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Exact solution and dynamic buckling analysis of a beam-column system having the elliptic type loading*

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Abstract This paper presents a closed form solution to the dynamic stability problem of a beam-column system with hinged ends loaded by an axial periodically time-varying compressive force of an elliptic type, i.e., $a_1\mathrm{cn}^2(\tau, k^2) + a_2\mathrm{sn}^2(\tau, k^2) + a_3\mathrm{dn}^2(\tau, k^2)$. The solution to the governing equation is obtained in the form of Fourier sine series. The resulting ordinary differential equation is solved analytically. Finding the exact analytical solutions to the dynamic buckling problems is difficult. However, the availability of exact solutions can provide adequate understanding for the physical characteristics of the system. In this study, the frequency-response characteristics of the system, the effects of the static load, the driving forces, and the frequency ratio on the critical buckling load are also investigated.

Key words dynamic buckling, exact solution, stability-instability, Jacobi elliptic functions

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1 Introduction

The analysis of the dynamic behavior of beam-column systems is of great importance in structural and engineering design. If a structural component is subjected to a dynamic loading, the dynamic buckling problem can be observed. The dynamic stability depends on the applied force and the geometry of the structure. In this study, the exact solution of the beam-column system subjected to a pulsating load of an elliptic type is considered. The pulsating load can often be approximate to the action of rotating machinery, i.e., turbines and power generators, on the columns of structures[1].

A few researchers have studied the analytical solution to dynamic stability problems. The stability and the dynamic analysis of a two-dimensional shear beam-column with generalized boundary conditions have been studied by Aristizabal-Ochoa[2]. The analytical results showed that the stability and the dynamic behavior of shear beams and shear beam-columns were governed by the bending moment equation. Pavlovic et al.[3] considered the dynamic stability problem of thin-walled beams subjected to the combined action of axial loads and end moments.

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Closed form analytical solutions were obtained for simply supported boundary conditions. By using the direct Lyapunov method, almost sure asymptotic stability and uniform stochastic stability conditions were obtained as a function of stochastic process variance, damping coefficient, geometric, and physical parameters of the beam. The inclined beam-column theory, incorporating distributed weight, and the inclination of the beam-column system have been investigated by Sampaio and Hundhausen[4]. They used an energy method to obtain the differential equation of the system, and a closed form solution was obtained in terms of hypergeometric functions. Zuniga[5] studied the problem of the dynamic response of a beam-column system with hinged ends subjected to an axial pulsating force of an elliptic type, i.e., $a\mathrm{cn}^2(\tau, k^2)$. The general exact solution to the resulting equation, that is, the well-known Lamé equation, was obtained for the hinged-hinged beam-column system.

The objectives of the present study are as follows:

(i) to find the analytical solution to the Lamé equation,

$$\frac{d^2u_1}{d\tau^2} + u_1(\tau)(d_1 + d_{2\mathrm{c}}\mathrm{cn}^2(\tau, k^2) + d_{2\mathrm{s}}\mathrm{sn}^2(\tau, k^2) + d_{2\mathrm{d}}\mathrm{dn}^2(\tau, k^2)) = 0;$$

(ii) to investigate the dynamic response characteristics of a beam-column system with hinged ends subjected to the time-varying force of an elliptic type,

$$a_1\mathrm{cn}^2(\tau, k^2) + a_2\mathrm{sn}^2(\tau, k^2) + a_3\mathrm{dn}^2(\tau, k^2).$$

2 Elastic beam-column with hinged ends loaded by an axial time-varying force

A dynamic stability of a uniform beam-column system with hinged ends is considered as shown in Fig. 1.

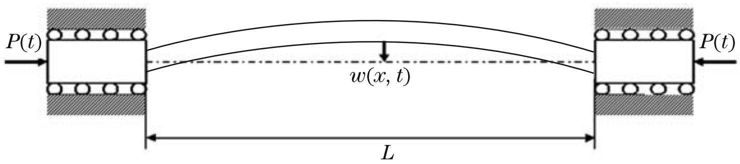


Fig. 1 Beam-column system subjected to a time-varying force

Both ends of this system are subjected to an axial time-varying compressive force of

$$P(t) = P_0 + P_1(t), \tag{1}$$

where P_0 is a stationary force, and $P_1(t)$ is a time-varying force of an elliptic type. The governing equation of the beam-column system is

$$\rho A\frac{\partial^2 w}{\partial t^2} + P(t)\frac{\partial^2 w}{\partial x^2} + EI\frac{\partial^4 w}{\partial x^4} = 0. \tag{2}$$

Equation (2) should be solved together with the following boundary conditions:

$$w(0, t) = w(L, t) = 0, \tag{3}$$

$$\frac{\partial^2}{\partial x^2}w(0, t) = \frac{\partial^2}{\partial x^2}w(L, t) = 0, \tag{4}$$

where $w(x,t)$ defines the lateral deflection. ρ is the density per unit length. E, I, and A represent the modulus of elasticity, the moment of inertia, and the cross sectional area, respectively. The solution to the problem can be obtained conveniently by taking $w(x,t)$ as Fourier sine series

$$w(x,t)=\sum_{n=1}^{\infty}u_n(t)\sin\Big(\frac{n\pi x}{L}\Big). \tag{5}$$

Substitute Eq. (5) into Eq. (2). $u_n(t)$ can be obtained by the following system of the linear differential equation:

$$\sum_{n=1}^{\infty}\Big(\rho A\ddot{u}_n(t)-P(t)\Big(\frac{n\pi}{L}\Big)^2u_n(t)+EI\Big(\frac{n\pi}{L}\Big)^4u_n(t)\Big)\sin\Big(\frac{n\pi x}{L}\Big)=0. \tag{6}$$

Since only the lowest eigenvalue has physical significance in buckling problems, Eq. (6) should be rewritten for the fundamental mode (n=1) as the following form:

$$\Big(\rho A\ddot{u}_1(t)-P(t)\Big(\frac{\pi}{L}\Big)^2u_1(t)+EI\Big(\frac{\pi}{L}\Big)^4u_1(t)\Big)\sin\Big(\frac{\pi x}{L}\Big)=0. \tag{7}$$

To have a non-trivial solution to Eq. (7), the terms in the parentheses must be equal to zero, i.e.,

$$\rho A\ddot{u}_1(t)-P(t)\Big(\frac{\pi}{L}\Big)^2u_1(t)+EI\Big(\frac{\pi}{L}\Big)^4u_1(t)=0. \tag{8}$$

Therefore, the problem is reduced to a second-order ordinary differential equation. Equation (8) may be non-dimensionalized for convenience in the numerical scheme by introducing dimensionless variable τ=Ωt and defining the Euler load $P_{\rm c}$=π^2EI/L^2 and the frequency ω_0^2=$\pi^4EI/(\rho AL^4)$. Then, the following differential equation with variable coefficients can be obtained:

$$\frac{d^2u_1}{d\tau^2}+u_1(\tau)(d_1+\overline{d}_1P_1(\tau))=0, \tag{9}$$

where

$$d_1=\frac{\omega_0^2}{\Omega^2}\Big(1-\frac{P_0}{P_{\rm c}}\Big),\quad \overline{d}_1=\frac{-\omega_0^2}{\Omega^2P_{\rm c}}. \tag{10}$$

3 Solution

The periodically varying force of $P_1(\tau)$ appearing in Eq. (9) is assumed as the linear combination of squares of Jacobi elliptic functions $\mathrm{cn}(\tau,k^2)$, $\mathrm{sn}(\tau,k^2)$, and $\mathrm{dn}(\tau,k^2)$ in the form of (the elementary definition of the elliptic functions is given in [6])

$$P_1(\tau)=a_1\mathrm{cn}^2(\tau,k^2)+a_2\mathrm{sn}^2(\tau,k^2)+a_3\mathrm{dn}^2(\tau,k^2), \tag{11}$$

where a_1, a_2, and a_3 are arbitrary real constants defining the magnitude of driving forces, and k is the modulus. Note that k should take the value between 0 and 1, otherwise the elliptic functions will become complex. Therefore, the loading described by Eq. (11) will be physically meaningless. The behavior of $P(\tau)$ is shown in Fig. 2 for different values of loading parameters a_1, a_2, and a_3. When a_1=1 and a_2=a_3=0, $P(\tau)$ agrees with the force studied in [5].

Now, substituting Eq. (11) into Eq. (9), one obtains

$$\frac{d^2u_1}{d\tau^2}+u_1(\tau)(d_1+d_{2\rm c}\mathrm{cn}^2(\tau,k^2)+d_{2\rm s}\mathrm{sn}^2(\tau,k^2)+d_{2\rm d}\mathrm{dn}^2(\tau,k^2))=0, \tag{12}$$

where

$$d_{2\rm c}=\overline{d}_1a_1,\quad d_{2\rm s}=\overline{d}_1a_2,\quad d_{2\rm d}=\overline{d}_1a_3. \tag{13}$$

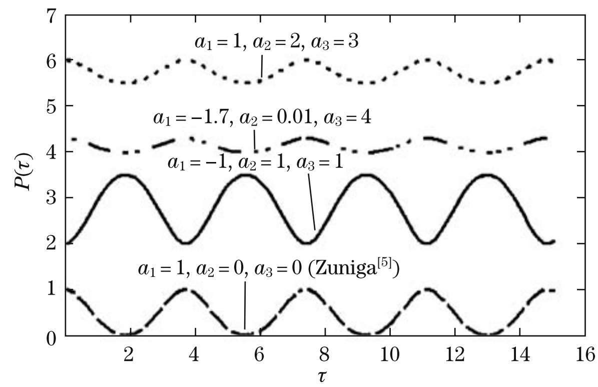


Fig. 2 Behavior of $P(\tau)$ for different values of a_1, a_2, and a_3 with a static load P_0=2 and modulus $k = \sqrt{2}/2$

If $d_{2\mathrm{s}}$=$d_{2\mathrm{d}}$=0, Eq. (13) reduces to the Lamé equation studied in [5].

To find the general solution to Eq. (12), the formulation given in [7] can be used. In this formulation, a second-order linear ordinary differential equation should be in the form of

$$f_2(x)\frac{d^2u}{dx^2} + f_1(x)\frac{du}{dx} + f_0(x)u(x) = 0, \tag{14}$$

and the general solution can be found as

$$u(x) = u_{\mathrm{p}}\Big(c_1 + c_2\int\frac{\mathrm{e}^{-F}}{u_{\mathrm{p}}^2}dx\Big), \tag{15}$$

where

$$F = \int\frac{f_1(x)}{f_2(x)}dx. \tag{16}$$

If a non-trivial particular solution u_{p} is known[7], a particular solution can be found according to Eq. (15), which is a general solution to Eq. (12). To find a particular solution, the following criteria should be satisfied.

(i) It should be a function of Jacobi elliptic functions cn, sn, and dn, since the derivatives of these three functions are always expressed in terms of the same elliptic functions.

(ii) The integrand given in Eq. (15) should be taken analytically in order to express the general solution expilicitly.

Considering all the criteria mentioned above, by trial and error, the following form can be introduced as a particular solution to Eq. (12):

$$u_{1\mathrm{p}}(\tau) = \sqrt{\mathrm{cn}(\tau,k^2) + \mathrm{dn}(\tau,k^2)}. \tag{17}$$

Note that although the particular solution given in [5] also satisfies these criteria ((i) and (ii)), the resulting algebraic equation for Eq. (12) cannot be solved.

Substituting Eq. (17) into Eq. (12) and using the following relations:

$$\mathrm{sn}(\tau,k^2) = \sqrt{1-\mathrm{cn}^2(\tau,k^2)}, \quad \mathrm{dn}(\tau,k^2) = \sqrt{1-k^2+k^2\mathrm{cn}^2(\tau,k^2)}, \tag{18}$$

one obtains Eq. (12) as the function of $\mathrm{cn}^2(\tau,k^2)$ in the following form:

$$\begin{aligned}&(-1+4d_1+4d_{2\mathrm{s}}+4d_{2\mathrm{d}}+2k^2-4d_{2\mathrm{d}}k^2+4d_{2\mathrm{c}}\mathrm{cn}^2(\tau,k^2)-4d_{2\mathrm{s}}\mathrm{cn}^2(\tau,k^2)\\&-3k^2\mathrm{cn}^2(\tau,k^2)+4d_{2\mathrm{d}}k^2\mathrm{cn}^2(\tau,k^2))(1-k^2+\mathrm{cn}^2(\tau,k^2)+k^2\mathrm{cn}^2(\tau,k^2)\\&+2\mathrm{cn}(\tau,k^2)\sqrt{1-k^2+k^2\mathrm{cn}^2(\tau,k^2)})/\big(4\mathrm{cn}(\tau,k^2)+4\sqrt{1-k^2+k^2\mathrm{cn}^2(\tau,k^2)}\big)^{\frac{3}{2}}=0.\end{aligned} \tag{19}$$

The above algebraic equation is to be valid for all values of τ if the following conditions are satisfied:

$$-1 + 4d_1 + 4d_{2\mathrm{s}} + 4d_{2\mathrm{d}} + 2k^2 - 4d_{2\mathrm{d}}k^2 = 0, \tag{20}$$

$$4d_{2\mathrm{c}} - 4d_{2\mathrm{s}} - 3k^2 + 4d_{2\mathrm{d}}k^2 = 0. \tag{21}$$

Then, the general solution satisfying the conditions given by Eqs. (20) and (21) can now be obtained in the form of

$$u_1(\tau) = u_{1\mathrm{p}}\Big(c_1 + c_2 \int \frac{\mathrm{e}^{-m}}{u_{1\mathrm{p}}^2(\tau)} d\tau\Big), \tag{22}$$

where m is an arbitrary constant.

The integral appearing in Eq. (22) is obtained[8]. Therefore, the general solution is

$$u_1(\tau) = \sqrt{\mathrm{cn}(\tau, k^2) + \mathrm{dn}(\tau, k^2)}\Big(c_1 + c_0 \frac{\mathrm{sn}(\tau, k^2)}{\mathrm{cn}(\tau, k^2) + \mathrm{dn}(\tau, k^2)}\Big), \tag{23}$$

where c_0=$\mathrm{e}^{-m}c_2$.

From the initial conditions

$$u_1(0) = u_{1\mathrm{c}}, \quad \frac{du_1}{d\tau}(0) = \overline{u}_{1\mathrm{c}}, \tag{24}$$

where $u_{1\mathrm{c}}$ and $\overline{u}_{1\mathrm{c}}$ are arbitrary constants, we find the exact solution

$$u_1(\tau) = \sqrt{\mathrm{cn}(\tau, k^2) + \mathrm{dn}(\tau, k^2)}\Big(\frac{u_{1\mathrm{c}}}{\sqrt{2}} + \overline{u}_{1\mathrm{c}}\sqrt{2} \frac{\mathrm{sn}(\tau, k^2)}{\mathrm{cn}(\tau, k^2) + \mathrm{dn}(\tau, k^2)}\Big). \tag{25}$$

Equation (25) includes bounded and unbounded terms. In order to make the solution physically meaningful, unbounded terms need to be vanished. For this purpose, appropriate initial conditions are introduced as $u_1(0)$=$u_{1\mathrm{c}}$ and $du_1(0)/d\tau$ =0.

Therefore, the solution to the Lamé equation becomes bounded, and is in the form of

$$u_{1\mathrm{b}} = \frac{u_{1\mathrm{c}}}{\sqrt{2}}\sqrt{\mathrm{cn}(\tau, k^2) + \mathrm{dn}(\tau, k^2)}. \tag{26}$$

In the second part of this study, the frequency-response characteristics of the system are discussed. For this purpose, using Eqs. (10), (13), (20), and (21), the relation related to the Euler load P_c, driving forces a_1, a_2, and a_3, the static load P_0, and the frequency ratio ω_0/Ω are obtained after expanding algebraic manipulations as

$$\frac{3P_\mathrm{c}^2 + 4(2a_1 + a_2 + 4a_3)P_\mathrm{c}\Big(\frac{\omega_0}{\Omega}\Big)^2 + 16a_3(a_1 + a_3)\Big(\frac{\omega_0}{\Omega}\Big)^4}{4P_\mathrm{c}\Big(3P_\mathrm{c} + 4a_\mathrm{c}\Big(\frac{\omega_0}{\Omega}\Big)^2\Big)} = \Big(\frac{\omega_0}{\Omega}\Big)^2\Big(1 - \frac{P_0}{P_\mathrm{c}}\Big). \tag{27}$$

The equation given above is rearranged to observe the effect of loading parameters on the change of frequency ratio as

$$\begin{aligned}&3P_\mathrm{c}^2/(4a_1P_\mathrm{c} + 2a_2P_\mathrm{c} + 8a_3P_\mathrm{c} + 6P_0P_\mathrm{c} - 3P_\mathrm{c}^2\\&+2\sqrt{((2a_1 + a_2 + 4a_3 + 3P_0 - 3P_\mathrm{c})^2 - 12a_3(a_1 + a_3 + P_0 - P_\mathrm{c}))P_\mathrm{c}^2})\\&= \frac{\Big(\frac{\omega_0}{\Omega}\Big)^2}{\Big(\frac{\omega_0}{\Omega}\Big)^2 - 1}.\end{aligned} \tag{28}$$

The left-hand side of Eq. (28) will be represented by $F(P_c, P_0, a_1, a_2, a_3)$ for better presentation in the related table and figures. Frequency response characteristics for different values of a_1, a_2, and a_3 are tabulated in Table 1. The values in the first line of the table agree with the frequency reponse discussed in [5]. It is seen that the difference in the frequency response characteristics between the present study and that of Zuniga[5] results from different analytical solutions.

Table 1 Special cases for frequency response characteristic $F(P_c, P_0, a_1, a_2, a_3)$

a_1	a_2	a_3	$F(P_c, P_0, a_1, a_2, a_3)$
a_1	0	0	$3P_c(8a_1+12P_0-9P_c)^{-1}$ (Zuniga[5])
0	a_2	a_3	$3P_c^2(2a_2P_c+8a_3P_c+6P_0P_c-3P_c^2 +2\sqrt{[(a_2+4a_3+3P_0-3P_c)^2-12a_3(a_3+P_0-P_c)]P_c^2})^{-1}$
a_1	0	a_3	$3P_c^2(4a_1P_c+8a_3P_c+6P_0P_c-3P_c^2 +2\sqrt{[(2a_1+4a_3+3P_0-3P_c)^2-12a_3(a_1+a_3+P_0-P_c)]P_c^2})^{-1}$
a_1	a_2	0	$3P_c(8a_1 4a_2+12P_0-9P_c)^{-1}$

4 Results and discussions

The frequency response characteristics of the beam-column system with hinged ends loaded by an axial periodically time-varying compressive force of an elliptic type given by Eq. (28) are shown in Fig. 3. In the interval (0,1) for the ratio ω_0/Ω, $F(P_c, P_0, a_1, a_2, a_3)$ asymptotically increases and goes to infinity at ω_0/Ω =1, where the resonance occurs. With the increase of ω_0/Ω, $F(P_c, P_0, a_1, a_2, a_3)$ approaches asymptotically to one.

The effects of driving forces a_1, a_2, and a_3 on P_c with the values of the frequency ratio ω_0/Ω=0.6 and the static load P_0=1 are shown in Fig. 4. P_c shows linearly increasing behavior with respect to a_1, a_2, and a_3. It is observed from the figure that change in P_c with a_3 is more obvious compared with the other driving forces a_1 and a_2.

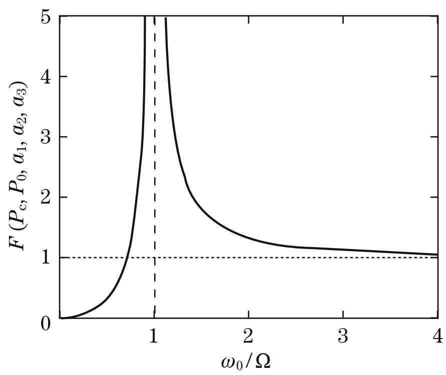


Fig. 3 Frequency response diagram

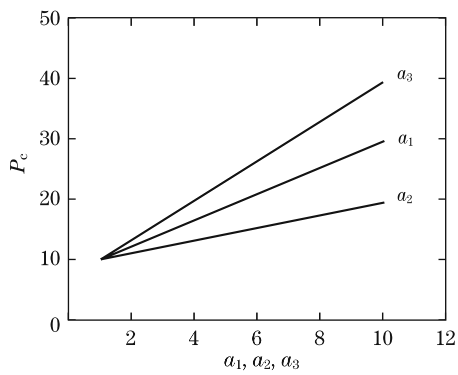


Fig. 4 Variation of P_c with a_1, a_2 and a_3 for frequency ratio ω_0/Ω=0.6 and static load P_0=1

It can be observed from Fig. 5 that the static load P_0 and a_3 have the same influence on P_c. Furthermore, as expected, the maximum value P_c is reached at the maximum values of P_0 and a_3. It is obvious that in Fig. 6, P_c decreases as the frequency ratio increases, and a_3 increases with the increase of P_c. It can also be observed that the maximum P_c is reached at the lowest frequency ratio and at the maximum a_3 value.

To clarify the effects of system parameters on the stability, the stability-instability charts (see Fig. 7) related with the static parameter d_1 and dynamic parameters d_{2c}, d_{2s}, and d_{2d} when k=0.5 are obtained by using the numerical solution to Eq. (12). Shaded regions appearing in the figures represent the stable solution to Eq. (12). In Fig. 7(c), the stability region is considerably different from Figs. 7(a) and 7(b) because of the most effective parameter a_3. In such a case, the effect of a_3 increases, and the stable region is enlarged. It is also noted that the stable regions are moderately different from those given in [5] since the driving force is assumed as the linear combination of square of the elliptic functions $\text{cn}(\tau, k^2)$, $\text{sn}(\tau, k^2)$, and $\text{dn}(\tau, k^2)$ in the present study.

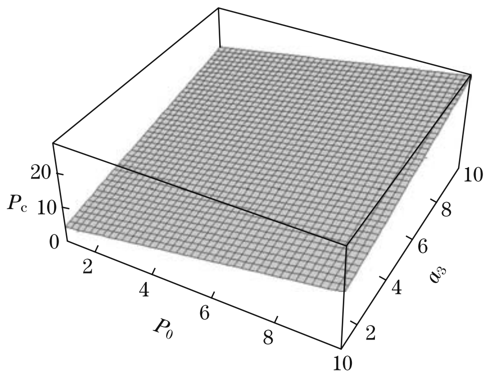


Fig. 5 Effects of P_0 and a_3 on P_c with driving forces a_1=1, a_2=1, and ω_0/Ω =1

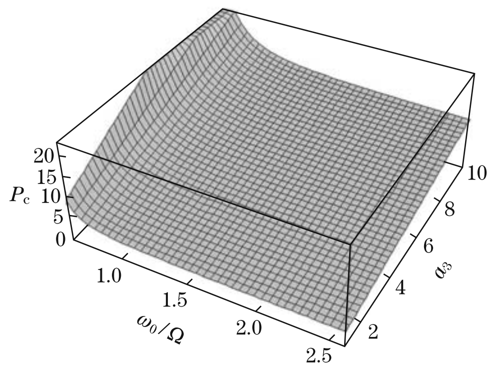
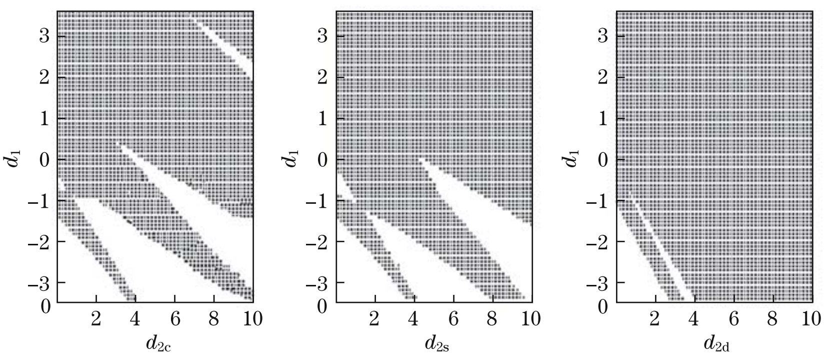


Fig. 6 Effects of frequency ratio ω_0/Ω and driving force a_3 on P_c with a_1=1, a_2=1



(a) d_{2s}=1, d_{2d}=1, k=0.5 (b) d_{2c}=1, d_{2d}=1, k=0.5 (c) d_{2c}=1, d_{2s}=1, k=0.5

Fig. 7 Stability-instability charts for Eq. (12)

5 Conclusions

In this paper, a closed form solution to the dynamic stability problem of a beam-column system loaded by an axial force of an elliptic type is presented. The frequency response characteristics of the system are obtained for different cases. The effects of the static load, driving forces, and frequency ratio on the Euler load are discussed. Stability-instability charts are also investigated for the Lamé equation (Eq. (12)). It can be concluded that the particular solution in [5] can only be used in the case of a_2=a_3=0 corresponding to the equation

$$\frac{d^2u_1}{d\tau^2} + u_1(\tau)(d_1 + d_{2c}\text{cn}^2(\tau, k^2)) = 0.$$

However, the particular solution given in this study is an appropriate solution to both the following two equations:

$$\frac{d^2u_1}{d\tau^2} + u_1(\tau)(d_1 + d_{2\mathrm{c}}\mathrm{cn}^2(\tau, k^2) + d_{2\mathrm{s}}\mathrm{sn}^2(\tau, k^2) + d_{2\mathrm{d}}\mathrm{dn}^2(\tau, k^2)) = 0,$$

$$\frac{d^2u_1}{d\tau^2} + u_1(\tau)(d_1 + d_{2\mathrm{c}}\mathrm{cn}^2(\tau, k^2)) = 0.$$

It is also found that a_3 is the most effective loading parameter for the presented problem.

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