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Exact and explicit solutions to some nonlinear evolution equations by utilizing the (G'/G)-expansion method

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ABSTRACT

In this paper, we demonstrate the effectiveness of the so-called (G'/G)-expansion method by examining some nonlinear evolution equations with physical interest. Our work is motivated by the fact that the (G'/G)-expansion method provides not only more general forms of solutions but also periodic and solitary waves. If we set the parameters in the obtained wider set of solutions as special values, then some previously known solutions can be recovered. The method appears to be easier and faster by means of a symbolic computation system.

1. Introduction

Nonlinear evolution equations (NLEEs) have been the subject of study in various branches of mathematical–physical sciences such as physics, biology, chemistry, etc. The analytical solutions of such equations are of fundamental importance since a lot of mathematical–physical models are described by NLEEs. Among the possible solutions to NLEEs, certain special form solutions may depend only on a single combination of variables such as traveling wave variables. In the literature, there is a wide variety of approaches to nonlinear problems for constructing traveling wave solutions. Some of these approaches are the Jacobi elliptic function method [1], inverse scattering method [2], Hirota's bilinear method [3], homogeneous balance method [4], homotopy perturbation method [5], Weierstrass function method [6], symmetry method [7], Adomian decomposition method [8], sine/cosine method [9], tanh/coth method [10], the F-expansion method [11], the Exp-function method [12,13] and so on. But, most of the methods may sometimes fail or can only lead to a kind of special solution and the solution procedures become very complex as the degree of nonlinearity increases.

Recently, the (G'/G)-expansion method, firstly introduced by Wang et al. [14], has become widely used to search for various exact solutions of NLEEs [15–26]. The value of the (G'/G)-expansion method is that one treats nonlinear problems by essentially linear methods. The method is based on the explicit linearization of NLEEs for traveling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. Moreover, it transforms a nonlinear equation to a simple algebraic computation.

Very lately, to enhance the (G'/G)-expansion method and expand the range of its applicability, further research has been carried out by several authors. Some generalizations of the method have been made by Zhang et al. [27,28]. Zhang et al. [29] improved the method to deal with evolution equations with variable coefficients. Zhang et al. [30] devised an algorithm for using the method to solve nonlinear differential-difference equations. Yu-Bin et al. [31] modified the method to derive traveling wave solutions for Whitham–Broer–Kaup-Like equations. Zhang [32] explored a new application of this method to some special nonlinear evolution equations, the balance numbers of which are not positive integers. For studying the Vakhnenko equation, Wen-An et al. [33] presented a new function expansion method which can be thought of as the general-

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ization of the (G'/G)-expansion method. Still, substantial work has to be done in order for the (G'/G)-expansion method be well established since every nonlinear equation has its own physically significant rich structure.

Although many efforts have been devoted to find various methods to solve (integrable or non-integrable) NLEEs, there is no a unified method. The main merits of the (G'/G)-expansion method over the other methods are that it gives more general solutions with some free parameters which, by suitable choice of the parameters, turn out to be some known solutions gained by the existing methods. Besides, (i) in all finite difference and finite element methods, it is necessary to have boundary and initial conditions. However, the (G'/G)-expansion method handles NLEEs in a direct manner with no requirement for initial/boundary condition or initial trial function at the outset. It obtains a general solution with free parameters that can be determined via boundary and/or initial conditions, (ii) most of the methods give solutions in a series form and it becomes essential to investigate the convergence of approximation series. For example, the Adomian decomposition method depends only on the initial conditions and obtains a solution in a series which converges to the exact solution of the problem. But, with the (G'/G)-expansion method, one may obtain a general solution without approximation, (iii) it serves as a powerful technique to integrate the NLEEs, even if the Painleve test of integrability fails, (iv) the solution procedure, using a computer algebra system like Mathematica, is of utter simplicity.

Our aim in this paper is to present an application of the (G'/G)-expansion method to some nonlinear problems that is solved by this method for the first time. The rest of this paper is organized as follows. In Section 2, we describe briefly the (G'/G)-expansion method. In Section 3, we apply the method to the modified Degasperis–Procesi (mDP) equation, Burgers–KdV equation and modified Benjamin–Bona–Mahony (mBBM) equation, respectively. In Section 4, some conclusions are given.

2. The (G'/G)-expansion method

The objective of this section is to outline the use of the (G'/G)-expansion method for solving certain nonlinear partial differential equations (PDEs). Suppose we have a nonlinear PDE for $u(x,t)$ in the form

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0, \tag{1}$$

where P is a polynomial in its arguments, which includes nonlinear terms and the highest order derivatives. The transformation $u(x,t) = U(\xi), \xi = kx + wt$ reduces Eq. (1) to the ordinary differential equation (ODE)

$$P(U, kU', wU', k^2U'', kwU'', w^2U'', \ldots) = 0, \tag{2}$$

where $U = U(\xi)$ and prime denotes derivative with respect to ξ. We assume that the solution of Eq. (2) can be expressed by a polynomial in (G'/G) as follows:

$$U = a_n\left(\frac{G'}{G}\right)^n + \cdots, \tag{3}$$

where $G = G(\xi)$ is the solution of the auxiliary linear second-order ordinary differential equation

$$G'' + \lambda G' + \mu G = 0, \tag{4}$$

where $G' = \frac{dG}{d\xi},\ G'' = \frac{d^2G}{d\xi^2},\ a_n \neq 0, \ldots, a_1, a_0, \lambda$ and μ are constants to be determined later. The unwritten part in (3) is also a polynomial in (G'/G), but the degree of which is generally equal to or less than $n-1$. The positive integer n can be determined by applying the homogeneous balancing method to the highest order derivatives and nonlinear terms appearing in Eq. (2). Then substituting (3) into Eq. (2) under the consideration of Eq. (4) yields a system of nonlinear algebraic equations for a_i, λ, μ, k, and w. Suppose that these constants can be determined by solving the simultaneous algebraic equations with the aid of a symbolic computation system such as Mathematica. On the other hand, depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$, the solutions of Eq. (4) can be readily written. As a result, more traveling wave solutions of Eq. (1) can be derived. Of course, the correctness of the obtained results must be assured by substituting them back into the original equation.

3. Applications

In this section, we will demonstrate the validity and reliability of (G'/G)-expansion method in detail with some nonlinear evolution equations arising in applications.

3.1. The modified Degasperis–Procesi equation

Let us consider the celebrated mDP equation

$$u_t - u_{xxt} + 4u^2u_x = 3u_xu_{xx} + uu_{xxx}, \tag{5}$$

which is the modified form of Degasperis–Procesi equation [34]. Eq. (5) can be considered as a model for shallow-water dynamics and is well-known to be integrable and possesses multi-soliton solutions with peaks (multi-peakons). A general form of Eq. (5), which is called the general modified DP–CH equation, has been studied by Wazwaz [35] and Biswas et al. [36] recently. Now, we introduce the variable $\xi = kx + wt$ and make the transformation $u(x,t) = U(\xi)$ to reduce Eq. (5) to the ODE

$$w(U - k^2U'') + \frac{4k}{3}U^3 - k^3UU'' - k^3(U')^2 + C = 0, \tag{6}$$

where $U = U(\xi)$, prime denotes derivative with respect to ξ, and $C$is an integration constant. Assume that the solution of Eq. (6) can be expressed as an ansatz (3) together with (4). Then, by using (3) and (4), it can be seen that

$$U^3 = a_n^3\left(\frac{G'}{G}\right)^{3n} + \cdots, \tag{7}$$

$$U'' = n(n+1)a_n\left(\frac{G'}{G}\right)^{n+2} + \cdots \tag{8}$$

Balancing the terms U^3 and UU'' in Eq. (6), from (7) and (8), we get $3n = n + (n+2)$ which yields the leading order $n = 2$. Therefore, in view of the (G'/G)-expansion method, we can assume the solution of Eq. (6) in the form

$$U = a_2\left(\frac{G'}{G}\right)^2 + a_1\left(\frac{G'}{G}\right) + a_0, \quad a_2 \neq 0. \tag{9}$$

It follows, from (4) and (9), that

$$U' = -2a_2\left(\frac{G'}{G}\right)^3 - (a_1 + 2a_2\lambda)\left(\frac{G'}{G}\right)^2 - (a_1\lambda + 2a_2\mu)\left(\frac{G'}{G}\right) - a_1\mu, \tag{10}$$

$$U'' = 6a_2\left(\frac{G'}{G}\right)^4 + (2a_1 + 10a_2\lambda)\left(\frac{G'}{G}\right)^3 + (8a_2\mu + 3a_1\lambda + 4a_2\lambda^2)\left(\frac{G'}{G}\right)^2 + (6a_2\lambda\mu + 2a_1\mu + a_1\lambda^2)\left(\frac{G'}{G}\right) + 2a_2\mu^2 + a_1\lambda\mu, \tag{11}$$

Substituting (9)–(11) into (6) and setting the coefficients of $(G'/G)^i, \ 0 \leqslant i \leqslant 6$, to zero, we get the system of nonlinear algebraic equations for $a_0, a_1, a_2, \lambda, \mu, k$, and w:

$$\left(\frac{G'}{G}\right)^0: \quad C + wa_0 + \frac{4ka_0^3}{3} - k^2w\lambda\mu a_1 - k^3\lambda\mu a_0a_1 - k^3\mu^2a_1^2 - 2k^2w\mu^2a_2 - 2k^3\mu^2a_0a_2 = 0, \tag{12a}$$

$$\left(\frac{G'}{G}\right)^1: \quad wa_1 - k^2w\lambda^2a_1 - 2k^2w\mu a_1 - k^3\lambda^2a_0a_1 - 2k^3\mu a_0a_1 + 4ka_0^2a_1 - 3k^3\lambda\mu a_1^2$$
$$- 6k^2w\lambda\mu a_2 - 6k^3\lambda\mu a_0a_2 - 6k^3\mu^2a_1a_2 = 0, \tag{12b}$$

$$\left(\frac{G'}{G}\right)^2: \quad -3k^2w\lambda a_1 - 3k^3\lambda a_0a_1 - 2k^3\lambda^2a_1^2 - 4k^3\mu a_1^2 + 4ka_0a_1^2 + wa_2 - 4k^2w\lambda^2a_2$$
$$- 8k^2w\mu a_2 - 4k^3\lambda^2a_0a_2 - 8k^3\mu a_0a_2 + 4ka_0^2a_2 - 15k^3\lambda\mu a_1a_2 - 6k^3\mu^2a_2^2 = 0, \tag{12c}$$

$$\left(\frac{G'}{G}\right)^3: \quad -2k^2wa_1 - 2k^3a_0a_1 - 5k^3\lambda a_1^2 + \frac{4ka_1^3}{3} - 10k^2w\lambda a_2 - 10k^3\lambda a_0a_2 - 9k^3\lambda^2a_1a_2$$
$$- 18k^3\mu a_1a_2 + 8ka_0a_1a_2 - 14k^3\lambda\mu a_2^2 = 0, \tag{12d}$$

$$\left(\frac{G'}{G}\right)^4: \quad -3k^3a_1^2 - 6k^2wa_2 - 6k^3a_0a_2 - 21k^3\lambda a_1a_2 + 4ka_1^2a_2 - 8k^3\lambda^2a_2^2 - 16k^3\mu a_2^2 + 4ka_0a_2^2 = 0, \tag{12e}$$

$$\left(\frac{G'}{G}\right)^5: \quad -12k^3a_1a_2 - 18k^3\lambda a_2^2 + 4ka_1a_2^2 = 0, \tag{12f}$$

$$\left(\frac{G'}{G}\right)^6: \quad -10k^3a_2^2 + \frac{4ka_2^3}{3} = 0. \tag{12g}$$

Solving the system ((12a)–(12g)) simultaneously, we get the solution sets:

$$\left\{\begin{array}{l} C = \frac{1}{96}k\left(32\left(-4 \pm \sqrt{16 - 15k^4(\lambda^2 - 4\mu)^2}\right) + 5k^4\left(24 \mp 3\sqrt{16 - 15k^4(\lambda^2 - 4\mu)^2} + 5k^2(\lambda^2 - 4\mu)\right)(\lambda^2 - 4\mu)^2\right), \\ w = \frac{1}{2}\left(-4k \pm k\sqrt{16 - 15k^4(\lambda^2 - 4\mu)^2}\right), a_0 = \frac{1}{8}\left(-4 + 5k^2\lambda^2 + 40k^2\mu \pm \sqrt{16 - 15k^4(\lambda^2 - 4\mu)^2}\right), \\ a_2 = \frac{15k^2}{2}, a_1 = \frac{15k^2\lambda}{2}. \end{array}\right\} \tag{13}$$

Now, substituting (13) together with the solutions of Eq. (4) into (9), we obtain the hyperbolic function traveling wave solutions to Eq. (5) as

$$u_{1,2}(x,t)=\frac{15a}{8}\left(\frac{C_1\cosh\frac{\sqrt{a}}{2}\left(x+\frac{1}{2}\left(-4\pm\sqrt{16-15a^2}\right)t\right)+C_2\sinh\frac{\sqrt{a}}{2}\left(x+\frac{1}{2}\left(-4\pm\sqrt{16-15a^2}\right)t\right)}{C_1\sinh\frac{\sqrt{a}}{2}\left(x+\frac{1}{2}\left(-4\pm\sqrt{16-15a^2}\right)t\right)+C_2\cosh\frac{\sqrt{a}}{2}\left(x+\frac{1}{2}\left(-4\pm\sqrt{16-15a^2}\right)t\right)}\right)^2\pm\frac{1}{8}\times\sqrt{16-15a^2}-\frac{5}{4}a-\frac{1}{2}, \tag{14}$$

where $a=k^2(\lambda^2-4\mu)>0, C_1$ and C_2 are arbitrary constants; the trigonometric function traveling wave solutions to Eq. (5) as

$$u_{3,4}(x,t)=-\frac{15a}{8}\left(\frac{-C_1\sin\frac{\sqrt{-a}}{2}\left(x+\frac{1}{2}\left(-4\pm\sqrt{16-15a^2}\right)t\right)+C_2\cos\frac{\sqrt{-a}}{2}\left(x+\frac{1}{2}\left(-4\pm\sqrt{16-15a^2}\right)t\right)}{C_1\cos\frac{\sqrt{-a}}{2}\left(x+\frac{1}{2}\left(-4\pm\sqrt{16-15a^2}\right)t\right)+C_2\sin\frac{\sqrt{-a}}{2}\left(x+\frac{1}{2}\left(-4\pm\sqrt{16-15a^2}\right)t\right)}\right)^2\pm\frac{1}{8}\times\sqrt{16-15a^2}-\frac{5}{4}a-\frac{1}{2}, \tag{15}$$

where $a=k^2(\lambda^2-4\mu)<0, C_1$ and C_2 are arbitrary constants; the rational function traveling wave solutions to Eq. (5) as

$$u_5(x,t)=\frac{15C_1^2}{2(C_1x+C_3)^2}, \tag{16}$$

$$u_6(x,t)=\frac{15C_1^2}{2(C_1(x-4t)+C_3)^2}-1, \tag{17}$$

where C_1 and $C_3=C_2/k$ are arbitrary constants.

In particular, if we take $C_2\neq0,\ C_1^2<C_2^2$, then (14) leads the formal solitary wave solutions to Eq. (5) as

$$u_{7,8}(x,t)=\frac{15a}{8}\tanh^2\left(\frac{\sqrt{a}}{2}\left(x+\frac{1}{2}\left(-4\pm\sqrt{16-15a^2}\right)t\right)+\xi_0\right)\pm\frac{1}{8}\sqrt{16-15a^2}-\frac{5}{4}a-\frac{1}{2}, \tag{18}$$

where $a>0$ and $\xi_0=\tanh^{-1}(C_1/C_2)$; and (15) gives the periodic wave solutions to Eq. (5) as

$$u_{9,10}(x,t)=-\frac{15a}{8}\cot^2\left(\frac{\sqrt{-a}}{2}\left(x+\frac{1}{2}\left(-4\pm\sqrt{16-15a^2}\right)t\right)+\xi_0\right)\pm\frac{1}{8}\sqrt{16-15a^2}-\frac{5}{4}a-\frac{1}{2}, \tag{19}$$

where $a<0$ and $\xi_0=\tan^{-1}(C_1/C_2)$.

Now, we compare our results with some others:

(i) We observe that the results (19), (20), (40), and (41) in Wazwaz [35] are particular cases of our results (14) and (15). For example, if we take $a=1$ and $\xi_0=0$ in our result (18), then we get two solutions with the wave speeds $\frac{5}{2}$ and $\frac{3}{2}$ of which the former one coincides with the Wazwaz's results (19) and (40). Moreover, in [35], our rational solutions (16) and (17) do not appear and traveling wave solutions are obtained only for one particular wave speed $\frac{5}{2}$.

(ii) If we take $a=1, C_1=\sqrt{35}$ and $C_2=\sqrt{15}$ in our result (14), then we get the Liu and Ouyang's result, peakon (10), in [37].

(iii) First, taking $a=1/2$ and $\xi_0=0$ in our result (18), then we get the result (1.9) of Wang and Tang [38]. Second, plugging $C_1=\sqrt{2}$ and $C_3=\sqrt{3}$ into our result (17) leads to the result (1.10) in [38]. Similarly, (1.11) and (1.12) in [38] can be derived from our results.

3.2. The Burgers–KdV equation

Next, we consider the Burgers–KdV equation

$$u_t+\alpha uu_x+\beta u_{xx}+\gamma u_{xxx}=0, \tag{20}$$

where α,β, and γ are arbitrary real constants with $\alpha\beta\gamma\neq0$. This equation is a model for the propagation of waves on an elastic tube [39]. Gibbon et al. [40] showed that Eq. (20) does not have the Painleve property. It can be regarded as a combination of the Burger's equation $(\alpha\neq0,\beta\neq0,\gamma=0)$ and the KdV equation $(\alpha\neq0,\beta=0,\gamma\neq0)$. Now, letting $u(x,t)=U(\xi),\xi=kx+wt$ in (20) and integrating the resulting ODE once, one obtains

$$\gamma k^3U''+k^2\beta U'+\frac{\alpha k}{2}U^2+wU+C=0, \tag{21}$$

where $U=U(\xi)$, prime denotes derivative with respect to ξ, and C is an integration constant. The approach is similar to the scheme used in previous section, so we will skip some details for simplicity. Balancing the terms U'' and U^2 in Eq. (21) yields the leading order $n=2$. Therefore, by substituting (3) with $n=2$ into (21) and solving the resulting system of five nonlinear algebraic equations for $a_0,a_1,a_2,\lambda,\mu,k$, and w, we find the solution set:

$$\left\{\begin{array}{l} d=\frac{-36k^2\beta^4+625w^2\gamma^2}{1250k\alpha\gamma^2},\ a_0=\frac{3k\beta^2-25w\gamma+30k^2\beta\gamma\lambda-75k^3\gamma^2\lambda^2}{25k\alpha\gamma},\\ \mu=\frac{-\beta^2+25k^2\gamma^2\lambda^2}{100k^2\gamma^2},\ a_2=-\frac{12k^2\gamma}{\alpha},\ a_1=-\frac{12(-k\beta+5k^2\gamma\lambda)}{5\alpha}. \end{array}\right\} \tag{22}$$

Substituting (22) into (3) with $n=2$ and using the solutions of Eq. (4), we derive the hyperbolic function traveling wave solution to Eq. (20) as

$$\begin{aligned} u_1(x,t)=&-\frac{3\beta^2}{25\alpha\gamma}\left(\frac{C_1\cosh\left|\frac{\beta}{10\gamma}\right|(x+ct)+C_2\sinh\left|\frac{\beta}{10\gamma}\right|(x+ct)}{C_1\sinh\left|\frac{\beta}{10\gamma}\right|(x+ct)+C_2\cosh\left|\frac{\beta}{10\gamma}\right|(x+ct)}\right)^2\\ &+\frac{6\beta}{25\alpha}\left|\frac{\beta}{10\gamma}\right|\left(\frac{C_1\cosh\left|\frac{\beta}{10\gamma}\right|(x+ct)+C_2\sinh\left|\frac{\beta}{10\gamma}\right|(x+ct)}{C_1\sinh\left|\frac{\beta}{10\gamma}\right|(x+ct)+C_2\cosh\left|\frac{\beta}{10\gamma}\right|(x+ct)}\right)+\frac{3\beta^2-25c\gamma}{25\alpha\gamma}, \end{aligned} \tag{23}$$

where $c=\frac{w}{k}$, C_1 and C_2 are arbitrary constants.

In particular, if we take $C_2\neq 0,\ C_1^2<C_2^2$, then (23) leads the formal solitary wave solution to Eq. (20) as

$$u_2(x,t)=-\frac{3\beta^2}{25\alpha\gamma}\tanh^2\left(\left|\frac{\beta}{10\gamma}\right|(x+ct)+\xi_0\right)+\frac{6\beta}{25\alpha}\left|\frac{\beta}{10\gamma}\right|\tanh\left(\left|\frac{\beta}{10\gamma}\right|(x+ct)+\xi_0\right)+\frac{3\beta^2-25c\gamma}{25\alpha\gamma}, \tag{24}$$

where $\xi_0=\tanh^{-1}(C_1/C_2)$ and c is an arbitrary constant.

For comparison purposes, we observe that our solutions (23) and (24) include the solutions (64) and (65) of Wazwaz [41] and so do the solutions (18) and (20) of Feng [42]. Besides, the wave speeds in [41,42] are derived only in terms of the parameters of the Burgers–KdV equation in contrast to the more general form wave speed in our result (23).

3.3. The modified Benjamin–Bona–Mahony equation

The regularized long-wave equation, also known as Benjamin–Bona–Mahony (BBM) equation, in the form

$$u_t+u_x+uu_x-u_{xxt}=0, \tag{25}$$

has been investigated, for the first time, by Benjamin et al. [43] as an alternative model to the Korteweg–de Vries equation for long waves and it plays an important role in the modeling of nonlinear dispersive systems. The BBM equation is applicable to the study of drift waves in plasma or the Rossby waves in rotating fluids. In this section, we investigate a variant of the BBM Eq. (25), which is also known as the modified Benjamin–Bona–Mahony (mBBM) equation, as follows:

$$u_t+\alpha u_x+\beta u^2u_x-\gamma u_{xxt}=0, \tag{26}$$

where α,β, and γ are arbitrary real constants. Similarly, the mBBM Eq. (26) may be viewed as an alternative to the modified Korteweg-de Vries equation. Recently, Wazwaz [44] and Biswas et al. [45] investigated a variant of Eq. (25), which is called the generalized BBM equation. Now, letting $u(x,t)=U(\xi),\xi=kx+wt$ in (26), we obtain

$$-\gamma k^2wU''+\frac{k\beta}{3}U^3+(w+\alpha k)U+C=0, \tag{27}$$

where $U=U(\xi)$, prime denotes derivative with respect to ξ, and $C$is an integration constant. Balancing the terms U'' and U^3 in Eq. (27) yields the leading order $n=1$. Therefore, we can write the solution of (27) in the form

$$U=a_1\left(\frac{G'}{G}\right)+a_0,\quad a_1\neq 0, \tag{28}$$

and thus we have

$$U^3=a_1^3\left(\frac{G'}{G}\right)^3+3a_1^2a_0\left(\frac{G'}{G}\right)^2+3a_1a_0^2\left(\frac{G'}{G}\right)+a_0^3, \tag{29}$$

$$U''=2a_1\left(\frac{G'}{G}\right)^3+3a_1\lambda\left(\frac{G'}{G}\right)^2+(a_1\lambda^2+2a_1\mu)\left(\frac{G'}{G}\right)+a_1\lambda\mu. \tag{30}$$

Substituting (28)–(30) into Eq. (27) and setting the coefficients of $(G'/G)^i$ $(i=0,1,2,3)$ to zero, we get a system of nonlinear algebraic equations for a_0,a_1,λ,μ,k, and w:

$$\left(\frac{G'}{G}\right)^0:\quad C+wa_0+k\alpha a_0+\frac{1}{3}k\beta a_0^3-k^2w\gamma\lambda\mu a_1=0, \tag{31a}$$

$$\left(\frac{G'}{G}\right)^1:\quad wa_1+k\alpha a_1-k^2w\gamma\lambda^2a_1-2k^2w\gamma\mu a_1+k\beta a_0^2a_1=0, \tag{31b}$$

$$\left(\frac{G'}{G}\right)^2: \quad -3k^2 w\gamma\lambda a_1 + k\beta a_0 a_1^2 = 0, \tag{31c}$$

$$\left(\frac{G'}{G}\right)^3: \quad -2k^2 w\gamma a_1 + \frac{1}{3}k\beta a_1^3 = 0, \tag{31d}$$

Solving the system (31a)–(31d) simultaneously, we end up with the solution sets:

$$\left\{d = 0, a_1 = \mp\frac{2k\sqrt{3\alpha\gamma}}{\sqrt{-\beta(2 + k^2\gamma(\lambda^2 - 4\mu))}}, w = -\frac{2k\alpha}{2 + k^2\gamma(\lambda^2 - 4\mu)}, a_0 = \mp\frac{k\lambda\sqrt{3\alpha\gamma}}{\sqrt{-\beta(2 + k^2\gamma(\lambda^2 - 4\mu))}}\right\}, \tag{32}$$

Now, substituting (32) together with the solutions of Eq. (4) into (28), we obtain the hyperbolic function traveling wave solutions to Eq. (26) as

$$u_{1,2}(x,t) = \mp\frac{\sqrt{3\alpha\gamma a}}{\sqrt{-\beta(2+\gamma a)}}\left(\frac{C_1\cosh\frac{\sqrt{a}}{2}\left(x - \frac{2\alpha}{2+\gamma a}t\right) + C_2\sinh\frac{\sqrt{a}}{2}\left(x - \frac{2\alpha}{2+\gamma a}t\right)}{C_1\sinh\frac{\sqrt{a}}{2}\left(x - \frac{2\alpha}{2+\gamma a}t\right) + C_2\cosh\frac{\sqrt{a}}{2}\left(x - \frac{2\alpha}{2+\gamma a}t\right)}\right), \tag{33}$$

where $a = k^2(\lambda^2 - 4\mu) > 0$, C_1 and C_2 are arbitrary constants; the trigonometric function traveling wave solutions to Eq. (26) as

$$u_{3,4}(x,t) = \mp\frac{\sqrt{3\alpha\gamma a}}{\sqrt{\beta(2+\gamma a)}}\left(\frac{-C_1\sin\frac{\sqrt{-a}}{2}\left(x - \frac{2\alpha}{2+\gamma a}t\right) + C_2\cos\frac{\sqrt{-a}}{2}\left(x - \frac{2\alpha}{2+\gamma a}t\right)}{C_1\cos\frac{\sqrt{-a}}{2}\left(x - \frac{2\alpha}{2+\gamma a}t\right) + C_2\sin\frac{\sqrt{-a}}{2}\left(x - \frac{2\alpha}{2+\gamma a}t\right)}\right), \tag{34}$$

where $a = k^2(\lambda^2 - 4\mu) < 0$, C_1 and C_2 are arbitrary constants; the rational function traveling wave solutions to Eq. (26) as

$$u_{5,6}(x,t) = \mp\frac{\sqrt{6\alpha\gamma}C_3}{\sqrt{-\beta}(C_3(x - \alpha t) + C_2)}, \tag{35}$$

where C_2 and $C_3 = kC_1$ are arbitrary constants.

In particular, if we take $C_2 \neq 0$, $C_1^2 < C_2^2$, then (33) leads the formal solitary wave solutions to Eq. (26) as

$$u_{7,8}(x,t) = \mp\frac{\sqrt{3\alpha\gamma a}}{\sqrt{-\beta(2+\gamma a)}}\tanh\left(\frac{\sqrt{a}}{2}\left(x - \frac{2\alpha}{2+\gamma a}t\right) + \xi_0\right), \tag{36}$$

where $a = k^2(\lambda^2 - 4\mu) > 0$ and $\xi_0 = \tanh^{-1}(C_1/C_2)$, and (34) gives the periodic wave solutions to Eq. (26) as

$$u_{9,10}(x,t) = \mp\frac{\sqrt{3\alpha\gamma a}}{\sqrt{\beta(2+\gamma a)}}\cot\left(\frac{\sqrt{-a}}{2}\left(x - \frac{2\alpha}{2+\gamma a}t\right) + \xi_0\right), \tag{37}$$

where $a = k^2(\lambda^2 - 4\mu) < 0$ and $\xi_0 = \tan^{-1}(C_1/C_2)$.

Now, we compare our results with others: (1) Yusufoğlu and Bekir's results [46]: To match the equations, first we take $\alpha = \beta = 1$, and $\gamma = -1$ in (26) and so do in (33)–(37). Then if we take $\xi_0 = 0$ and $a = 4\alpha^2$ in the new form of (36), our result will be the same as the first expression of (5.6) in [46]. Moreover, by taking $C_1 \neq 0$, $C_1^2 > C_2^2$, and $\xi_0 = \tanh^{-1}(C_2/C_1)$ in our more general result (33) and following the same approach leads to the second expression of (5.6) in [46]. It can be shown that our solutions (34) and (37) include the solutions (5.9) and (5.10) of [46]. (2) By a similar discussion, we can observe that Wazwaz's [47] results (68) and (69) are special cases of our results (33) and (36). It is worth to note that our rational solutions (35) are not derived in [46,47].

4. Conclusion

This study shows that the (G'/G)-expansion method is quite efficient and practically well suited for use in finding exact solutions for the problems considered here. The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. Though the obtained solutions represent only a small part of the large variety of possible solutions for the equations considered, they might serve as seeding solutions for a class of localized structures existing in the physical systems. Furthermore, our solutions are in more general forms, and many known solutions to these equations are only special cases of them. With the aid of Mathematica, we have assured the correctness of the obtained solutions by putting them back into the original equations. We hope that they will be useful for further studies in applied sciences.

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