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FINITE NUMBER OF KALUZA–KLEIN MODES, ALL WITH ZERO MASSES

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Kaluza–Klein modes of fermions in a five-dimensional toy model are considered. The number of Kaluza–Klein modes that survive after integration over extra dimensions is finite in this space. Moreover, the extra dimensional piece of the kinetic part of the Lagrangian in this space induces no mass for the higher Kaluza–Klein modes on contrary to the standard lore.

Keywords: Kaluza–Klein modes; extra dimensions.

The use of extra dimension(s) is a popular tool in high energy physics[1–10] because it gives a more tidy picture of nature, that ranges from geometrization of all forces of nature in the spirit of general relativity to a better understanding of the cosmological constant problem, hierarchy problem, fermion generations, Yukawa couplings and flavor etc. The world we live in is apparently four-dimensional. Hence extra dimensions (if exist) must be hidden at present (relatively low) energies. The standard way to ensure this is to take the extra dimension(s) be compact and tiny (e.g. a tiny circle). Then, by Fourier theorem, a field in the whole space can be expanded in a tower of particles that are identical except their masses and their profile in the extra dimension(s). Such a tower of particle (or field) is called a Kaluza–Klein (KK) tower of that particle (or field), and it is an infinite series except in some cases that need complicated boundary conditions to be satisfied.[11–14] Depending on the boundary conditions, the KK tower may contain a zero mode (i.e. a mode that does not depend on the extra dimension(s)) or not. A zero mode does not acquire a mass from the extra dimensional piece of the kinetic part of the Lagrangian while all other modes gain masses of order of $\frac{1}{L}$ where L is the size of the extra dimension. Phenomenological considerations require the masses of the higher KK modes to be at least in TeV scale, and usually in the order of Planck mass for standard model particles.[15] So KK modes except the zero mode cannot be identified with the usual particles. Therefore a scheme where the number of KK modes is finite and all gain zero masses from the kinetic part of the Lagrangian would be highly desirable. In this study we present a toy model for fermions where the number of observed KK modes is finite at current energies, and all modes are massless as long as the kinetic part of the Lagrangian is considered.

In the vein of a framework proposed for cosmological constant problem,[16–19] we consider the following five-dimensional metric

$$\begin{aligned} ds^2 &= g_{BC}\, dx^B\, dx^C = \cos kz (g_{\bar{B}\bar{C}}\, dx^{\bar{B}}\, dx^{\bar{C}}) \\ &= \cos k\, z [g_{\bar{\mu}\bar{\nu}}(x) dx^{\bar{\mu}}\, dx^{\bar{\nu}} - dz^2]\,, \\ & B, C, \bar{B}, \bar{C} = 0,1,2,3,4\,, \qquad \bar{\mu}, \bar{\nu} = 0,1,2,3\,, \end{aligned} \tag{1}$$

where the symbol x with no indices stands for the four-dimensional coordinates $x^{\bar{\mu}}$. We take the extra dimension to be compact and its size be L and $k = \frac{2\pi}{L}$. Although this metric has singularity at $kz = \frac{\pi}{2}$, this singularity does not survive after integration over the extra dimension z (i.e. at the scales larger than the size of the extra dimension). Moreover, the location of the singularity at the sharp value, $kz = \frac{\pi}{2}$ suggests that this singularity may be removed by the metric fluctuations in quantum gravity.[20] So given the toy model nature of this study we will not dwell on this technical point further for the sake of a relatively simple framework to study.

We take $g_{\bar{B}\bar{C}} = \eta_{\bar{B}\bar{C}} = \mathrm{diag}(1,-1,-1,-1,-1)$ (i.e. $g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}}$) to have a simple model where one can focus on the essential points of the model. The action for (free) fermionic fields for this space is

$$\begin{aligned} S_f &= \int (\cos kz)^{\frac{5}{2}} \mathcal{L}_f\, d^4x\, dz \\ &= \int (\cos kz)^2 i\bar{\chi}\gamma^a \left(\partial_a + \frac{k}{8}\tan kz [\gamma_4, \gamma_a]\right) \chi\, d^4x\, dz + \mathrm{H.C.}\,, \\ & \{\gamma^a, \gamma^b\} = 2\eta^{ab}\,, \qquad (\eta^{ab}) = \mathrm{diag}(1,-1,-1,-1,-1) \end{aligned} \tag{2}$$

where H.C. stands for Hermitian conjugate, and the second term is spin connection term (see Appendix A). The small Latin indices a, b, etc. refer to the tangent space while the capital Latin indices A, B, etc. refer to the space defined by (1). The tangent space in this case coincides with $g_{\bar{B}\bar{C}}\, dx^{\bar{B}} dx^{\bar{C}} = \eta_{\bar{B}\bar{C}}\, dx^{\bar{B}}\, dx^{\bar{C}}$. So the indices with a bar above also refer to the tangent space in this paper. The action is required to be invariant under the five-dimensional spacetime reflections, namely,

$$x^a \to -x^a\,, \qquad a = 0,1,2,3,4 \tag{3}$$

where all coordinates are spacetime reflected simultaneously.

χ may be Fourier decomposed in the extra dimension as

$$\chi = \chi_{\mathcal{A}} + \chi_{\mathcal{S}}\,, \tag{4}$$

$$\chi_{\mathcal{A}}(x,z) = \sum_{n=-\infty}^{\infty} \chi_n^{\mathcal{A}}(x) \sin\left(\frac{1}{2} n\, kz\right) = \sum_{|n|=1}^{\infty} \tilde{\chi}_{|n|}^{\mathcal{A}}(x) \sin\left(\frac{1}{2}|n| kz\right), \tag{5}$$

$$\chi_{\mathcal{S}}(x,z) = \sum_{n=-\infty}^{\infty} \chi_n^{\mathcal{S}}(x) \cos\left(\frac{1}{2} n\, kz\right) = \chi_0(x) + \sum_{|n|=1}^{\infty} \tilde{\chi}_{|n|}^{\mathcal{S}}(x) \cos\left(\frac{1}{2}|n|kz\right), \tag{6}$$

$$\tilde{\chi}_{|n|}^{\mathcal{A}}(x) = \chi_n^{\mathcal{A}}(x) - \chi_{-n}^{\mathcal{A}}(x)\,, \qquad \tilde{\chi}_{|n|}^{\mathcal{S}}(x) = \chi_n^{\mathcal{S}}(x) + \chi_{-n}^{\mathcal{S}}(x)$$

(where the absolute value signs in $|n|$ is used to emphasize the positiveness of n in those terms, and half-fractional values in the sum correspond to anti-periodic boundary conditions). The form of $\chi_n^{\mathcal{A}(\mathcal{S})}$ is determined by the requirement of covariance under (the spinor representation of) $\mathrm{SO}(3,1)$ and is given by

$$\begin{gathered} \chi_n^{\mathcal{A}(\mathcal{S})} = \chi_{0\,n}^{\mathcal{A}(\mathcal{S})} + \sum \Gamma^4 \chi_{4n}^{\mathcal{A}(\mathcal{S})}\,, \\ \{\Gamma^B, \Gamma^C\} = \frac{2}{\cos kz}\eta^{BC}\,, \qquad B, C = 0, 1, 2, 3, 4\,, \end{gathered} \tag{7}$$

where $\Gamma^{B(C)}$'s are the gamma matrices of (1). However, we let $\chi_n^{\mathcal{A}(\mathcal{S})}$ simply be $\chi_{0\,n}^{\mathcal{A}(\mathcal{S})}$ for the sake of simplicity and it does not essentially change the result as we shall mention when we discuss the masses of the KK modes. Let us return to the main subject after this remark. We take $\chi_n^{\mathcal{A}(\mathcal{S})}$ to transform under (3) as

$$\chi_n(x) \to (-1)^{\lambda_n} \mathcal{CPT}\, \chi_n(-x)\,, \qquad \lambda_n = \frac{1}{2}(-1)^{\frac{n}{2}}\,, \tag{8}$$

where the upper indices $\mathcal{A}$ and $\mathcal{S}$ are suppressed, and $\mathcal{CPT}$ denotes the usual four-dimensional CPT operator (acting on the spinor part of the field). Only the positions of fields (i.e. x^a's) are multiplied by -1 while the orientation of the fields in the spacetime remain essentially the same, i.e. the spinor part of χ remains essentially the same. In this respect (3) is the analog of CPT transformation rather than PT transformation in four dimensions. The invariance of (2) under (3) requires $i\bar{\chi}\gamma^a\partial_a\chi$ in $\mathcal{L}_f$ be invariant under (3). $i\bar{\chi}\gamma^\mu\partial_\mu\chi$ is invariant under four-dimensional CPT. These together imply that $i\bar{\chi}\gamma^\mu\partial_\mu\chi$ (i.e. $i\bar{\chi}\gamma^\mu\chi$) is even under the extra dimensional part of (3). So the possible form of $\mathcal{L}_f$ (after requiring it be odd under (3)) is

$$i\bar{\chi}_{\mathcal{S}}\gamma^a\partial_a\chi_{\mathcal{S}} \quad \text{and/or} \quad i\bar{\chi}_{\mathcal{A}}\gamma^a\partial_a\chi_{\mathcal{A}}\,. \tag{9}$$

In other words, (3) requires $\mathcal{L}_f$ to be either of the terms in (9) or their linear combination.

Further, the invariance of the action under an extra dimensional reflection similar to the one given in Refs. 16 and 17

$$kz \to \pi + kz \tag{10}$$

is imposed. Under (10) the volume element in (2) transforms as

$$(\cos kz)^{\frac{5}{2}}\, d^4x\, dz \to \sqrt{-1}(\cos kz)^{\frac{5}{2}}\, d^4x\, dz\,. \tag{11}$$

Then invariance of (2) under (10) requires $i\bar{\chi}\gamma^a\partial_a\chi$ to be even under the same transformation. We impose χ satisfy anti-periodic boundary conditions,[21] i.e. $\chi(z=0) = -\chi(z=L)$. This sets n in (5), (6) to be odd. Then, the invariance of action (including the quantum paths) under (3), (8) and (10) requires the four-dimensional part of S_f be (see Appendix B)

$$\begin{aligned}
&\sum_{r,s=0}^{\infty}\int d^4x\, i\bar{\chi}_{(2|r|+1)}\gamma^{\bar{\mu}}\partial_{\bar{\mu}}\chi_{(2|s|+1)}\\
&\qquad\times 2\int dz(\cos kz)^2\left[\cos\frac{2|r|+1}{2}kz\cos\frac{2|s|+1}{2}kz\right.\\
&\qquad\quad\left.-\sin\frac{2|r|+1}{2}kz\sin\frac{2|s|+1}{2}kz\right]+\mathrm{H.C.}\\
&\quad=\sum_{r,s=0}^{\infty}\int d^4x\, i\bar{\chi}_{(2|r|+1)}\gamma^{\bar{\mu}}\partial_{\bar{\mu}}\chi_{(2|s|+1)}\\
&\qquad\times\int_0^L dz(\cos 2kz+1)\cos(|r|+|s|+1)kz+\mathrm{H.C.}\\
&\quad=\frac{1}{2}\sum_{r,s=0}^{\infty}\int d^4x\, i\bar{\chi}_{(2|r|+1)}\gamma^{\bar{\mu}}\partial_{\bar{\mu}}\chi_{(2|s|+1)}\int_0^L dz[\cos\left(|r|+|s|-1\right)kz]+\mathrm{H.C.}\,,
\end{aligned} \tag{12}$$

where $2r+1 = 4l+1$, $2s+1 = 4p+3$ $(l,p = 0,1,2,\ldots)$ or vice versa. Because of the periodicity of cosine function, the terms in (12) give nonzero contributions after integration over z only if the arguments of cosines are zero. This is possible only when

$$|r|+|s|-1 = 0 \Rightarrow r=0\,,\quad s=1 \quad \text{or} \quad s=1\,,\quad r=0\,. \tag{13}$$

The result of z integration in (12) is

$$\frac{L}{2}\int d^4x[i\bar{\chi}_1\gamma^{\bar{\mu}}\partial_{\bar{\mu}}\chi_3 + i\bar{\chi}_3\gamma^{\bar{\mu}}\partial_{\bar{\mu}}\chi_1]+\mathrm{H.C.} \tag{14}$$

The diagonalization of (14) results in

$$\frac{1}{2}L\int d^4x[i\bar{\psi}\gamma^{\bar{\mu}}\partial_{\bar{\mu}}\psi - i\bar{\tilde{\psi}}\gamma^{\bar{\mu}}\partial_{\bar{\mu}}\tilde{\psi}]+\mathrm{H.C.} \tag{15}$$

$$\psi = \frac{1}{\sqrt{2}}(\chi_1+\chi_3)\,,\qquad \tilde{\psi} = \frac{1}{\sqrt{2}}(\chi_1-\chi_3)\,. \tag{16}$$

Hence there are one usual fermion and one ghost fermion in the spectrum.

The $i\bar{\chi}\gamma^4\partial_4\chi$ part of $\mathcal{L}_f$ reduces to $\sin\frac{|n|-|m|}{2}kz$ type of terms as a result of the action of the derivative operator ∂_4 in (2) (see Appendix A.2). This, in turn, results in odd number of sine terms (in the action) that leads to zero after integration over z. The number of modes that survive after integration may be increased by changing the extra dimension dependent conformal factor and/or the dimension of the space. For example if the conformal factor in (1) is changed to $\cos^2 kz$, then the condition (13) is changed into $r + s - 3 = 0$. The kinetic term induces no mass term in this case as well because the extra dimensional derivatives induce odd number of sine terms in this case as well. The same second term in (2), that is, the spin connection term also induces no mass term because it contains odd number of sine terms as well (see Appendix A.1). A similar conclusion should be expected for more complicated conformal terms or higher dimensional spaces. In other words, no mass is induced for Kaluza–Klein modes through the extra dimensional part of the kinetic term in this model, and similar results are expected for more complicated situations with similar conformal terms and symmetries. Here we have taken χ_n's to be simply given by the first terms in (7). However, taking the general form does not change the conclusion because vanishing of the extra dimensional kinetic term after integration follows directly from the extra dimensional coordinates rather than the extra dimensional form of the spinor.

We have introduced an extra dimensional model where only two modes of Kaluza–Klein tower appear at low energies. These modes correspond to a fermion and a ghost fermion. These fermions are massless provided we do not introduce a bulk mass term explicitly. The ghost fermion may be identified by a Lee–Wick[23] or Pauli–Villars[22] type regularization field. These results are quite nonstandard both in the emergence of a finite number of Kaluza–Klein modes and the modes higher than zero mode gaining no masses through the extra dimensional piece of kinetic term at low energies where the extra dimensions become directly unobservable. In fact this is also the basic tool to distinguish this scheme from the usual Kaluza–Klein prescription. If nature behaves in the way described here, then all Kaluza–Klein modes of a fermion will be observed at short distances smaller than the size of the corresponding extra dimension while only a finite number of these modes will be detected at larger scales after they are produced (even when they are stable or long living so that they can travel large distances before decay). Moreover, the coupling of fermions to other particles would vary nonlinearly with distance at the scales smaller than the size of the extra dimension since the screening effect of the conformal factor $\cos kz$ changes nonlinearly at distances below the size of the extra dimension. This would be another characteristic of this type of models. In fact one may easily find different metrics of different form and in different dimensions with finite number of Kaluza–Klein modes (obtained after integration over extra dimensions) and all with zero masses. The aim of this study is to show the possibility of obtaining finite number of Kaluza–Klein modes at low energies, and the possibility of massless Kaluza–Klein modes higher than zero mode. So a relatively simple model where these properties can be observed is studied here rather than a detailed model that is in agreement with phenomenology. We hope that different variations of such models with more realistic spectra may be found in the future.

Appendix A. Possible Contributions to Masses Due to the Spin Connection and the Extra Dimensional Part of the Kinetic Term

A.1. *Contribution due to spin connection*

The vielbeins, e^a_B, corresponding to the metric, g_{BC} in (1), and those corresponding to its inverse g^{BC} are determined from

$$g_{BC} = \eta_{ab} e^a_B e^b_C \,, \qquad g^{BC} = \eta^{ab} e_a^B e_b^C \,, \tag{A.1}$$

where the lower indices a, b stand for the tangent space of the original space (e.g. the one defined by (1)). The vielbeins corresponding to the metric in (1) are found to be

$$e^a_B = \sqrt{\cos kz}\, \delta^a_B \,, \qquad e_a^B = \frac{1}{\sqrt{\cos kz}} \delta_a^B \,, \tag{A.2}$$

where δ^a_B, δ_a^B are the Kronecker delta, and $(\eta_{AB}) = \mathrm{diag}(1, -1, -1, -1, -1)$. In curved spaces the derivative term ∂_B when acting on spinors is replaced by[24] D_B

$$D_B = \partial_B + \frac{i}{2} J_{bc} \omega_B^{bc} \,, \tag{A.3}$$

where $J_{bc} = -\frac{i}{4}[\gamma_b, \gamma_c]$.

Here $\gamma_{b(c)}$ are the (flat) tangent space Dirac gamma matrices that are related to the gamma matrices of the original space Γ_B by

$$\Gamma_B = e^a_B \gamma_a \,, \qquad \{\Gamma_B, \Gamma_C\} = 2g_{AB} = \frac{2}{\cos kz} \eta_{BC} \,, \qquad \{\gamma_a, \gamma_b\} = 2\eta_{ab} \,,$$

$$B, C = 0, 1, 2, 3, 4 \,, \qquad a, b = \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4} \,,$$

where the bars over the integers are used to emphasize that they belong to the tangent space, and ω_B^{bc}'s are the spin connections, that are given by

$$\omega_B^{bc} = \left[e^b_K \left(\frac{\partial e^c_P}{\partial x^B} \right) - \Gamma^F_{PB} e^b_K e^c_F \right] g^{KP} \,, \tag{A.4}$$

where $\Gamma^F_{PB} = \frac{1}{2} g^{FG}(g_{PG,B} + g_{BG,P} - g_{PB,G})$ denotes Christoffel symbols, and the commas denote the usual derivative with respect to that coordinate. The nonvanishing Γ^F_{PB}'s in the space defined by Eq. (1) are

$$\Gamma^\mu_{\nu 4} = -\frac{k}{2} \delta^\mu_\nu \tan kz \,, \qquad \Gamma^4_{\mu\nu} = \frac{k}{2} \eta_{\mu\nu} \tan kz \,, \qquad \Gamma^4_{44} = \frac{k}{2} \tan kz \,. \tag{A.5}$$

So the spin connection that gives a nonzero contribution is found to be

$$\omega_\mu^{bc} = \frac{k}{2} \tan kz [\delta^b_4 \delta^c_\mu - \delta^b_\mu \delta^c_4] \,. \tag{A.6}$$

The ω_4^{bc} element of spin connection is found to be zero. Then

$$D_\mu = \partial_\mu + \frac{i}{2} J_{bc}\omega_\mu^{bc} = \partial_\mu + \frac{k}{8}\tan kz[\gamma_{\bar{4}}, \gamma_{\bar{\mu}}]\,, \qquad D_4 = \partial_4\,, \tag{A.7}$$

$$\Gamma^B D_B = e_a^\mu \gamma^a D_\mu + e_a^4 \gamma^a \partial_4 = \frac{1}{\sqrt{\cos kz}}[\gamma^{\bar{\mu}} D_{\bar{\mu}} + \gamma^{\bar{4}}\partial_{\bar{4}}]\,, \tag{A.8}$$

where B, C, F etc. denote the spacetime coordinates while a, b, c etc. and the indices with a bar above e.g. $\bar{B}$, $\bar{4}$ etc. denote the tangent space. Although there is a bar over 4 in (A.7), that bar is omitted in (2) to simplify the notation. Thus the result may be written in a more compact form as in Eq. (2) where ω_4^{ab} gives null contribution.

After using Eqs. (A.7) and (A.8) one obtains Eq. (2). It is evident from (2) and Eq. (A.6) that the integration of the spin connection term $e_a^\mu \gamma^a \omega_\mu^{bc} J_{bc}$ over the extra dimension z is proportional to

$$\int_0^{2\pi} (\cos kz)^2 \tan kz\, d(kz) = 0\,. \tag{A.9}$$

In other words, the spin connection term does not contribute to the masses of ψ, $\tilde{\psi}$ of Eq. (16) at (relatively low energies) where the extra dimension cannot be seen.

A.2. *Contribution due to the extra dimensional part of the kinetic term*

The extra dimensional part of the kinetic term for the action of the field χ is

$$\begin{aligned}
&\int (\cos kz)^{\frac{5}{2}} i\bar{\chi}\gamma^4 \partial_4 \chi\, d^4x\, dz = \sum_{r,s=0}^{\infty} \int d^4x\, i\bar{\chi}_{(2|r|+1)}\gamma^4 \chi_{(2|s|+1)} \int dz (\cos kz)^2 \\
&\quad \times \left\{ \left(\cos\frac{2|r|+1}{2}kz + \sin\frac{2|r|+1}{2}kz \right) \partial_4 \left(\cos\frac{2|s|+1}{2}kz - \sin\frac{2|s|+1}{2}kz \right) \right. \\
&\quad \left. + \left(\cos\frac{2|r|+1}{2}kz - \sin\frac{2|r|+1}{2}kz \right) \partial_4 \left(\cos\frac{2|s|+1}{2}kz + \sin\frac{2|s|+1}{2}kz \right) \right\} \\
&\quad = -k \sum_{r,s=0}^{\infty} (2|s|+1) \int d^4x\, i\bar{\chi}_{(2|r|+1)}\gamma^4 \chi_{(2|s|+1)} \int dz (\cos kz)^2 \\
&\qquad \times \left[\cos\frac{2|r|+1}{2}kz \sin\frac{2|s|+1}{2}kz + \sin\frac{2|r|+1}{2}kz \cos\frac{2|s|+1}{2}kz \right] = 0\,,
\end{aligned} \tag{A.10}$$

where H.C. (in Eq. (12)) that stands for the addition of the Hermitian conjugate of the preceding term is suppressed. So the extra dimensional piece of the kinetic term in this paper does not contribute to the masses of ψ or $\tilde{\psi}$ at length scales larger than the size of the extra dimension.

Appendix B. Derivation of Eq. (12)

It is observed that

as $kz \to \pi + kz$

$$
\begin{aligned}
&\text{(i) if} \quad n = 4l+1 \Rightarrow \left(\cos\frac{n}{2}kz + \sin\frac{n}{2}kz\right) \to \left(\cos\frac{n}{2}kz - \sin\frac{n}{2}kz\right)\\
&\qquad\qquad\qquad\quad\ \left(\cos\frac{n}{2}kz - \sin\frac{n}{2}kz\right) \to -\left(\cos\frac{n}{2}kz + \sin\frac{n}{2}kz\right)\\
&\text{(ii) if} \quad n = 4l+3 \Rightarrow \left(\cos\frac{n}{2}kz + \sin\frac{n}{2}kz\right) \to -\left(\cos\frac{n}{2}kz - \sin\frac{n}{2}kz\right)\\
&\qquad\qquad\qquad\quad\ \left(\cos\frac{n}{2}kz - \sin\frac{n}{2}kz\right) \to \left(\cos\frac{n}{2}kz + \sin\frac{n}{2}kz\right)\\
&\qquad\qquad\qquad\quad\ l = 0, 1, 2, \ldots\,.
\end{aligned}
\tag{B.1}
$$

The requirement that the action (2) be invariant under (8) requires $n = 4l+1$ type of modes couple to $m = 4p+3$ type of modes. In the light of this observation the combination that is invariant under (10) is

$$
\begin{aligned}
&\left\{\left(\cos\frac{2|r|+1}{2}kz + \sin\frac{2|r|+1}{2}kz\right)\left(\cos\frac{2|r|+1}{2}kz - \sin\frac{2|r|+1}{2}kz\right)\right.\\
&\qquad \left. + \left(\cos\frac{2|r|+1}{2}kz - \sin\frac{2|r|+1}{2}kz\right)\left(\cos\frac{2|s|+1}{2}kz + \sin\frac{2|s|+1}{2}kz\right)\right\}\\
&\quad = 2\left[\cos\frac{2|r|+1}{2}kz\cos\frac{2|s|+1}{2}kz - \sin\frac{2|r|+1}{2}kz\sin\frac{2|s|+1}{2}kz\right],
\end{aligned}
\tag{B.2}
$$

where

$$
\begin{aligned}
&2|r|+1 = 4l+1 \quad \text{and} \quad 2|s|+1 = 4p+3\\
&\text{or} \quad 2|r|+1 = 4l+3 \quad \text{and} \quad 2|s|+1 = 4p+1\\
&l, p = 0, 1, 2, 3, \ldots\,.
\end{aligned}
\tag{B.3}
$$

So the four-dimensional part of S_f becomes

$$
\begin{aligned}
&\sum_{r,s=0}^{\infty}\int d^4x\, i\bar{\chi}_{(2|r|+1)}\gamma^{\bar{\mu}}\partial_{\bar{\mu}}\chi_{(2|s|+1)}\int dz(\cos kz)^2\\
&\quad \times\left\{\left(\cos\frac{2|r|+1}{2}kz + \sin\frac{2|r|+1}{2}kz\right)\left(\cos\frac{2|r|+1}{2}kz - \sin\frac{2|r|+1}{2}kz\right)\right.\\
&\qquad \left. + \left(\cos\frac{2|r|+1}{2}kz - \sin\frac{2|r|+1}{2}kz\right)\left(\cos\frac{2|s|+1}{2}kz + \sin\frac{2|s|+1}{2}kz\right)\right\},
\end{aligned}
\tag{B.4}
$$

where the H.C. symbol (as in Eq. (A.10)) is suppressed. (B.4) after using (B.2) results in (12). It is evident from (12) that the resulting Lagrangian has the form required by (9) as well.

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References

1. T. Kaluza, *Sitzungsber. Preuss. Akad. Wiss. Berlin* (*Math. Phys.*) **K41**, 966 (1921).
2. O. Klein, *Z. Phys.* **37**, 895 (1926).
3. M. J. Duff, Kaluza–Klein theory in perspective, in *Stockholm 1994*, *The Oskar Klein Centenary*, preprint arXiv:hep-th/9410046.
4. N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *Phys. Lett. B* **429**, 263 (1998).
5. I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *Phys. Lett. B* **436**, 257 (1998).
6. N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *Phys. Rev. D* **59**, 086004 (1999).
7. L. Randall and R. Sundrum, *Phys. Rev. Lett.* **83**, 3370 (1999).
8. L. Randall and R. Sundrum, *Phys. Rev. Lett.* **83**, 4690 (1999).
9. Y. Uehara, *Mod. Phys. Lett. A* **17**, 1551 (2002).
10. T. Flacke, D. Hooper and J. Marc-Russell, *Phys. Rev. D* **73**, 095002 (2006) [Erratum-*ibid.* **74**, 019902 (2006)].
11. F. Grard and J. Nuyts, *Phys. Rev. D* **74**, 124013 (2006).
12. F. Grard and J. Nuyts, *Phys. Rev. D* **76**, 124022 (2007).
13. F. Grard and J. Nuyts, arXiv:0712.3692.
14. F. Grard and J. Nuyts, arXiv:0803.1741.
15. I. Antoniadis, *Phys. Lett. B* **246**, 377 (1990).
16. R. Erdem, *Phys. Lett. B* **621**, 11 (2005).
17. R. Erdem, *Phys. Lett. B* **639**, 348 (2006).
18. R. Erdem, *J. Phys. A* **40**, 6945 (2007).
19. R. Erdem, *J. Phys. A* **41**, 235401 (2008).
20. E. I. Guendelman and A. B. Kaganovich, *Class. Quantum Grav.* **25**, 235015 (2008).
21. P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory* (Springer, 1997), pp. 168–171.
22. W. Pauli and F. Villars, *Rev. Mod. Phys.* **21**, 434 (1949).
23. T. D. Lee and G. C. Wick, *Phys. Rev. D* **2**, 1033 (1970).
24. This may be found in many textbooks although the conventions may vary a little. Our convention is that of S. Weinberg, *Quantum Theory of Fields* (Cambridge Univ. Press, 2000), Vol. 3.