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Short communication

A discrete generalization of the extended simplest equation method

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ABSTRACT

We modified the so-called extended simplest equation method to obtain discrete traveling wave solutions for nonlinear differential-difference equations. The Wadati lattice equation is chosen to illustrate the method in detail. Further discrete soliton/periodic solutions with more arbitrary parameters, as well as discrete rational solutions, are revealed. We note that using our approach one can also find in principal highly accurate exact discrete solutions for other lattice equations arising in the applied sciences.

1. Introduction

As it is well known, the investigation of nonlinear differential-difference equations (NDDEs) is still of interest since the original work of Fermi, Pasta and Ulam in the 1950s [1]. NDDEs, playing an important role in the study of nonlinear physical phenomena, have become the focus of common concern in various branches of applied sciences such as condensed matter physics, biophysics, and mechanical engineering, etc., and in different physical problems such as molecular crystals, currents in electrical networks, atomic chains, etc. [2–5]. In high energy physics, one can also encounter NDDEs in numerical simulation of soliton dynamics. Contrary to difference equations which are being fully discretized, NDDEs are semi-discretized with some (or all) of their space variables discretized while time is usually kept continuous.

In the last four decades or so, searching for exact discrete analytic solutions of NDDEs by using a range of different analytic methods has been the main purpose of many researchers. To this end, some attractive powerful methods primarily developed for solving nonlinear evolution equations (NEEs) are generalized to a considerable number of NDDEs. How to extend a method for NEEs to solve NDDEs is an interesting and important issue. For instance, Hirota's bilinear method is considered by Hu and Ma [6] to construct special soliton like solutions of the Toeplitz lattice. Liu et al. [7] have found explicit and exact travelling wave solutions to three NDDEs by using the Jacobi elliptic function expansion method. Baldwin and his team [8], with the aid of a computer algebra system, developed an algorithm for discrete nonlinear models in terms of a tanh function. Their work can be thought as a breakthrough for solving NDDEs symbolically. Dai et al. [9] presented an extended Jacobian elliptic function algorithm for NDDEs. The Exp-function method is extended by Zhu [10,11] to obtain some physically important solutions. Xie et al. [12] investigated the discrete sine-Gordon equation by applying a method which is based on Riccati equation expansion. By using the so-called ADM-Padé technique, Yang et al. [13] studied two nonlinear lattice equations. Hu et al. [14], implementing the homotopy perturbation method, analyzed a nonlinear differential-difference equation arising in nanotechnology. More recently, Zhen [15] devised a discrete tanh method for NDDEs, and so on.

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Generally, it is hard to extend an analytic method for NDDEs because series obstacles arise in searching for iterative relations from indices n to $n \pm i$. However, there is plenty of substantial work still to be done for the applicability of distinct analytic methods to NDDEs for it is obvious that no method can solve all types of NDDEs. As far as we could verify, little work is being done to symbolically compute exact discrete solutions of NDDEs while there has been a considerable amount of work done on finding exact solutions to NEEs. Hence, it will make sense to do more research on exactly solving NDDEs by improving known methods.

In 2005, Kudryashov [16,17] proposed the simplest equation method to search for exact solutions of nonlinear differential equations in the form of solitary and periodic waves. Two basic ideas are taken into consideration for the proposed method. The first idea is to use the simplest *nonlinear* ordinary differential equation (having *lesser* order then the equation studied) with known general solution to construct new special solutions. For example, as the simplest equation, one can use the Riccati equation, the equation for the Jacobi elliptic function, the equation for the Weierstrass elliptic function, etc. The second idea is to account all possible singularities of the solutions of the equation studied. Later, in 2008, Kudryashov and Loguinova [18] modified the simplest equation method, called the extended simplest equation method, by considering a *higher* (third) order *linear* ordinary differential equation as another simplest equation. Henceforth, by inspiring their pioneer work, our goal in this study is to further generalize the extended simplest equation method for exactly solving NDDEs.

The rest of this paper is organized as follows. In Section 2, we describe our method, which is originated from the extended simplest equation method, for finding exact discrete traveling wave solutions of NDDEs, and state the main steps. In Section 3, we illustrate the method in detail by studying the so-called Wadati lattice equation. Finally, some conclusions are given in Section 4.

2. Methodology

Assume that we have a system of M polynomial NDDEs in the form

$$\Delta\left(\mathbf{u}_{\mathbf{n}+\mathbf{p}_1}(\mathbf{x}), \ldots, \mathbf{u}_{\mathbf{n}+\mathbf{p}_k}(\mathbf{x}), \ldots, \mathbf{u}'_{\mathbf{n}+\mathbf{p}_1}(\mathbf{x}), \ldots, \mathbf{u}'_{\mathbf{n}+\mathbf{p}_k}(\mathbf{x}), \ldots, \mathbf{u}^{(r)}_{\mathbf{n}+\mathbf{p}_1}(\mathbf{x}), \ldots, \mathbf{u}^{(r)}_{\mathbf{n}+\mathbf{p}_k}(\mathbf{x})\right) = 0, \tag{1}$$

in which the dependent variable $\mathbf{u_n}$ have M components $u_{i,\mathbf{n}}$ and so do its shifts; the continuous variable $\mathbf{x}$ has N components x_i; the discrete variable $\mathbf{n}$ has Q components n_j; the k shift vectors $\mathbf{p}_i \in \mathbb{Z}^Q$; and $\mathbf{u}^{(r)}(\mathbf{x})$ denotes the collection of mixed derivative terms of order r.

Step 1. To search for travelling wave solutions of Eq. (1), we first take into consideration the wave transformation

$$\mathbf{u}_{\mathbf{n}+\mathbf{p}_s}(\mathbf{x}) = \mathbf{U}_{\mathbf{n}+\mathbf{p}_s}(\xi_\mathbf{n}), \xi_\mathbf{n} = \sum_{i=1}^{Q} d_i n_i + \sum_{j=1}^{N} c_j x_j + \zeta, \quad (s = 1, 2, \ldots, k), \tag{2}$$

where the coefficients $c_1, c_2, \ldots, c_N, d_1, d_2, \ldots, d_Q$ and the phase ζ are all constants. Then, Eq. (1) changes into

$$\Delta\left(\mathbf{U}_{\mathbf{n}+\mathbf{p}_1}(\xi_\mathbf{n}), \ldots, \mathbf{U}_{\mathbf{n}+\mathbf{p}_k}(\xi_\mathbf{n}), \ldots, \mathbf{U}'_{\mathbf{n}+\mathbf{p}_1}(\xi_\mathbf{n}), \ldots, \mathbf{U}'_{\mathbf{n}+\mathbf{p}_k}(\xi_\mathbf{n}), \ldots, \mathbf{U}^{(r)}_{\mathbf{n}+\mathbf{p}_1}(\xi_\mathbf{n}), \ldots, \mathbf{U}^{(r)}_{\mathbf{n}+\mathbf{p}_k}(\xi_\mathbf{n})\right) = 0. \tag{3}$$

Step 2. We suppose that the solution of Eq. (3) is in the finite series expansion form

$$\mathbf{U_n}(\xi_\mathbf{n}) = \sum_{l=-m}^{m} a_i \left(\frac{\psi'(\xi_\mathbf{n})}{\psi(\xi_\mathbf{n})}\right)^l, \tag{4}$$

where m is a positive integer, a_is are constants to be determined later, $\psi(\xi_\mathbf{n})$ is the general solution of the simplest equation. Here, we would like to take the full advantage of linear theory, and thus we let the simplest equation in $\psi(\xi_\mathbf{n})$ be the second-order linear ordinary differential equation

$$\psi''(\xi_\mathbf{n}) + k\psi(\xi_\mathbf{n}) = 0, \tag{5}$$

where k is an arbitrary constant and prime denotes derivative with respect to $\xi_\mathbf{n}$. We also point out that the power of the extended simplest equation method lies in the fact that it has the flexibility of choosing such equation. The general solution of the simplest Eq. (5) is well known for us. Thus, we get the following cases:

$$\frac{\psi'(\xi_\mathbf{n})}{\psi(\xi_\mathbf{n})} = \sqrt{-k}\left(\frac{A_1 \cosh\left(\sqrt{-k}\xi_\mathbf{n}\right) + A_2 \sinh\left(\sqrt{-k}\xi_\mathbf{n}\right)}{A_1 \sinh\left(\sqrt{-k}\xi_\mathbf{n}\right) + A_2 \cosh\left(\sqrt{-k}\xi_\mathbf{n}\right)}\right), \quad k < 0, \tag{6a}$$

$$\frac{\psi'(\xi_\mathbf{n})}{\psi(\xi_\mathbf{n})} = \sqrt{k}\left(\frac{-A_1 \sin\left(\sqrt{k}\xi_\mathbf{n}\right) + A_2 \cos\left(\sqrt{k}\xi_\mathbf{n}\right)}{A_1 \cos\left(\sqrt{k}\xi_\mathbf{n}\right) + A_2 \sin\left(\sqrt{k}\xi_\mathbf{n}\right)}\right), \quad k > 0, \tag{6b}$$

$$\frac{\psi'(\xi_{\mathbf{n}})}{\psi(\xi_{\mathbf{n}})}=\frac{A_1}{A_1\xi_{\mathbf{n}}+A_2},\quad k=0, \tag{6c}$$

where A_1 and A_2 are arbitrary constants.

Step 3. By a simple calculation, we can get the identity

$$\xi_{\mathbf{n}+\mathbf{p}_s}=\xi_{\mathbf{n}}+\varphi_s,\quad \varphi_s=p_{s1}d_1+p_{s2}d_2+\cdots+p_{sQ}d_Q, \tag{7}$$

where p_{sj} is the jth component of the shift vector $\mathbf{p}_s$. Hence, considering the trigonometric/hyperbolic function identities and using the functions (6a)–(6c) as well as (7), we derive the uniform formulas

$$\mathbf{U}_{\mathbf{n}+\mathbf{p}_s}(\xi_{\mathbf{n}})=\sum_{l=-m}^{m}a_i\left(\frac{\sqrt{-k}\psi'(\xi_{\mathbf{n}})-k\tanh\left(\sqrt{-k}\varphi_s\right)\psi(\xi_{\mathbf{n}})}{\sqrt{-k}\psi(\xi_{\mathbf{n}})+\tanh\left(\sqrt{-k}\varphi_s\right)\psi'(\xi_{\mathbf{n}})}\right)^l,\quad k<0, \tag{8a}$$

$$\mathbf{U}_{\mathbf{n}+\mathbf{p}_s}(\xi_{\mathbf{n}})=\sum_{l=-m}^{m}a_i\left(\frac{\sqrt{k}\psi'(\xi_{\mathbf{n}})-k\tan\left(\sqrt{k}\varphi_s\right)\psi(\xi_{\mathbf{n}})}{\sqrt{k}\psi(\xi_{\mathbf{n}})+\tan\left(\sqrt{k}\varphi_s\right)\psi'(\xi_{\mathbf{n}})}\right)^l,\quad k>0, \tag{8b}$$

$$\mathbf{U}_{\mathbf{n}+\mathbf{p}_s}(\xi_{\mathbf{n}})=\sum_{l=-m}^{m}a_i(\psi'(\xi_{\mathbf{n}})/(\psi(\xi_{\mathbf{n}})+\varphi_s\psi'(\xi_{\mathbf{n}})))^l,\quad k=0. \tag{8c}$$

Step 4. Balancing the highest order nonlinear term(s) and the highest order derivative term in $\mathbf{U}_{\mathbf{n}}(\xi_{\mathbf{n}})$ as in the continuous case, we can easily determine the degree m of Eqs. 4 and (8a)–(8c) from Eq. (3). Since $\mathbf{U}_{\mathbf{n}+\mathbf{p}_s}$ can be interpreted as being of degree zero in $(\psi'(\xi_{\mathbf{n}})/\psi(\xi_{\mathbf{n}}))$, the leading terms of $\mathbf{U}_{\mathbf{n}+\mathbf{p}_s}(\mathbf{p}_s\neq 0)$will not have any effect on the balancing procedure.

Step 5. Substituting the ansätze 4 and (8a)–(8c) together with (5) into Eq. (3), equating the coefficients of $(\psi'(\xi_{\mathbf{n}})/\psi(\xi_{\mathbf{n}}))^l(l=0,1,2,\ldots)$ to zero, we obtain a system of nonlinear algebraic equations from which the undetermined constants $a_i,\quad d_i,c_j$, and k can be explicitly found. Substituting these results into (4), we can derive varies kind of exact discrete solutions to Eq. (1). Finally, it is essential to substitute the obtained solutions back into the original Eq. (1) to assure their correctness.

3. Application

An important model for discrete solitons is the lattice equation

$$\frac{du_n(t)}{dt}=\left(\alpha+\beta u_n(t)+\gamma u_n^2(t)\right)(u_{n-1}(t)-u_{n+1}(t)), \tag{9}$$

where $u_n(t)=u(n,t)$ is the displacement of the nth particle from the equilibrium position, $n\in \mathrm{Z},\alpha,\beta$, and $\gamma\neq 0$ are arbitrary parameters. For convenience, we call Eq. (9) as the Wadati lattice equation (WLE) since it was first introduced by Wadati [19] in 1976. It is obvious that the WLE includes the famous NDDEs; the Hybrid lattice equation [8], the modified Volterra lattice equation [20], and the discretized mKdV equation [8,21]. Moreover, the WLE can be thought as a discrete version of the nonlinear partial differential equation

$$u_t+6\alpha uu_x+6\beta u^2u_x+u_{xxx}=0.$$

Now, for solving the WLE, we first introduce the traveling wave transformation

$$u_n=U_n(\xi_n),\quad \xi_n=d_1n+c_1t+\zeta, \tag{10}$$

where d_1,c_1 are constants to be determined later, and ζ is an arbitrary phase constant. Then, Eq. (9) can be converted into

$$c_1\frac{dU_n(\xi_n)}{d\xi_n}=\left(\alpha+\beta U_n(\xi_n)+\gamma U_n^2(\xi_n)\right)(U_{n-1}(\xi_n)-U_{n+1}(\xi_n)). \tag{11}$$

We expand the solution of Eq. (11) in the frame (4), and balancing the linear term of the highest order with the highest nonlinear term in (11) leads to $m=1$. Thus, we consider the ansatz

$$U_n(\xi_n)=a_0+a_1\left(\frac{\psi'(\xi_{\mathbf{n}})}{\psi(\xi_{\mathbf{n}})}\right)+a_{-1}\left(\frac{\psi'(\xi_{\mathbf{n}})}{\psi(\xi_{\mathbf{n}})}\right)^{-1}, \tag{12}$$

for the discrete travelling wave solutions of Eq. (9). Now, the case analysis follows:

Case 1: When $k<0$, from (8a), we have

$$U_{n\pm 1}(\xi_n)=\sum_{l=-1}^{1}a_i\left(\frac{\sqrt{-k}\psi'(\xi_{\mathbf{n}})\mp k\tanh\left(\sqrt{-k}\varphi_s\right)\psi(\xi_{\mathbf{n}})}{\sqrt{-k}\psi(\xi_{\mathbf{n}})\pm\tanh\left(\sqrt{-k}\varphi_s\right)\psi'(\xi_{\mathbf{n}})}\right)^l. \tag{13}$$

Substituting (12) and (13) along with (5) into (11), clearing the denominator and setting the coefficients of all powers like $(\psi'(\xi_{\mathbf{n}})/\psi(\xi_{\mathbf{n}}))^l (0 \leqslant l \leqslant 8)$ to zero, we derive a system of nonlinear algebraic equations for $a_0, a_1, a_{-1}, d_1, c_1$, and k. Solving the set of algebraic equations (from now on, we omit to display them for the sake of saving space) simultaneously, we get the following solution sets (denoted in curly brackets) and the corresponding discrete hyperbolic function traveling wave solutions of Eq. (9):

Case 1.1:

$$\left\{c_1 = \frac{(\beta^2 - 4\alpha\gamma)}{2\sqrt{-k}\gamma}\tanh\left(\sqrt{-k}d_1\right),\ a_0 = -\frac{\beta}{2\gamma}, a_1 = \mp\frac{\sqrt{\beta^2 - 4\alpha\gamma}}{2\sqrt{-k}\gamma}\tanh\left(\sqrt{-k}d_1\right),\ a_{-1} = 0\right\}, \tag{14}$$

$$u^{\mp}_{n,1}(t) = -\frac{\beta}{2\gamma} \mp \frac{\sqrt{\beta^2 - 4\alpha\gamma}\tanh\left(\sqrt{-k}d_1\right)}{2\gamma}\left(\frac{A_1\cosh\left(\sqrt{-k}\xi_n\right) + A_2\sinh\left(\sqrt{-k}\xi_n\right)}{A_1\sinh\left(\sqrt{-k}\xi_n\right) + A_2\cosh\left(\sqrt{-k}\xi_n\right)}\right), \tag{15}$$

where $\xi_n = d_1 n + \frac{(\beta^2 - 4\alpha\gamma)}{2\sqrt{-k}\gamma}\tanh\left(\sqrt{-k}d_1\right)t + \zeta$ and A_1, A_2 are arbitrary constants.

Case 1.2:

$$\left\{c_1 = \frac{(\beta^2 - 4\alpha\gamma)}{2\sqrt{-k}\gamma}\tanh\left(\sqrt{-k}d_1\right),\ a_0 = -\frac{\beta}{2\gamma},\ a_1 = 0, a_{-1} = \frac{\mp\sqrt{-k}\sqrt{\beta^2 - 4\alpha\gamma}}{2\gamma}\tanh\left(\sqrt{-k}d_1\right)\right\}, \tag{16}$$

$$u^{\mp}_{n,2}(t) = -\frac{\beta}{2\gamma} \mp \frac{\sqrt{\beta^2 - 4\alpha\gamma}\tanh\left(\sqrt{-k}d_1\right)}{2\gamma}\left(\frac{A_1\cosh\left(\sqrt{-k}\xi_n\right) + A_2\sinh\left(\sqrt{-k}\xi_n\right)}{A_1\sinh\left(\sqrt{-k}\xi_n\right) + A_2\cosh\left(\sqrt{-k}\xi_n\right)}\right)^{-1}, \tag{17}$$

where $\xi_n = d_1 n + \frac{(\beta^2 - 4\alpha\gamma)}{2\sqrt{-k}\gamma}\tanh\left(\sqrt{-k}d_1\right)t + \zeta$ and A_1, A_2 are arbitrary constants.

Case 1.3:

$$\left\{c_1 = \frac{(\beta^2 - 4\alpha\gamma)\sinh\left(2\sqrt{-k}d_1\right)}{4\sqrt{-k}\gamma}, a_0 = -\frac{\beta}{2\gamma}, a_1 = \pm\frac{\sqrt{\beta^2 - 4\alpha\gamma}\sinh\left(2\sqrt{-k}d_1\right)}{4\sqrt{-k}\gamma}, a_{-1} = \mp\frac{\sqrt{-k}\sqrt{\beta^2 - 4\alpha\gamma}\sinh\left(2\sqrt{-k}d_1\right)}{4\gamma}\right\}, \tag{18}$$

$$\begin{aligned} u^{\mp}_{n,3}(t) = {} & -\frac{\beta}{2\gamma} \mp \frac{\sqrt{\beta^2 - 4\alpha\gamma}\sinh\left(2\sqrt{-k}d_1\right)}{4\gamma}\left(\frac{A_1\cosh\left(\sqrt{-k}\xi_n\right) + A_2\sinh\left(\sqrt{-k}\xi_n\right)}{A_1\sinh\left(\sqrt{-k}\xi_n\right) + A_2\cosh\left(\sqrt{-k}\xi_n\right)}\right)^{-1} \\ & - \frac{\sqrt{\beta^2 - 4\alpha\gamma}\sinh\left(2\sqrt{-k}d_1\right)}{4\gamma}\left(\frac{A_1\cosh\left(\sqrt{-k}\xi_n\right) + A_2\sinh\left(\sqrt{-k}\xi_n\right)}{A_1\sinh\left(\sqrt{-k}\xi_n\right) + A_2\cosh\left(\sqrt{-k}\xi_n\right)}\right), \end{aligned} \tag{19}$$

where $\xi_n = d_1 n + \frac{(\beta^2 - 4\alpha\gamma)\sinh\left(2\sqrt{-k}d_1\right)}{4\sqrt{-k}\gamma}t + \zeta$ and A_1, A_2 are arbitrary constants.

Case 1.4.:

$$\left(c_1 = \frac{(\beta^2 - 4\alpha\gamma)\tanh\left(2\sqrt{-k}d_1\right)}{4\sqrt{-k}\gamma}, a_0 = -\frac{\beta}{2\gamma}, a_1 = \mp\frac{\sqrt{\beta^2 - 4\alpha\gamma}\tanh\left(2\sqrt{-k}d_1\right)}{4\sqrt{-k}\gamma}, a_{-1} = \mp\frac{\sqrt{-k}\sqrt{\beta^2 - 4\alpha\gamma}\tanh\left(2\sqrt{-k}d_1\right)}{4\gamma}\right), \tag{20}$$

$$\begin{aligned} u^{\mp}_{n,4}(t) = {} & -\frac{\beta}{2\gamma} \mp \frac{\sqrt{\beta^2 - 4\alpha\gamma}\tanh\left(2\sqrt{-k}d_1\right)}{4\gamma}\left(\frac{A_1\cosh\left(\sqrt{-k}\xi_n\right) + A_2\sinh\left(\sqrt{-k}\xi_n\right)}{A_1\sinh\left(\sqrt{-k}\xi_n\right) + A_2\cosh\left(\sqrt{-k}\xi_n\right)}\right)^{-1} \\ & + \frac{\sqrt{\beta^2 - 4\alpha\gamma}\tanh\left(2\sqrt{-k}d_1\right)}{4\gamma}\left(\frac{A_1\cosh\left(\sqrt{-k}\xi_n\right) + A_2\sinh\left(\sqrt{-k}\xi_n\right)}{A_1\sinh\left(\sqrt{-k}\xi_n\right) + A_2\cosh\left(\sqrt{-k}\xi_n\right)}\right), \end{aligned} \tag{21}$$

where $\xi_n = d_1 n + \frac{(\beta^2 - 4\alpha\gamma)\tanh\left(2\sqrt{-k}d_1\right)}{4\sqrt{-k}\gamma}t + \zeta$ and A_1, A_2 are arbitrary constants.

Remark 1. It appears that we can recover some previously known solutions from our results. To be more clear, we take our Case 1.1 for example, and make a comparison analysis with other works as follows:

(i) One of the exponential function solutions to the WLE presented by Dai et al. [22] is

$$u_n(t) = \frac{-\frac{\beta}{2\gamma}(\exp(\xi_n) + b_{-1}\exp(-\xi_n)) + \frac{\sqrt{\beta^2-4\alpha\gamma}}{2\gamma}\tanh(d)(\exp(\xi_n) - b_{-1}\exp(-\xi_n))}{b_{-1}\exp(-\xi_n) + \exp(\xi_n)}, \tag{22}$$

where $\xi_n = dn + \frac{(\beta^2-4\alpha\gamma)}{2\gamma}\tanh(d)t + \zeta$. We extracted (22) from the formula (11) in there. An equivalent form of (22), using the identity $\exp(2x) = (1+\tanh(x))/(1-\tanh(x))$, is that

$$u_n(t) = -\frac{\beta}{2\gamma} + \frac{\sqrt{\beta^2-4\alpha\gamma}\tanh(d)}{2\gamma}\left(\frac{(1-b_{-1}) + (1+b_{-1})\tanh(\xi_n)}{(1+b_{-1}) + (1-b_{-1})\tanh(\xi_n)}\right). \tag{23}$$

Now, if we substitute $d_1 = d, k = -1, A_1 = 1 - b_{-1}$ and $A_2 = 1 + b_{-1}$ into our solution function $u^+_{n,1}(t)$ of (15), then we get the same result as (23).

(ii) One of the solitary wave solutions to the WLE obtained by Xie and Wang [23] is

$$u_n(t) = -\frac{\beta}{2\gamma} + \frac{\sqrt{\beta^2-4\alpha\gamma}\tanh(d)}{2\gamma}\tanh\left(dn + \frac{(\beta^2-4\alpha\gamma)\tanh(d)}{2\gamma}t + \xi_0\right), \tag{24}$$

where $\xi_n = dn + \frac{(\beta^2-4\alpha\gamma)\tanh(d)}{2\gamma}t + \xi_0$. If we take $A_2 \neq 0, A_1^2 < A_2^2$ in our solution function $u^+_{n,1}(t)$ of (15), then we get the formal discrete solitary wave solution to the WLE as

$$u^+_{n,1}(t) = -\frac{\beta}{2\gamma} + \frac{\sqrt{\beta^2-4\alpha\gamma}\tanh\left(\sqrt{-k}d_1\right)}{2\gamma}\tanh\left(\sqrt{-k}\left(d_1 n + \frac{(\beta^2-4\alpha\gamma)}{2\sqrt{-k}\gamma}\tanh\left(\sqrt{-k}d_1\right)t + \zeta\right) + \zeta_0\right), \tag{25}$$

where $\zeta_0 = \tanh^{-1}(A_1/A_2)$. Now, letting $d_1 = d, k = -1$, and $\zeta = \xi_0 - \zeta_0$ in (25) leads to the same result (24).

(iii) Finally, if we take $A_2 \neq 0, A_1^2 < A_2^2$ in our solution function $u^-_{n,1}(t)$ of (15), then we get the formal discrete solitary wave solution to the WLE as

$$u^-_{n,1}(t) = -\frac{\beta'}{2\gamma} - \frac{\sqrt{\beta^2-4\alpha\gamma}\tanh\left(\sqrt{-k}d_1\right)}{2\gamma}\tanh\left(\sqrt{-k}\left(d_1 n + \frac{(\beta^2-4\alpha\gamma)}{2\sqrt{-k}\gamma}\tanh\left(\sqrt{-k}d_1\right)t + \zeta\right) + \zeta_0\right), \tag{26}$$

where $\zeta_0 = \tanh^{-1}(A_1/A_2)$. Now, letting $d_1 = d, k = -1$, and $\zeta = \xi_0 - \zeta_0$ in (26) leads to the same result stated as Case 2 in Wu and Xia [24].

Consequently, our results are wider in the sense that they contain more arbitrary parameters.

Remark 2. There have been precedents when "solutions" derived by the Exp-function method do not satisfy the original differential equation, see Kudryashov and Loguinova [25]. Unintentionally, we observed that the result (11) of Dai et al. [22] contains a superfluous (constant) solution to the WLE. Hence, the derivation of (12) and (13) from (11) are not correctly performed in there.

Case 2 When $k > 0$, from (8b), we have

$$U_{n\pm1}(\xi_{\mathbf{n}}) = \sum_{l=-1}^{1} a_i\left(\frac{\sqrt{k}\psi'(\xi_{\mathbf{n}}) \mp k\tan\left(\sqrt{k}\varphi_s\right)\psi(\xi_{\mathbf{n}})}{\sqrt{k}\psi(\xi_{\mathbf{n}}) \pm \tan\left(\sqrt{k}\varphi_s\right)\psi'(\xi_{\mathbf{n}})}\right)^l. \tag{27}$$

Substituting (12) and (27) along with (5) into (11), clearing the denominator and setting the coefficients of all powers like $(\psi'(\xi_{\mathbf{n}})/\psi(\xi_{\mathbf{n}}))^l (0 \leqslant l \leqslant 8)$ to zero, we derive a system of nonlinear algebraic equations for $a_0, a_1, a_{-1}, d_1, c_1$, and k. Solving the set of algebraic equations simultaneously, we get the following solution sets and the corresponding discrete trigonometric function traveling wave solutions of Eq. (9):

Case 2.1:

$$\left\{c_1 = \frac{(\beta^2-4\alpha\gamma)}{2\sqrt{k}\gamma}\tan\left(\sqrt{k}d_1\right), a_0 = -\frac{\beta}{2\gamma}, a_1 = 0, a_{-1} = \mp\frac{\sqrt{k}\sqrt{\beta^2-4\alpha\gamma}}{2\gamma}\tan\left(\sqrt{k}d_1\right)\right\}, \tag{28}$$

$$u^{\mp}_{n,5}(t) = -\frac{\beta}{2\gamma} \mp \frac{\sqrt{\beta^2 - 4\alpha\gamma}\tan\left(\sqrt{k}d_1\right)}{2\gamma}\left(\frac{-A_1\sin\left(\sqrt{k}\xi_n\right) + A_2\cos\left(\sqrt{k}\xi_n\right)}{A_1\cos\left(\sqrt{k}\xi_n\right) + A_2\sin\left(\sqrt{k}\xi_n\right)}\right)^{-1}, \tag{29}$$

where $\xi_n = d_1 n + \frac{(\beta^2-4\alpha\gamma)}{2\sqrt{k}\gamma}\tan\left(\sqrt{k}d_1\right)t + \zeta$ and A_1, A_2 are arbitrary constants.

Case 2.2:

$$\left\{c_1 = \frac{(\beta^2 - 4\alpha\gamma)}{2\sqrt{k}\gamma}\tan\left(\sqrt{k}d_1\right),\ a_0 = -\frac{\beta}{2\gamma},\ a_1 = \mp\frac{\sqrt{\beta^2 - 4\alpha\gamma}}{2\sqrt{k}\gamma}\tan\left(\sqrt{k}d_1\right),\ a_{-1} = 0\right\}, \tag{30}$$

$$u^{\mp}_{n,6}(t) = -\frac{\beta}{2\gamma} \mp \frac{\sqrt{\beta^2 - 4\alpha\gamma}\tan\left(\sqrt{k}d_1\right)}{2\gamma}\left(\frac{-A_1\sin\left(\sqrt{k}\xi_n\right) + A_2\cos\left(\sqrt{k}\xi_n\right)}{A_1\cos\left(\sqrt{k}\xi_n\right) + A_2\sin\left(\sqrt{k}\xi_n\right)}\right), \tag{31}$$

where $\xi_n = d_1 n + \frac{(\beta^2-4\alpha\gamma)}{2\sqrt{k}\gamma}\tan\left(\sqrt{k}d_1\right)t + \zeta$ and A_1, A_2 are arbitrary constants.

Remark 3. As in the previous case, for instance, by modifying our solution (31) and assigning appropriate arbitrary values to the parameters, we can obtain the result given in Case 3 of Xie and Wang [23]. Unfortunately, no trigonometric function solutions to the WLE appear in both Dai et al. [22] and Wu and Xia [24].

Case 3: When $k = 0$, from (8c), we have

$$U_{n\pm1}(\xi_n) = \sum_{l=-1}^{1} a_i(\psi'(\xi_{\mathbf{n}})/(\psi(\xi_{\mathbf{n}}) \pm \varphi_s\psi'(\xi_{\mathbf{n}})))^l. \tag{32}$$

Substituting (12) and (32) along with (5) into (11), clearing the denominator and setting the coefficients of all powers like $(\psi'(\xi_{\mathbf{n}})/\psi(\xi_{\mathbf{n}}))^l (0 \leqslant l \leqslant 6)$ to zero, we derive a system of nonlinear algebraic equations for a_0, a_1, a_{-1}, d_1, and c_1. Solving the set of algebraic equations simultaneously, we get the following solution sets and the corresponding discrete rational function traveling wave solutions of Eq. (9):

$$\left\{c_1 = \frac{(\beta^2 - 4\alpha\gamma)d_1}{2\gamma},\ a_0 = -\frac{\beta}{2\gamma},\ a_1 = \mp\frac{\sqrt{\beta^2 - 4\alpha\gamma}d_1}{2\gamma},\ a_{-1} = 0\right\}, \tag{33}$$

$$u^{\mp}_{n,7}(t) = -\frac{\beta}{2\gamma} \mp \frac{\sqrt{\beta^2 - 4\alpha\gamma}d_1}{2\gamma}\left(\frac{A_1}{A_1\left(d_1 n + \frac{(\beta^2-4\alpha\gamma)}{2\gamma}d_1 t + \zeta\right) + A_2}\right), \tag{34}$$

where A_1, A_2 are arbitrary constants.

Remark 4. Our rational function solutions (34) are not derived by the authors [22–24].

4. Conclusion

We solved the Wadati lattice equation by proposing a variant of the extended simplest equation method for NDDEs. In solving the problem, we considered all the three cases that arises in the analysis, in details. For each case, we studied the sub-cases exhaustively and thus came up with a complete spectrum of discrete solutions to this equation. To our knowledge, some of these solutions are found for the first time. The obtained results with more free parameters include most of the solutions in the open literature as special cases. We have assured the correctness of the solutions by putting them back into the original equation. Our method does not require a large amount of CPU time to solve NDDEs when it is implemented with the aid of a computer algebra system such as Mathematica.

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