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Weakly distributive modules. Applications to supplement submodules

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Abstract. In this paper, we define and study weakly distributive modules as a proper generalization of distributive modules. We prove that, weakly distributive supplemented modules are amply supplemented. In a weakly distributive supplemented module every submodule has a unique coclosure. This generalizes a result of Ganesan and Vanaja. We prove that π-projective duo modules, in particular commutative rings, are weakly distributive. Using this result we obtain that in a commutative ring supplements are unique. This generalizes a result of Camillo and Lima. We also prove that any weakly distributive $\oplus$-supplemented module is quasi-discrete.

Keywords. Distributive module; supplement submodule.

1. Introduction

In this paper, we study a generalization of distributive modules, namely weakly distributive modules. Distributive rings and modules are studied extensively in the literature, e.g. see [3], [7], [9], [10]. Let M be an R-module and $U \subseteq M$. The submodule U is said to be a *distributive* submodule (of M) if $U = U \cap X + U \cap Y$ for all submodules $X, Y \subseteq M$. And M is called *distributive* if each submodule of M is a distributive submodule.

We shall call U a *weak distributive* submodule of M if $U = U \cap X + U \cap Y$ for all submodules $X, Y \subseteq M$ such that $X + Y = M$. A module M is said to be *weakly distributive* if every submodule of M is a weak distributive submodule of M. A ring R is weakly distributive if R is a weakly distributive left R-module. Weakly distributive modules are proper generalization of distributive modules (see, Example 3.3). We obtain that, a weakly distributive module is distributive if and only if every submodule is weakly distributive.

The paper is organized as follows. In §2, it is shown that homomorphic images of weakly distributive modules are weakly distributive. We prove that any π-projective duo module is weakly distributive. In particular any commutative ring is weakly distributive.

In §3, we characterize weakly distributive semisimple modules. We show that, the notions of being, weakly distributive, distributive and duo coincide for semisimple modules.

In §4, we prove that the sum and intersection of two direct summands of a weakly distributive module is again a direct summand. As a consequence we obtain that, if e and f are idempotent endomorphisms of a weakly distributive module then ef and fe are idempotent endomorphisms.

In §5, we deal with weakly distributive supplemented modules. We prove that supplement submodules are unique in weakly distributive modules, and any weakly distributive supplemented module is amply supplemented. We prove that any weakly distributive supplemented module is a UCC module, this generalizes Theorem 4.8 of [4]. π-projective duo modules are weakly distributive. Using this, we obtain that in a commutative ring supplements are unique, and this generalizes Proposition 5(1) of [2]. We also prove that, any weakly distributive $\oplus$-supplemented module is quasi-discrete.

Throughout, R is a ring with an identity element and all modules are unital left R-modules.

2. Weakly distributive modules

It is well-known that, if $f: M \to N$ is an isomorphism, then there is a one-to-one correspondence between the submodules of M and the submodules of N. Therefore, any module (lattice) isomorphic to a weakly distributive module, is itself weakly distributive.

Lemma 2.1. *Let M be a weakly distributive module and $f: M \to N$ be a homomorphism. Then* Im f *is a weakly distributive module.*

Proof. Let $f: M \to N$ be a homomorphism. Then Im $f \cong M/K$ where $K = \operatorname{Ker} f$. Let $U/K + V/K = M/K$ for some $U, V \subseteq M$ and $X/K \subseteq M/K$. Then we have $U + V = M$. Since M is weakly distributive, $X = X \cap U + X \cap V$. Therefore,

$$X/K = (X \cap U + X \cap V)/K = [(X/K) \cap (U/K)] + [(X/K) \cap (V/K)].$$

Hence Im $f \cong M/K$ is weakly distributive. □

Examples of weak distributive submodules can be found in the following lemma. Recall that an R-module M is called *π-projective* if whenever $M = X + Y$, there exists $\alpha \in \operatorname{End}(M)$ such that $\alpha(M) \subseteq X$ and $(1 - \alpha)(M) \subseteq Y$.

Lemma 2.2. *Let M be a π-projective module. Every fully invariant submodule of M is a weak distributive submodule of M.*

Proof. Let U be a fully invariant submodule of M and suppose $M = X + Y$. Then there exists an endomorphism $f \in \operatorname{End}(M)$ such that $f(M) \subseteq X$ and $(1 - f)(M) \subseteq Y$. Since U is a fully invariant submodule of M, we have

$$f(U) \subseteq U \cap X, \quad (1 - f)(U) \subseteq U \cap Y.$$

Then

$$U \subseteq f(U) + (1 - f)(U) \subseteq U \cap X + U \cap Y$$

and so $U = U \cap X + U \cap Y$. That is U is a weak distributive submodule of M. □

A module M is called *duo* module if every submodule of M is a fully invariant submodule of M. Therefore the following corollary is a consequence of Lemma 2.2.

COROLLARY 2.3

Any π-projective duo module is a weakly distributive module. In particular, any left (right) duo ring is weakly distributive.

The following example shows that weakly distributive modules need not be duo, in general. Recall that a module is called uniserial if its lattice of submodules is totally ordered by inclusion.

Example 2.4. Let R be a valuation domain (i.e. discrete valuation ring) and K be the field of quotients of R. Then K is a uniserial module, and hence K is a weakly distributive module. Let pR be the unique maximal ideal of R. Then p is not a unit of R, and so $k = \frac{1}{p} \in K \backslash R$.

Consider the homomorphism $f: K \to K$ defined as $f(a) = ka$. Then $f(R) = kR \nsubseteq R$, that is, R is not a fully invariant submodule of K. Hence K is not a duo module.

3. Semisimple weakly distributive modules

In this section we shall characterize weakly distributive semisimple modules. We begin with the following lemma.

Lemma 3.1. *If M is any non-zero module, then $M \oplus M$ is not weakly distributive.*

Proof. For any $0 \neq m \in M$, the submodule $N = R(m, m)$ of $M \oplus M$ is not a weak distributive submodule, because $N \cap (M \oplus \{0\}) = \{(0, 0)\} = N \cap (\{0\} \oplus M)$. □

PROPOSITION 3.2

Let $M = \oplus_{i \in I} S_i$ be a semisimple module where S_i is a simple module for each $i \in I$. Then M is a weakly distributive module if and only if $\mathrm{Hom}(S_i, S_j) = 0$ *for every $i, j \in I$ such that $i \neq j$.*

Proof. Let $i, j \in I$ and $i \neq j$. Then the submodule $S_i \oplus S_j$ of M is a direct summand of M and hence a weakly distributive module. And so by Lemma 3.1, S_i and S_j are not isomorphic. Clearly this implies that $\mathrm{Hom}(S_i, S_j) = 0$. For the converse let A be a submodule of M. First we shall prove that A is a fully invariant submodule of M. Since M is semisimple $A = \oplus_{j \in J} S_j$ for some $J \subset I$. Let $f \in \mathrm{End}(M)$. Then $f(S_j) \subset S_j$ for each $j \in J$, by hypothesis. Therefore $f(A) \subset A$. That is A is a fully invariant submodule of M. Since M is semisimple it is self-projective, hence M is π-projective. Then Lemma 2.2 implies that A is a weak distributive submodule of M. □

From the proof of Proposition 3.2, it is clear that, the notions of being weakly distributive, distributive and duo coincide for semisimple modules.

Example 3.3. Let F be any field and V be a vector space over F. Consider the trivial extension $R = F \times V$ in which the multiplication is defined as $(a, v)(b, w) = (ab, aw + bv)$, where $a, b \in F$ and $v, w \in V$. The ring R is a local ring whose Socle is $(0 \times V)$. Then ${}_R R$ is a weakly distributive module while the submodule $(0 \times V)$ of R is not unless the dimension of V is one by Lemma 3.1.

PROPOSITION 3.4

An R-module M is distributive if and only if every submodule of M is a weakly distributive module.

Proof. The necessity part is clear. For sufficiency, let A, B and C be submodules of M. Then,

$$\begin{aligned} A \cap (B + C) &= [A \cap (B + C)] \cap (B + C) \\ &= [A \cap (B + C)] \cap B + [A \cap (B + C)] \cap C \\ &= A \cap B + A \cap C, \end{aligned}$$

since $B + C$ is weakly distributive. □

4. Endomorphism ring of weakly distributive modules

A module M is said to satisfy the *summand sum property* if $U + V$ is a direct summand of M whenever U and V are direct summands of M. M satisfies the *summand intersection property* if $U \cap V$ is a direct summand of M whenever U and V are direct summands of M.

Lemma 4.1. *Weakly distributive modules satisfy the summand sum property and the summand intersection property.*

Proof. Let X and Y be direct summands of M. Suppose $M = X \oplus X' = Y \oplus Y'$. Then $X = X \cap Y \oplus X \cap Y'$. We get $M = X \oplus X' = X \cap Y \oplus X \cap Y' \oplus X'$. Hence $X \cap Y$ is a direct summand of M, and so M has the summand intersection property.

To prove that M has the summand sum property, we need to show that $X + Y$ is a direct summand of M. We have

$$\begin{aligned} X + Y &= X \cap Y + X \cap Y' + Y \cap X + Y \cap X' \\ &= (X \cap Y + X' \cap Y) + X \cap Y' \\ &= Y \oplus X \cap Y'. \end{aligned}$$

Now we get $M = Y \oplus Y' = Y \oplus Y' \cap X \oplus Y' \cap X' = (X + Y) \oplus Y' \cap X'$. This completes the proof. □

PROPOSITION 4.2

Let M be a weakly distributive module and let $e, f \in \mathrm{End}(M)$ be idempotent endomorphisms. Then $\mathrm{Im}(ef) = \mathrm{Im}(fe) = \mathrm{Im}(e) \cap \mathrm{Im}(f)$ and fe is an idempotent.

Proof. Since e and f are idempotent endomorphisms of M, we have

$$M = e(M) \oplus (1 - e)(M) = f(M) \oplus (1 - f)(M).$$

Using the fact that M is weakly distributive, $e(M) = e(M) \cap f(M) \oplus e(M) \cap (1 - f)(M)$. Applying f we get

$$f(e(M)) = f(e(M) \cap f(M)) = e(M) \cap f(M).$$

Now applying e to $f(M) = f(M) \cap e(M) \oplus f(M) \cap (1 - e)(M)$, we get

$$e(f(M)) = e(f(M) \cap e(M)) = f(M) \cap e(M).$$

We obtained $\mathrm{Im}(ef) = \mathrm{Im}(fe) = \mathrm{Im}(e) \cap \mathrm{Im}(f)$. Since $\mathrm{Im}(f)$ and $\mathrm{Im}(e)$ are direct summands of M, then $\mathrm{Im}(e) \cap \mathrm{Im}(f)$ is a direct summand of M by Lemma 4.1.

For any $m \in M$, since $\mathrm{Im}(fe) = \mathrm{Im}(e) \cap \mathrm{Im}(f)$, we have

$$efe(m) = e(fe(m)) = fe(m) \in \mathrm{Im}(f)$$

and therefore

$$fefe(m) = f(efe(m)) = efe(m) = fe(m).$$

Hence fe is an idempotent endomorphism of M. □

COROLLARY 4.3

Let M be a weakly distributive module and let $e, f \in \mathrm{End}(M)$ be idempotent endomorphisms. If $\mathrm{Im}(e)$ and $\mathrm{Im}(f)$ are indecomposable submodules of M, then either $\mathrm{Im}(f) = \mathrm{Im}(e)$ or $ef = fe = 0$.

Proof. Suppose that both ef and fe are nonzero. Then by Proposition 4.2, ef is an idempotent and so $\mathrm{Im}(ef) = \mathrm{Im}(e) \cap \mathrm{Im}(f)$ is a (nonzero) direct summand of M. Then $\mathrm{Im}(e) \cap \mathrm{Im}(f)$ is a direct summand of both the indecomposable submodules $\mathrm{Im}(e)$ and $\mathrm{Im}(f)$. Hence $\mathrm{Im}(e) = \mathrm{Im}(e) \cap \mathrm{Im}(f) = \mathrm{Im}(f)$. □

5. Applications to supplement submodules

In this section we shall obtain several results related with weakly distributive and some variation of supplemented modules.

First we deal with weakly supplemented modules. The condition of weak distributivity gives nice structure on maximal submodules of such modules.

We begin with the following lemma which is trivial. We include it for completeness.

Lemma 5.1. *For an R-module M and submodules A, B, C of M the following are equivalent.*

(1) $A \cap (B + C) = A \cap B + A \cap C$.
(2) $A + B \cap C = (A + B) \cap (A + C)$.

DEFINITION 5.2

Let M be an R-module and $\{N_\lambda\}_\Lambda$ a family of proper submodules. Then $\{N_\lambda\}_\Lambda$ is called *completely coindependent* if for every $\lambda \in \Lambda$,

$$N_\lambda + \bigcap_{\mu \neq \lambda} N_\mu = M$$

holds.

In general, in a weakly supplemented module M, the family of maximal submodules of M need not be completely coindependent. For example, let S be a simple R-module and $M = S \oplus S$. Let $0 \neq x \in S$. Then the submodules $S \oplus 0$, $0 \oplus S$ and $R(x, x)$ are maximal submodules of M. We have $R(x, x) + [(S \oplus 0) \cap (0 \oplus S)] = R(x, x) \neq M$, so that the set of maximal submodules of M is not completely coindependent. In case the module is also weakly distributive we shall see that the family of maximal submodules is completely coindependent. First we need the following lemma.

Lemma 5.3. *Let M be an R-module and A a maximal submodule of M. If $A \cap (B + C) = A \cap B + A \cap C$ and $B \cap C \subseteq A$ hold for some $B, C \subseteq M$, then $B \subseteq A$ or $C \subseteq A$.*

Proof. By Lemma 5.1, we have $A + B \cap C = (A + B) \cap (A + C)$. If $B \nsubseteq A$, then

$$A = A + B \cap C = (A + B) \cap (A + C) = M \cap (A + C) = A + C.$$

Therefore $C \subseteq A$. □

A submodule K of M is *small* in M, denoted as $K \ll M$, if $K + L = M$ implies $L = M$ for each submodule L of M. A submodule V of M is called a *weak supplement* if $U + V = M$ and $U \cap V \ll M$ for some submodule U of M. M is called *weakly supplemented* if every submodule of M has (is) a weak supplement.

PROPOSITION 5.4

Let M be a weakly supplemented module. Suppose every maximal submodule of M is a weak distributive submodule of M. Then the set $\{N_i\}_{i\in I}$ of all maximal submodules of M is a completely coindependent family.

Proof. Let N be a maximal submodule of M and V be a weak supplement of N in M. Since $N \cap V \ll M$, we have $N \cap V \subseteq \mathrm{Rad}\, M$. So $N \cap V$ is contained in every maximal submodule of M. Let K be a maximal submodule of M such that $K \neq N$. Then by Lemma 5.3, $V \subseteq K$. Therefore, $M = N + V \subseteq N + \bigcap_{\substack{i\in I\\ N_i \neq N}} N_i$. □

As a consequence, we obtain the following.

COROLLARY 5.5

Let M be a weakly distributive and weakly supplemented module. Then the family of maximal submodules of M is completely coindependent.

Let M be an R-module and $U, V \subseteq M$. The submodule V is called a *supplement* of U in M if $U + V = M$ and V is minimal with respect to this property. That is, if $U + V' = M$ for some $V' \subseteq V$, then $V' = V$. By [11], V is a supplement of U if and only if $U + V = M$ and $U \cap V \ll V$. A module M is called *supplemented* if every submodule of M has a supplement in M. Clearly any direct summand of M is a supplement submodule. Supplement of a submodule need not be unique, in general. For example, if R is a field and $M = R \oplus R$, then the submodules $R(1, 0)$ and $R(0, 1)$ are both supplements of $R(1, 1)$ in M. Examples of modules in which supplements are unique and some characterization of these modules are given in [4].

A module is said to be *hollow* if every proper submodule is small. A module is said to be *local* if it has a largest proper submodule. Every local module is hollow.

Let M be a module. The sum $M = \sum_{i\in I} N_i$ is said to be *irredundant* if $M \neq \sum_{i\neq j} N_i$ for each $j \in I$. A finitely generated module is supplemented if and only if it can be written as an irredundant sum of local submodules (see, Corollary 11 of [6]). A representation of a module as an irredundant sum of local modules is not unique in general. For example, let p be a prime integer, then the $\mathbb{Z}$-module $\mathbb{Z}/\mathbb{Z}p$ is a simple module. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}/\mathbb{Z}p \oplus \mathbb{Z}/\mathbb{Z}p$. Let $x = (1 + \mathbb{Z}p,\ 1 + \mathbb{Z}p)$ and $y = (0,\ 1 + \mathbb{Z}p)$. Then $\mathbb{Z}x \simeq \mathbb{Z}y \simeq \mathbb{Z}/\mathbb{Z}p$ is a simple $\mathbb{Z}$-module. Now $M = \mathbb{Z}/\mathbb{Z}p \oplus \mathbb{Z}/\mathbb{Z}p = \mathbb{Z}x \oplus \mathbb{Z}y$ and each direct summand is local. For weakly distributive modules we have the following.

Theorem 5.6. *Let M be a weakly distributive module and let $M = \sum_{i\in I} L_i = \sum_{j\in J} H_j$ be two irredundant sum of hollow submodules of M. Then $L_i = H_{\sigma(i)}$ for each $i \in I$, where σ is a permutation of J.*

Proof. Consider the submodules L_i and $N_i = \sum_{j \in I \setminus \{i\}} L_j$. Since the sum $\sum_{i \in I} L_i$ is irredundant, N_i is a proper submodule of M. So there exists an index $t \in J$ such that $H_t \nsubseteq N_i$ (otherwise, $H_j \subseteq N_i$ for all $j \in J$ and so $\sum_{j \in J} H_j \subseteq N_i$, a contradiction). Since $M = L_i + N_i$ and M is weakly distributive, we have $H_t = H_t \cap L_i + H_t \cap N_i$. Now $H_t \cap N_i$ is a proper submodule of H_t and so $H_t \cap N_i \ll H_t$. So $H_t = H_t \cap L_i$, that is $H_t \subseteq L_i$. If $H_t \subsetneq L_i$, then $H_t \ll L_i$ and so $H_t \ll M$. Since $M = H_t + \sum_{j \in J\{t\}} H_j$, we get $M = \sum_{j \in J \setminus \{t\}} H_j$, a contradiction. Therefore $L_i = H_t$. This implies that for each $i \in I$, $L_i = H_{\sigma(i)}$, where σ is a permutation of J. □

PROPOSITION 5.7

Let M be an R-module and U, V, W be submodules of M. Then

(1) *If V is weak distributive submodule of M and $M = U + V = U + W$, then $M = U + V \cap W$.*
(2) *If M is weakly distributive and W is a supplement of U and V, then W is a supplement of $U \cap V$.*
(3) *If V is a weak distributive submodule of M and $M = U + V$, then every supplement of U is contained in V.*

Proof.

(1) Since $M = U + W$ and V is weak distributive, we have $V = U \cap V + V \cap W$. Then $M = U + V = U + V \cap U + V \cap W = U + V \cap W$ as claimed.
(2) By hypothesis $M = U + W = V + W$. Now by (1), $M = W + U \cap V$ and $W \cap U \cap V \subseteq U \cap W \ll W$. Hence W is a supplement of $U \cap V$.
(3) Suppose Z is a supplement of U in M. Then $M = U + Z$ and so $V = V \cap U + V \cap Z$. This implies $M = U + V = U + V \cap U + V \cap Z = U + V \cap Z$. Since Z is a supplement of U and $V \cap Z \subseteq Z$, we have $V \cap Z = Z$, that is $Z \subseteq V$. □

A submodule K of M is said to be *nonsmall* in M if K is not a small submodule of M.

PROPOSITION 5.8

Let M be weakly distributive module and let $M = \sum_{i \in I} L_i$ be an irredundant sum of local submodules. If Rad $M \ll M$, *then $\{L_i\}_{i \in I}$ is the set of all nonsmall local submodules of M.*

Proof. Let L be a nonsmall local submodule of M. Since Rad $M \ll M$ and L is nonsmall. $L \nsubseteq \operatorname{Rad} M$. So there is a maximal submodule K of M such that $L \nsubseteq K$. Since K is maximal, $K + L = M$ and also $K \cap L \ll L$, because L is local. In this case L is a supplement of K in M.

On the other hand, since K is a proper submodule of M and $M = \sum_{i \in I} L_i$ there is an index $i \in I$ such that $L_i \nsubseteq K$. Then by the first part of the proof we have that L_i is a supplement of K in M. Now by Proposition 5.7(3) we have $L \subseteq L_i$ and $L_i \subseteq L$, that is, $L = L_i$. This completes the proof. □

COROLLARY 5.9

Let M be a weakly distributive and finitely generated module. If M is supplemented, then M has only finitely many nonsmall local submodules.

Camillo and Lima proved in Proposition 5 of [2] that, if a principal ideal I, in a commutative ring, has a supplement then I has a unique supplement. From Proposition 5.7, we obtain the following corollary. Since commutative ring is weakly distributive by Lemma 2.2, the following corollary is a generalization of Proposition 5 of [2].

COROLLARY 5.10

Let R be a weakly distributive ring and I, J be two ideals of R. Suppose I is a supplement of J in R, then any supplement of J is equal to I, that is J has a unique supplement in R.

Proof. Let I' be a supplement of J in R. Then $I' \subseteq I$ and $I' \subseteq I$ by Proposition 5.7(3). Hence $I' = I$. □

M is said to be *amply supplemented* if whenever $M = A + B$, then A has a supplement in M which is contained in B. Amply supplemented modules are supplemented.

Theorem 5.11. *Let M be a weakly distributive supplemented module. Then M is amply supplemented. Moreover every submodule has a unique supplement.*

Proof. Let $M = A + B$. For the proof, we must show that A has a supplement in B. By hypothesis, M is supplemented, and so A has a supplement, say C, in M. Then by Proposition 5.7(3) the submodule C is contained in B. Hence M is an amply supplemented module. Now suppose D is also a supplement of A in M. Then by Proposition 5.7(1) $M = A + C \cap D$, and so $C = C \cap D$ and $D = C \cap D$. Therefore $C = D$. □

Let M be a module and $N \subseteq M$. The submodule N is said to be *coclosed* in M, if $N/L \ll M/L$ implies $L = N$ for each submodule L of M. A submodule $K \subseteq N$ is called a *coclosure* of N if $N/K \ll M/K$ and K is a coclosed submodule in M. In [4] a module M is called a *unique coclosure module,* denoted by UCC, if every submodule of M has a unique coclosure in M. Modules whose submodules have a unique coclosure are studied in [4] extensively, and they proved the following:

Theorem 5.12 (Theorem 4.8 of [4]). *An amply supplemented distributive module is a UCC module.*

The following lemma is clear from the definitions and we include its proof for completeness.

Lemma 5.13. *Let M be an amply supplemented module. If every submodule of M has a unique supplement then M is a UCC module.*

Proof. Let A be a submodule of M and B be a supplement of A. Suppose $A', A'' \subseteq A$ are two coclosures of A. Since $A' \cap B \subseteq A \cap B$, $A'' \cap B \subseteq A \cap B$ and $A \cap B \ll M$. Then $A' \cap B \ll A'$ and $A'' \cap B \ll A''$, because A' and A'' are coclosed submodules of M. So both A' and A'' are supplements of B. By hypothesis $A' = A''$. Hence A has a unique coclosure, and so M is a UCC module. □

The following is a generalization of Theorem 5.12.

COROLLARY 5.14

Let M be a weakly distributive and supplemented module. Then M is a UCC-module.

Let M be an R-module. M is called *$\oplus$-supplemented* if M is supplemented and every submodule of M has a supplement that is a direct summand of M. M is said to be *completely $\oplus$-supplemented* if every direct summand of M is $\oplus$-supplemented. M is called *lifting* if M is amply supplemented and every supplement submodule of M is a direct summand (see [1]). Clearly, lifting modules are $\oplus$-supplemented.

Lemma 5.15. *Let M be a weakly distributive module and K be a direct summand of M. Then $(K + L)/L$ is a direct summand of M/L for every $L \subseteq M$.*

Proof. Suppose $M = K \oplus K'$ for some $K' \subseteq M$. Since M is weakly distributive, we have $L = K \cap L \oplus K' \cap L$ and by modular law:

$$\begin{aligned}(K + L) \cap (K' + L) &= (K + (K' \cap L)) \cap (K' + (K \cap L)) \\ &= (K \cap (K' + (K \cap L)) + (K' \cap L) \\ &= (K \cap K') + (K \cap L) + (K' \cap L) = L.\end{aligned}$$

Therefore, $M/L = [(K + L)/L] \oplus [(K' + L)/L]$. □

Factor modules of $\oplus$-supplemented modules need not be $\oplus$-supplemented in general (see Example 2.2 of [5]). When the module is weakly distributive we have the following.

Theorem 5.16. *Let M be a weakly distributive module. If M is $\oplus$-supplemented, then every factor module of M is $\oplus$-supplemented. In particular, every direct summand of M is $\oplus$-supplemented, hence M is completely $\oplus$-supplemented.*

Proof. Let $L \subseteq M$ and $L \subseteq N$. Since M is $\oplus$-supplemented, N has a supplement K in M and K is a direct summand of M. Now, $(K + L)/L$ is a supplement of N/L by 41.1(7) of [11]. It remains to see that $(K + L)/L$ is a direct summand of M/L, and this is clear by Lemma 5.15. This proves that M/L is $\oplus$-supplemented. The rest of the theorem is a consequence of the first part. □

A module M is said to have the property (D3) if whenever M_1 and M_2 are direct summands of M with $M_1 + M_2 = M$, then $M_1 \cap M_2$ is a direct summand of M. A module M is called *quasi-discrete* if M is lifting and has (D3) (see [8]).

COROLLARY 5.17

Any weakly distributive $\oplus$-supplemented module is quasi-discrete.

Proof. By Theorem 5.11, M is lifting because M is $\oplus$-supplemented. By Lemma 4.1, M also satisfies summand intersection property, hence it has the property (D3). Therefore M is quasi-discrete. □

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