The Discrete (G'/G)-Expansion Method Applied to the Differential-Difference Burgers Equation and the Relativistic Toda Lattice System

İsmail Aslan
Department of Mathematics, Izmir Institute of Technology, Urla, İzmir 35430, Turkey

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We introduce the discrete (G'/G)-expansion method for solving nonlinear differential–difference equations (NDDEs). As illustrative examples, we consider the differential–difference Burgers equation and the relativistic Toda lattice system. Discrete solitary, periodic, and rational solutions are obtained in a concise manner. The method is also applicable to other types of NDDEs. © 2010 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 28: 127–137, 2012

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I. INTRODUCTION

Nonlinear differential–difference equations (NDDEs) play a crucial role in many branches of applied physical sciences such as condensed matter physics, biophysics, atomic chains, molecular crystals, and discretization in solid-state and quantum physics. They also play an important role in numerical simulation of soliton dynamics in high-energy physics because of their rich structures. Therefore, researchers have shown a wide interest in studying NDDEs since the original work of Fermi et al. [1] in the 1950s. Contrary to difference equations that are being fully discretized, NDDEs are semidiscretized, with some (or all) of their space variables discretized, while time is usually kept continuous. As far as we could verify, little work has been done to search for exact solutions of NDDEs. Hence, it would make sense to do more research on solving NDDEs.

On the other hand, many well-known analytical methods initially developed for solving nonlinear evolution equations (NEEs) have been successfully extended to NDDEs. For instance, Tsuchida et al. [2, 3] extended the inverse scattering method to study some NDDEs. Hu and Ma [4] applied Hirota's bilinear method to the Toeplitz lattice equation. Baldwin et al. [5], with the aid of MATHEMATICA, presented an algorithm to find exact traveling wave solutions of NDDEs

Correspondence to: İsmail Aslan, Department of Mathematics, Izmir Institute of Technology, Urla, İzmir 35430, Turkey (e-mail: ismailaslan@iyte.edu.tr)

in terms of the hyperbolic tangent function. Their work can be considered as a breakthrough for solving NDDEs via symbolic computation. Liu et al. [6] found exact and explicit traveling wave solutions to three NDDEs by means of the Jacobi elliptic function expansion method. Dai et al. [7] presented an extended Jacobian elliptic function algorithm for solving NDDEs. Zhu [8] implemented the Exp-function method to NDDEs. A NDDE arising in nanotechnology has been analyzed by Zhu et al. [9] with the homotopy perturbation method. Xie et al. [10] considered the discrete sine-Gordon equation by a method based on the Riccati equation expansion. Zhen [11] devised a discrete hyperbolic tangent method for solving NDDEs. Yang et al. [12] investigated two nonlinear lattice equations via the so-called ADM-Padé technique. Aslan [13] demonstrated the applicability of the extended simplest equation method to NDDEs, etc. However, it is a challenging task to generalize one method for NEEs to solve NDDEs because series obstacles usually arise in searching for iterative relations from indices n to $n \pm i$.

Recently, Wang et al. [14] proposed the so-called (G'/G)-expansion method to solve NEEs arising in mathematical physics. Right after their pioneer work, the (G'/G)-expansion method became popular in the research community, and a number of studies refining the initial idea have been published [15–28]. The value of the (G'/G)-expansion method is that one treats nonlinear problems by essentially linear methods. It originated from the well-known F-expansion method and the homogeneous balance method.

Our main goal in this study is to present the discrete (G'/G)-expansion method for solving NDDEs. We apply our method to two physically important NDDEs for the first time. The rest of this article is organized as follows: In Section II, we express the discrete (G'/G)-expansion method in detail. In Section III, we apply the method to the differential–difference Burgers equation and the relativistic Toda lattice system, respectively. In Section IV, we give some concluding remarks.

II. THE DISCRETE (G'/G)-EXPANSION METHOD

Let us consider a system of M polynomial NDDEs in the form

$$P\left(\mathbf{u}_{\mathbf{n}+\mathbf{p}_1}(\mathbf{x}),\ldots,\mathbf{u}_{\mathbf{n}+\mathbf{p}_k}(\mathbf{x}),\ldots,\mathbf{u}'_{\mathbf{n}+\mathbf{p}_1}(\mathbf{x}),\ldots,\mathbf{u}'_{\mathbf{n}+\mathbf{p}_k}(\mathbf{x}),\ldots,\mathbf{u}^{(r)}_{\mathbf{n}+\mathbf{p}_1}(\mathbf{x}),\ldots,\mathbf{u}^{(r)}_{\mathbf{n}+\mathbf{p}_k}(\mathbf{x})\right)=0, \quad (1)$$

where the dependent variable $\mathbf{u}_\mathbf{n}$ have M components $u_{i,\mathbf{n}}$ and so do its shifts, the continuous variable $\mathbf{x}$ has N components x_i, the discrete variable $\mathbf{n}$ has Q components n_j, the k shift vectors $\mathbf{p}_i \in \mathbb{Z}^Q$, and $\mathbf{u}^{(r)}(\mathbf{x})$ denotes the collection of mixed derivative terms of order r.

Step 1. By means of the wave transformation

$$\mathbf{u}_{\mathbf{n}+\mathbf{p}_s}(\mathbf{x})=\mathbf{U}_{\mathbf{n}+\mathbf{p}_s}(\xi_\mathbf{n}),\xi_\mathbf{n}=\sum_{i=1}^{Q}d_i n_i+\sum_{j=1}^{N}c_j x_j+\zeta(s=1,2,\ldots,k), \quad (2)$$

where the coefficients $c_1,c_2,\ldots,c_N,d_1,d_2,\ldots,d_Q$ and the phase ζ are all constants, Eq. (1) can be reduced to

$$P\left(\mathbf{U}_{\mathbf{n}+\mathbf{p}_1}(\xi_\mathbf{n}),\ldots,\mathbf{U}_{\mathbf{n}+\mathbf{p}_k}(\xi_\mathbf{n}),\ldots,\mathbf{U}'_{\mathbf{n}+\mathbf{p}_1}(\xi_\mathbf{n}),\ldots,\mathbf{U}'_{\mathbf{n}+\mathbf{p}_k}(\xi_\mathbf{n}),\ldots,\mathbf{U}^{(r)}_{\mathbf{n}+\mathbf{p}_1}(\xi_\mathbf{n}),\ldots,\mathbf{U}^{(r)}_{\mathbf{n}+\mathbf{p}_k}(\xi_\mathbf{n})\right)=0, \quad (3)$$

where the primes denote derivatives with respect to $\xi_\mathbf{n}$.

Step 2. We predict the solution of Eq. (3) in the finite series form

$$\mathbf{U_n}(\xi_\mathbf{n}) = \sum_{l=0}^{m} a_l \left(\frac{G'(\xi_\mathbf{n})}{G(\xi_\mathbf{n})}\right)^l, \quad a_m \neq 0, \tag{4}$$

where m (a positive integer) and a_ls are constants to be specified, and $G(\xi_\mathbf{n})$ is the general solution of the auxiliary equation

$$G''(\xi_\mathbf{n}) + \lambda G'(\xi_\mathbf{n}) + \mu G(\xi_\mathbf{n}) = 0, \tag{5}$$

where λ and μ are constants to be determined, and the primes denote derivatives with respect to $\xi_\mathbf{n}$. The general solution of Eq. (5) is well known for us. Thus, we have the following cases:

$$\frac{G'(\xi_\mathbf{n})}{G(\xi_\mathbf{n})} = -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1 \cos h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_\mathbf{n}\right)+C_2 \sin h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_\mathbf{n}\right)}{C_1 \sin h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_\mathbf{n}\right)+C_2 \cos h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_\mathbf{n}\right)}\right), \quad \lambda^2-4\mu > 0, \tag{6a}$$

$$\frac{G'(\xi_\mathbf{n})}{G(\xi_\mathbf{n})} = -\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_\mathbf{n}\right)+C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_\mathbf{n}\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_\mathbf{n}\right)+C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_\mathbf{n}\right)}\right), \quad \lambda^2-4\mu < 0, \tag{6b}$$

$$\frac{G'(\xi_\mathbf{n})}{G(\xi_\mathbf{n})} = -\frac{\lambda}{2} + \frac{C_1}{C_1\xi_\mathbf{n}+C_2}, \quad \lambda^2-4\mu = 0, \tag{6c}$$

where C_1 and C_2 are arbitrary constants.

Step 3. By an easy computation, we can get the identity

$$\xi_{\mathbf{n}+\mathbf{p}_s} = \xi_\mathbf{n} + \varphi_s, \quad \varphi_s = p_{s1}d_1 + p_{s2}d_2 + \cdots + p_{sQ}d_Q, \tag{7}$$

where p_{sj} is the jth component of the shift vector $\mathbf{p}_s$. Hence, we derive the following expressions:

$$\mathbf{U}_{\mathbf{n}+\mathbf{p}_s}(\xi_\mathbf{n}) = \sum_{l=0}^{m} a_l \left(-\frac{\lambda}{2} + \frac{\frac{\lambda}{2}+\frac{G'(\xi_\mathbf{n})}{G(\xi_\mathbf{n})}+\frac{\sqrt{\lambda^2-4\mu}}{2}\tan h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\varphi_s\right)}{1+\frac{2}{\sqrt{\lambda^2-4\mu}}\left(\frac{\lambda}{2}+\frac{G'(\xi_\mathbf{n})}{G(\xi_\mathbf{n})}\right)\tan h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\varphi_s\right)}\right)^l, \quad a_m \neq 0, \quad \lambda^2-4\mu > 0, \tag{8a}$$

$$\mathbf{U}_{\mathbf{n}+\mathbf{p}_s}(\xi_\mathbf{n}) = \sum_{l=0}^{m} a_l \left(-\frac{\lambda}{2} + \frac{\frac{\lambda}{2}+\frac{G'(\xi_\mathbf{n})}{G(\xi_\mathbf{n})}-\frac{\sqrt{4\mu-\lambda^2}}{2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\varphi_s\right)}{1+\frac{2}{\sqrt{4\mu-\lambda^2}}\left(\frac{\lambda}{2}+\frac{G'(\xi_\mathbf{n})}{G(\xi_\mathbf{n})}\right)\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\varphi_s\right)}\right)^l, \quad a_m \neq 0, \quad \lambda^2-4\mu < 0, \tag{8b}$$

$$\mathbf{U}_{\mathbf{n}+\mathbf{p}_s}(\xi_\mathbf{n}) = \sum_{l=0}^{m} a_l \left(-\frac{\lambda}{2} + \left(\left(\frac{\lambda}{2}+\frac{G'(\xi_\mathbf{n})}{G(\xi_\mathbf{n})}\right) \Big/ \left(1+\left(\frac{\lambda}{2}+\frac{G'(\xi_\mathbf{n})}{G(\xi_\mathbf{n})}\right)\varphi_s\right)\right)\right)^l, \quad a_m \neq 0, \quad \lambda^2-4\mu = 0. \tag{8c}$$

Step 4. We can easily determine the degree m of Eq. (4) and Eqs. (8a)–(8c) from Eq. (3) by using the homogeneous balance principle for the highest order nonlinear term(s) and the highest order derivative term in $\mathbf{U_n}(\xi_\mathbf{n})$. Because $\mathbf{U_{n+p_s}}$ can be interpreted as being of degree zero in $(G'(\xi_\mathbf{n})/G(\xi_\mathbf{n}))$, the leading terms of $\mathbf{U_{n+p_s}}$ ($\mathbf{p}_s \neq 0$) will not affect the balancing procedure.

Step 5. As a final step, substituting the ansatz (4) and (8a)–(8c) along with (5) into Eq. (3) and setting the coefficients of $(G'(\xi_\mathbf{n})/G(\xi_\mathbf{n}))^l$ ($l = 0, 1, 2, \ldots$) to zero, we obtain a system of nonlinear algebraic equations from which the parameters a_l, d_i, c_j, λ, and μ can be explicitly found. Substituting these values into (4), we can get various kinds of discrete exact solutions to Eq. (1).

III. APPLICATIONS

In this section, we apply the discrete (G'/G)-expansion method to two NDDEs and obtain a variety of exact discrete solutions. Among those, to our knowledge, some solutions appear to be novel.

A. The Differential–Difference Burgers Equation

The differential–difference Burgers equation [29] reads

$$\frac{du_n(t)}{dt} = (1 + u_n(t))(u_{n+1}(t) - u_n(t)), \tag{9}$$

where $u_n(t) = u(n, t)$, $n \in \mathbb{Z}$, is the displacement of the nth particle from the equilibrium position. Equation (9) is a soliton equation and exhibits a solution structure distinct from that of the modified Volterra lattice equation, the Hybrid lattice equation, and the mKdV lattice equation [30]. Now, to solve Eq. (9), we first make the transformation

$$u_n(t) = U_n(\xi_n), \quad \xi_n = dn + ct + \zeta, \tag{10}$$

where d and c are constants to be determined and ζ is an arbitrary phase constant. Then, Eq. (9) can be reduced to

$$cU_n'(\xi_n) = (1 + U_n(\xi_n))(U_{n+1}(\xi_n) - U_n(\xi_n)), \tag{11}$$

where the prime denotes the derivative with respect to ξ_n. According to the methodology described in the previous section, we search for solutions of Eq. (11) in the frame (4). Balancing the linear term of the highest order with the highest nonlinear term in (11) leads to $m = 1$. Thus, we assume the solutions of Eq. (11) in the form

$$U_n(\xi_n) = a_0 + a_1 \left(\frac{G'(\xi_n)}{G(\xi_n)} \right), \quad a_1 \neq 0, \tag{12}$$

where a_0 and a_1 are constants to be determined.

Case 1. When $\lambda^2 - 4\mu > 0$, from (8a), we have

$$U_{n+1}(\xi_n) = a_0 + a_1 \left(-\frac{\lambda}{2} + \frac{\frac{\lambda}{2} + \frac{G'(\xi_n)}{G(\xi_n)} + \frac{\sqrt{\lambda^2-4\mu}}{2} \tan h \left(\frac{\sqrt{\lambda^2-4\mu}}{2} d \right)}{1 + \frac{2}{\sqrt{\lambda^2-4\mu}} \left(\frac{\lambda}{2} + \frac{G'(\xi_n)}{G(\xi_n)} \right) \tan h \left(\frac{\sqrt{\lambda^2-4\mu}}{2} d \right)} \right). \tag{13}$$

Substituting (12) and (13) along with (5) into (11), clearing the denominator and setting the coefficients of $(G'(\xi_n)/G(\xi_n))^l$ $(0 \le l \le 4)$ to zero, we derive a system of nonlinear algebraic equations for a_0, a_1, d, c, λ, and μ. Solving the system, we get the following solution set

$$\left\{ a_0 = \left(-2 + c\lambda + c\sqrt{\lambda^2 - 4\mu} \cot h \left(\sqrt{\lambda^2 - 4\mu} d/2 \right) \right) /2, a_1 = c \right\} \tag{14}$$

and the corresponding hyperbolic function solution to Eq. (9) as

$$u_{n,1}(t) = -1 + \frac{c\sqrt{\lambda^2 - 4\mu}}{2} \left(\begin{array}{l} \cot h \left(\frac{\sqrt{\lambda^2-4\mu}}{2} d \right) \\ + \left(\frac{C_1 \cos h \left(\frac{\sqrt{\lambda^2-4\mu}}{2} (dn+ct+\zeta) \right) + C_2 \sin h \left(\frac{\sqrt{\lambda^2-4\mu}}{2} (dn+ct+\zeta) \right)}{C_1 \sin h \left(\frac{\sqrt{\lambda^2-4\mu}}{2} (dn+ct+\zeta) \right) + C_2 \cos h \left(\frac{\sqrt{\lambda^2-4\mu}}{2} (dn+ct+\zeta) \right)} \right) \end{array} \right), \tag{15}$$

where λ, μ, d, c, ζ, C_1, and C_2 are arbitrary constants.

Remark 1. In particular, if we take $C_2 \neq 0$ and $C_1^2 < C_2^2$ in (15) then we obtain a formal discrete solitary wave solution to Eq. (9) as

$$\begin{aligned} u_{n,2}(t) = -1 + \frac{c\sqrt{\lambda^2 - 4\mu}}{2} \Bigg(& \cot h \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} d \right) \\ & + \tan h \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (dn + ct + \zeta) + \tan h^{-1} \left(\frac{C_1}{C_2} \right) \right) \Bigg). \end{aligned} \tag{16}$$

Case 2. When $\lambda^2 - 4\mu < 0$, from (8b), we have

$$U_{n+1}(\xi_n) = a_0 + a_1 \left(-\frac{\lambda}{2} + \frac{\frac{\lambda}{2} + \frac{G'(\xi_n)}{G(\xi_n)} - \frac{\sqrt{4\mu-\lambda^2}}{2} \tan \left(\frac{\sqrt{4\mu-\lambda^2}}{2} d \right)}{1 + \frac{2}{\sqrt{4\mu-\lambda^2}} \left(\frac{\lambda}{2} + \frac{G'(\xi_n)}{G(\xi_n)} \right) \tan \left(\frac{\sqrt{4\mu-\lambda^2}}{2} d \right)} \right). \tag{17}$$

Substituting (12) and (17) along with (5) into (11), clearing the denominator and setting the coefficients of $(G'(\xi_n)/G(\xi_n))^l$ $(0 \leq l \leq 4)$ to zero, we derive a system of nonlinear algebraic equations for a_0, a_1, d, c, λ, and μ. Solving the system, we get the following solution set

$$\left\{a_0 = \left(-2 + c\lambda + c\sqrt{4\mu - \lambda^2}\cot\left(\sqrt{4\mu - \lambda^2}d/2\right)\right)/2, a_1 = c\right\} \tag{18}$$

and the corresponding trigonometric function solution to Eq. (9) as

$$u_{n,3}(t) = -1 + \frac{c\sqrt{4\mu - \lambda^2}}{2} \times \left(\cot\left(\frac{\sqrt{4\mu - \lambda^2}}{2}d\right) + \left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn + ct + \zeta)\right) + C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn + ct + \zeta)\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn + ct + \zeta)\right) + C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn + ct + \zeta)\right)}\right)\right), \tag{19}$$

where λ, μ, d, c, ζ, C_1, and C_2 are arbitrary constants.

Remark 2. In particular, if we take $C_2 \neq 0$ and $C_1^2 < C_2^2$ in (19), then we get a discrete periodic wave solution to Eq. (9) as

$$u_{n,4}(t) = -1 + \frac{c\sqrt{4\mu - \lambda^2}}{2}\left(\cot\left(\frac{\sqrt{4\mu - \lambda^2}}{2}d\right) + \cot\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(dn + ct + \zeta) + \tan^{-1}\left(\frac{C_1}{C_2}\right)\right)\right). \tag{20}$$

Case 3. When $\lambda^2 - 4\mu = 0$, from (8c), we have

$$U_{n+1}(\xi_n) = a_0 + a_1\left(-\frac{\lambda}{2} + \left(\left(\frac{\lambda}{2} + \frac{G'(\xi_n)}{G(\xi_n)}\right)/\left(1 + \left(\frac{\lambda}{2} + \frac{G'(\xi_n)}{G(\xi_n)}\right)d\right)\right)\right). \tag{21}$$

Substituting (12) and (21) along with (5) into (11), clearing the denominator and setting the coefficients of $(G'(\xi_n)/G(\xi_n))^l$ $(0 \leq l \leq 4)$ to zero, we derive a system of nonlinear algebraic equations for a_0, a_1, d, c, λ, and μ. Solving the system, we get the following solution set

$$\left\{a_0 = \left(-2 + \frac{2c}{d} + \lambda c\right)/2, a_1 = c\right\} \tag{22}$$

and the corresponding rational function solution to Eq. (9) as

$$u_{n,5}(t) = -1 + c\left(\frac{C_1}{C_1(dn + ct + \zeta) + C_2} + \frac{1}{d}\right), \tag{23}$$

where d, c, ζ, C_1, and C_2 are arbitrary constants.

B. The Relativistic Toda Lattice System

The relativistic Toda lattice system [31] reads

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = (1+\alpha u_n)(v_n - v_{n-1}), \\ \dfrac{\partial v_n}{\partial t} = v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}). \end{cases} \tag{24}$$

To solve Eq. (24), we first make the transformation

$$u_n = U_n(\xi_n), \quad v_n = V_n(\xi_n), \quad \xi_n = dn + ct + \zeta, \tag{25}$$

where c, d, and ζ are constants. Then, we reduce Eq. (24) to

$$\begin{cases} cU_n'(\xi_n) = (1+\alpha U_n(\xi_n))(V_n(\xi_n) - V_{n-1}(\xi_n)), \\ cV_n'(\xi_n) = V_n(\xi_n)(U_{n+1}(\xi_n) - U_n(\xi_n) + \alpha V_{n+1}(\xi_n) - \alpha V_{n-1}(\xi_n)), \end{cases} \tag{26}$$

where the prime denotes the derivative with respect to ξ_n. Because the procedure is similar to the scheme used in Section IIIA, we will omit most of the details here. By the homogeneous balance principle, we assume the solution of Eq. (26) in the form

$$\begin{cases} U_n(\xi_n) = a_0 + a_1\left(\dfrac{G'(\xi_n)}{G(\xi_n)}\right), \quad a_1 \neq 0, \\ V_n(\xi_n) = b_0 + b_1\left(\dfrac{G'(\xi_n)}{G(\xi_n)}\right), \quad b_1 \neq 0, \end{cases} \tag{27}$$

where a_0, a_1, b_0, and b_1 are constants to be determined.

Case 1. $\lambda^2 - 4\mu > 0$.

In this case, we first derive the expressions $U_{n+1}(\xi_n)$ and $V_{n\pm1}(\xi_n)$ in accordance with (8a) and substitute them into Eq. (26). Then, clearing the denominator and setting the coefficients of $(G'(\xi_n)/G(\xi_n))^l$ $(0 \leq l \leq 4)$ to zero, we derive a system of nonlinear algebraic equations for a_0, a_1, b_0, b_1, d, c, λ, and μ. Solving the system, we get the following solution set

$$\left\{ a_0 = \frac{1}{2\alpha}\left(-2 - c\alpha\left(-\lambda + \sqrt{\lambda^2-4\mu}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}d\right)\right)\right), \right.$$
$$\left. a_1 = c, b_0 = -\frac{c}{2\alpha}\left(\lambda - \sqrt{\lambda^2-4\mu}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}d\right)\right), b_1 = \frac{-c}{\alpha} \right\} \tag{28}$$

and the corresponding hyperbolic function solution to Eq. (24) as

$$\begin{cases} u_{n,1}(t) = -\dfrac{1}{\alpha} - \dfrac{c\sqrt{\lambda^2-4\mu}}{2}\left(\cot h\left(\dfrac{\sqrt{\lambda^2-4\mu}}{2}d\right)\right. \\ \qquad \left. - \left(\dfrac{C_1\cos h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right)+C_2\sin h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right)}{C_1\sin h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right)+C_2\cos h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right)}\right)\right), \\ v_{n,1}(t) = \dfrac{c\sqrt{\lambda^2-4\mu}}{2\alpha}\left(\cot h\left(\dfrac{\sqrt{\lambda^2-4\mu}}{2}d\right)\right. \\ \qquad \left. - \left(\dfrac{C_1\cos h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right)+C_2\sin h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right)}{C_1\sin h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right)+C_2\cos h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right)}\right)\right), \end{cases} \tag{29}$$

where λ, μ, d, c, ζ, C_1, and C_2 are arbitrary constants.

Remark 3. In particular, if we take $C_2 \neq 0$ and $C_1^2 < C_2^2$ in (29) then we get a formal solitary wave solution to Eq. (24) as

$$\begin{cases} u_{n,2}(t) = -\frac{1}{\alpha} - \frac{c\sqrt{\lambda^2-4\mu}}{2}\left(\cot h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}d\right) - \tan h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta) + \tan h^{-1}\left(\frac{C_1}{C_2}\right)\right)\right), \\ v_{n,2}(t) = \frac{c\sqrt{\lambda^2-4\mu}}{2\alpha}\left(\cot h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}d\right) - \tan h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta) + \tan h^{-1}\left(\frac{C_1}{C_2}\right)\right)\right). \end{cases} \tag{30}$$

Moreover, letting $\delta = \sqrt{\lambda^2-4\mu}\zeta/2 + \tan h^{-1}(C_1/C_2)$, $d = 2d_1/\sqrt{\lambda^2-4\mu}$, and $c = 2c_1/\sqrt{\lambda^2-4\mu}$ in (30) leads to

$$\begin{cases} u_{n,3}(t) = -\dfrac{1}{\alpha} - c_1\cot h(d_1) + c_1\tan h(d_1 n + c_1 t + \delta), \\ v_{n,3}(t) = \dfrac{c_1}{\alpha}\cot h(d_1) - \dfrac{c_1}{\alpha}\tan h(d_1 n + c_1 t + \delta), \end{cases} \tag{31}$$

which coincides with the result of Baldwin et al. [5].

Case 2. $\lambda^2 - 4\mu < 0$.

In this case, we first derive the expressions $U_{n+1}(\xi_n)$ and $V_{n\pm1}(\xi_n)$ in accordance with (8b) and substitute them into Eq. (26). Then, clearing the denominator and setting the coefficients of $(G'(\xi_n)/G(\xi_n))^l$ $(0 \leq l \leq 3)$ to zero, we derive a system of nonlinear algebraic equations for a_0, a_1, b_0, b_1, d, c, λ, and μ. Solving the system, we get the following solution set

$$\left\{a_0 = \frac{-1}{2\alpha}\left(2 + c\alpha\left(-\lambda + \sqrt{4\mu-\lambda^2}\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}d\right)\right)\right),\right.$$
$$\left. a_1 = c, b_0 = -\frac{c}{2\alpha}\left(\lambda - \sqrt{4\mu-\lambda^2}\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}d\right)\right), b_1 = \frac{-c}{\alpha}\right\} \quad (32)$$

and the corresponding trigonometric function solution to Eq. (24) as

$$\begin{cases} u_{n,4}(t) = -\frac{1}{\alpha} - \frac{c\sqrt{4\mu-\lambda^2}}{2}\left(\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}d\right)\right. \\ \qquad \left. - \left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right) + C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right) + C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)}\right)\right), \\ v_{n,4}(t) = \frac{c\sqrt{4\mu-\lambda^2}}{2\alpha}\left(\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}d\right)\right. \\ \qquad \left. - \left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right) + C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right) + C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)}\right)\right), \end{cases} \quad (33)$$

where λ, μ, d, c, ζ, C_1, and C_2 are arbitrary constants.

Remark 4. As a special example, if we take $C_2 \neq 0$ and $C_1^2 < C_2^2$ in (33), then we obtain a formal periodic wave solution to Eq. (24) as

$$\begin{cases} u_{n,5}(t) = -\frac{1}{\alpha} - \frac{c\sqrt{4\mu-\lambda^2}}{2}\left(\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}d\right) - \cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta) + \tan^{-1}\left(\frac{C_1}{C_2}\right)\right)\right), \\ v_{n,5}(t) = \frac{c\sqrt{4\mu-\lambda^2}}{2\alpha}\left(\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}d\right) - \cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta) + \tan^{-1}\left(\frac{C_1}{C_2}\right)\right)\right). \end{cases} \quad (34)$$

In addition, letting $\theta_0 = \sqrt{4\mu-\lambda^2}\zeta/2 + \tan^{-1}(C_1/C_2)$, $d = 2h/\sqrt{4\mu-\lambda^2}$, and $c = 2\theta_1/\sqrt{4\mu-\lambda^2}$ in (34) leads to

$$\begin{cases} u_{n,6}(t) = -\frac{1}{\alpha} - \theta_1\cot(h) + \theta_1\cot(hn + \theta_1 t + \theta_0), \\ v_{n,6}(t) = \frac{\theta_1}{\alpha}\cot(h) - \frac{\theta_1}{\alpha}\cot(hn + \theta_1 t + \theta_0), \end{cases} \quad (35)$$

which coincides with the Case 2 [formulas (4.6.1) and (4.6.2)] of Yaxuan et al. [32].

Case 3. $\lambda^2 - 4\mu = 0$.

In this case, we first derive the expressions $U_{n+1}(\xi_n)$ and $V_{n\pm1}(\xi_n)$ in accordance with (8c) and substitute them into Eq. (26). Then, clearing the denominator and setting the coefficients of

$(G'(\xi_n)/G(\xi_n))^l$ $(0 \leq l \leq 4)$ to zero, we derive a system of nonlinear algebraic equations for a_0, a_1, b_0, b_1, d, c, λ, and μ. Solving the system, we get the following solution set

$$\left\{a_0 = -\frac{1}{\alpha} - \frac{(2-d\lambda)c}{2d}, a_1 = c, b_0 = \frac{(2-d\lambda)c}{2d\alpha}, b_1 = -\frac{c}{\alpha}\right\} \tag{36}$$

and the corresponding rational function solution to Eq. (24) as

$$\begin{cases} u_{n,7}(t) = -\dfrac{c}{d} - \dfrac{1}{\alpha} + \dfrac{cC_1}{C_1(dn+ct+\zeta)+C_2}, \\ v_{n,7}(t) = \dfrac{c}{d\alpha} - \dfrac{cC_1}{\alpha(C_1(dn+ct+\zeta)+C_2)}, \end{cases} \tag{37}$$

where d, c, ζ, C_1, and C_2 are arbitrary constants.

Remark 5. We note that our solution (37) is not derived in Baldwin et al. [5] and Yaxuan et al. [32].

IV. CONCLUSION

We analyzed the differential–difference Burgers equation and the relativistic Toda lattice system by the so-called discrete (G'/G)-expansion method. Three types of discrete traveling wave solutions with more arbitrary parameters are obtained. We observed that some of the previously known solutions are particular cases of our solutions. We checked the correctness of the obtained results by putting them back into the original equation with the aid of MATHEMATICA; this provides an extra measure of confidence in the results. We conclude that the discrete (G'/G)-expansion method has the potential of discovering many new and interesting solutions for other types of NDDEs.

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