

Analytic solutions to nonlinear differential-difference equations by means of the extended (G'/G) -expansion method

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Abstract

In this paper, a discrete extension of the (G'/G) -expansion method is applied to a relativistic Toda lattice system and a discrete nonlinear Schrödinger equation in order to obtain discrete traveling wave solutions. Closed form solutions with more arbitrary parameters, which reduce to solitary and periodic waves, are exhibited. New rational solutions are also obtained. The method is straightforward and concise, and its applications in physical sciences are promising.

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1. Introduction

Since the original work of Fermi *et al* [1] in the 1950s, there has been an explosion of interest to the study of wave phenomena arising in nonlinear differential-difference equations (NDDEs) which are at the very heart of almost any many-particle system. Generally, the dynamics of a many-particle system can be considered as a discrete analog of certain continuous systems. Condensed matter physics is a particular research area of physical sciences where discreteness plays an important role, and the same could also be said of topics in biophysics, chemistry or mechanical engineering. In the last four decades or so, paying more attention to such equations, researchers proposed many physically important NDDEs [2–5].

Recently, Wang *et al* [6] proposed the so-called (G'/G) -expansion method to seek for exact solutions of nonlinear evolution equations in the form of solitary and periodic waves. The essential observation about the (G'/G) -expansion method is that it reveals further results with more arbitrary parameters and is powerful in the sense that it takes full advantage of linear theory by assuming a second-order linear equation as the ansatz. The solution procedure is easy, reliable and efficient, as well as does not require a large amount of run-time with the help of a computer algebra system such as MATHEMATICA. Naturally, the (G'/G) -expansion

method has been applied to various kinds of nonlinear problems in science and engineering, and lately more attention is paid to its adaptation and generalization [7–23].

Our objective in this paper is to perform an analytic study on a relativistic Toda lattice system and a discrete nonlinear Schrödinger equation by using the extended (G'/G)-expansion method. The rest of this paper is organized as follows. The method of solution is described in section 2. We analyze two physically important equations in section 3. Finally, some concluding remarks are given in section 4.

2. The extended (G'/G)-expansion method for NDDEs

Let us consider a system of M polynomial NDDEs in the form

$$\Delta(\mathbf{u}_{\mathbf{n}+\mathbf{p}_1}(\mathbf{x}), \dots, \mathbf{u}_{\mathbf{n}+\mathbf{p}_k}(\mathbf{x}), \dots, \mathbf{u}'_{\mathbf{n}+\mathbf{p}_1}(\mathbf{x}), \dots, \mathbf{u}'_{\mathbf{n}+\mathbf{p}_k}(\mathbf{x}), \dots, \mathbf{u}^{(r)}_{\mathbf{n}+\mathbf{p}_1}(\mathbf{x}), \dots, \mathbf{u}^{(r)}_{\mathbf{n}+\mathbf{p}_k}(\mathbf{x})) = 0, \tag{1}$$

in which the dependent variable $\mathbf{u}_{\mathbf{n}}$ have M components $u_{i,\mathbf{n}}$ and so do its shifts, the continuous variable \mathbf{x} has N components x_i , the discrete variable \mathbf{n} has Q components n_j , and the k shift vectors $\mathbf{p}_i \in \mathbb{Z}^Q$ and $\mathbf{u}^{(r)}(\mathbf{x})$ denote the collection of mixed derivative terms of order r .

Step 1. For traveling wave solutions to equation (1), we first make the wave transformation

$$\mathbf{u}_{\mathbf{n}+\mathbf{p}_s}(\mathbf{x}) = \mathbf{U}_{\mathbf{n}+\mathbf{p}_s}(\xi_{\mathbf{n}}), \quad \xi_{\mathbf{n}} = \sum_{i=1}^Q d_i n_i + \sum_{j=1}^N c_j x_j + \zeta \quad (s = 1, 2, \dots, k), \tag{2}$$

where the coefficients $c_1, c_2, \dots, c_N, d_1, d_2, \dots, d_Q$ and the phase ζ are all constants. Then, equation (1) reduces to

$$\Delta(\mathbf{U}_{\mathbf{n}+\mathbf{p}_1}(\xi_{\mathbf{n}}), \dots, \mathbf{U}_{\mathbf{n}+\mathbf{p}_k}(\xi_{\mathbf{n}}), \dots, \mathbf{U}'_{\mathbf{n}+\mathbf{p}_1}(\xi_{\mathbf{n}}), \dots, \mathbf{U}'_{\mathbf{n}+\mathbf{p}_k}(\xi_{\mathbf{n}}), \dots, \mathbf{U}^{(r)}_{\mathbf{n}+\mathbf{p}_1}(\xi_{\mathbf{n}}), \dots, \mathbf{U}^{(r)}_{\mathbf{n}+\mathbf{p}_k}(\xi_{\mathbf{n}})) = 0. \tag{3}$$

Step 2. We assume that the solution of equation (3) is in the finite series expansion form

$$\mathbf{U}_{\mathbf{n}}(\xi_{\mathbf{n}}) = \sum_{l=-m}^m a_l \left(\frac{G'(\xi_{\mathbf{n}})}{G(\xi_{\mathbf{n}})} \right)^l, \quad a_{-m}^2 + a_m^2 \neq 0, \tag{4}$$

where m (a positive integer) and a_i 's are constants to be determined, and $G(\xi_{\mathbf{n}})$ is the general solution of the equation

$$G''(\xi_{\mathbf{n}}) + \mu G(\xi_{\mathbf{n}}) = 0, \tag{5}$$

where μ is an arbitrary constant and the prime denotes the derivative with respect to $\xi_{\mathbf{n}}$. The general solution of equation (5) is well known for us. Thus, we have the following cases:

$$\frac{G'(\xi_{\mathbf{n}})}{G(\xi_{\mathbf{n}})} = \sqrt{-\mu} \left(\frac{C_1 \cosh(\sqrt{-\mu}\xi_{\mathbf{n}}) + C_2 \sinh(\sqrt{-\mu}\xi_{\mathbf{n}})}{C_1 \sinh(\sqrt{-\mu}\xi_{\mathbf{n}}) + C_2 \cosh(\sqrt{-\mu}\xi_{\mathbf{n}})} \right), \quad \mu < 0, \tag{6a}$$

$$\frac{G'(\xi_{\mathbf{n}})}{G(\xi_{\mathbf{n}})} = \sqrt{\mu} \left(\frac{-C_1 \sin(\sqrt{\mu}\xi_{\mathbf{n}}) + C_2 \cos(\sqrt{\mu}\xi_{\mathbf{n}})}{C_1 \cos(\sqrt{\mu}\xi_{\mathbf{n}}) + C_2 \sin(\sqrt{\mu}\xi_{\mathbf{n}})} \right), \quad \mu > 0, \tag{6b}$$

$$\frac{G'(\xi_{\mathbf{n}})}{G(\xi_{\mathbf{n}})} = \frac{C_1}{C_1 \xi_{\mathbf{n}} + C_2}, \quad \mu = 0, \tag{6c}$$

where C_1 and C_2 are arbitrary constants.

Step 3. By a straightforward calculation, we can get the identity

$$\xi_{\mathbf{n}+\mathbf{p}_s} = \xi_{\mathbf{n}} + \varphi_s, \quad \varphi_s = p_{s1}d_1 + p_{s2}d_2 + \dots + p_{sQ}d_Q, \tag{7}$$

where p_{sj} is the j th component of the shift vector \mathbf{p}_s . Hence, considering the trigonometric/hyperbolic function identities and using the functions (6a)–(6c) together with (7), we derive the uniform formulas:

$$\mathbf{U}_{\mathbf{n}+\mathbf{p}_s}(\xi_{\mathbf{n}}) = \sum_{l=-m}^m a_l \left(\frac{\sqrt{-\mu}G'(\xi_{\mathbf{n}}) - \mu \tanh(\sqrt{-\mu}\varphi_s)G(\xi_{\mathbf{n}})}{\sqrt{-\mu}G(\xi_{\mathbf{n}}) + \tanh(\sqrt{-\mu}\varphi_s)G'(\xi_{\mathbf{n}})} \right)^l, \quad a_{-m}^2 + a_m^2 \neq 0, \quad \mu < 0, \tag{8a}$$

$$\mathbf{U}_{\mathbf{n}+\mathbf{p}_s}(\xi_{\mathbf{n}}) = \sum_{l=-m}^m a_l \left(\frac{\sqrt{\mu}G'(\xi_{\mathbf{n}}) - \mu \tan(\sqrt{\mu}\varphi_s)G(\xi_{\mathbf{n}})}{\sqrt{\mu}G(\xi_{\mathbf{n}}) + \tan(\sqrt{\mu}\varphi_s)G'(\xi_{\mathbf{n}})} \right)^l, \quad a_{-m}^2 + a_m^2 \neq 0, \quad \mu > 0, \tag{8b}$$

$$\mathbf{U}_{\mathbf{n}+\mathbf{p}_s}(\xi_{\mathbf{n}}) = \sum_{l=-m}^m a_l \left(\frac{G'(\xi_{\mathbf{n}})}{G(\xi_{\mathbf{n}}) + \varphi_s G'(\xi_{\mathbf{n}})} \right)^l, \quad a_{-m}^2 + a_m^2 \neq 0, \quad \mu = 0. \tag{8c}$$

Step 4. By means of the ansatz (4), we define the degree of $\mathbf{U}_{\mathbf{n}}(\xi_{\mathbf{n}})$ as $D[\mathbf{U}_{\mathbf{n}}(\xi_{\mathbf{n}})] = m$ which gives rise to the degree of other expressions as

$$D[\mathbf{U}_{\mathbf{n}}^{(r)}(\xi_{\mathbf{n}})] = m + r, \quad D[(\mathbf{U}_{\mathbf{n}}^{(r)}(\xi_{\mathbf{n}}))^\beta] = \beta(m + r), \\ D[(\mathbf{U}_{\mathbf{n}}(\xi_{\mathbf{n}}))^\alpha (\mathbf{U}_{\mathbf{n}}^{(r)}(\xi_{\mathbf{n}}))^\beta] = \alpha m + \beta(m + r).$$

Balancing the highest order nonlinear term(s) and the highest order derivative term in $\mathbf{U}_{\mathbf{n}}(\xi_{\mathbf{n}})$, we can easily determine the degree m of equations (4) and (8a)–(8c) from equation (3). Since $\mathbf{U}_{\mathbf{n}+\mathbf{p}_s}$ can be interpreted as being of degree zero in $(G'(\xi_{\mathbf{n}})/G(\xi_{\mathbf{n}}))$, the leading terms of $\mathbf{U}_{\mathbf{n}+\mathbf{p}_s}$ ($\mathbf{p}_s \neq 0$) will not have any affect on the balancing procedure.

Step 5. Substituting the ansätze (4) and (8a)–(8c) together with (5) into equation (3), equating the coefficients of $(G'(\xi_{\mathbf{n}})/G(\xi_{\mathbf{n}}))^l$ ($l = 0, 1, 2, \dots$) to zero, we obtain a system of nonlinear algebraic equations from which the undetermined constants a_l , d_i , c_j and μ can be explicitly found. Substituting these results into (4), we can derive various kinds of discrete exact solutions to equation (1).

Note 1. It is worth to mention that there are three improved computational steps in our algorithm to obtain more wider results in a concise manner. First, the extended method leads to the solution of the form (4) in which the sum goes from $l = -m$ to $l = m$ instead of from $l = 0$ to $l = m$. Second, the standard method [6, 20] uses the auxiliary equation $G'' + \lambda G' + \mu G = 0$ as the ansatz. Without loss of generality, we consider the auxiliary equation (5) by taking $\lambda = 0$. This approach provides equivalent results with the original method. However, it is more advantageous since it minimizes the number of parameters, see [24]. Third, contrary to the procedure [20], we consider another case (namely, the case (8c)) for the inclusion of rational solutions.

3. Applications

In this section, we apply the algorithm described in the preceding section to some NDDEs.

3.1. Relativistic Toda lattice system

One of the most famous models for discrete solitons is the integrable Toda lattice system [25]:

$$\begin{cases} \frac{du_n}{dt} = (u_{n+1} - v_n)v_n - (u_{n-1} - v_{n-1})v_{n-1}, \\ \frac{dv_n}{dt} = v_n(u_{n+1} - u_n). \end{cases} \quad (9)$$

For solving equation (9), we first let

$$u_n = U_n(\xi_n), \quad v_n = V_n(\xi_n), \quad \xi_n = dn + ct + \zeta, \quad (10)$$

where c and d are constants to be determined and ζ denotes the phase shift. Then, equation (9) can be reduced to

$$\begin{cases} cU'_n - (U_{n+1} - V_n)V_n + (U_{n-1} - V_{n-1})V_{n-1} = 0, \\ cV'_n - V_n(U_{n+1} - U_n) = 0, \end{cases} \quad (11)$$

where $U_n = U_n(\xi_n)$, $V_n = V_n(\xi_n)$ and the prime denotes the derivative with respect to ξ_n . We expand the solution of (11) in the frame (4). Balancing the linear term of the highest order with the highest nonlinear term in (11) yields to $m = 1$. Thus, we look for solutions of equation (11) in the form

$$\begin{cases} U_n = a_0 + a_1 \left(\frac{G'}{G}\right) + a_{-1} \left(\frac{G'}{G}\right)^{-1}, & a_{-1}^2 + a_1^2 \neq 0, \\ V_n = b_0 + b_1 \left(\frac{G'}{G}\right) + b_{-1} \left(\frac{G'}{G}\right)^{-1}, & b_{-1}^2 + b_1^2 \neq 0, \end{cases} \quad (12)$$

where $G = G(\xi_n)$ satisfies equation (5), and $a_0, a_1, a_{-1}, b_0, b_1, b_{-1}$ are arbitrary constants to be specified.

Case 1. When $\mu < 0$, from (8a), we have

$$U_{n\pm 1} = \sum_{l=-1}^1 a_l \left(\frac{\sqrt{-\mu}G' \mp \mu \tanh(\sqrt{-\mu}\varphi_s)G}{\sqrt{-\mu}G \pm \tanh(\sqrt{-\mu}\varphi_s)G'} \right)^l, \quad (13a)$$

$$V_{n\pm 1} = \sum_{l=-1}^1 b_l \left(\frac{\sqrt{-\mu}G' \mp \mu \tanh(\sqrt{-\mu}\varphi_s)G}{\sqrt{-\mu}G \pm \tanh(\sqrt{-\mu}\varphi_s)G'} \right)^l. \quad (13b)$$

Substituting the ansatz (12) and the expressions (13a) and (13b) along with equation (5) into equation (11), clearing the denominator and equating the coefficients of $(G'/G)^l$ ($0 \leq l \leq 10$) to zero, we obtain a system of nonlinear algebraic equations for $a_0, a_1, a_{-1}, b_0, b_1, b_{-1}, c, d$ and μ . Solving the system (we will omit to display them for simplicity) simultaneously, we get the following solution sets:

$$\{a_0 = c\sqrt{-\mu}\coth(d\sqrt{-\mu}), b_0 = c\sqrt{-\mu}\coth(d\sqrt{-\mu}), a_{-1} = 0, a_1 = c, b_{-1} = 0, b_1 = c\}, \quad (14a)$$

$$\{a_0 = 2c\sqrt{-\mu}\coth(2d\sqrt{-\mu}), b_0 = 2c\sqrt{-\mu}\coth(2d\sqrt{-\mu}), a_{-1} = -c\mu, a_1 = c, b_{-1} = -c\mu, b_1 = c\}, \quad (14b)$$

and the corresponding discrete hyperbolic function solutions to equation (9) as

$$\begin{cases} u_{n,1}(t) = c\sqrt{-\mu}(w_n(t) + \coth(d\sqrt{-\mu})), \\ v_{n,1}(t) = c\sqrt{-\mu}(w_n(t) + \coth(d\sqrt{-\mu})), \end{cases} \quad (15a)$$

$$\begin{cases} u_{n,2}(t) = c\sqrt{-\mu} \left(\frac{1}{w_n(t)} + w_n(t) + 2 \coth(2d\sqrt{-\mu}) \right), \\ v_{n,2}(t) = c\sqrt{-\mu} \left(\frac{1}{w_n(t)} + w_n(t) + 2 \coth(2d\sqrt{-\mu}) \right), \end{cases} \quad (15b)$$

where

$$w_n(t) = \frac{C_1 \cosh(\sqrt{-\mu}(dn + ct + \zeta)) + C_2 \sinh(\sqrt{-\mu}(dn + ct + \zeta))}{C_1 \sinh(\sqrt{-\mu}(dn + ct + \zeta)) + C_2 \cosh(\sqrt{-\mu}(dn + ct + \zeta))}, \quad (16)$$

in which $\mu < 0, d, c, \zeta, C_1, C_2$ are arbitrary constants.

As a special example, if we let ' $C_1 = 0$ and $C_2 \neq 0$ ' or ' $C_1 \neq 0$ and $C_2 = 0$ ' in (15a) respectively, then we get formal discrete solitary wave solutions to equation (9) as follows:

$$\begin{cases} u_{n,3}(t) = c\sqrt{-\mu}(\tanh(\sqrt{-\mu}(dn + ct + \zeta)) + \coth(d\sqrt{-\mu})), \\ v_{n,3}(t) = c\sqrt{-\mu}(\tanh(\sqrt{-\mu}(dn + ct + \zeta)) + \coth(d\sqrt{-\mu})), \end{cases} \quad (17)$$

$$\begin{cases} u_{n,4}(t) = c\sqrt{-\mu}(\coth(\sqrt{-\mu}(dn + ct + \zeta)) + \coth(d\sqrt{-\mu})), \\ v_{n,4}(t) = c\sqrt{-\mu}(\coth(\sqrt{-\mu}(dn + ct + \zeta)) + \coth(d\sqrt{-\mu})), \end{cases} \quad (18)$$

where $\mu < 0, d, c$ and ζ are arbitrary constants.

Case 2. When $\mu > 0$, from (8b), we have

$$U_{n\pm 1} = \sum_{l=-1}^1 a_l \left(\frac{\sqrt{\mu}G' \mp \mu \tan(\sqrt{\mu}\phi_s)G}{\sqrt{\mu}G \pm \tan(\sqrt{\mu}\phi_s)G'} \right)^l, \quad (19a)$$

$$V_{n\pm 1} = \sum_{l=-1}^1 b_l \left(\frac{\sqrt{\mu}G' \mp \mu \tan(\sqrt{\mu}\phi_s)G}{\sqrt{\mu}G \pm \tan(\sqrt{\mu}\phi_s)G'} \right)^l. \quad (19b)$$

Substituting the ansatz (12) and the expressions (19a), (19b) along with equation (5) into equation (11), clearing the denominator and equating the coefficients of $(G'/G)^l (0 \leq l \leq 10)$ to zero, we obtain a system of nonlinear algebraic equations for $a_0, a_1, a_{-1}, b_0, b_1, b_{-1}, c, d$ and μ . Solving the system simultaneously, we get the following solution sets:

$$\{a_0 = c\sqrt{\mu}\cot(d\sqrt{\mu}), b_0 = c\sqrt{\mu}\cot(d\sqrt{\mu}), a_{-1} = 0, a_1 = c, b_{-1} = 0, b_1 = c\}, \quad (20a)$$

$$\{a_0 = 2c\sqrt{\mu}\cot(2d\sqrt{\mu}), b_0 = 2c\sqrt{\mu}\cot(2d\sqrt{\mu}), a_{-1} = -c\mu, a_1 = c, b_{-1} = -c\mu, b_1 = c\}, \quad (20b)$$

and the corresponding discrete trigonometric function solutions to equation (9) as

$$\begin{cases} u_{n,5}(t) = c\sqrt{\mu}(w_n(t) + \cot(d\sqrt{\mu})), \\ v_{n,5}(t) = c\sqrt{\mu}(w_n(t) + \cot(d\sqrt{\mu})), \end{cases} \quad (21a)$$

$$\begin{cases} u_{n,6}(t) = c\sqrt{\mu} \left(-\frac{1}{w_n(t)} + w_n(t) + 2 \cot(2d\sqrt{\mu}) \right), \\ v_{n,6}(t) = c\sqrt{\mu} \left(-\frac{1}{w_n(t)} + w_n(t) + 2 \cot(2d\sqrt{\mu}) \right), \end{cases} \quad (21b)$$

where

$$w_n(t) = \frac{C_2 \cos(\sqrt{\mu}(dn + ct + \zeta)) - C_1 \sin(\sqrt{\mu}(dn + ct + \zeta))}{C_1 \cos(\sqrt{\mu}(dn + ct + \zeta)) + C_2 \sin(\sqrt{\mu}(dn + ct + \zeta))}, \quad (22)$$

in which $\mu > 0$, d, c, ζ, C_1 and C_2 are arbitrary constants.

As a special example, if we take ‘ $C_1 = 0$ and $C_2 \neq 0$ ’ or ‘ $C_1 \neq 0$ and $C_2 = 0$ ’ in the expression (21a) respectively, then we get formal discrete periodic wave solutions to equation (9) as follows:

$$\begin{cases} u_{n,7}(t) = c\sqrt{\mu}(\cot(\sqrt{\mu}(dn + ct + \zeta)) + \cot(d\sqrt{\mu})), \\ v_{n,7}(t) = c\sqrt{\mu}(\cot(\sqrt{\mu}(dn + ct + \zeta)) + \cot(d\sqrt{\mu})), \end{cases} \quad (23)$$

$$\begin{cases} u_{n,8}(t) = c\sqrt{\mu}(-\tan(\sqrt{\mu}(dn + ct + \zeta)) + \cot(d\sqrt{\mu})), \\ v_{n,8}(t) = c\sqrt{\mu}(-\tan(\sqrt{\mu}(dn + ct + \zeta)) + \cot(d\sqrt{\mu})), \end{cases} \quad (24)$$

where $\mu > 0$, d, c and ζ are arbitrary constants.

Case 3. When $\mu = 0$, from (8c), we have

$$U_{n\pm 1} = \sum_{l=-1}^1 a_l \left(\frac{G'}{G \pm \varphi_s G'} \right)^l, \quad (25a)$$

$$V_{n\pm 1} = \sum_{l=-1}^1 b_l \left(\frac{G'}{G \pm \varphi_s G'} \right)^l. \quad (25b)$$

Substituting the ansatz (12) and the expressions (25a) and (25b) along with equation (5) into equation (11), clearing the denominator and equating the coefficients of $(G'/G)^l$ ($0 \leq l \leq 10$) to zero, we obtain a system of nonlinear algebraic equations for $a_0, a_1, a_{-1}, b_0, b_1, b_{-1}, c$ and d . Solving the system simultaneously, we get the following solution set:

$$\left\{ a_0 = \frac{c}{d}, a_1 = c, b_1 = c, b_0 = \frac{c}{d}, a_{-1} = 0, b_{-1} = 0 \right\} \quad (26)$$

and the corresponding discrete rational function solution to equation (9) as

$$\begin{cases} u_{n,9}(t) = c \left(\frac{1}{d} + \frac{C_1}{C_1(dn+ct+\zeta)+C_2} \right), \\ v_{n,9}(t) = c \left(\frac{1}{d} + \frac{C_1}{C_1(dn+ct+\zeta)+C_2} \right), \end{cases} \quad (27)$$

where c, d, ζ, C_1 and C_2 are arbitrary constants.

Note 2. We observe that our solution (17) coincides with the solution of Baldwin *et al* [25] while our other solutions do not appear there. To the best of our knowledge, our rational solution (27) is presented here for the first time.

3.2. The discrete nonlinear Schrödinger equation

We now consider the integrable discrete nonlinear Schrödinger equation [26]

$$i \frac{du_n(t)}{dt} + \alpha(u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)) + \beta|u_n(t)|^2(u_{n+1}(t) + u_{n-1}(t)) = 0, \quad (28)$$

where $u_n(t) = u(n, t)$ denotes the displacement of the n th particle from the equilibrium position, $i = \sqrt{-1}$, and α , and β are nonzero real constants.

For solving equation (28), we first make the traveling wave transformation

$$u_n = e^{i\theta_n} \phi_n(\xi_n), \quad \theta_n = d_1 n + c_1 t + \zeta_1, \quad \xi_n = d_2 n + c_2 t + \zeta_2, \quad (29)$$

and

$$u_{n+1} = e^{i\theta_n} e^{id_1} \phi_{n+1}(\xi_n), \quad u_{n-1} = e^{i\theta_n} e^{-id_1} \phi_{n-1}(\xi_n), \quad (30)$$

where $\phi_n = \phi_n(\xi_n)$ is a real-valued function, d_1 and c_1 are the wave number of the carrier wave and the frequency, c_2 and d_2 are related to the group velocity and the pulse width, ζ_1 and ζ_2 denote the initial phases. Now, using the Euler formula $e^{\pm id_1} = \cos d_1 \pm i \sin d_1$, equation (28) can be reduced to the system

$$\begin{cases} -c_1 \phi_n + \cos(d_1)(\alpha + \beta \phi_n^2)(\phi_{n+1} + \phi_{n-1}) - 2\alpha \phi_n = 0, \\ c_2 \phi'_n + \sin(d_1)(\alpha + \beta \phi_n^2)(\phi_{n+1} - \phi_{n-1}) = 0, \end{cases} \quad (31)$$

where the prime denotes the derivative with respect to ξ_n . We expand the solution of (31) in the form of (4). Balancing the linear term of the highest order with the highest nonlinear term in (31) leads to $m = 1$. Thus, for the traveling wave solutions of (31), we assume the ansatz

$$\phi_n = a_0 + a_1 \left(\frac{G'}{G}\right) + a_{-1} \left(\frac{G'}{G}\right)^{-1}, \quad a_{-1}^2 + a_1^2 \neq 0, \quad (32)$$

where $G = G(\xi_n)$ satisfies equation (5), and a_0 , a_1 , and a_{-1} are arbitrary constants to be determined. Because the procedure is similar to the scheme used in section 3.1, we will omit most of the details here.

Case 1. $\mu < 0$.

In this case, we first derive the expressions $\phi_{n\pm 1}$ in accordance with (8a) and substitute them along with (32) into equation (31). Then, clearing the denominator and setting the coefficients of $(G'/G)^l$ ($0 \leq l \leq 8$) to zero, we derive a system of nonlinear algebraic equations for a_0 , a_1 , a_{-1} , d_1 , d_2 , c_1 , c_2 and μ . Solving the system, we get the following solution set:

$$\left\{ \begin{array}{l} c_1 = 2\alpha(-1 + \cos(d_1)\text{sech}^2(\sqrt{-\mu}d_2)), \quad c_2 = -2\alpha \sin(d_1)\tanh(\sqrt{-\mu}d_2)/\sqrt{-\mu}, \\ a_0 = 0, \quad a_{-1} = 0 \\ a_1 = \mp \sqrt{-\alpha/\beta} \tanh(\sqrt{-\mu}d_2)/\sqrt{-\mu} \end{array} \right\} \quad (33)$$

and the corresponding discrete hyperbolic function solution to equation (28) as

$$u_{n,1}^\mp(t) = \mp \frac{\sqrt{-\alpha} \tanh(\sqrt{-\mu}d_2)}{\sqrt{\beta}} \left(\frac{C_1 \cosh(\sqrt{-\mu}\xi_n) + C_2 \sinh(\sqrt{-\mu}\xi_n)}{C_1 \sinh(\sqrt{-\mu}\xi_n) + C_2 \cosh(\sqrt{-\mu}\xi_n)} \right) \times \exp(i(d_1n + 2\alpha(-1 + \cos(d_1)\text{sech}^2(\sqrt{-\mu}d_2))t + \zeta_1)), \quad \frac{\alpha}{\beta} < 0, \quad (34)$$

where $\xi_n = d_2n - \frac{2\alpha \sin(d_1)\tanh(\sqrt{-\mu}d_2)}{\sqrt{-\mu}}t + \zeta_2$, and $\mu < 0$, d_1 , d_2 , ζ_1 , ζ_2 , C_1 and C_2 remain arbitrary.

As a particular example, if we let 'C₁ = 0 and C₂ ≠ 0' or 'C₁ ≠ 0 and C₂ = 0' in (34) respectively, then we get formal discrete solitary wave solutions to equation (28) as follows:

$$u_{n,2}^\mp(t) = \mp \frac{\sqrt{-\alpha} \tanh(\sqrt{-\mu}d_2)}{\sqrt{\beta}} \tanh \left(\sqrt{-\mu} \left(d_2n - \frac{2\alpha \sin(d_1)\tanh(\sqrt{-\mu}d_2)}{\sqrt{-\mu}}t + \zeta_2 \right) \right) \times \exp(i(d_1n + 2\alpha(-1 + \cos(d_1)\text{sech}^2(\sqrt{-\mu}d_2))t + \zeta_1)), \quad \frac{\alpha}{\beta} < 0, \quad (35)$$

$$u_{n,3}^\mp(t) = \mp \frac{\sqrt{-\alpha} \tanh(\sqrt{-\mu}d_2)}{\sqrt{\beta}} \coth \left(\sqrt{-\mu} \left(d_2n - \frac{2\alpha \sin(d_1)\tanh(\sqrt{-\mu}d_2)}{\sqrt{-\mu}}t + \zeta_2 \right) \right) \times \exp(i(d_1n + 2\alpha(-1 + \cos(d_1)\text{sech}^2(\sqrt{-\mu}d_2))t + \zeta_1)), \quad \frac{\alpha}{\beta} < 0, \quad (36)$$

where $\mu < 0$, d_1 , d_2 , ζ_1 and ζ_2 are arbitrary constants.

Case 2. $\mu > 0$.

In this case, we first derive the expressions $\phi_{n\pm 1}$ in accordance with (8b) and substitute them along with (32) into equation (31). Then, clearing the denominator and setting the coefficients of $(G'/G)^l$ ($0 \leq l \leq 8$) to zero, we derive a system of nonlinear algebraic equations for $a_0, a_1, a_{-1}, d_1, d_2, c_1, c_2$ and μ . Solving the system, we get the following solution set:

$$\left\{ \begin{array}{l} c_1 = 2\alpha(-1 + \cos(d_1)\sec^2(\sqrt{\mu}d_2)), \quad c_2 = -2\alpha \sin(d_1)\tan(\sqrt{\mu}d_2)/\sqrt{\mu}, \\ a_0 = 0, \quad a_{-1} = 0, \\ a_1 = \mp\sqrt{-\alpha/\beta}\tan(\sqrt{\mu}d_2)/\sqrt{\mu} \end{array} \right\} \quad (37)$$

and the corresponding discrete trigonometric function solution to equation (28) as

$$u_{n,4}^\mp(t) = \mp \frac{\sqrt{-\alpha}\tan(\sqrt{\mu}d_2)}{\sqrt{\beta}} \left(\frac{-C_1 \sin(\sqrt{\mu}\xi_n) + C_2 \cos(\sqrt{\mu}\xi_n)}{C_1 \cos(\sqrt{\mu}\xi_n) + C_2 \sin(\sqrt{\mu}\xi_n)} \right) \times \exp(i(d_1n + 2\alpha(-1 + \cos(d_1)\sec^2(\sqrt{\mu}d_2))t + \zeta_1)), \quad \frac{\alpha}{\beta} < 0, \quad (38)$$

where $\xi_n = d_2n - \frac{2\alpha \sin(d_1)\tan(\sqrt{\mu}d_2)}{\sqrt{\mu}}t + \zeta_2$, and $\mu > 0, d_1, d_2, \zeta_1, \zeta_2, C_1$ and C_2 remain arbitrary.

As a particular example, if we take ‘ $C_1 = 0$ and $C_2 \neq 0$ ’ or ‘ $C_1 \neq 0$ and $C_2 = 0$ ’ in the expression (38) respectively, then we get formal discrete periodic wave solutions to equation (28) as follows:

$$u_{n,5}^\mp(t) = \mp \frac{\sqrt{-\alpha}\tan(\sqrt{\mu}d_2)}{\sqrt{\beta}} \cot \left(\sqrt{\mu} \left(d_2n - \frac{2\alpha \sin(d_1)\tan(\sqrt{\mu}d_2)}{\sqrt{\mu}}t + \zeta_2 \right) \right) \times \exp(i(d_1n + 2\alpha(-1 + \cos(d_1)\sec^2(\sqrt{\mu}d_2))t + \zeta_1)), \quad \frac{\alpha}{\beta} < 0, \quad (39)$$

$$u_{n,6}^\mp(t) = \pm \frac{\sqrt{-\alpha}\tan(\sqrt{\mu}d_2)}{\sqrt{\beta}} \tan \left(\sqrt{\mu} \left(d_2n - \frac{2\alpha \sin(d_1)\tan(\sqrt{\mu}d_2)}{\sqrt{\mu}}t + \zeta_2 \right) \right) \times \exp(i(d_1n + 2\alpha(-1 + \cos(d_1)\sec^2(\sqrt{\mu}d_2))t + \zeta_1)), \quad \frac{\alpha}{\beta} < 0, \quad (40)$$

where $\mu > 0, d_1, d_2, \zeta_1$ and ζ_2 are arbitrary constants.

Case 3. $\mu = 0$.

In this case, we first derive the expressions $\phi_{n\pm 1}$ in accordance with (8c) and substitute them along with (32) into equation (31). Then, clearing the denominator and setting the coefficients of $(G'/G)^l$ ($0 \leq l \leq 7$) to zero, we derive a system of nonlinear algebraic equations for $a_0, a_1, a_{-1}, d_1, d_2, c_1$ and c_2 . Solving the system, we get the following solution set:

$$\left\{ \begin{array}{l} c_2 = -2\alpha d_2 \sin(d_1), \quad c_1 = 2(-\alpha + \alpha \cos(d_1)), \quad a_1 = \mp \frac{\sqrt{-\alpha}d_2}{\sqrt{\gamma}}, \quad a_0 = 0, \quad a_{-1} = 0 \end{array} \right\} \quad (41)$$

and the corresponding discrete rational function solution to equation (28) as

$$u_{n,7}^\mp(t) = \mp \frac{\sqrt{-\alpha}d_2}{\sqrt{\beta}} \left(\frac{C_1}{C_1(d_2n - 2\alpha d_2 \sin(d_1)t + \zeta_2) + C_2} \right) \times \exp(i(d_1n + 2(-\alpha + \alpha \cos(d_1))t + \zeta_1)), \quad \frac{\alpha}{\beta} < 0, \quad (42)$$

where $d_1, d_2, \zeta_1, \zeta_2, C_1$ and C_2 remain arbitrary.

Note 3. We observe that our solution (35) matches the solution (45) of Huang and Liu [27]. However, our rational solution (42) is not presented in there and derived here for the first time.

Note 4. It is an important fact that one should be aware of the limitations of each of the existing methods. There is no guarantee that they will succeed for a specialized nonlinear

problem. Any of them can have some advantages and disadvantages. If treated rigorously, the (G'/G) -expansion method provides exact traveling wave solutions in a neat form from which one can construct solitary and periodic waves, as well as rational ones. One of the pitfalls of the (G'/G) -expansion method, by assuming the solution of the equation in the polynomial form with many parameters, is that it sometimes leads to inconsistent nonlinear algebraic systems. Another one is that it is entirely algorithmic and involves a large amount of tedious calculations which can become virtually unmanageable if attempted manually. We have encountered no difficulty while working on the relativistic Toda lattice system (9). However, we could not get results for the constraint $\alpha/\beta > 0$ while working on the discrete nonlinear Schrödinger equation (28).

4. Conclusion

We systematically illustrated the solution procedure of the extended (G'/G) -expansion method for NDDEs. We obtained a rich variety of discrete traveling wave solutions to a relativistic Toda lattice system and a discrete nonlinear Schrödinger equation. Using a single method, three types of exact solutions are observed: hyperbolic function solutions, trigonometric function solutions and rational function solutions. These obtained solutions with arbitrary parameters may be important to explain some physical phenomena. We would like to point out here that the rational solutions (27) and (42) cannot be obtained by other methods. To the best of our knowledge, these solutions with arbitrary parameters are new; this fact illustrates that our algorithm is effective and more powerful for NDDEs. All solutions are derived here with less algebraic expansion computations with the help of MATHEMATICA. We assured the correctness of our solutions by putting them back into the original equation. More applications of the extended (G'/G) -expansion method to other types of NDDEs deserve further investigation.

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