theta) ||u_h^m||_\delta ||u_h^{m-1}||_\delta . \\
\leq \quad & ((1-\theta)^2 + \frac{1}{4}) ||u_h^m||_\delta^2 + (\theta^2 + \frac{1}{4} ||u_h^{m-1}||_\delta^2)
\end{aligned}
$$

Using this inequality in (6.25),

$$
\begin{aligned}
\frac{1}{2} ||u_h^n||_\delta^2 \quad + \quad & 2\tau \sum_{m=1}^{n} |||u_h^m|||^2 \leq \frac{\tau t^n s}{2} \sum_{m=1}^{n} ||f(t^{m-\theta})||^2 + \frac{\tau t^n s}{2} \sum_{m=2}^{n} \delta^2 ||\partial_\tau f(t^{m-\theta})||^2 \\
+ \quad & 6\delta^2 \sup_{t\in[0,t^n]} ||f(t)||^2 + \frac{4\tau}{t^n s} \sum_{m=1}^{n} \left[\left((1-\theta^2) + \frac{1}{4} \right) ||u_h^m||_\delta^2 \right. \\
+ \quad & \left. \left(\theta^2 + \frac{5}{4} \right) ||u_h^{m-1}||_\delta^2 \right] + 2||u_h^0||_\delta^2 . \qquad (6.26)
\end{aligned}
$$

Since $\theta \in [0, 1/2]$, (6.26) implies

$$\begin{aligned} ||u_h^n||_\delta^2 &+ 4\tau \sum_{m=1}^n |||u_h^m|||^2 \le \tau t^n s \sum_{m=1}^n ||f(t^{m-\theta})||^2 + \tau t^n s \sum_{m=1}^n \delta^2 ||\partial_\tau f(t^{m-\theta})||^2 \\ &+ 12\delta^2 \sup_{t\in[0,t^n]} ||f(t)||^2 + \frac{16\tau}{t^n s} \sum_{m=0}^n ||u_h^m||_\delta^2 + 4||u_h^0||_\delta^2. \end{aligned} \quad (6.27)$$

We can apply the lemma 6.4 for $s > 16$ to (6.27)

$$\begin{aligned} ||u_h^n||_\delta^2 &+ 4\tau \sum_{m=1}^n |||u_h^n|||^2 \le \exp\left(\tau \sum_{m=1}^n \frac{1/st^n}{1 - 16\tau/st^n}\right) \left[\tau \sum_{m=1}^n st^n ||f(t^{m-\theta})||^2 \right. \\ &+ \left. st^n \tau \sum_{m=2}^n \delta^2 ||\partial_\tau f(t^{m-\theta})||^2 + 12\delta^2 \sup_{t\in[0,t^n]} ||f(t)||^2 + 4||u_h^0||_\delta^2 \right]. \end{aligned}$$

By the definition of the symbol $\lesssim$, we obtain the desired result (6.14).

If the source function f belongs to the space $C^0(0, T; L^2(\Omega))$, the summation by parts in (6.14) may not be bounded uniformly. Hence for this case the estimate should be reanalyzed starting from the inequality (6.21).

$$||u_h^n||_\delta^2 + 2\tau \sum_{m=1}^n |||u_h^n|||^2 \le 2\tau \left| \sum_{m=1}^n F^{m-\theta}(w_h(u_h^{m-\theta} + \delta\partial_\tau u_h^m)) \right| + ||u_h^0||_\delta^2.$$

Applying the inequality for $F^{n-\theta}$ in lemma 6.3, we have

$$||u_h^n||_\delta^2 + 2\tau \sum_{m=1}^n |||u_h^m|||^2 \le 2\sqrt{2}\tau \sum_{m=1}^n ||f(t^{m-\theta})||.||u_h^{m-\theta} + \delta\partial_\tau u_h^m||_\delta + ||u_h^0||_\delta^2. \quad (6.28)$$

Using the norm equivalence (6.12), there holds

$$||w_h(u_h^{m-\theta} + \delta\partial_\tau u_h^m)|| \le \sqrt{2}||u_h^{m-\theta} + \delta\partial_\tau u_h^m||_\delta \le \sqrt{2}\left(1 + \frac{\delta}{\tau}\right)(||u_h^{m-1}||_\delta + ||u_h^m||_\delta).$$

Then (6.28) implies that

$$||u_h^n||_\delta^2 + \tau \sum_{m=1}^{n} |||u_h^n|||^2 \leq 2\sqrt{2}\tau \sum_{m=1}^{n} \left(1 + \frac{\delta}{\tau}\right) ||f(t^{m-\theta})||(||u_h^{m-1}||_\delta + ||u_h^m||_\delta) + ||u_h^0||_\delta^2.$$

It follows from the Young's inequality with $\kappa = 2\sqrt{2}t^n$,

$$\begin{aligned} ||u_h^n||_\delta^2 + \tau \sum_{m=1}^{n} |||u_h^n|||^2 &\leq \tau \sum_{m=1}^{n} \left[\frac{1}{2t^n} (||u_h^{m-1}||_\delta + ||u_h^m||_\delta)^2 \right. \\ &+ \left. 4t^n \left(1 + \frac{\delta}{\tau}\right)^2 ||f(t^{m-\theta})||^2 \right] + ||u_h^0||_\delta^2. \end{aligned}$$

Finally applying the lemma 6.4 and the definition of the symbol $\lesssim$, we get the desired result (6.15) such that

$$\begin{aligned} ||u_h^n||_\delta^2 + \tau \sum_{m=1}^{n} |||u_h^m|||^2 &\leq \exp\left(\tau \sum_{m=1}^{n} \frac{1/(2t^n)}{1 - \tau/(2t^n)} \right) \\ &\quad \left[\tau \sum_{m=1}^{n} 4t^n \left(1 + \frac{\delta}{\tau}\right)^2 ||f(t^{m-\theta})||^2 + ||u_h^0||_\delta^2 \right] \\ &\lesssim \tau \sum_{m=1}^{n} t^n \left(1 + \frac{\delta}{\tau}\right)^2 ||f(t^{m-\theta})||^2 + ||u_h^0||_\delta^2. \end{aligned}$$

□

In the case $f \in C^0(0, T; L^2(\Omega))$, stability is obtained if $\delta \leq \tau$ that is strong requirement. When the Backward Euler method reanalyzed, this condition can be stretched to $\delta^2 \leq \tau$.

Corollary 6.1 *Let $\{u_h^n\}_{n=0}^N$ be the solution of the scheme (6.9) discretized using backward Euler method, assume that $\delta^2 \leq \tau$, then at every time level $1 \leq n \leq N$ there holds*

$$||u_h^n||_\delta^2 + \tau \sum_{m=1}^{n} |||u_h^m|||^2 \lesssim \tau(1 + t^n) \sum_{m=1}^{n} ||f(t^m)||^2 + ||u_h^0||_\delta^2 \qquad (6.29)$$

Proof For the backward Euler method, relation (6.19) is expressed using the formula:

$$\begin{aligned} \tau F^m(w_h(u_h^m + \delta\partial_\tau u_h^m)) &= (u_h^m - u_h^{m-1}, u_h^m) + \epsilon\delta(\nabla u_h^m, \nabla(u_h^m - u_h^{m-1})) \\ &+ \tau|||u_h^m|||^2 + \delta^2(\beta.\nabla u_h^m, \beta\nabla(u_h^m - u_h^{m-1})). \end{aligned}$$

Using (6.10) with $\theta = 0$ we obtain

$$\begin{aligned} \tau F^m(w_h(u_h^m + \delta\partial_\tau u_h^m)) &= \frac{1}{2}(||u_h^m||^2 - ||u_h^{m-1}||^2 + ||u_h^m - u_h^{m-1}||^2) + \tau|||u_h^m|||^2 \\ &+ \frac{\epsilon\delta}{2}(||\nabla u_h^m||^2 - ||\nabla u_h^{m-1}||^2 + ||\nabla(u_h^m - u_h^{m-1})||^2) \\ &+ \frac{\delta^2}{2}(||\beta\nabla u_h^m||^2 - ||\beta\nabla u_h^{m-1}||^2 + ||\beta\nabla(u_h^m - u_h^{m-1})||^2). \end{aligned}$$

By the definition of $||.||_\delta$, it follows that

$$\begin{aligned} &\tau F^m(w_h(u_h^m + \delta\partial_\tau u_h^m)) \\ &\quad = \tau|||u_h^m|||^2 + \frac{1}{2}(||u_h^m||_\delta^2 - ||u_h^{m-1}||_\delta^2 + ||u_h^m - u_h^{m-1}||_\delta^2). \end{aligned} \tag{6.30}$$

Let us now give a bound for the term $F^m(w_h(u_h^m + \delta\partial_\tau u_h^m))$. Using the lemma 6.3 with triangular inequality, we have

$$F^m(w_h(u_h^m + \delta\partial_\tau u_h^m)) \leq \sqrt{2}\Big(\underbrace{||f(t^m)||\;||u_h^m||_\delta}_{I} + \underbrace{||f(t^m)||\frac{\delta}{\tau}||u_h^m - u_h^{m-1}||_\delta}_{II}\Big) \tag{6.31}$$

By the Young's inequality with $\kappa_I = 2\sqrt{2}t^n$ and $\kappa_{II} = \sqrt{2}$ for the I and II terms in (6.31) respectively, we may write

$$\begin{aligned} F^m(w_h(u_h^m + \delta\partial_\tau u_h^m)) &\leq (1 + 2t^n)||f(t^m)||^2 + \frac{1}{4t^n}||u_h^m||_\delta^2 \\ &+ \frac{\delta^2}{2\tau^2}||u_h^m - u_h^{m-1}||_\delta^2. \end{aligned} \tag{6.32}$$

The term $||u_h^m - u_h^{m-1}||_\delta^2$ can be absorbed by the same term in (6.30) provided that $\tau \geq \delta^2$. Hence from (6.30) and (6.32) we obtain that

$$2\tau|||u_h^m|||^2 + ||u_h^m||_\delta^2 - ||u_h^{m-1}||_\delta^2 \leq 2\tau(1+2t^n)||f(t^m)||^2 + \frac{\tau}{2t^n}||u_h^m||_\delta^2. \quad (6.33)$$

Now sum over $m = 1, ..., n$, we have

$$||u_h^n||_\delta^2 + \tau\sum_{m=1}^{n}|||u_h^m|||^2 \leq \tau\sum_{m=1}^{n}\left[\frac{1}{2t^n}||u_h^m||_\delta^2 + 2(1+2t^n)||f(t^m)||^2\right] + ||u_h^0||_\delta^2.$$

Applying the lemma 6.4, there holds

$$||u_h^n||_\delta^2 + \tau\sum_{m=1}^{n}|||u_h^m|||^2 \leq \exp\left(\frac{n}{2n-1}\right)\left[\tau\sum_{m=1}^{n}4(1+t^n)||f(t^m)||^2 + ||u_h^0||_\delta^2\right]. \quad (6.34)$$

By the definition of the symbol $\lesssim$, we obtain the desired result (6.29). $\square$

In order to escape growing the right hand side of the estimate (6.14), the factor t^n can be eliminated when $||.||_\delta$ norm considered. For the Crank-Nicolson method, we now state this analysis in the following corollary.

Corollary 6.2 *Let $\{u_h^n\}_{n=0}^N$ be the solution of the scheme (6.9) discretized using Crank-Nicolson method then at every time level $1 \leq n \leq N$ there holds*

$$||u_h^n||_\delta \lesssim \tau\left(1 + \frac{\delta}{\tau}\right)\sum_{m=1}^{n}||f(t^{m-1/2})|| + ||u_h^0||_\delta. \quad (6.35)$$

Proof For the Crank-Nicolson method, relation (6.19) is expressed using the formula:

$$\begin{aligned}
2\tau F^{m-1/2}(w_h(u_h^{m-1/2} &+ \delta\partial_\tau u_h^m)) \\
&= (u_h^m - u_h^{m-1}, u_h^m + u_h^{m-1}) \\
&+ \epsilon\delta(\nabla(u_h^m + u_h^{m-1}), \nabla(u_h^m - u_h^{m-1})) \\
&+ 2\tau|||u_h^m|||^2 + \delta^2(\beta.\nabla(u_h^m - u_h^{m-1}), \beta\nabla(u_h^m - u_h^{m-1})).
\end{aligned}$$

Using (6.10) with $\theta = 1/2$, we obtain

$$\begin{aligned} 4\tau F^{m-1/2}(w_h(u_h^{m-1/2} &+ \delta\partial_\tau u_h^m)) \\ &= (||u_h^m||^2 - ||u_h^{m-1}||^2) + \epsilon\delta(||\nabla u_h^m||^2 - ||\nabla u_h^{m-1}||^2) \\ &+ \delta^2(||\beta\nabla u_h^m||^2 - ||\beta\nabla u_h^{m-1}||^2) + 4\tau|||u_h^m|||^2. \end{aligned}$$

By the definition of $||.||_\delta$, it follows that

$$4\tau F^{m-1/2}(w_h(u_h^{m-1/2} + \delta\partial_\tau u_h^m)) = 4\tau|||u_h^m|||^2 \quad + \quad (||u_h^m||_\delta^2 - ||u_h^{m-1}||_\delta^2). \tag{6.36}$$

Let us now give a bound for the term $F^{m-1/2}(w_h(u_h^{m-1/2} + \delta\partial_\tau u_h^m))$. Using the lemma 6.3 with triangular inequality, we have

$$\begin{aligned} F^{m-1/2}(w_h(u_h^{m-1/2} &+ \delta\partial_\tau u_h^m)) \\ &\leq \sqrt{2}||f(t^{m-1/2})||\left(\frac{1}{2}||u_h^m + u_h^{m-1}||_\delta + \frac{\delta}{\tau}||u_h^m - u_h^{m-1}||_\delta\right) \\ &\leq ||f(t^{m-1/2})||\left(||u_h^m||_\delta + ||u_h^{m-1}||_\delta + \frac{\delta}{\tau}||u_h^m||_\delta + \frac{2\delta}{\tau}||u_h^{m-1}||_\delta\right) \\ &\leq ||f(t^{m-1/2})||\left(1 + \frac{2\delta}{\tau}\right)(||u_h^m||_\delta + ||u_h^{m-1}||_\delta). \end{aligned} \tag{6.37}$$

Now combine with (6.36) and (6.37), we obtain

$$4\tau|||u_h^m|||^2 + (||u_h^m||_\delta^2 - ||u_h^{m-1}||_\delta^2) \leq 4\tau||f(t^{m-1/2})||\left(1 + \frac{2\delta}{\tau}\right)(||u_h^m||_\delta + ||u_h^{m-1}||_\delta).$$

It can be written that

$$\begin{aligned} (||u_h^m||_\delta - ||u_h^{m-1}||_\delta)(||u_h^m||_\delta &+ ||u_h^{m-1}||_\delta) \\ &\leq 8\tau||f(t^{m-1/2})||\left(1 + \frac{\delta}{\tau}\right)(||u_h^m||_\delta + ||u_h^{m-1}||_\delta). \end{aligned}$$

Using the definition of the symbol $\lesssim$ and cancel the term $||u_h^m||_\delta + ||u_h^{m-1}||_\delta$ from both side,

$$||u_h^m||_\delta - ||u_h^{m-1}||_\delta \lesssim \tau\left(1 + \frac{\delta}{\tau}\right)||f(t^{m-1/2})||.$$

Finally taking sum over $m = 1, ..., n$ gives the desired result. □

6.5. Convergence Analysis

In this part convergence analysis for the SUPG/$\theta = 0$ and SUPG/$\theta = 1/2$ are investigated. In order to apply stability estimates obtained in previous section, we have assumed that exact solution u is sufficiently smooth. We also use the notation $U_m^{i,j}$ as introduced in (Burman, 2010), that simplifies the estimates, for different norms of the exact solution. Here the index $m \in \{1, 2, \infty\}$ refers to the norm in time and the indices i and j refer to numbers of derivatives applied to u in time and space, respectively. Moreover

$$U_1^{i,j} := \int_0^{t^n} |\partial_t^i u|_j dt \ , \ U_2^{i,j} := \int_0^{t^n} |\partial_t^i u|_j^2 dt \ , \ U_\infty^{i,j} := \sup_{t\in[0,t^n]} |\partial_t^i u|_j^2.$$

The following a priori error estimates for Ritz-projection in (Burman and Smith, 2011) is widely used in our analysis:

Lemma 6.5 *For $t \in [0, T]$ and $Pe > 1$, let $u(t) \in H^{k+1}(\Omega)$ be the solution of (6.3) and $r_h u(t) \in V_h^0$ be the solution of*

$$a(r_h u(t), v_h) = a(u(t), v_h) \ \ \forall v_h \in V_h^0.$$

Then

$$||(r_h u - u)(t)||_*^2 \lesssim h^{2k+1}||\beta||_\infty |u(t)|_{k+1}^2 \tag{6.38}$$

and for $u \in H^m(0,T;H^{k+1}(\Omega))$

$$||\partial_t^m(r_h u - u)(t)||_*^2 \lesssim h^{2k+1}||\beta||_\infty |\partial_t^m u(t)|_{k+1}^2 \qquad (6.39)$$

where k *is the degree of the polynomial functions in finite element space* V^h *and the *-norm is defined by*

$$||v||_*^2 = ||v||^2 + \delta||\beta\nabla v||^2 + \epsilon||\nabla v||^2.$$

Note that since we restrict the finite element space to the continuous piecewise linear polynomial functions space, we take $k = 1$ in our analysis.

Let us take $\theta = 0$ in (6.9) then we obtain backward Euler time stepping formulation such that

For $1 \le n \le N$ find $u_h^n \in V_h^0$ such that

$$(\partial_\tau u_h^n, w_h(v_h)) + a(u_h^n, v_h) = F^n(w_h(v_h)) \;\; \forall v_h \in V_h^0 \qquad (6.40)$$

with $u_h^0 = r_h u_0$.
Now we can give our main result for this method.

Theorem 6.2 *Let* $\{u_h^n\}_{n=0}^N$ *be the solution of (6.40) and* u *be the solution of (6.1). Then for* $1 \le n \le N$ *there holds*

$$\begin{aligned} ||u_h^n - u(t^n)||_\delta^2 &+ \tau\sum_{m=1}^n |||u_h^m - u(t^m)|||^2 \\ &\lesssim h^3||\beta||_\infty\Big[(t^n+\delta)U_2^{1,2} + t^n\delta^2 U_2^{2,2} + \delta^2 U_\infty^{1,2} + \tau^2\delta^2 U_\infty^{2,2} \\ &+ (1+||\beta||_\infty t^n)U_\infty^{0,2}\Big] + \tau^2\Big[t^n U_2^{2,0} + t^n\delta^2 U_2^{3,0} + \delta^2 U_\infty^{2,0}\Big]. \end{aligned} \qquad (6.41)$$

Proof First we decompose the error in a discrete error and a projection error.

$$u_h^n - u(t^n) = u_h^n - r_h u(t^n) + r_h u(t^n) - u(t^n) = \phi^n + \eta^n$$

where $\phi^n = u_h^n - r_h u(t^n)$ and $\eta^n = r_h u(t^n) - u(t^n)$. By the definition of Ritz-projection and (6.40), it follows that for $n \geq 1$

$$\begin{aligned}(\partial_\tau \phi^n, w_h(v_h)) + a(\phi^n, v_h) &= (\partial_\tau \phi^n, w_h(v_h)) + a(u_h^n - u(t^n), v_h) \\ &= (\partial_\tau \phi^n, w_h(v_h)) - (\partial_\tau u_h^n - \partial_t u(t^n), w_h(v_h)) \\ &= -(\lambda^n, w_h(v_h)) \;\; \forall v_h \in V_h^0. \qquad (6.42)\end{aligned}$$

where

$$\lambda^n = \underbrace{(r_h - I)\partial_\tau u(t^n)}_{\lambda_1^n} + \underbrace{\partial_\tau u(t^n) - \partial_t u(t^n)}_{\lambda_2^n}.$$

As a consequence of the stability estimate (6.27) there holds

$$\begin{aligned}||\phi^n||_\delta^2 + \tau \sum_{m=1}^n |||\phi^m|||^2 &\lesssim t^n \tau \sum_{m=1}^n ||\lambda^m||^2 + t^n \tau \sum_{m=2}^n \delta^2 ||\partial_\tau \lambda^m||^2 \\ &+ \delta^2(||\lambda^1||^2 + ||\lambda^n||^2) + \frac{\tau}{t^n} \sum_{m=1}^n ||\phi^m||_\delta^2 + ||\phi^0||_\delta. \qquad (6.43)\end{aligned}$$

Since $u_h^0 = r_h u(t^0)$, $\phi^0 = 0$. Applying lemma 6.4, we get

$$\begin{aligned}||\phi^n||_\delta^2 + \tau \sum_{m=1}^n |||\phi^m|||^2 &\lesssim \tau t^n \sum_{m=1}^n ||\lambda^m||^2 + \tau t^n \sum_{m=2}^n \delta^2 ||\partial_\tau \lambda^m||^2 \\ &+ \delta^2(||\lambda^1||^2 + ||\lambda^n||^2). \qquad (6.44)\end{aligned}$$

By the standard Taylor development, there exists $\xi^n \in [t^{n-1}, t^n]$ such that

$$\partial_\tau u(t^n) = \partial_t u(t^n) - \frac{\tau}{2}\partial_t^2 u(\xi^n).$$

Now we obtain the upper bound for the last two terms in (6.44).

$$\begin{aligned}||\lambda^n||^2 &\leq ||(r_h - I)\partial_\tau u(t^n) + \partial_\tau u(t^n) - \partial_t u(t^n)||^2 \\ &\lesssim ||(r_h - I)(u_t(t^n) - \frac{\tau}{2}u_{tt}(\xi^n))||^2 + \tau^2 \sup_{0 \leq t \leq t^n} ||u_{tt}(t)||^2 \qquad (6.45)\end{aligned}$$

and by the lemma 6.5 it follows

$$||\lambda^n||^2 \lesssim h^3||\beta||_\infty |u_t(t^n)|_2^2 + \tau^2 h^3 ||\beta||_\infty |u_{tt}(\xi^n)|_2^2 + \tau^2 \sup_{0\le t\le t^n} ||u_{tt}(t)||^2.$$

Then

$$\begin{aligned} ||\lambda^1||^2 + ||\lambda^n||^2 &\lesssim h^3||\beta||_\infty \left[\sup_{0\le t\le t^n} |u_t(t)|_2^2 + \tau^2 \sup_{0\le t\le t^n} |u_{tt}(t)|_2^2 \right] \\ &+ \tau^2 \sup_{0\le t\le t^n} ||u_{tt}(t)||^2. \end{aligned} \tag{6.46}$$

Since β and ϵ are constant in time r_h commutes with time derivative, we therefore have

$$|\lambda_1^m| = \left| \frac{1}{\tau} \int_{t^{m-1}}^{t^m} (r_h - I) u_t(t) dt \right| \le \frac{1}{\sqrt{\tau}} \left(\int_{t^{m-1}}^{t^m} |(r_h - I) u_t(t)|^2 dt \right)^{1/2}. \tag{6.47}$$

Then

$$\begin{aligned} \tau \sum_{m=1}^n ||\lambda_1^m||^2 &= \tau \sum_{m=1}^n \int_\Omega |\lambda_1^m|^2 dx \\ &\le \sum_{m=1}^n \int_\Omega \int_{t^{m-1}}^{t^m} |(r_h - I)u_t(t)|^2 dt dx = \int_0^{t^n} ||\partial_t \eta||^2 dt \\ &\lesssim h^3 ||\beta||_\infty \int_0^{t^n} |\partial_t u(t)|_2^2 dt = h^3 ||\beta||_\infty U_2^{1,2}. \end{aligned} \tag{6.48}$$

Now consider

$$\begin{aligned} |\partial_\tau \lambda_1^m| &= \left| (r_h - I)\partial_\tau \left[\frac{u(t^m) - u(t^{m-1})}{\tau} \right] \right| = |(r_h - I)\partial_\tau u_t(\xi^m)| \\ &\le \frac{1}{\sqrt{\tau}} \left(\int_{\xi^m - \tau}^{\xi^m} |(r_h - I)u_{tt}(t)|^2 dt \right)^{1/2} \quad \text{for some } \ \xi^m \in [t^{m-1}, t^m] \\ &\le \frac{1}{\sqrt{\tau}} \left(\int_{t^{m-2}}^{t^m} |(r_h - I)u_{tt}(t)|^2 dt \right)^{1/2}. \end{aligned} \tag{6.49}$$

Then

$$\begin{aligned}\tau\sum_{m=2}^{n}||\partial_\tau\lambda_1^m||^2 &\leq \sum_{m=2}^{n}\int_\Omega\int_{t^{m-2}}^{t^m}|(r_h-I)u_{tt}(t)|^2dtdx \\ &\lesssim \int_0^{t^n}||\partial_t^2\eta(t)||^2dt \lesssim h^3||\beta||_\infty\int_0^{t^n}|u_{tt}(t)|_2^2dt \\ &\lesssim h^3||\beta||_\infty U_2^{2,2}. \end{aligned} \tag{6.50}$$

The term related to λ_2^m can be similarly estimated as λ_1^m such that

$$\lambda_2^m = -\frac{1}{\tau}\int_{t^{m-1}}^{t^m}(t-t^{m-1})u_{tt}(t)dt \leq \sqrt{\tau}\left(\int_{t^{m-1}}^{t^m}|u_{tt}(t)|^2dt\right)^{1/2}. \tag{6.51}$$

Then

$$\begin{aligned}\tau\sum_{m=1}^{n}||\lambda_2^m||^2 &= \tau\sum_{m=1}^{n}\int_\Omega\tau\int_{t^{m-1}}^{t^m}|u_{tt}(t)|^2dtdx = \tau^2\int_0^{t^n}\int_\Omega|u_{tt}(t)|^2dxdt \\ &= \tau^2\int_0^{t^n}||u_{tt}(t)||^2dt = \tau^2U_2^{2,0}. \end{aligned} \tag{6.52}$$

Similarly we can bound the term $\tau\sum_{m=2}^{n}||\partial_\tau\lambda_2^m||$ such that

$$\tau\sum_{m=2}^{n}||\partial_\tau\lambda_2^m||^2 \lesssim \tau^2\int_0^{t^n}||u_{ttt}(t)||^2dt = \tau^2U_2^{3,0}. \tag{6.53}$$

Hence we obtain the following estimate from (6.44)

$$\begin{aligned}||\phi^n||_\delta^2+\tau\sum_{m=1}^{n}|||\phi^m|||^2 &\lesssim h^3||\beta||_\infty\Big[t^nU_2^{1,2}+t^n\delta^2U_2^{2,2}+\delta^2U_\infty^{1,2} \\ &+ \tau^2\delta^2U_\infty^{2,2}\Big]+\tau^2\Big[t^nU_2^{2,0}+t^n\delta^2U_2^{3,0}+\delta^2U_\infty^{2,0}\Big]. \end{aligned} \tag{6.54}$$

We now estimate the approximation error of Rietz-projection in the following terms

$$||\eta^n||_\delta \;\text{ and }\; \tau\sum_{m=1}^{n}|||\eta^m|||$$

Using the definition of the norm $||.||_\delta$, inverse inequality and lemma 6.5 we obtain

$$\begin{aligned}||\eta^n||_\delta^2 &= ||(r_h-I)u(t^n)||^2+\delta^2||\beta.\nabla(r_h-I)u(t^n)||^2+\epsilon\delta||\nabla(r_h-I)u(t^n)||^2\\ &\lesssim \left(1+\frac{\delta^2||\beta||_\infty^2+\epsilon\delta}{h^2}\right)||(r_h-I)u(t^n)||^2.\end{aligned}$$

Since $\delta\overline{\beta}=\mu h$ and $\epsilon<\overline{\beta}h$, then $\delta||\beta||_\infty\lesssim h$ and $\epsilon\delta\lesssim h^2$. Therefore we may write

$$||\eta^n||_\delta^2\lesssim h^3||\beta||_\infty U_\infty^{0,2} \tag{6.55}$$

Let us consider

$$\begin{aligned}\tau\sum_{m=1}^{n}|||\eta^m|||^2 &= \tau\sum_{m=1}^{n}\delta||\partial_\tau\eta^m+\beta\nabla\eta^m||^2+\epsilon||\nabla\eta^m||^2\\ &\lesssim \tau\sum_{m=1}^{n}\delta||\partial_\tau(r_h-I)u(t^m)||^2+(\delta||\beta||_\infty^2+\epsilon)||\nabla(r_h-I)u(t^m)||^2\\ &\lesssim h^3||\beta||_\infty\delta\int_0^{t^n}|u_t|_2^2dt+(\delta||\beta||_\infty^2+\epsilon)\sum_{m=1}^{n}\frac{\tau}{h^2}||(r_h-I)u(t^m)||^2\\ &\lesssim h^3||\beta||_\infty(\delta U_2^{1,2}+t^n||\beta||_\infty\sup_{0<t\le t^n}|u(t)|_2^2)\\ &\lesssim h^3||\beta||_\infty(\delta U_2^{1,2}+t^n||\beta||_\infty U_\infty^{0,2}).\end{aligned} \tag{6.56}$$

Then we obtain the following estimate

$$||\eta^n||_\delta^2+\tau\sum_{m=1}^{n}|||\eta^m|||^2\lesssim h^3||\beta||_\infty\left[(1+||\beta||_\infty t^n)U_\infty^{0,2}+\delta U_2^{1,2}\right]. \tag{6.57}$$

Combination of (6.54) and (6.57) gives the desired result. □

When the solution u is not sufficiently smooth, the right hand side of (6.41) may fail to be bounded. In this case we have still optimal convergence rates in space and time of SUPG/$\theta = 0$-method by using the stability estimate (6.1).

Corollary 6.3 *Let $\{u_h^n\}_{n=0}^N$ be the solution of (6.40), with $\delta^2 \leq \tau$, and u be the solution of (6.1). Then for n such that $1 \leq n \leq N$ there holds*

$$\begin{aligned} ||u_h^n - u(t^n)||_\delta^2 &+ \tau \sum_{m=1}^{n} |||u_h^m - u(t^m)|||^2 \\ &\lesssim h^3 ||\beta||_\infty \Big[(1 + t^n + \delta) U_2^{1,2} + (1 + ||\beta||_\infty t^n) U_\infty^{0,2} \Big] \\ &+ \tau^2 (1 + t^n) U_2^{2,0}. \end{aligned} \tag{6.58}$$

Proof As a consequence of the stability analysis in the corollary (6.1) for the term ϕ^n we obtain

$$\begin{aligned} ||\phi^n||_\delta^2 + \tau \sum_{m=1}^{n} |||\phi^m|||^2 &\lesssim \tau(1 + t^n) \sum_{m=1}^{n} ||\lambda^m||^2 \\ &\lesssim (1 + t^n)[h^3 ||\beta||_\infty U_2^{1,2} + \tau^2 U_2^{2,0}]. \end{aligned} \tag{6.59}$$

Using the estimate (6.57) and (6.59) proof can be easily completed. □

Let us take $\theta = 1/2$ in (6.9) then we obtain Crank-Nicolson time stepping formulation such that

For $1 \leq n \leq N$ find $u_h^n \in V_h^0$ such that

$$(\partial_\tau u_h^n, w_h(v_h)) + \frac{1}{2} a(u_h^n + u_h^{n-1}, v_h) = F^{n-1/2}(w_h(v_h)) \quad \forall v_h \in V_h^0 \tag{6.60}$$

with $u_h^0 = r_h u_0$. Now we can give our main result for this method.

Theorem 6.3 *Let $\{u_h^n\}_{n=0}^N$ be the solution of (6.60) and u be the solution of (6.1). Then* *for* $1 \leq n \leq N$ *there holds*

$$
\begin{aligned}
||u_h^n - u(t^n)||_\delta^2 &+ \tau \sum_{m=1}^{n} |||u_h^m - u(t^m)|||^2 \\
&\lesssim h^3 \Big[(t^n ||\beta||_\infty + \delta) U_2^{1,2} + t^n \delta^2 ||\beta||_\infty U_2^{2,2} + \delta^2 (U_\infty^{1,2} + \tau^4 U_\infty^{3,2}) \\
&+ (1 + ||\beta||_\infty) U_\infty^{0,2} \Big] + \tau^4 \Big[t^n U_2^{3,0} + t^n ||\beta||_\infty^2 U_2^{2,1} + t^n \epsilon^2 U_2^{2,2} \\
&+ t^n \delta^2 U_2^{4,0} + t^n \delta^2 ||\beta||_\infty^2 U_2^{3,1} + t^n \delta^2 \epsilon^2 U_2^{3,2} \\
&+ \delta^2 U_\infty^{3,0} + \delta^2 ||\beta||_\infty^2 U_\infty^{2,1} + \delta^2 \epsilon^2 U_\infty^{2,2} \Big]. \qquad (6.61)
\end{aligned}
$$

Proof First we decompose the error in a discrete error and a projection error.

$$
u_h^n - u(t^n) = u_h^n - r_h u(t^n) + r_h u(t^n) - u(t^n) = \phi^n + \eta^n
$$

where $\phi^n = u_h^n - r_h u(t^n)$ and $\eta^n = r_h u(t^n) - u(t^n)$. By the definition of Ritz-projection and (6.60), it follows that for $1 \leq n \leq N$:

$$
\begin{aligned}
(\partial_\tau \phi^n, w_h(v_h)) &+ \frac{1}{2} a(\phi^n + \phi^{n-1}, v_h) \\
&= (\partial_\tau (u_h^n - r_h u(t^n)), w_h(v_h)) + \frac{1}{2} a(u_h^n - u(t^n) + u_h^{n-1} - u(t^{n-1}), v_h) \\
&= -\Big[(r_h \partial_\tau u(t^n), w_h(v_h)) - (\partial_\tau u_h^n, w_h(v_h)) - (\partial_t u(t^{n-1/2}), w_h(v_h)) \\
&- a(u(t^{n-1/2}), v_h) + (f(t^{n-1/2}), w_h(v_h)) - a(u_h^{n-1/2}, v_h) \\
&+ \frac{1}{2} a(u(t^n) + u(t^{n-1}), v_h) \Big] \\
&= -\Big[((r_h - I) \partial_\tau u(t^n), w_h(v_h)) + (\partial_\tau u(t^n) - \partial_t u(t^{n-1/2}), w_h(v_h)) \\
&- a(u(t^{n-1/2}), v_h) + \frac{1}{2} a(u(t^n) + u(t^{n-1}), v_h) \Big] \\
&= -(\lambda^n, w_h(v_h)) \ \ \forall v_h \in V_h^0. \qquad (6.62)
\end{aligned}
$$

where

$$\begin{aligned}
\lambda^n &= \underbrace{(r_h - I)\partial_\tau u(t^n)}_{\lambda_1^n} + \underbrace{\partial_\tau u(t^n) - \partial_t u(t^{n-1/2})}_{\lambda_2^n} \\
&+ \underbrace{\frac{1}{2}(\beta.\nabla u(t^n) + \beta.\nabla u(t^{n-1})) - \beta.\nabla u(t^{n-1/2})}_{\lambda_3^n} \\
&+ \underbrace{\epsilon \triangle u(t^{n-1/2}) - \frac{1}{2}(\epsilon \triangle u(t^n) + \epsilon \triangle u(t^{n-1}))}_{\lambda_4^n} \\
&= \lambda_1^n + \lambda_2^n + \lambda_3^n + \lambda_4^n.
\end{aligned}$$

As a consequence of the stability estimate (6.22) and (6.24) there holds

$$\begin{aligned}
||\phi^n||_\delta^2 + \tau \sum_{m=1}^n |||\phi^m|||^2 &\lesssim t^n \tau \sum_{m=1}^n ||\lambda^m||^2 + t^n \tau \sum_{m=2}^n \delta^2 ||\partial_\tau \lambda^m||^2 \\
&+ \delta^2(||\lambda^1||^2 + ||\lambda^n||^2) + \frac{\tau}{t^n} \sum_{m=1}^n ||\phi^m||_\delta^2 + ||\phi^0||_\delta.
\end{aligned}$$

Since $u_h^0 = r_h u(t^0)$, $\phi^0 = 0$. Applying lemma 6.4, we get

$$\begin{aligned}
||\phi^n||_\delta^2 + \tau \sum_{m=1}^n |||\phi^m|||^2 &\lesssim \tau t^n \sum_{m=1}^n ||\lambda^m||^2 + \tau t^n \sum_{m=2}^n \delta^2 ||\partial_\tau \lambda^m||^2 \\
&+ \delta^2(||\lambda^1||^2 + ||\lambda^n||^2).
\end{aligned} \tag{6.63}$$

By the standard Taylor development, there exists $\xi^n \in [t^{n-1}, t^n]$ such that

$$\partial_\tau u(t^n) = \partial_t u(t^{n-1/2}) + \frac{\tau^2}{24} \partial_t^3 u(\xi^n).$$

Then we may write that

$$\epsilon \triangle \left(u(t^{n-1/2}) - \frac{u(t^n) + u(t^{n-1})}{2} \right) = -\frac{\tau^2}{4} \epsilon \triangle u_{tt}(\xi^n)$$

and

$$-\beta.\nabla\left(u(t^{n-1/2})+\frac{u(t^n)+u(t^{n-1})}{2}\right)=\frac{\tau^2}{4}\beta\nabla u_{tt}(\xi^n).$$

Now we obtain the upper bound for the last two terms in (6.63).

$$\begin{aligned}||\lambda^n||^2 &\lesssim ||(r_h-I)(u_t(t^{n-1/2})+\frac{\tau^2}{24}u_{ttt}(\xi^n))||^2\\ &+ \tau^4\left[\sup_{0\le t\le t^n}||u_{ttt}(t)||^2+\sup_{0\le t\le t^n}||\beta.\nabla u_{tt}(t)||^2+\sup_{0\le t\le t^n}||\epsilon\triangle u_{tt}(t)||^2\right].\end{aligned}$$

and by the lemma (6.5) it follows

$$\begin{aligned}||\lambda^n||^2 &\lesssim h^3\left[|u_t(t^{n-1/2})|_2^2+\tau^4\sup_{0\le t\le t^n}|u_{ttt}(t)|_2^2\right]\\ &+ \tau^4\left[\sup_{0\le t\le t^n}||u_{ttt}(t)||^2+\sup_{0\le t\le t^n}||\beta.\nabla u_{tt}(t)||^2+\sup_{0\le t\le t^n}||\epsilon\triangle u_{tt}(t)||^2\right].\end{aligned}$$

Then

$$\begin{aligned}||\lambda^1||^2+||\lambda^n||^2 &\lesssim h^3\left[\sup_{0\le t\le t^n}|u_t(t)|_2^2+\tau^4\sup_{0\le t\le t^n}|u_{ttt}(t)|_2^2\right]\\ &+ \tau^4\left[\sup_{0\le t\le t^n}||u_{ttt}(t)||^2+\sup_{0\le t\le t^n}||\beta.\nabla u_{tt}(t)||^2+\sup_{0\le t\le t^n}||\epsilon\triangle u_{tt}(t)||^2\right]\\ &\lesssim h^3(U_\infty^{1,2}+\tau^4U_\infty^{3,2})+\tau^4(U_\infty^{3,0}+||\beta||_\infty^2U_\infty^{2,1}+\epsilon^2U_\infty^{2,2}). \qquad (6.64)\end{aligned}$$

Now we need to find an estimate for the terms for $i=1,2,3,4$

$$\tau\sum_{m=1}^{n}||w_i^m||^2 \text{ and } \tau\sum_{m=1}^{n}||\partial_\tau w_i^m||^2$$

$\tau\sum_{m=1}^{n}||\lambda_1^m||^2$ and $\tau\sum_{m=2}^{n}||\partial_\tau\lambda_1^m||^2$ are estimated in a similar manner as for backward scheme such that

$$\tau\sum_{m=1}^{n}||\lambda_1^m||^2 = h^3||\beta||_\infty U_2^{1,2} \text{ and } \tau\sum_{m=2}^{n}||\partial_\tau\lambda_1^m||^2 = h^3||\beta||_\infty U_2^{2,2}. \qquad (6.65)$$

Now let us consider the terms related to λ_2^m:

$$\begin{aligned} \lambda_2^m &= \frac{1}{2\tau}\Big[\int_{t^{m-1}}^{t^{m-1/2}}(t-t^{m-1})^2u_{ttt}(t)dt + \int_{t^{m-1/2}}^{t^m}(t-t^m)^2u_{ttt}(t)dt\Big] \\ ||\lambda_2^m|| &= \frac{1}{2\tau}\frac{\tau^2}{4}\int_{t^{m-1}}^{t^m}||u_{ttt}(t)||dt \lesssim \tau^{3/2}\Big(\int_{t^{m-1}}^{t^m}||u_{ttt}(t)||^2dt\Big)^{1/2}. \end{aligned}$$

Then

$$\begin{aligned} \tau\sum_{m=1}^{n}||\lambda_2^m||^2 &= \tau\sum_{m=1}^{n}\tau^3\int_{t^{m-1}}^{t^m}||u_{ttt}(t)||^2dt \\ &\lesssim \tau^4\int_0^{t^n}||u_{ttt}(t)||^2dt = \tau^4U_2^{3,0}, \end{aligned} \qquad (6.66)$$

and similarly we obtain

$$\tau\sum_{m=2}^{n}||\partial_\tau\lambda_2^m||^2 \lesssim \tau^4U_2^{4,0}. \qquad (6.67)$$

Now consider the terms related to λ_3^m.

$$\begin{aligned} \lambda_3^m &= \frac{1}{2}(\beta\nabla u(t^m) + \beta\nabla u(t^{m-1})) - \beta\nabla u(t^{m-1/2}) \\ &= \frac{1}{2}\Big[\int_{t^{n-1}}^{t^{n-1/2}}(t-t^{n-1})\beta.\nabla u_{tt}(t)dt + \int_{t^{n-1/2}}^{t^n}(t^n-t)\beta.\nabla u_{tt}(t)dt\Big]. \\ ||\lambda_3^m|| &\leq \frac{\tau}{4}\int_{t^{m-1}}^{t^m}||\beta.\nabla u_{tt}(t)||dt \lesssim \tau^{3/2}\Big(\int_{t^{m-1}}^{t^m}||\beta.\nabla u_{tt}(t)||^2dt\Big)^{1/2}. \end{aligned}$$

Then we may write

$$\tau \sum_{m=1}^{n} ||\lambda_3^m||^2 \lesssim \tau^4 ||\beta||_\infty^2 U_2^{2,1}. \tag{6.68}$$

For the contribution $\tau \sum_{m=2}^{n} ||\partial_\tau \lambda_3^m||^2$ it can be written that

$$\begin{aligned}
\partial_\tau \lambda_3^m &= -(\partial_\tau \beta . \nabla u(t^{m-1/2}) - \partial_t \beta . \nabla u(t^{m-1})) + \frac{1}{2}(\partial_\tau \beta . \nabla u(t^m) - \partial_t \beta . \nabla u(t^{m-1/2})) \\
&+ \frac{1}{2}(\partial_\tau \beta . \nabla u(t^{m-1}) - \partial_t \beta . \nabla u(t^{m-3/2})) - \partial_t \lambda_3^{m-1/2}.
\end{aligned}$$

The first three terms on the right hand side are on the same form as w_2 and the last term is on the same form as w_3. Using the results on w_2 and w_3 we conclude

$$\tau \sum_{m=2}^{n} ||\partial_\tau \lambda_3^m||^2 \lesssim \tau^4 ||\beta||_\infty^2 U_2^{3,1}. \tag{6.69}$$

Consider

$$\begin{aligned}
\lambda_4^m &= \epsilon \triangle u(t^{m-1/2}) - \frac{1}{2}(\epsilon \triangle u(t^m) + \epsilon \triangle u(t^{m-1})) \\
&= \frac{1}{2}\Big[\int_{t^{n-1}}^{t^{n-1/2}} (t^{n-1} - t)\epsilon \triangle u_{tt}(t)dt + \int_{t^{n-1/2}}^{t^n} (t - t^n)\epsilon \triangle u_{tt}(t)dt\Big]. \\
||\lambda_4^m|| &\le \frac{\tau}{4}\int_{t^{m-1}}^{t^m} ||\epsilon \triangle u_{tt}(t)||dt \lesssim \tau^{3/2}\Big(\int_{t^{m-1}}^{t^m} ||\epsilon \triangle u_{tt}(t)||^2 dt\Big)^{1/2}.
\end{aligned}$$

Then we may write

$$\tau \sum_{m=1}^{n} ||\lambda_4^m||^2 \lesssim \tau^4 \epsilon^2 U_2^{2,2}. \tag{6.70}$$

Contribution of $\tau \sum_{m=2}^{n} ||\partial_\tau \lambda_4^m||^2$ is same as the term $\tau \sum_{m=2}^{n} ||\partial_\tau \lambda_3^m||^2$ such that

$$\tau \sum_{m=2}^{n} ||\partial_\tau \lambda_4^m||^2 \lesssim \tau^4 \epsilon^2 U_2^{3,2}. \tag{6.71}$$

Now using (6.64)-(6.71) in (6.63)

$$
\begin{aligned}
||\phi^n||_\delta^2 + \tau \sum_{m=1}^{n} |||\phi^m|||^2 &\lesssim h^3 \Big[t^n ||\beta||_\infty U_2^{1,2} + t^n \delta^2 ||\beta||_\infty U_2^{2,2} + \delta^2 (U_\infty^{1,2} + \tau^4 U_\infty^{3,2}) \Big] \\
&+ \tau^4 \Big[t^n U_2^{3,0} + t^n ||\beta||_\infty^2 U_2^{2,1} + t^n \epsilon^2 U_2^{2,2} \\
&+ t^n \delta^2 U_2^{4,0} + t^n \delta^2 ||\beta||_\infty^2 U_2^{3,1} + t^n \delta^2 \epsilon^2 U_2^{3,2} \\
&+ \delta^2 U_\infty^{3,0} + \delta^2 ||\beta||_\infty^2 U_\infty^{2,1} + \delta^2 \epsilon^2 U_\infty^{2,2} \Big]. \qquad (6.72)
\end{aligned}
$$

Projection error can be bounded as similar as the analysis for the backward Euler scheme. Therefore we obtain desiered result by means of (6.57) and (6.72). □

If the solution is not smooth enough to satisfy the uniform bounds we still have optimal convergence under the inverse CFL condition $\delta \le \tau$ for the Crank-Nicholson method.

Corollary 6.4 *Let $\{u_h^n\}_{n=0}^N$ be the solution of (6.60), and u be the solution of (6.1). Then for n such that $1 \le n \le N$ there holds*

$$
\begin{aligned}
||u_h^n - u(t^n)||_\delta &\lesssim h^{3/2} ||\beta||_\infty^{1/2} \left[\left(1 + \frac{\delta}{\tau}\right) U_1^{1,2} + U_\infty^{0,2} \right] \\
&+ \tau^2 \left(1 + \frac{\delta}{\tau}\right) \left[U_1^{3,0} + ||\beta||_\infty U_1^{2,1} + \epsilon U_1^{2,2} \right]. \qquad (6.73)
\end{aligned}
$$

Proof As a consequence of the stability analysis in the corollary (6.2) for the term ϕ^n we obtain

$$
\begin{aligned}
||\phi^n||_\delta &\lesssim (\tau + \delta) \sum_{m=1}^{n} ||\lambda^m|| \\
&\lesssim \left(1 + \frac{\delta}{\tau}\right) \tau \sum_{m=1}^{n} (||\lambda_1^m|| + ||\lambda_2^m|| + ||\lambda_3^m|| + ||\lambda_4^m||).
\end{aligned}
$$

Using the approximation properties of the Ritz-projection, leading to the upper bounds (Burman, 2010)

$$\tau \sum_{m=1}^{n} ||\lambda_1^m|| \lesssim h^{3/2}||\beta||_\infty^{1/2} U_1^{1,2} \quad , \quad \tau \sum_{m=1}^{n} ||\lambda_2^m|| \lesssim \tau^2 U_1^{3,0} \, ,$$
$$\tau \sum_{m=1}^{n} ||\lambda_3^m|| \lesssim \tau^2 ||\beta||_\infty U_1^{2,1} \quad , \quad \tau \sum_{m=1}^{n} ||\lambda_4^m|| \lesssim \tau^2 \epsilon U_1^{2,2}.$$

Proof is completed by these bounds and (6.55). □

Numerical experiments that validate the order of convergence of methods considered here can be found in (Burman and Smith, 2011).

CHAPTER 7

AN OPERATOR SPLITTING APPROACH COMBINED WITH THE SUPG METHOD FOR THE TRANSPORT EQUATIONS

The mathematical models describing the transport phenomena are time dependent advection diffusion reaction equations. This kind of equation with linear or nonlinear reaction term is one for which approximate solution procedures continue to exhibit significant limitations for certain problems of physical interest. The most interesting cases are appeared when advection is dominated.

In this chapter we advocate an operator splitting method which is widely used to simulate the models come from environmental processes (Zlatev, 1995), (Geiser, 2008), (Levine, Pamuk, Sleeman and Hamilton, 2010), (Ewing, 2002), (Frolkovič and Geiser, 2000). We split the transport equation into two unsteady subproblems. The main advantage of splitting is that each subproblem can be discretized separately by the convenient method independently from the other subproblem.

7.1. Transport Problem and Operator Splitting

In this section we consider a model equation for simulating the transport and decay of particles in a fluid:

$$\begin{aligned} u_t + \mathcal{L}u &= R(u) + f \text{ in } \Omega_t := \Omega \times (0,T] \\ u &= 0 \text{ on } \partial\Omega \times [0,T] \\ u &= u^0 \text{ on } \Omega \times \{0\} \end{aligned} \tag{7.1}$$

where elliptic operator $\mathcal{L}$, source function f and initial datum u^0 are defined in previous chapter and R(u) is a nonlinear reaction term comes from the following models:

- Radioactive decay model: $R(u) = -au$.
- Logistic model : $R(u) = au - bu^2$.

- Bio-remediation model : $R(u) = \dfrac{au}{u+b}$.

Here a and b are nonnegative real numbers for each model. In order to simplify the notation, let us define

$$\mathcal{L}_f(u) := f - \mathcal{L}u$$

Then the equation (7.1) can be read

$$u_t = \mathcal{L}_f(u) + R(u)$$

An efficient approach for finding the approximate solution of (7.1) is based on an operator splitting strategy. The principle of this procedure is starting from u_h^n, an approximation $u(t^n,.)$, construct u_h^{n+1} through two or more intermediate values, each one obtained by solving a boundary value problem related to only one of the separating operators. In the literature authors generally prefer to separate diffusion from advection (Quarteroni and Valli, 1996), (Geiser, Ewing and Liu, 2005). Unlike to this prevailing approach we separate the non-linear reaction term from advection diffusion term such that

$$w_t = R(w). \qquad (7.2)$$

$$z_t = \mathcal{L}_f(z) \qquad (7.3)$$

Since R and $\mathcal{L}_f$ are not commute operator except for radioctive decay model, we obtain a splitting error first order ($o(\tau)$). On the other hand our splitting has two important advantages that we can apply stabilized finite element method SUPG with backward Euler time stepping to (7.3), which is done in previous section, and exact solution of (7.2) can be easily obtained.

$$w_t = R(w) \text{ in } \Omega_t := \Omega \times (0,T) \qquad (7.4)$$

$$w = \phi \text{ on } \Omega \times \{0\}$$

Exact solution of the equation (7.4) can be given for each reaction terms described above such that

- Radioactive decay model : $w(x,t) = e^{-at}\phi(x)$
- Logistic model: $w(x,t) = \dfrac{a\phi(x)}{b\phi(x)(1-e^{-at}) + ae^{-at}}$
- Bio-remediation model : $w(x,t) + bln|w(x,t)| = at + \phi(x) + bln|\phi(x)|$

For more complex cases, one may use an appropriate time integrator for instance generalized Euler or Runge Kutta (RK) methods instead of their exact expressions.

We also use the two step Yanenko splitting strategy (see Fig. 7.1) which is first order accurate and unconditionally stable if the discrete counterparts of the differential operators are non-negative definite matrices (Marchuk, 1990). More formal description

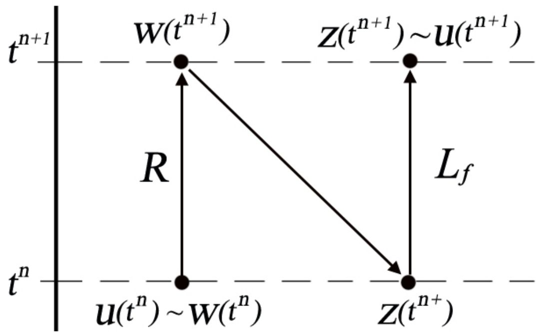

Figure 7.1. Two step Yanenko splitting scheme.

of two step Yanenko splitting method can be given in the following algorithm. Starting with $z(t^0) = u^0$, then two subproblems are sequentially solved on the sub-intervals $(t^n, t^{n+1}],\ n = 0, ..., N-1$:
Given $z(t^n)$ find $w : \Omega \times (t^n, t^{n+1}] \to \mathbb{R}$ such that

$$\begin{aligned} w_t &= R(w) \text{ in } \Omega \times (t^n, t^{n+1}] \\ w(t^n) &= z(t^n) \text{ on } \Omega. \end{aligned} \tag{7.5}$$

Find $z : \Omega \times (t^n, t^{n+1}] \to \mathbb{R}$ such that

$$\begin{aligned} z_t &= \mathcal{L}_f(z) \text{ in } \ \Omega \times (t^n, t^{n+1}] \\ z &= 0 \text{ on } \ \partial\Omega \times [t^n, t^{n+1}] \\ z(t^{n+}) &= w(t^{n+1}) \text{ on } \ \Omega. \end{aligned} \tag{7.6}$$

This two step splitting algorithm presents $z(t^n), \ n = 1, ..., N$ which is an approximation of $u(t^n)$.

7.2. Numerical Experiments

We firstly test our method for the following one-dimensional transport problems:

$$\begin{aligned} u_t - 0.0001 u_{xx} + u_x &= R(u) + 1 \text{ in } \ \Omega_t := (0,1) \times (0,2] \\ u(0,t) = u(1,t) &= 0 \text{ for } \ t \in [0,2] \\ u(x,0) &= 0 \text{ for } \ x \in (0,1). \end{aligned} \tag{7.7}$$

where the reaction term is chosen as follows:

- Radioactive decay model : $R(u) = -15u$.
- Logistic model : $R(u) = 15u - u^2$.
- Bio-remediation model : $R(u) = \dfrac{15u}{u+1}$.

For all numerical simulation $N = 400$ uniform time steps are used and space discretization $\mathfrak{T}_h$ of $\Omega = (0,1)$ is made by 20 quasi-uniform subintervals. For all numerical methods we obtain a sequence of continuous piecewise linear approximation u_L^n $(n = 1, 2, ..., 400)$ then we only compare the final time results obtained by different schemes. In Fig. (7.2) and Fig (7.3)(left), the curve labeled with splitting with SUPG and the curve labeled with standard splitting are obtained by our algorithm (7.5)-(7.6) and the algorithm proposed in (Geiser, Ewing and Liu, 2005), respectively. The reference approximations were computed with the Galerkin method on a very fine mesh and with sufficiently small time steps. The curve labeled with Standard Galerkin in the Fig. 7.2(left) also illustrates the result obtained by the Galerkin method on the coarse mesh for radioactive decay model .

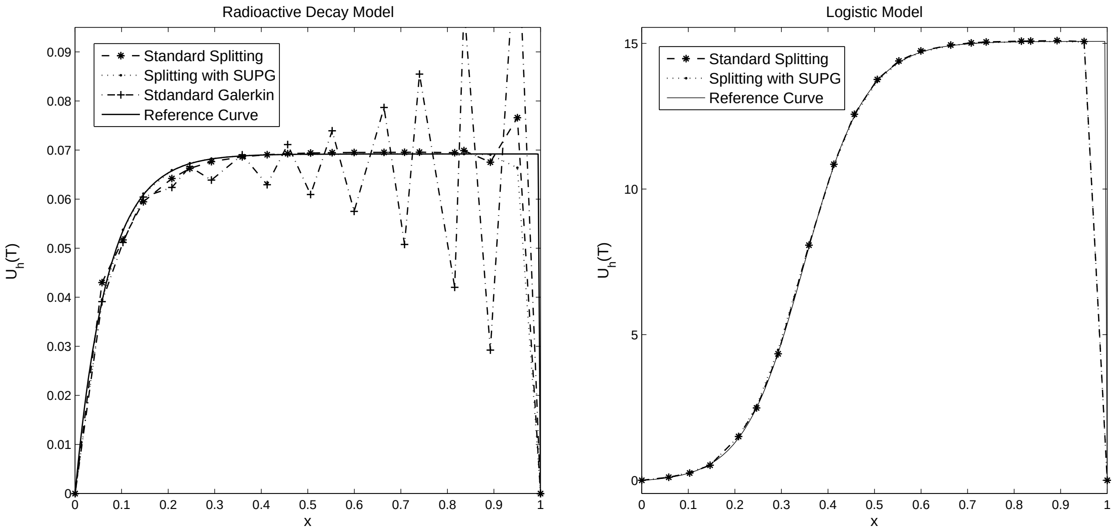

Figure 7.2. Numeric simulations for radioactive decay model (left) and logistic model (right).

The convergence plot in Fig. 7.3 (right) is presented by using 400-450-500-550-600-650 time steps for the radioactive decay test model:

$$\begin{aligned} u_t - 0.01u_{xx} + u_x &= -15u \text{ in } \ \Omega_t := (0,1) \times (0,2] \\ u(0,t) = u(1,t) &= 0 \text{ for } \ t \in [0,2] \\ u(x,0) &= \exp(50x)\sin(\pi x). \end{aligned} \tag{7.8}$$

In this case exact solution of the problem (7.8) can be written:

$$u(x,t) = \exp(50x - 40t - 0.01\pi^2 t)\sin(\pi x).$$

As we see in Fig. 7.3 (right), our splitting algorithm presents first order convergence with respect to the time step size.

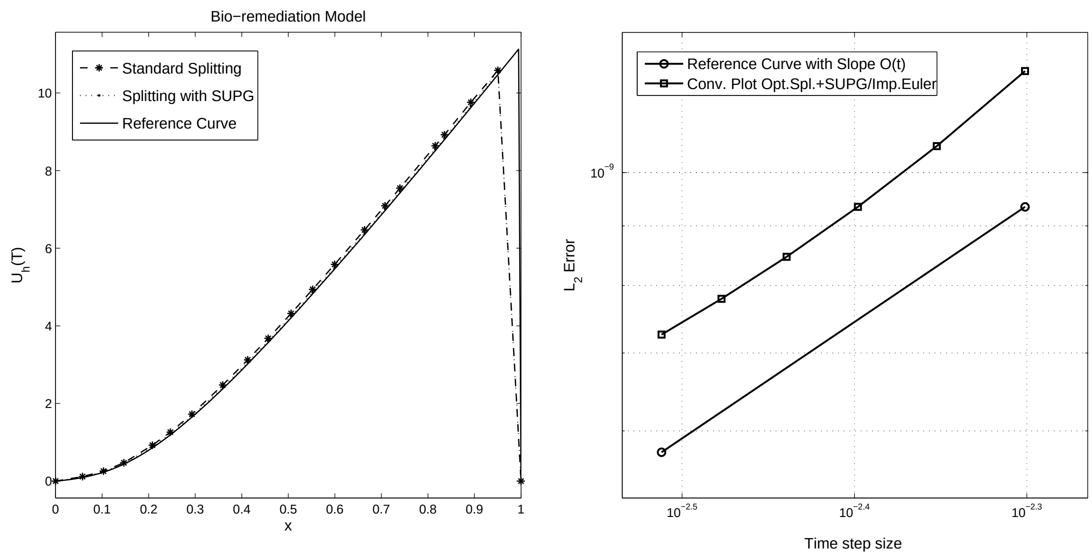

Figure 7.3. Numeric simulation for bio-remediation model (left) and error rate for the radioactive decay model (right).

Finally we also illustrate the numerical performance of our splitting strategy for two dimensional problems:

$$\begin{aligned} u_t - 0.0001\triangle u + (1,1).\nabla u &= R(u) \text{ in } \Omega_t := \Omega \times (0,T] \\ u &= 0 \text{ on } \partial\Omega \times [0,T] \\ u &= u^0 \text{ on } \Omega \times \{0\} \end{aligned} \tag{7.9}$$

where the reaction term is chosen as follows:

- Radioactive decay model : $R(u) = -3u$.
- Logistic model : $R(u) = 3u - u^2$.
- Bio-remediation model : $R(u) = \dfrac{3u}{u+1}$

and space discretization $\mathfrak{T}_h$ of $\Omega = (0,1)^2$ is made by 800 quasi-uniform triangles described in Fig. 7.4 (left). We choose final time $T = 1/2$ and fixed time step size $\tau = T/400$. We also choose a discrete initial data whose form is square prism of height 1 such that

$$u_0(x,y) = \begin{cases} 1, & (x,y) \in [\frac{3}{16}, \frac{6}{16}]^2 \\ 0, & \text{otherwise} \end{cases} \tag{7.10}$$

The contour-lines of the interpolant of the initial data is shown in Fig. 7.4(right). We

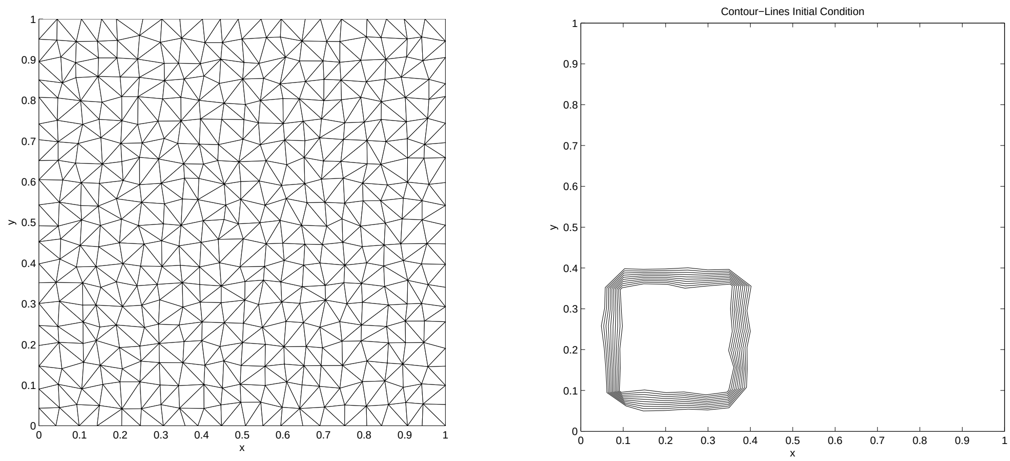

Figure 7.4. Quasi-uniform mesh (left) and contour-lines of the initial data (right).

also compare the final time results obtained by the Galerkin approximation and the present splitting algorithm in Fig. 7.5. As we see in this figure, although the Galerkin approximation is completely contaminated by spurious oscillations all over the whole domain Ω, our splitting strategy provides oscillation-free approximations. The satisfactory results of our splitting algorithm are presented for the logistic and bio-remediation model in Fig. 7.6.

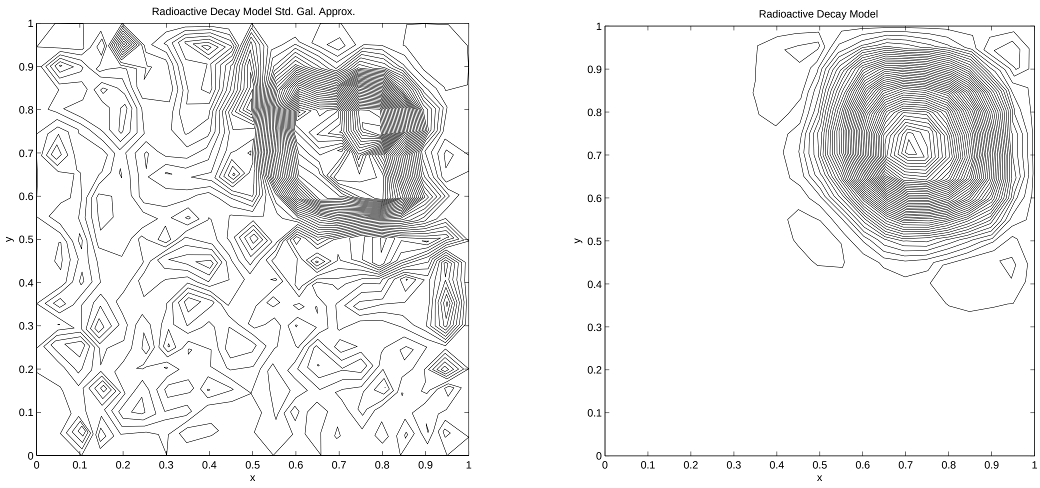

Figure 7.5. Contour-lines of Galerkin approximation (left) and our splitting algorithm (right) for the radioactive decay model at the final time.

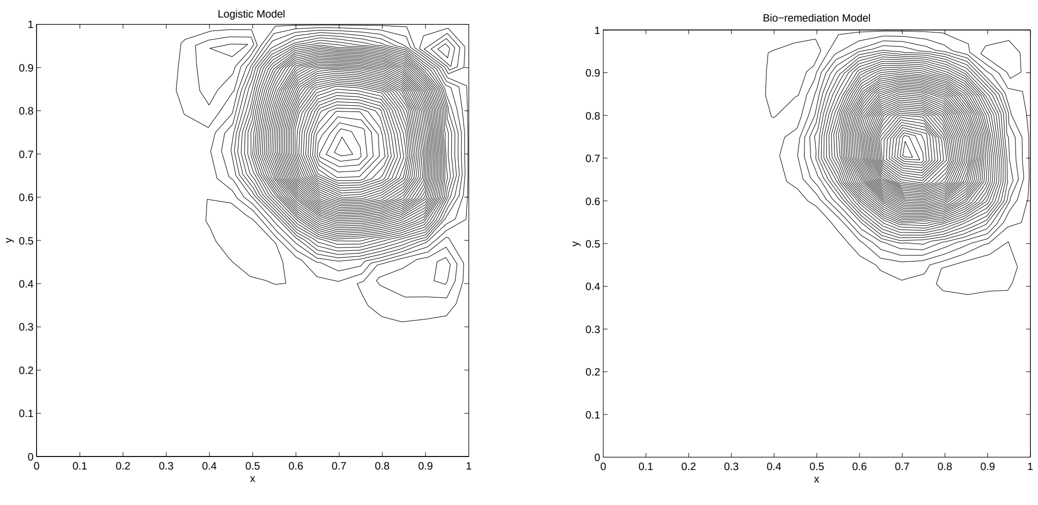

Figure 7.6. Contour-lines of our operator splitting approximation for the logistic (left) and bio-remediation (right) models at the final time.

CHAPTER 8

CONCLUSION

Here we studied enriched finite element methods for both steady and unsteady convection diffusion equations. For the unsteady problems, the methods based on separating spatial and temporal discretization was considered. In detail we primarily applied the stabilization techniques to the steady problem then adapted these algorithms to the unsteady problem in combination with the θ method. Above all our goal was the construct efficient and practical approximations for both the steady and unsteady problems. Therefore we utilized the pseudo approximation techniques for solving the residual equation which appears as a result of bubble elimination procedure.

For the case of continuous piecewise linear elements, the stability and convergence analysis of the SUPG/θ method for the unsteady convection diffusion equation have been studied. We managed to extend analysis for the pure convection equation given in (Burman, 2010) to the convection diffusion equation. Consequently for both A-stable cases $\theta = 0$ and $\theta = 1/2$, assuming regularity conditions on the data we proved uniform stability and quasi optimal convergence of the algorithm that allow safely using this method to more general transport problems.

For piecewise bilinear finite element discretization on rectangular grid, enriched finite element methods RFB, MS and MIX have been considered for the steady problem. Then the pseudo approximation techniques, which employ only a few nodes in each element, for evaluating the enriching basis functions were suggested. Next, for the unsteady problem we suggested a proper adaptation recipe, to combine these methods developed for the steady equation with the θ method. Several numerical tests support the good performance of the corresponding methods for both the steady and unsteady equations.

Lastly, we suggested an operator splitting strategy for the transport equations which includes nonlinear reaction terms. In our splitting strategy the first part becomes a first order nonlinear differential equation without space derivatives and the second part becomes an unsteady linear convection diffusion equation. The former problem can be solved exactly by using simple analytical techniques. The latter one is problematic when convection is dominated. In this regime, we have used the SUPG method for space and θ method for time discretization. Numerical tests indicate our splitting strategy provides oscillation-free approximations.

REFERENCES

Asensio M.I., Ayuso B., and Sangalli G., 2007: Coupling Stabilized Finite Element methods with finite difference time integration for Advection-Diffusion-Reaction Problems. *Comput. Methods Appl. Mech. Engrg.,* vol. 196, pp. 3475-3491.

Baiocchi C., Brezzi F. and Franca L.P., 1993: Virtual Bubbles and the Galerkin Least-Squares Method. *Comput. Methods Appl. Mech. Engrg.,*, vol. 105, pp. 125-141.

Brezzi F., Baiocchi C. and Franca L.P., 1992: A Relationship Between Stabilized Finite Element Methods and the Galerkin Method with Bubble Functions. *Comput. Methods Appl. Mech. Engrg.,*, vol. 96, pp. 117-129.

Brezzi F. and Douglas J., 1988: Stabilized mixed methods for the Stokes problem. *Numerische Mathematik*, vol. 53(1-2), pp. 225-235.

Brezzi F., Franca L.P. and Russo A., 1998: Further considerations on residual-free bubbles for advective-diffusive equations, *Comput. Methods Appl. Mech. Engrg.,* vol. 166, pp. 25-33.

Bochev P.B., Gunzburger M.D., Shadid J.N, 2004: Stability of the SUPG finite element method for transient advection-diffusion problems. *Comput. Methods Appl. Mech. Engrg.,* vol. 193, pp. 2301-2323.

Brooks A.N. and Hughes T.J.R., 1982: Streamline upwind/Petrov - Galerkin Formulations for Convection Dominated Flows with Particular Emphasis on the Incompressible Navier-Stokes Equations, *Comput. Methods Appl. Mech. Engrg.,* vol. 32, pp. 199-259, 1982.

Brezzi F., Hughes T.J.D., Marini L.D., Russo A. and Süli E., 1999: A Priori Error Analysis of Residual-Free Bubbles for Advection-Diffusion Problems, *SIAM J. Numer. Anal.,* vol. 36, pp. 1933-1948.

Brezzi F., Hauke G., Marini L.D. and Sangalli G., 2003: Link-Cutting Bubbles for the Stabilization of Convection-Diffusion-Reaction Problems, *Mathematical Models and Methods in Applied Sciences,* vol. 13, pp. 3445-461.

Brezzi F., Marini D. and Russo A., 1998: Applications of the Pseudo Residual-Free Bubbles to the Stabilization of Convection-Diffusion Problems, *Comput. Methods Appl. Mech. Engrg.,* vol. 166, pp. 51-63.

Brezzi F., Marini D. and Russo A., 2000: The residual-free bubbles for advection-diffusion problems: General error analysis, *Numer. Math.,* vol. 85, pp. 31-47.

Brezzi F., Marini D. and Russo A., 2005: On the Choice of a Stabilizing Subgrid for Convection-Diffusion Problems, *Comput. Methods Appl. Mech. Engrg.,* vol. 194, pp. 127-148.

Brezzi F. and Russo A., 1994: Choosing Bubbles for Advection-Diffusion Problems, *Math. Models Methods Appl. Sci.,* vol. 4, pp. 571-587.

Burman E., 2010: Consistent SUPG-method for transient transport problems: stability and convergence. *Comput. Methods Appl. Mech. Engrg.,* vol. 199, 17-20, pp. 1114-1123.

Burman E. and Smith G., 2011: Analysis of the space semi-discretized SUPG method for transient convection-diffusion equations, *Mathematical Models and Methods in Applied Science*, vol. 21(10) pp. 2049-2068.

Celia M.A., Kindred J.S. and Herrera I., 1989: Contaminant Transport and Biodegradation a Numerical Model for Reactive Transport in Porous Media, *Water Resources Research*, vol. 25(6), pp. 1141-1148.

Dettmer W. and Peric D., 2003: An analysis of the time integration algorithms for the finite element solutions of incompressible Navier-Stokes equations based on a stabilised formulation, *Comput. Methods Appl. Mech. Engrg.*, vol. 192 (9-10), pp. 1177-1226.

Ewing R.E., 2002: Up-scaling of biological processes and multiphase flow in porous media, *IIMA Volumes in Mathematics and its Applications, Springer-Verlag,* vol. 295 pp. 195-215.

Frutos J., Garcia-Archilla B. and Novo J., 2010: Stabilization of Galerkin finite element approximation to transient convection-diffusion problems, *Siam Journal on Numerical Analysis*, vol. 48 (3), pp. 953-979.

Franca L.P. and Frey S.L., 1992: Stabilized Finite Element Methods: II. The incompressible Navier-Stokes equations, *Comput. Methods Appl. Mech. Engrg.,* vol. 99(2-3), pp. 209-233.

Franca L.P., Frey S.L. and Hughes T.J.R., 1992: Stabilized Finite Element Methods: I. Application to the Advective-Diffusion Model, *Comput. Methods Appl. Mech. Engrg.,* vol. 95, pp. 253-276.

Franca L.P. and Russo A., 1996: Deriving upwinding, mass lumping and selective reduced integration by residual-free bubbles, *Appl. Math. Lett.,* vol. 9 pp. 83-88.

Frolkovič P., Geiser J., 2000: Numerical Simulation of Radionuclides Transport in Double Porosity Media with Sorption, *Proceedings of Algorithmy, Conference of Scientific Computing*, pp. 28-36.

Franca L.P., Madureira A.L., Tobiska L. and Valentin F., 2005: Convergence Analysis of a Multiscale Finite Element Method for Singularly Perturbed Problems, *International Computational Engineering,* vol. 3(3), pp. 297-312.

Franca L.P., Madureira A.L. and Valentin F., 2005: Towards multiscale functions: Enriching finite element spaces with local but not bubble-like functions, *Comp. Methods Appl. Mech. Engrg.,* vol. 194, pp. 3006-3021.

Frutos J. and Novo J., 2008: Bubble stabilization of linear finite element methods for nonlinear evolutionary convection-diffusion equations, *Comput. Methods Appl. Mech. Engrg.,* vol. 197, pp. 3988-3999.

Franca L.P., Neslitürk A.I. and Stynes M., 1998: On the Stability of Residual-Free Bubbles for Convection-Diffusion Problems and Their Approximation by Two-Level Finite Element Method, *Comput. Methods Appl. Mech. Engrg.,* vol. 166, pp. 35-49.

Franca L.P., Ramalho J.V.A. and Valentin F., 2005: Multiscale and Residual-Free Bubble Functions for Reaction-Advection-Diffusion Problems. *International Computational Engineering,* vol. 3(3), pp. 297-312.

Franca L.P., Ramalho J.V.A. and Valentin F., 2006: Enriched Finite Element Methods for Unsteady Reaction-Diffusion Problems. *Commun. Numer. Meth. Engng.,* vol. 22(6), pp. 619-625.

Franca L.P. and Tobiska L., Stability of The Residual Free Bubble Method for Bilinear Finite Elements on Rectangular Grids, *IMA Journal of Numerical Analysis,* vol. 22, pp. 73-87.

Franca L.P. and Valentin F., 2000: On an Improved Unusual Stabilized Finite Element Method for the Advective-Reactive-Diffusive Equation, *Comp. Methods Appl. Mech. Engrg.,* vol. 190, pp. 1785-1800, 2000.

Geiser J., 2008: Decomposition methods for differential equations: Theory and application, *CRC Press, Taylor and Francis Group.*

Geiser J., Ewing R.E. and Liu. J., 2005: Operator Splitting Methods for Transport Equations with Nonlinear Reactions, *Proceeding of the Third MIT Conference on Computational Fluid and Solid Mechanic*.

Harari I., 2004: Stability of semidiscrete formulations for parabolic problems at small time steps, *Comput. Methods Appl. Mech. Engrg.,* vol. 193, pp. 1491-1516.

Harari I. and Hughes T.J.R., 1994: Stabilized Finite Element Methods for Steady Advection-Diffusion with Production, *Comput. Methods Appl. Mech. Engrg.,* vol. 115, pp. 165-191.

Heywood J.G., Rannacher R., 1990: Finite-element approximation of the nonstationary Navier-Stokes' problem. IV. Error analysis for second-order time discretization, *SIAM J. Numer. Anal.,* vol. 27(2), pp. 353-384.

Hughes T.J.R., Franca L.P. and Balestra M., 1986: A new finite element formulation for computational fluid dynamics. 5. Circumventing the Babuska-Brezzi condition - A stable Petrov-Galerkin formulation of the Stokes problem accommodating equal-order interpolations, *Comput. Methods Appl. Mech. Engrg.,* vol. 59(1), pp. 85-99.

John V., Novo J., 2011: Error analysis of the SUPG finite element discretization of evolutionary convection-diffusion-reaction equations, *Siam Journal on Numerical Analysis,* vol. 49(3), pp. 1149-1176.

Johnson C., Nävert U. and Pitkäranta J., 1984: Finite element methods for linear hyperbolic problems, *Comput. Methods Appl. Mech. Engrg.*, vol. 45(1-3) pp. 285-312.

Lube G. and Weiss D., 1995: Stabilized Finite element methods for singularly perturbed parabolic problems, *Appl. Numer. Math.*, vol. 17, pp. 431-459.

Marchuk G.I., 1990: Splitting and alternating direction methods, in Handbook of Numerical Analysis, *I. Ph.G. Ciarlet and J.-L. Lions eds., North-Holland*, pp. 197-462

Neslitürk A.I., 2006: A Stabilizing Subgrid for Convection-Diffusion Problem, *Mathematical Models and Methods in Applied Science,* vol. 16(2), pp. 211-231.

Neslitürk A.I., 2010: On the Choice of Stabilizing Subgrid for Convection-Diffusion Problem on Rectangular Grids, *Computers and Mathematics with Applications,* vol. 59, pp. 3687-3699.

Levine H.A., Pamuk S., Sleeman B.D., Hamilton M.N., 2010: Mathematical model of capillary formation and development in tumor angiogenesis: penetration into the stroma, *Bull. Math. Biol.* vol. 63(5) pp. 801-863.

Quarteroni A. and Valli A., 1996: Numerical Approximation of Partial Differential Equations, *Springer.*

Ross H.-G., Stynes M. and Tobiska L., 2008: Robust Numerical Methods for Singularly Perturbed Differential Equations, *Springer.*

Raviart P.A. and Thomas P.A., 1992: Introduction álanalyse numerique des equations aux derivees partielles, *Masson, Paris*, 1992.

Sangalli G., 2000: Global and local error analysis for the residual-free bubbles method applied to advection-dominated problems, *SIAM J. Numer.,* vol. 38, pp. 1496-1522.

Thomee V., 2006: Galerkin finite element methods for parabolic problems, *Springer.*

Zlatev Z., 1995: Computer Treatment of Large Air Pollution Models, *Kluwer Academic Publishers, Dordrecht-Boston-London.*

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Neslitürk, A.İ. and Baysal, O., "Pseudo Multiscale Functions For The Stabilization of Convection-Diffusion Equations on Rectangular Grids", Journal for Multiscale Computational Engineering., Accepted 2012.