Applied Mathematics and Computation 217 (2010) 1421–1429

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Generalized solitary and periodic wave solutions to a (2 + 1)-dimensional Zakharov–Kuznetsov equation

İsmail Aslan

Department of Mathematics, Izmir Institute of Technology, Urla, İzmir 35430, Turkey

ARTICLE INFO

Keywords:
Exp-function method
Zakharov–Kuznetsov equation
Solitary wave solutions

ABSTRACT

In this paper, the Exp-function method is employed to the Zakharov–Kuznetsov equation as a (2 + 1)-dimensional model for nonlinear Rossby waves. The observation of solitary wave solutions and periodic wave solutions constructed from the exponential function solutions reveal that our approach is very effective and convenient. The obtained results may be useful for better understanding the properties of two-dimensional coherent structures such as atmospheric blocking events.

1. Introduction

Solving nonlinear evolution equations (NLEEs) has become a valuable task in many scientific areas including applied mathematics as well as the physical sciences and engineering in the last four decades or so. For this purpose, some accurate techniques have been presented in the open literature; for example, inverse scattering transform method [1], Jacobi elliptic function method [2], tanh–coth function method [3], sine–cosine function method [4], symmetry method [5], F-expansion method [6], Hirota's bilinear method [7], Painlevé expansion method [8], homogeneous balance method [9], Bäcklund transformation method [10], Adomian decomposition method [11], variational iteration method [12], homotopy analysis method [13], homotopy perturbation method [14] and so on. On the other hand, with the development of computer algebra systems (they allow us to perform the tedious and complicated algebraic calculations on a computer) in recent years, many direct and effective methods using symbolic computation are also presented such as the (G'/G)-expansion method [15–19] and the Exp-function method [20–24].

It is well known that many important dynamics processes can be described by specific nonlinear partial differential equations. In 1974, Zakharov and Kuznetsov [25] derived an equation which describes weakly nonlinear ion-acoustic waves in a strongly magnetized lossless plasma composed of cold ions and hot isothermal electrons. The Zakharov–Kuznetsov (ZK) equation is also known as one of two-dimensional generalizations of the KdV equation, another one being the Kadomtsev–Petviashvili (KP) equation for example. In contrast to the KP equation, the ZK equation is non-integrable by the inverse scattering transform method, though Shivamoggi [26] showed that it posses the Painlevé property by making a Painlevé analysis of the ZK equation. The ZK equation has also been derived in the context of plasma physics [27,28]. Biswas and Zerrad [29] considered the ZK equation with dual-power law nonlinearity and obtained 1-soliton solution by using the solitary wave ansatze.

The ZK equation is a very attractive model equation for the study of vortices in geophysical flows since it supports stable lump solitary waves [30]. Thus, more recently, to study the dynamics of two-dimensional coherent structures in planetary atmospheres and oceans, Gottwald [31] derived the ZK equation for large scale motion from the barotropic quasigeostrophic

E-mail address: ismailaslan@iyte.edu.tr

0096-3003/$ - see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2009.05.037

equation as a two-dimensional model for Rossby waves. The (2 + 1)-dimensional Zakharov–Kuznetsov ((2 + 1)D-ZK) equation [31] reads

$$u_t + \delta u_x + \alpha u u_x + \beta u_{xxx} + \gamma u_{xyy} = 0, \quad (1)$$

where δ, α, β, and γ are nonzero arbitrary constants and $u = u(x,y,t)$. He [32] applied the homotopy perturbation method to the (2 + 1)D-ZK equation (1) to search for traveling wave solutions. Using a sub-equation method (the elliptic equation is taken as a transformation), the traveling wave solutions for the (2 + 1)D-ZK equation (1) are also studied by Fu et al. [33]. There has not been much research on this special form (2 + 1)D-ZK equation (1) in the literature. The main goal of our present work is to further analyze this less studied form of the ZK equation by using the so-called Exp-function method.

Based on He and Wu's pioneer work [20] and his followers the Exp-function method has found some popularity in a research community, and there has been a number of papers refining the initial idea [34–39]. The Exp-function "method" consists of trying rational combinations of exponential functions as an "ansatz" to find exact solutions of the ODE for traveling waves of the original equation. The method is powerful for it can take full advantage of computer algebra systems, the solution procedure is actually almost impossible without using a computer.

2. The Exp-function method based on the symbolic computation

To begin with, suppose that we have a nonlinear partial differential equation for $u(x,y,t)$ in the form

$$P(u, u_t, u_x, u_y, u_{tt}, u_{tx}, u_{ty}, u_{xx}, u_{xy}, u_{yy}, \ldots) = 0, \quad (2)$$

where P is a polynomial in its arguments. We give an algorithmic description of our method as follows:

Step 1. (*Reduce NPDE to nonlinear ODE*) By taking $u(x,y,t) = U(\zeta)$, $\zeta = kx + my + wt$, where k, m, and w are arbitrary non-zero constants, look for traveling wave solutions of Eq. (2), and transform it to the ordinary differential equation

$$Q(U, U', U'', \ldots) = 0, \quad (3)$$

where prime denotes the derivative with respect to ζ.

Step 2. (*Simplify the nonlinear ODE*) Integrate Eq. (3), if possible, term by term one or more times. This yields constant(s) of integration. The integration constant(s) can be set to zero for simplicity.

Step 3. (*Make an ansatz*) Suppose the solution $U(\zeta)$ of Eq. (3) can be expressed in the form

$$U(\zeta) = \frac{a_c \exp(c\zeta) + \cdots + a_{-d} \exp(-d\zeta)}{b_p \exp(p\zeta) + \cdots + b_{-q} \exp(-q\zeta)}, \quad (4)$$

where c, d, p and q are unknown positive integers to be determined, a_i and b_j are unknown constants.

Step 4. (*Determine the parameters*) Determine the highest order nonlinear term and the linear term of highest order in Eq. (3) and express them in terms of (4). Then, in the resulting terms, balance the highest order Exp-function to determine c and p, and the lowest order Exp-function to determine d and q.

Step 5. (*Generate a set of algebraic equations*) Substitute (4) into Eq. (3) and equate the coefficients of $\exp(l\eta)$ to zero, obtain a *system* of algebraic equations for a_i, b_j, k, m and w. Then, to determine these constants, solve the system with the aid of a computer algebra system such as Mathematica.

Step 6. (*Obtain exact solutions*) Substitute the values solved in Step 5 into expression (4) and find the traveling wave solutions of Eq. (2). Then, ıt is necessary to substitute them into the original Eq. (2) to assure the correctness of the solutions.

3. Analytic solutions to the (2 + 1)D-ZK equation

To seek for the traveling wave solutions to the (2 + 1)D-ZK equation (1), we make the transformation $u(x,y,t) = V(\zeta)$, $\zeta = kx + my + wt$, where k, m and w are constants to be determined later. Then, integrating the resulting ODE once and setting the constant of integration to zero, we get

$$(\beta k^3 + \gamma k m^2) V'' + \frac{\alpha k}{2} V^2 + (\delta k + w) V = 0, \quad (5)$$

where primes denote the derivatives with respect to ζ. Now, we make an ansatz

$$V(\zeta) = \frac{a_c \exp(c\zeta) + \cdots + a_{-d} \exp(-d\zeta)}{b_p \exp(p\zeta) + \cdots + b_{-q} \exp(-q\zeta)} \quad (6)$$

for the solution of Eq. (5) and balance the terms V'' and V^2. By a simple calculation, we have

$$V'' = \frac{k_1 \exp[(c + 3p)\zeta] + \cdots}{k_2 \exp[4p\zeta] + \cdots} \quad (7)$$

and

$$V^2 = \frac{k_3 \exp[2c\zeta] + \cdots}{k_4 \exp[2p\zeta] + \cdots} = \frac{k_3 \exp[2(c+p)\zeta] + \cdots}{k_4 \exp[4p\zeta] + \cdots}, \tag{8}$$

where k_i's are determined coefficients for simplicity. Balancing highest order of Exp-function in Eqs. (7) and (8), we have

$$c + 3p = 2(c+p), \tag{9}$$

which leads to the result

$$p = c. \tag{10}$$

Similarly, from the ansatz (6), we have

$$V'' = \frac{\cdots + l_1 \exp[-(d+3q)\zeta]}{\cdots + l_2 \exp[-4q\zeta]} \tag{11}$$

and

$$V^2 = \frac{\cdots + l_3 \exp[-2d\zeta]}{\cdots + l_4 \exp[-2q\zeta]} = \frac{\cdots + l_3 \exp[-2(d+q)\zeta]}{\cdots + l_4 \exp[-4q\zeta]}, \tag{12}$$

where l_i's are determined coefficients for simplicity. Balancing lowest order of Exp-function in Eqs. (11) and (12), we have

$$-(d+3q) = -2(d+q), \tag{13}$$

which leads to the result

$$q = d. \tag{14}$$

We can freely choose the values of c and d in general. However, the final solution does not strongly depend on the values of c and d [20,23].

Case 1: $p = c = 1$, $d = q = 1$

In this case, the solution of Eq. (5) can be expressed as

$$V(\zeta) = \frac{a_1 \exp(\zeta) + a_0 + a_{-1} \exp(-\zeta)}{b_1 \exp(\zeta) + b_0 + b_{-1} \exp(-\zeta)}. \tag{15}$$

Substituting (15) into Eq. (5), we have

$$\frac{1}{A}[C_0 + C_1 \exp(\zeta) + C_2 \exp(2\zeta) + C_3 \exp(3\zeta) + C_4 \exp(4\zeta) + C_5 \exp(5\zeta) + C_6 \exp(6\zeta)] = 0, \tag{16}$$

where $A = 2(b_1\exp(2\zeta) + b_0\exp(\zeta) + b_{-1})^3$ and

$$\begin{aligned}
C_0 &= k\alpha a_{-1}^2 b_{-1} + 2wa_{-1}b_{-1}^2 + 2k\delta a_{-1}b_{-1}^2,\\
C_1 &= 2k\alpha a_{-1}a_0b_{-1} + 2wa_0b_{-1}^2 + 2k^3\beta a_0b_{-1}^2 + 2km^2\gamma a_0b_{-1}^2 + 2k\delta a_0b_{-1}^2 + k\alpha b_0a_{-1}^2 + 4wa_{-1}b_{-1}b_0\\
&\quad - 2k^3\beta a_{-1}b_{-1}b_0 - 2km^2\gamma a_{-1}b_{-1}b_0 + 4k\delta a_{-1}b_{-1}b_0,\\
C_2 &= k\alpha a_0^2b_{-1} + 2k\alpha a_{-1}a_1b_{-1} + 2wa_1b_{-1}^2 + 8k^3\beta a_1b_{-1}^2 + 8km^2\gamma a_1b_{-1}^2 + 2k\delta a_1b_{-1}^2 + 2k\alpha a_{-1}a_0b_0\\
&\quad + 4wa_0b_{-1}b_0 - 2k^3\beta a_0b_{-1}b_0 - 2km^2\gamma a_0b_{-1}b_0 + 4k\delta a_0b_{-1}b_0 + 2wa_{-1}b_0^2 + 2k^3\beta a_{-1}b_0^2\\
&\quad + 2km^2\gamma a_{-1}b_0^2 + 2k\delta a_{-1}b_0^2 + k\alpha a_{-1}^2b_1 + 4wa_{-1}b_{-1}b_1 - 8k^3\beta a_{-1}b_{-1}b_1 - 8km^2\gamma a_{-1}b_{-1}b_1\\
&\quad + 4k\delta a_{-1}b_{-1}b_1,\\
C_3 &= 2k\alpha a_0a_1b_{-1} + k\alpha a_0^2b_0 + 2k\alpha a_{-1}a_1b_0 + 4wb_{-1}a_1b_0 + 6k^3\beta b_{-1}a_1b_0 + 6km^2\gamma b_{-1}a_1b_0 + 4k\delta b_{-1}a_1b_0\\
&\quad + 2wa_0b_0^2 + 2k\delta a_0b_0^2 + 2k\alpha a_{-1}a_0b_1 + 4wb_{-1}a_0b_1 - 12k^3\beta b_{-1}a_0b_1 - 12km^2\gamma b_{-1}a_0b_1 + 4k\delta b_{-1}a_0b_1\\
&\quad + 4wa_{-1}b_0b_1 + 6k^3\beta a_{-1}b_0b_1 + 6km^2\gamma a_{-1}b_0b_1 + 4k\delta a_{-1}b_0b_1,\\
C_4 &= k\alpha a_1^2b_{-1} + 2k\alpha a_0a_1b_0 + 2wa_1b_0^2 + 2k^3\beta a_1b_0^2 + 2km^2\gamma a_1b_0^2 + 2k\delta a_1b_0^2 + k\alpha b_1a_0^2 + 2k\alpha a_{-1}a_1b_1\\
&\quad + 4wb_{-1}a_1b_1 - 8k^3\beta b_{-1}a_1b_1 - 8km^2\gamma b_{-1}a_1b_1 + 4k\delta b_{-1}a_1b_1 + 4wb_0a_0b_1 - 2k^3\beta b_0a_0b_1 - 2km^2\gamma b_0a_0b_1\\
&\quad + 4k\delta b_0a_0b_1 + 2wa_{-1}b_1^2 + 8k^3\beta a_{-1}b_1^2 + 8km^2\gamma a_{-1}b_1^2 + 2k\delta a_{-1}b_1^2,\\
C_5 &= k\alpha a_1^2b_0 + 2k\alpha a_0a_1b_1 + 4wa_1b_0b_1 - 2k^3\beta a_1b_0b_1 - 2km^2\gamma a_1b_0b_1 + 4k\delta a_1b_0b_1 + 2wa_0b_1^2\\
&\quad + 2k^3\beta a_0b_1^2 + 2km^2\gamma a_0b_1^2 + 2k\delta a_0b_1^2,\\
C_6 &= k\alpha a_1^2b_1 + 2wa_1b_1^2 + 2k\delta a_1b_1^2.
\end{aligned}$$

Equating the coefficients of $\exp(j\zeta)$ to zero in (16) and solving the resulting nonlinear algebraic system for a_1, a_0, a_{-1}, b_1, b_0, b_{-1}, k, m and w, we have the solution sets:

$$\left\{a_{-1}=0, a_0=\frac{6b_0\left(k^2\beta+m^2\gamma\right)}{\alpha}, a_1=0, b_{-1}=\frac{b_0^2}{4b_1}, w=-k\left(k^2\beta+m^2\gamma+\delta\right)\right\}, \tag{17}$$

$$\left\{a_{-1}=\frac{-b_0^2\left(k^2\beta+m^2\gamma\right)}{2\alpha b_1}, a_0=\frac{4b_0\left(k^2\beta+m^2\gamma\right)}{\alpha}, a_1=\frac{-2b_1\left(k^2\beta+m^2\gamma\right)}{\alpha}, b_{-1}=\frac{b_0^2}{4b_1}, w=k\left(k^2\beta+m^2\gamma-\delta\right)\right\}. \tag{18}$$

Now, substituting (17) into (15) yields a more general exponential function solution to Eq. (1) as

$$u_1(x,y,t)=\frac{24b_0b_1\left(k^2\beta+m^2\gamma\right)\exp\left(kx+my-k\left(k^2\beta+m^2\gamma+\delta\right)t\right)}{\alpha\left(b_0+2b_1\exp\left(kx+my-k\left(k^2\beta+m^2\gamma+\delta\right)t\right)\right)^2}, \tag{19}$$

where b_1, b_0, k and m are non-zero real constants. If we take $b_0=\pm 2b_1$ in (19), then we get the formal solitary wave solutions to Eq. (1) as

$$u_2(x,y,t)=\frac{3\left(k^2\beta+m^2\gamma\right)}{\alpha}\operatorname{sech}^2\left(\frac{1}{2}\left(kx+my-k\left(k^2\beta+m^2\gamma+\delta\right)t\right)\right), \tag{20}$$

$$u_3(x,y,t)=-\frac{3\left(k^2\beta+m^2\gamma\right)}{\alpha}\operatorname{csch}^2\left(\frac{1}{2}\left(kx+my-k\left(k^2\beta+m^2\gamma+\delta\right)t\right)\right), \tag{21}$$

where k and m are non-zero real constants.

Next, substituting (18) into (15) leads to another more general exponential function solution

$$u_4(x,y,t)=\frac{2\left(k^2\beta+m^2\gamma\right)}{\alpha}\left(-1+\frac{12b_0b_1\exp\left(kx+my+k\left(k^2\beta+m^2\gamma-\delta\right)t\right)}{\left(b_0+2b_1\exp\left(kx+my+k\left(k^2\beta+m^2\gamma-\delta\right)t\right)\right)^2}\right), \tag{22}$$

where b_1, b_0, k and m are non-zero real constants. If we take $b_0=\pm 2b_1$ in (22), then we have the formal solitary wave solutions to Eq. (1) as

$$u_5(x,y,t)=\frac{\left(k^2\beta+m^2\gamma\right)}{\alpha}(2-\cosh\zeta)\operatorname{sech}^2\frac{\zeta}{2}, \tag{23}$$

$$u_6(x,y,t)=\frac{-\left(k^2\beta+m^2\gamma\right)}{\alpha}(2+\cosh\zeta)\operatorname{csch}^2\frac{\zeta}{2}, \tag{24}$$

where $\zeta=kx+my+k\left(k^2\beta+m^2\gamma-\delta\right)t$, k and m are non-zero real constants.

When k, m and w are imaginary numbers in the complex variation $\zeta=kx+my+wt$, then it is possible to convert the obtained solitary solutions into periodic or compact-like solutions. To do so, we write $k=iK$, $m=iM$, $w=iW$ and use the transformations

$$\begin{aligned}\exp(\zeta)&=\exp\left(i(Kx+My+Wt)\right)=\cos\left(Kx+My+Wt\right)+i\sin\left(Kx+My+Wt\right),\\ \exp(-\zeta)&=\exp\left(i(-Kx-My-Wt)\right)=\cos\left(Kx+My+Wt\right)-i\sin\left(Kx+My+Wt\right).\end{aligned} \tag{25}$$

Now, plugging (25) into the solitary solution (19) gives

$$u_7(x,y,t)=\frac{-24b_0b_1\left(K^2\beta+M^2\gamma\right)}{\alpha\left[\left(4b_1^2+b_0^2\right)\cos\left(Kx+My+\left(K^3\beta+KM^2\gamma-K\delta\right)t\right)+4b_0b_1+\left(4b_1^2-b_0^2\right)i\sin\left(Kx+My+\left(K^3\beta+KM^2\gamma-K\delta\right)t\right)\right]}. \tag{26}$$

If we search for periodic or compact-like solution, then the imaginary part in (26) must be zero and thus we get

$$b_0=\mp 2b_1. \tag{27}$$

Finally, substituting (27) and (26) yields the periodic solutions to Eq. (1) as

$$u_{8,9}(x,y,t)=\frac{-6\left(K^2\beta+M^2\gamma\right)}{\alpha\left[1\mp\cos\left(Kx+My+\left(K^3\beta+KM^2\gamma-K\delta\right)t\right)\right]}, \tag{28}$$

where K and M are free real parameters. By a similar approach, from (22) we can obtain compact-like solutions to Eq. (1) in the form

$$u_{10,11}(x,y,t) = \frac{2\left(K^2\beta + M^2\gamma\right)}{\alpha}\left[1 - \frac{3}{1 \mp \cos\left(Kx + My - \left(K^3\beta + KM^2\gamma + K\delta\right)t\right)}\right], \tag{29}$$

where K and M are free real parameters.

Case 2: $p = c = 2$, $d = q = 2$

Then the trial function (6) becomes

$$V(\zeta) = \frac{a_2\exp(2\zeta) + a_1\exp(\zeta) + a_0 + a_{-1}\exp(-\zeta) + a_{-2}\exp(-2\zeta)}{b_2\exp(2\zeta) + b_1\exp(\zeta) + b_0 + b_{-1}\exp(-\zeta) + b_{-2}\exp(-2\zeta)}. \tag{30}$$

There are some free parameters in (30), so we set $b_2 = 1$, $b_1 = 0$, $b_{-1} = 0$ for simplicity and thus (30) takes the form

$$V(\zeta) = \frac{a_2\exp(2\zeta) + a_1\exp(\zeta) + a_0 + a_{-1}\exp(-\zeta) + a_{-2}\exp(-2\zeta)}{\exp(2\zeta) + b_0 + b_{-2}\exp(-2\zeta)}. \tag{31}$$

Substituting (31) into Eq. (5), we get

$$\frac{1}{2}(\exp(4\zeta) + b_0\exp(2\zeta) + b_{-2})^{-3}\sum_{j=0}^{12} C_j\exp(j\zeta) = 0. \tag{32}$$

We omit to display the coefficients C_j explicitly. Then, equating the coefficients of $\exp(j\zeta)$ to zero and solving the resulting nonlinear algebraic system for a_2, a_1, a_0, a_{-1}, a_{-2}, b_0, b_{-2}, k, m and w, we have the solution sets:

$$\left\{a_{-2} = 0, a_{-1} = 0, a_2 = 0, a_1 = 0, a_0 = \frac{24b_0\left(k^2\beta + m^2\gamma\right)}{\alpha}, b_{-2} = \frac{b_0^2}{4}, w = -k\left(4k^2\beta + 4m^2\gamma + \delta\right)\right\}, \tag{33}$$

$$\left\{\begin{array}{l} a_{-2} = \frac{-\left(k^2\beta+m^2\gamma\right)b_0^2}{2\alpha}, a_{-1} = \frac{-3\sqrt{2}}{\alpha}i\left(k^2\beta + m^2\gamma\right)b_0^{3/2}, a_2 = \frac{-2\left(k^2\beta+m^2\gamma\right)}{\alpha}, a_1 = \frac{6\sqrt{2b_0}}{\alpha}i\left(k^2\beta + m^2\gamma\right), \\ a_0 = \frac{10b_0\left(k^2\beta+m^2\gamma\right)}{\alpha}, b_{-2} = \frac{b_0^2}{4}, w = k\left(k^2\beta + m^2\gamma - \delta\right), i = \sqrt{-1}. \end{array}\right\}, \tag{34}$$

$$\left\{\begin{array}{l} a_{-2} = 0, a_{-1} = \frac{-3\sqrt{2}}{\alpha}i\left(k^2\beta + m^2\gamma\right)b_0^{3/2}, a_2 = 0, a_1 = \frac{6\sqrt{2b_0}}{\alpha}i\left(k^2\beta + m^2\gamma\right), a_0 = \frac{12b_0\left(k^2\beta+m^2\gamma\right)}{\alpha}, \\ b_{-2} = \frac{b_0^2}{4}, w = -k\left(k^2\beta + m^2\gamma + \delta\right), i = \sqrt{-1}. \end{array}\right\}, \tag{35}$$

$$\left\{\begin{array}{l} a_{-2} = \frac{-2b_0^2\left(k^2\beta+m^2\gamma\right)}{\alpha}, a_{-1} = 0, a_2 = \frac{-8\left(k^2\beta+m^2\gamma\right)}{\alpha}, a_1 = 0, a_0 = \frac{16b_0\left(k^2\beta+m^2\gamma\right)}{\alpha}, \\ b_{-2} = \frac{b_0^2}{4}, w = k\left(4k^2\beta + 4m^2\gamma - \delta\right). \end{array}\right\}. \tag{36}$$

Now, substituting (33) into (31) yields more general exponential function solution

$$u_{12}(x,y,t) = \frac{96b_0\left(k^2\beta + m^2\gamma\right)\exp\left(2\left(kx + my - k\left(4k^2\beta + 4m^2\gamma + \delta\right)t\right)\right)}{\alpha\left(b_0 + 2\exp\left(2\left(kx + my - k\left(4k^2\beta + 4m^2\gamma + \delta\right)t\right)\right)\right)^2}, \tag{37}$$

where b_0, k and m are non-zero real numbers. If we take $b_0 = \pm 2$ in (37), then we have the formal solitary wave solutions to Eq. (1) as

$$u_{13}(x,y,t) = \frac{12\left(k^2\beta + m^2\gamma\right)}{\alpha}\operatorname{sech}^2\left(kx + my - k\left(4k^2\beta + 4m^2\gamma + \delta\right)t\right), \tag{38}$$

$$u_{14}(x,y,t) = \frac{-12\left(k^2\beta + m^2\gamma\right)}{\alpha}\operatorname{csch}^2\left(kx + my - k\left(4k^2\beta + 4m^2\gamma + \delta\right)t\right), \tag{39}$$

where k and m are non-zero real constants.

Next, substituting (34) into (31) leads to the following more general exponential function solution

$$u_{15}(x,y,t) = \frac{2\left(k^2\beta + m^2\gamma\right)\left(-4\exp(4\zeta) + 12\sqrt{2b_0}i\exp(3\zeta) + 20b_0\exp(2\zeta) - 6\sqrt{2b_0^3}i\exp(\zeta) - b_0^2\right)}{\alpha(b_0 + 2\exp(2\zeta))^2}, \tag{40}$$

where $\zeta = kx + my + k\left(k^2\beta + m^2\gamma - \delta\right)t$, $i = \sqrt{-1}$, b_0, k and m are non-zero real numbers. If we take $b_0 = \pm 2$ in (40), then we have the formal solitary wave solutions to Eq. (1) as

$$u_{16}(x,y,t) = \frac{\left(k^2\beta + m^2\gamma\right)}{\alpha}(6i\tanh\zeta + (5 - \cosh 2\zeta)\operatorname{sech}\zeta)\operatorname{sech}\zeta, \tag{41}$$

$$u_{17}(x,y,t) = \frac{\left(k^2\beta + m^2\gamma\right)}{\alpha}(-6\coth\zeta - (5 + \cosh 2\zeta)\operatorname{csch}\zeta)\operatorname{csch}\zeta, \tag{42}$$

where $\zeta = kx + my + k\left(k^2\beta + m^2\gamma - \delta\right)t$, $i = \sqrt{-1}$, k and m are non-zero real numbers.

Similarly, substituting (35) into (31) results in another more general exponential function solution which reads

$$u_{18}(x,y,t) = \frac{12\left(k^2\beta + m^2\gamma\right)\sqrt{b_0}\exp(\zeta)\left(2\sqrt{2}i\exp(2\zeta) + 4\sqrt{b_0}\exp(\zeta) - \sqrt{2}b_0 i\right)}{\alpha(b_0 + 2\exp(2\zeta))^2}, \tag{43}$$

where $\zeta = kx + my - k(k^2\beta + m^2\gamma + \delta)t$, $i = \sqrt{-1}$, b_0, k and m are non-zero real numbers. If we take $b_0 = \pm 2$ in (43), then we have the formal solitary wave solutions to Eq. (1) as

$$u_{19}(x,y,t) = \frac{6\left(k^2\beta + m^2\gamma\right)}{\alpha}(\operatorname{sech}\zeta + i\tanh\zeta)\operatorname{sech}\zeta, \tag{44}$$

$$u_{20}(x,y,t) = -\frac{6\left(k^2\beta + m^2\gamma\right)}{\alpha}(\coth\zeta + \operatorname{csch}\zeta)\operatorname{csch}\zeta, \tag{45}$$

where $\zeta = kx + my - k(k^2\beta + m^2\gamma + \delta)t$, $i = \sqrt{-1}$, k and m are non-zero real numbers.

Finally, substituting (36) into (31) results in more general exponential function solution

$$u_{21}(x,y,t) = \frac{-8\left(k^2\beta + m^2\gamma\right)}{\alpha}\left(1 - \frac{12b_0\exp\left(2\left(kx + my + k\left(4k^2\beta + 4m^2\gamma - \delta\right)t\right)\right)}{\left(b_0 + 2\exp\left(2\left(kx + my + k\left(4k^2\beta + 4m^2\gamma - \delta\right)t\right)\right)\right)^2}\right), \tag{46}$$

where b_0, k and m are non-zero real numbers. If we take $b_0 = \pm 2$ in (46), then we have the formal solitary wave solutions to Eq. (1) as

$$u_{22}(x,y,t) = \frac{4\left(k^2\beta + m^2\gamma\right)}{\alpha}(2 - \cosh 2\zeta)\operatorname{sech}^2\zeta, \tag{47}$$

$$u_{23}(x,y,t) = \frac{4\left(k^2\beta + m^2\gamma\right)}{\alpha}(-2 - \cosh 2\zeta)\operatorname{csch}^2\zeta, \tag{48}$$

where $\zeta = kx + my + k\left(4k^2\beta + 4m^2\gamma - \delta\right)t$, k and m are non-zero real numbers.

Now, by the same procedure as in the previous case, applying the transformations (25) to the functions (37), (40), (43) and (46) respectively, we get periodic wave solutions to Eq. (1) as follows:

$$u_{24,25}(x,y,t) = \frac{-24\left(K^2\beta + M^2\gamma\right)}{\alpha\left[1 \mp \cos\left(2\left(Kx + My + (4K^3\beta + 4KM^2\gamma - K\delta)t\right)\right)\right]}, \tag{49}$$

$$u_{26,27}(x,y,t) = \frac{2\left(K^2\beta + M^2\gamma\right)}{\alpha}\left(1 - \frac{3}{1 \mp \sin\left(Kx + My - \left(K^3\beta + KM^2\gamma + K\delta\right)t\right)}\right), \tag{50}$$

$$u_{28,29}(x,y,t) = \frac{6\left(K^2\beta + M^2\gamma\right)}{\alpha\left(1 \mp \sin\left(Kx + My + \left(K^3\beta + KM^2\gamma - K\delta\right)t\right)\right)}, \tag{51}$$

$$u_{30,31}(x,y,t) = \frac{8\left(K^2\beta + M^2\gamma\right)}{\alpha}\left[1 - \frac{3}{1 \mp \cos\left(2\left(Kx + My - \left(4K^3\beta + 4KM^2\gamma + K\delta\right)t\right)\right)}\right], \tag{52}$$

where K and M are free real parameters. For the sake of the brevity, we do not give any further details here.

Case 3: $p = c = 3$, $d = q = 2$

Then the trial function (3) becomes

$$V(\zeta) = \frac{a_3\exp(3\zeta) + a_2\exp(2\zeta) + a_1\exp(\zeta) + a_0 + a_{-1}\exp(-\zeta) + a_{-2}\exp(-2\zeta)}{b_3\exp(3\zeta) + b_2\exp(2\zeta) + b_1\exp(\zeta) + b_0 + b_{-1}\exp(-\zeta) + b_{-2}\exp(-2\zeta)}. \tag{53}$$

There are some free parameters in (53), so we set $b_3 = 1$, $b_2 = 0$, $b_1 = 0$, $b_{-1} = 0$ for simplicity and thus (53) takes the form

$$V(\zeta) = \frac{a_3\exp(3\zeta) + a_2\exp(2\zeta) + a_1\exp(\zeta) + a_0 + a_{-1}\exp(-\zeta) + a_{-2}\exp(-2\zeta)}{\exp(3\zeta) + b_0 + b_{-2}\exp(-2\zeta)}. \tag{54}$$

By the same manipulation as illustrated above, we obtain the solution sets:

$$\left\{\begin{array}{l} a_{-2} = \frac{6\left(\frac{2}{5}\right)^{2/3}b_0^{5/3}\left(k^2\beta + m^2\gamma\right)}{5\alpha}, a_{-1} = \frac{-36\left(\frac{2}{5}\right)^{1/3}b_0^{4/3}\left(k^2\beta + m^2\gamma\right)}{5\alpha}, a_0 = \frac{62b_0\left(k^2\beta + m^2\gamma\right)}{5\alpha}, \\ a_2 = \frac{12\left(\frac{2}{5}\right)^{1/3}b_0^{1/3}\left(k^2\beta + m^2\gamma\right)}{\alpha}, a_3 = \frac{-2\left(k^2\beta + m^2\gamma\right)}{\alpha}, a_1 = \frac{-24\left(\frac{2}{5}\right)^{2/3}b_0^{2/3}\left(k^2\beta + m^2\gamma\right)}{\alpha}, \\ b_{-2} = -\frac{3}{5}\left(\frac{2}{5}\right)^{2/3}b_0^{5/3}, w = k\left(k^2\beta + m^2\gamma - \delta\right) \end{array}\right\}, \tag{55}$$

$$\left\{\begin{array}{l} a_{-2}=0, a_{-1}=\frac{-36\left(\frac{2}{5}\right)^{1/3} b_0^{4/3}\left(k^2\beta+m^2\gamma\right)}{5\alpha}, a_0=\frac{72 b_0\left(k^2\beta+m^2\gamma\right)}{5\alpha}, a_2=\frac{12\left(\frac{2}{5}\right)^{1/3} b_0^{1/3}\left(k^2\beta+m^2\gamma\right)}{\alpha}, \\ a_3=0, a_1=\frac{-24\left(\frac{2}{5}\right)^{2/3} b_0^{2/3}\left(k^2\beta+m^2\gamma\right)}{\alpha}, b_{-2}=-\frac{3}{5}\left(\frac{2}{5}\right)^{2/3} b_0^{5/3}, w=-k\left(k^2\beta+m^2\gamma+\delta\right) \end{array}\right\}. \tag{56}$$

Now, substituting (55) into (54) yields more general exponential function solution

$$u_{32}(x,y,t)=\frac{2\left(k^2\beta+m^2\gamma\right)}{\alpha}\left(-1+\frac{60\sqrt[3]{b_0}}{\sqrt[3]{2500}\exp(\zeta)+20\sqrt[3]{b_0}+\sqrt[3]{400b_0^2}\exp(-\zeta)}\right), \tag{57}$$

where $\zeta = kx + my + k(k^2\beta + m^2\gamma - \delta)t$, b_0, k and m are non-zero real numbers.

Next, substituting (56) into (54) leads to the following more general exponential function solution

$$u_{33}(x,y,t)=\frac{120\left(k^2\beta+m^2\gamma\right)\sqrt[3]{b_0}}{\alpha\left(\sqrt[3]{2500}\exp(\zeta)+20\sqrt[3]{b_0}+\sqrt[3]{400b_0^2}\exp(-\zeta)\right)}, \tag{58}$$

where $\zeta = kx + my - k(k^2\beta + m^2\gamma + \delta)t$, b_0, k and m are non-zero real numbers.

We note that if we take $b_0 = 5/2$ in (57) and (58), then we observe that the resulting formal solitary wave solutions to Eq. (1) are the same as (23) and (20), respectively. Similarly, if we apply the transformations (25) to (57) and (58) and take $b_0 = 5/2$, then we get periodic wave solutions to Eq. (1) which are included in (29) and (28), respectively.

Remark 1. There is no a universal method for nonlinear equations. Each of the existing methods presented in the open literature for solving nonlinear evolution equations has some advantages and disadvantages. However, there are some reasons to select the Exp-function method over the others: (i) The Exp-function method provides exponential function solutions from which we can construct solitary and periodic wave solutions by setting the parameters as special values. (ii) As mentioned in [40], Jacobi elliptic function method [2] cannot be applied to solve the NLEEs in which the odd and even-order derivative terms coexist. This observation is also true for tanh–coth function method [3] and F-expansion method [6]. However, in [41], it is shown that the Burgers' equation in which the odd and even-order derivative terms coexist can be solved by the Exp-function method. (iii) Being less restrictive, the Exp-function method can be extended to a wide class of equations such as NLEEs with variable coefficients [42,43], a KdV equation with forcing term [44], and nonlinear differential-difference equations [45–48]. Furthermore, the Exp-function method is more applicable to discrete NLEEs than hyperbolic function method. (iv) Instead of involving a sub-equation together with a long list of solutions, it can be applied to other sub-equations such as Ricatti equations [22], [35], [49–52]. Moreover, it can be combined with other methods such as the variational iteration method [53].

Remark 2. It is always better not to integrate the original equation and set the resulting constant(s) to zero in order to get more general form solutions. Otherwise, some types of solutions may be missed. The reason is that an integration constant which corresponds to a general form of solutions is arbitrary and therefore, if it is set to zero, only a special solution is derived. The Exp-function method, by assuming the solution of the equation in exponential form with many parameters, entails the solution of several sets of nonlinear algebraic equations which sometimes constitute inconsistent systems. In our case, in order the Exp-function method work for us properly, it has become obligatory to reduce the order of the transformed ODE by integration. However, no method is perfect and can promise finding all solutions of a given (integrable or non-integrable) nonlinear partial differential equation.

Remark 3. The obtained generalized solitary wave solutions with the free parameters b_0 and/or b_1 might imply some fascinating physical meanings hidden in the (2 + 1)D-ZK equation. Of course, we can set the parameters b_0 and b_1 equal to other values, resulting in different solitary wave shapes. These free parameters might be related to the initial and/or the boundary data for the problem, as well. Nevertheless, we have ensured the correctness of our solutions by putting them back into the original Eq. (1) with the aid of Mathematica.

4. Conclusion

Using the Exp-function method, we have successfully obtained various kinds of novel exact analytic solutions to a special form (2 + 1)D-ZK equation, which admits physical significance, derived by Gottwalld [31] recently. Our solutions include exponential function solutions, solitary wave solutions and periodic wave solutions. As a result, the power of the employed method is confirmed. Of course, we are unable to give further details for the real physical meaning of our analytic solutions due to the lack of experimental and theoretical basis related to these solutions. We believe that these solutions will also be of great importance for numerical simulation in applied mathematics.

Acknowledgement

I would like to thank the anonymous referees for their useful and valuable suggestions.

References

[1] M.J. Ablowitz, H. Segur, Solitons and Inverse Scattering Transform, SIAM, Philadelphia, 1981.
[2] G.T. Liu, T.Y. Fan, New applications of developed Jacobi elliptic function expansion methods, Phys. Lett. A 345 (2005) 161–166.
[3] W. Malfliet, W. Hereman, The tanh method I: Exact solutions of nonlinear evolution and wave equations, Phys. Scr. 54 (1996) 563–568.
[4] A.M. Wazwaz, Distinct variants of the KdV equation with compact and noncompact structures, Appl. Math. Comput. 150 (2004) 365–377.
[5] G.W. Bluman, S. Kumei, Symmetries and Differential Equations, Springer-Verlag, New York, 1989.
[6] M.A. Abdou, The extended F-expansion method and its application for a class of nonlinear evolution equations, Chaos Solitons Fract. 31 (2007) 95–104.
[7] R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge, 2004.
[8] J. Weiss, M. Tabor, G. Carnevale, The Painleve property for partial differential equations, J. Math. Phys. 24 (1983) 522–526.
[9] M.L. Wang, Exact solutions for a compound KdV–Burgers equation, Phys. Lett. A 213 (1996) 279–287.
[10] R.M. Miura, Backlund Transformation, Springer-Verlag, New York, 1996.
[11] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, Boston, 1994.
[12] J.H. He, A new approach to nonlinear partial differential equations, Commun. Nonlinear Sci. Numer. Simul. 2 (1997) 230–235.
[13] S.J. Liao, Homotopy analysis method-A kind of nonlinear analytical technique not depending on small parameters, Shanghai J. Mech. 18 (1997) 196–200.
[14] J.H. He, An approximate solution technique depending on an artificial parameter: a special example, Commun. Nonlinear Sci. Numer. Simul. 3 (1998) 92–97.
[15] M. Wang, X. Li, J. Zhang, The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, Phys. Lett. A 372 (2008) 417–423.
[16] İ. Aslan, T. Öziş, Analytic study on two nonlinear evolution equations by using the (G'/G)-expansion method, Appl. Math. Comput. 209 (2009) 425–429.
[17] İ. Aslan, T. Öziş, On the validity and reliability of the (G'/G)-expansion method by using higher-order nonlinear equations, Appl. Math. Comput. 211 (2009) 531–536.
[18] T. Öziş, İ. Aslan, Symbolic computation and construction of new exact traveling wave solutions to Fitzhugh–Nagumo and Klein–Gordon equations, Z. Naturforsch. 64a (2009) 15–20.
[19] T. Öziş, İ. Aslan, Symbolic computations and exact and explicit solutions of some nonlinear evolution equations in mathematical physics, Commun. Theor. Phys. 51 (2009) 577–580.
[20] J.H. He, X.H. Wu, Exp-function method for nonlinear wave equations, Chaos Solitons Fract. 30 (2006) 700–708.
[21] J.H. He, L.N. Zhang, Generalized solitary solution and compacton-like solution of the Jaulent–Miodek equations using the Exp-function method, Phys. Lett. A 372 (2008) 1044–1047.
[22] X.H. Wu, J.H. He, Solitary solutions, periodic solutions and compacton-like solutions using the Exp-function method, Comput. Math. Appl. 54 (2007) 966–986.
[23] J.H. He, M.A. Abdou, New periodic solutions for nonlinear evolution equations using Exp-function method, Chaos Solitons Fract. 34 (2007) 1421–1429.
[24] T. Öziş, İ. Aslan, Exact and explicit solutions to the (3 + 1)-dimensional Jimbo–Miwa equation via the Exp-function method, Phys. Lett. A 372 (2008) 7011–7015.
[25] V.E. Zakharov, E.A. Kuznetsov, On three-dimensional solitons, Soviet Phys. 39 (1974) 285–288.
[26] B.K. Shivamoggi, The Painlevé analysis of the Zakharov–Kuznetsov equation, Phys. Scr. 42 (1990) 641–642.
[27] S. Munro, E.J. Parkes, The derivation of a modified Zakharov–Kuznetsov equation and the stability of its solutions, J. Plasma Phys. 62 (1999) 305–317.
[28] J. Das, A. Bandyopadhyay, K.P. Das, Stability of an alternative solitary-wave solution of an ion-acoustic wave obtained from the MKdV–KdV–ZK equation in magnetized non-thermal plasma consisting of warm adiabatic ions, J. Plasma Phys. 72 (2006) 587–604.
[29] A. Biswas, E. Zerrad, 1-soliton solution of the Zakharov–Kuznetsov equation with dual-power law nonlinearity, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 3574–3577.
[30] H. Iwasaki, S. Toh, T. Kawahara, Cylindrical quasi-solitons of the Zakharov–Kuznetsov equation, Physica D 43 (1990) 293–303.
[31] G.A. Gottwalld, The Zakharov–Kuznetsov equation as a two-dimensional model for nonlinear Rossby wave, arXiv: nlin, 0312009, 2003.
[32] J.H. He, Application of homotopy perturbation method to nonlinear wave equations, Chaos Soliton Fract. 26 (2005) 695–700.
[33] Z. Fu, S. Liu, S. Liu, Multiple structures of 2-D nonlinear Rossby wave, Chaos Soliton Fract. 24 (2005) 383–390.
[34] X.W. Zhou, Y.X. Wen, J.H. He, Exp-function method to solve the nonlinear dispersive $K(m,n)$ equations, Int. J. Nonlinear Sci. Numer. Simul. 3 (2008) 301–306.
[35] X.H. Wu, J.H. He, EXP-function method and its application to nonlinear equations, Chaos Soliton Fract. 38 (2008) 903–910.
[36] M.A. Abdou, E.M. Abulwafa, Application of the Exp-function method to the Riccati equation and new exact solutions with three arbitrary functions of quantum Zakharov equations, Z. Naturforsch. 63a (2008) 646–652.
[37] V. Marinakis, The Exp-function method find n-soliton solutions, Z. Naturforsch. 63a (2008) 653–656.
[38] C.Q. Dai, Y.Y. Wang, Exact travelling wave solutions of Toda lattice equations obtained via the Exp-function method, Z. Naturforsch. 63a (2008) 657–662.
[39] S. Zhang, W. Wang, J.L. Tong, The Exp-function method for the Riccati equation and exact solutions of dispersive long wave equations, Z. Naturforsch. 63a (2008) 663–670.
[40] S. Liu, Z. Fu, S. Liu, Q. Zhao, Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, Phys. Lett. A 289 (2001) 69–74.
[41] A. Ebaid, Exact solitary wave solutions for some nonlinear evolution equations via Exp-function method, Phys. Lett. A 365 (2007) 213–219.
[42] S.A. El-Wakil, M.A. Madkour, M.A. Abdou, Application of Exp-function method for nonlinear evolution equations with variable coefficients, Phys. Lett. A 369 (2007) 62–69.
[43] S. Zhang, Application of Exp-function method to a KdV equation with variable coefficients, Phys. Lett. A 365 (2007) 448–453.
[44] S. Zhang, Exp-function method exactly solving a KdV equation with forcing term, Appl. Math. Comput. 197 (2008) 128–134.
[45] S. Zhang, Exp-function method for constructing explicit and exact solutions of a lattice equation, Appl. Math. Comput. 199 (2008) 242–249.
[46] S.D. Zhu, Discrete (2 + 1)-dimensional Toda lattice equation via Exp-function method, Phys. Lett. A 372 (2008) 654–657.
[47] C. Dai, X. Cen, S. Wu, Exact travelling wave solutions of the discrete sine–Gordon equation obtained via the exp-function method, Nonlinear Anal. 70 (2009) 58–63.
[48] C. Dai, X. Cen, S. Wu, The application of He's exp-function method to a nonlinear differential-difference equation, Chaos Solitons Fract., doi:10.1016/j.chaos.2008.02.021.
[49] S. Zhang, Application of Exp-function method to Riccati equation and new exact solutions with three arbitrary functions of Broer–Kaup–Kupershmidt equations, Phys. Lett. A 372 (2008) 1873–1880.
[50] C. Dai, New exact solutions of the (3 + 1)-dimensional Burgers system, Phys. Lett. A 373 (2009) 181–187.

[51] C. Dai, Y. Wang, New variable separation solutions of the (2 + 1)-dimensional asymmetric Nizhnik–Novikov–Veselov system, Nonlinear Anal., doi:10.1016/j.na.2008.12.052.
[52] C. Dai, J. Chen, Exact solutions of (2 + 1)-dimensional stochastic Broer–Kaup equation, Phys. Lett. A 373 (2009) 1218–1225.
[53] J. Zhu, New explicit exact solutions of the mKdV equation using the variational iteration method combined with Exp-function method, Chaos Solitons Fract., doi:10.1016/j.chaos.2007.08.051.