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COFINITELY WEAK SUPPLEMENTED LATTICES

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In this paper it is shown that an E-complemented complete modular lattice L with small radical is weakly supplemented if and only if it is semilocal. L is a cofinitely weak supplemented lattice if and only if every maximal element of L has a weak supplement in L. If $a/0$ is a cofinitely weak supplemented (weakly supplemented) sublattice and $1/a$ has no maximal element ($1/a$ is weakly supplemented and a has a weak supplement in L), then L is cofinitely weak supplemented (weakly supplemented).

Key Words: Cofinite element, weak supplement, weakly supplemented lattice, cofinitely weak supplemented lattice.

1. INTRODUCTION

Throughout L denotes an arbitrary complete modular lattice with smallest element 0 and greatest element 1; by a lattice we will mean a complete modular lattice. An element a of L is said to be small in L if $a \vee b \neq 1$ holds for every $b \neq 1$. It is denoted by $a \ll L$. An element a of L is called a supplement of an element b in L if $a \vee b = 1$ and a is minimal with respect to this property. Equivalently, an element a is a supplement of b in L if and only if $a \vee b = 1$ and $a \wedge b \ll a/0$. Reducing the last condition to $a \wedge b \ll L$ we obtain the definition of weak supplements. L is said to be supplemented (respectively, weakly supplemented) if every element a of L has a supplement (respectively, weak supplement) in L. Many properties of weak supplement submodules hold in an arbitrary lattice and sometimes the proofs can be obtained by slight modification of those for modules. We give examples of lattices showing that not all generalizations are true and give the proofs of the results for lattices in those situations when the proofs are essentially different from those in the module case. Some results proved for lattices provide new results or simpler proofs of known results for modules. An element a of L is said to be essential if $a \wedge b \neq 0$ for every nonzero element b in L. It is denoted by $a \trianglelefteq L$ (see [7]). An element b is called an E-complement of an element a of L if $a \wedge b = 0$ and $a \vee b \trianglelefteq L$. A lattice L is called E-complemented if every element of L has an E-complement in L (see [8]). If for every element a of L there is an element b of L such that $a \vee b = 1$ and $a \wedge b = 0$, then L is said to be complemented (see [6]). The radical $\mathrm{rad}(L)$ of L is the meet of all maximal elements of L (see [10]). If $1/\mathrm{rad}(L)$ is complemented, then L is called a semilocal lattice (cf. [4, 17.1]). In Section 2 weakly supplemented lattices are studied. We prove that an E-complemented lattice L with small radical is weakly supplemented if and only if it is semilocal. Also we show that an E-complemented weakly supplemented lattice L with zero radical is complemented.

A sublattice of the form $b/a = \{x \in L \mid a \leq x \leq b\}$ is called a quotient sublattice (see [5]). An element a of L is called cofinite in L if the quotient sublattice $1/a$ is compact, that is $1 = \bigvee_{i \in I} x_i$ for some elements $x_i \geq a$ implies that $1 = \bigvee_{i \in F} x_i$ for some finite subset F of I. If each element of L is a join of compact elements, then L is said to be compactly generated (see [10]). In Section 3 we study cofinitely weak supplemented lattices or briefly cws-lattices, that is lattices whose cofinite elements have weak supplements. It is proved that L is a cws-lattice if and only if every maximal element of L has a weak supplement. We give a condition under which a compactly generated cofinitely weak supplemented lattice is cofinitely supplemented.

2. Weakly Supplemented Lattices

The following example shows that a homomorphic image of a small element under a lattice morphism need not be small unlike the module case.

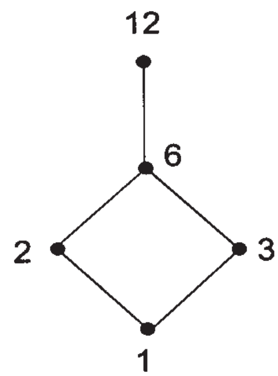
Example 2.1 — Let $A = \{1, 2, 3, 6, 12\}$ and $B = \{1, 2, 3, 6\}$. Consider the lattices $(A, |)$ and $(B, |)$ where $|$ is the divides relation: $x \mid y$ means x divides y.

Consider the lattice morphism $f : (A, |) \to (B, |)$ defined by $f(k) = k$ for $k = 1, 2, 3, 6$ and $f(12) = 6$. Clearly, $2 \ll A$ since $2 \vee x \neq 12$ for all $x \neq 12$. But $f(2) = 2 \not\ll B$ since $2 \vee 3 = 6$ whilst $3 \neq 6$.

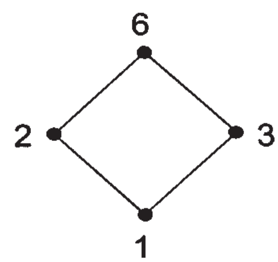
Nevertheless using the following properties of small sublattices we will show that the quotient sublattices $1/a$ of a weakly supplemented lattice is weakly supplemented. We will write $a < b$ if $a \leq b$ and $a \neq b$.

Lemma 2.2 — ([3, Lemma 7.2, Lemma 7.3 and Lemma 7.4]) Let $a < b$ be elements in L.

(1) If $a \ll b/0$, then $a \vee c \ll (b \vee c)/c$ for every c in L.



$(A, |)$



$(B, |)$

(2) $b \ll L$ if and only if $a \ll L$ and $b \ll 1/a$.

(3) If $a \ll b/0$, then $a \ll L$.

Proposition 2.3 — If L is a weakly supplemented lattice, then for every element a the quotient sublattice $1/a$ is also weakly supplemented.

PROOF : Let b be an element of $1/a$. Since L is weakly supplemented, there is a weak supplement x of b in L, i.e. $x \vee b = 1$ and $x \wedge b \ll L = 1/0$. Clearly $(a \vee x) \vee b = 1$. By Lemma 2.2(1), $(a \vee x) \wedge b = (b \wedge x) \vee a \ll (1 \vee a)/a = 1/a$. □

Small cover of a weakly supplemented module is weakly supplemented (see [4, 17.13]). The same is true for lattices.

Proposition 2.4 — If $1/a$ is a weakly supplemented sublattice of L for some element $a \ll L$, then L is also weakly supplemented.

PROOF : For every element x in L there exists a weak supplement y of $x \vee a$ in $1/a$, i.e. $y \vee (x \vee a) = 1$ and $y \wedge (x \vee a) \ll 1/a$. By Lemma 2.2(2), $y \wedge (x \vee a) \ll L$. Thus $y \wedge x \leq y \wedge (x \vee a) \ll L$. Hence y is a weak supplement of x in L. □

Proposition 2.5 — (cf. [9, Proposition 2.2(5)], see also [4, 17.13] and [4, 20.3]). If a is a supplement of some element of a weakly supplemented lattice L, then the quotient sublattice $a/0$ is also weakly supplemented.

PROOF : Let a be a supplement of b in L, i.e. $a \vee b = 1$ and $a \wedge b \ll a/0$. By Proposition 2.3, $1/b = (a \vee b)/b \cong a/(a \wedge b)$ is weakly supplemented. Thus by Proposition 2.4, $a/0$ is weakly supplemented. □

Proposition 2.6 — (cf. [1, Proposition 2.7]). If b is a weak supplement of a in L and $c \ll L$, then b is a weak supplement of $a \vee c$ in L.

PROOF : Clearly $(a \vee c) \vee b = 1$. Let $d = a \wedge b$ and $u = (a \vee c) \wedge b$. Suppose $u \vee y = 1$ for some y in L. Clearly $u \vee x = 1$ where $x = y \vee d$. Then $b = b \wedge 1 = b \wedge (u \vee x) = u \vee (b \wedge x)$ and $1 = a \vee b = a \vee u \vee (b \wedge x) = a \vee [(a \vee c) \wedge b] \vee (b \wedge x)$. By modular law,

$$1 = [(a \vee c) \wedge (a \vee b)] \vee (b \wedge x) = a \vee c \vee (b \wedge x).$$

Since $c \ll L$, $1 = a \vee (b \wedge x)$. Then $b = (b \wedge x) \vee (b \wedge a) = b \wedge x$, that is $b \leq x$. Now $1 = u \vee x \leq b \vee x \leq x$, so $x = 1$. Since $d \ll L$, $y = 1$. Thus b is a weak supplement of $a \vee c$ in L. □

The proofs of the following two propositions are the same as for modules (see [4, 17.9 (6) and 17.12]).

Proposition 2.7 — If $a \vee b = 1$ for some elements a, b of a weakly supplemented lattice L, then a has a weak supplement c in L such that $c \leq b$.

Proposition 2.8 — If $a_1 \vee a_2 = 1$ for some elements a_1, a_2 of L with $a_1/0$ and $a_2/0$ weakly supplemented, then L is weakly supplemented.

The following theorem generalizes [2, Theorem 2.1] to lattices.

***Theorem* 2.9** — *If $1/a$ and $a/0$ are weakly supplemented and a has a weak supplement in L, then L is also weakly supplemented.*

PROOF : Let b be a weak supplement of a in L. Since $a/0$ is weakly supplemented, $a/(a \wedge b)$ is weakly supplemented. The quotient sublattice $b/(a \wedge b)$ is also weakly supplemented since $b/(a \wedge b) \cong (a \vee b)/a \cong 1/a$. Then $1/(a \wedge b) = [a/(a \wedge b)] \vee [b/(a \wedge b)]$ is weakly supplemented by Proposition 2.8. Therefore L is weakly supplemented by Proposition 2.4. □

An element c is called a pseudo-complement of an element b in L if $b \wedge c = 0$ and c is maximal with respect to this property. L is said to be pseudo-complemented if every element of L has a pseudo-complement in L (see [11]). Pseudo-complemented lattices are E-complemented (see [8]). On the other hand if L is the lattice of all submodules of a weak supplemented module which is not supplemented, then the dual lattice L^0 is E-complemented but not pseudo-complemented.

It is well known that the lattice of submodules of every module is pseudo-complemented (see [11]) and therefore E-complemented. The following example shows that this fact need not be true in an arbitrary lattice.

Example 2.10 — Consider the interval $[0, 1]$ with usual topology. The set $\mathcal{C}$ of closed subsets of $[0,1]$ form a complete distributive lattice with respect to the operations: $\bigwedge_{i\in I} C_i = \bigcap_{i\in I} C_i$ and $\bigvee_{i\in I} C_i = \overline{\bigcup_{i\in I} C_i}$ (closure of $\bigcup_{i\in I} C_i$) for any family $\{C_i\}_{i\in I}$ from $\mathcal{C}$. Suppose that $\{0\}$ has an E-complement A in $\mathcal{C}$, i.e. $\{0\}\cap A = \phi$ and $\{0\}\cup A \trianglelefteq \mathcal{C}$. Since A is closed, $a = \inf A \in A$, therefore $a > 0$. Then $A \subseteq [a,1] \subset \left[\frac{a}{2}, 1\right]$ and $(\{0\}\cup A)\cap \left[\frac{a}{4}, \frac{a}{2}\right] = \phi$, that is $\{0\}\cup A$ is not essential in $\mathcal{C}$. This contradiction shows that the lattice $\mathcal{C}$ is not E-complemented.

Recall that if every element a of L is a complement of an element in L, i.e. $a \vee b = 1$ and $a \wedge b = 0$ for some b in L, then L is called a complemented lattice (see [6]).

Lemma 2.11 — If L is complemented, then $a/0$ is complemented for every element a of L.

PROOF : Let x be an element of $a/0$. Since L is complemented, there exists an element y of L such that $x \vee y = 1$ and $x \wedge y = 0$. Clearly $a \wedge (x \wedge y) = 0$. By modular law $a = a \wedge 1 = a \wedge (x \vee y) = x \vee (a \wedge y)$. So x is a complement of $(a \wedge y)$ in $a/0$. □

Lemma 2.12 — (see [3, Exercise 4.5]) If a is essential in L, then for every element b of L, $a \wedge b$ is essential in $b/0$.

PROOF : Suppose $(a \wedge b) \wedge c = 0$ for some c in $b/0$. Since $a \trianglelefteq L$, $c = b \wedge c = 0$. □

An element c of L is called compact, if for every subset $X = \{x_i \mid i \in I\}$ of L with $c \leq \bigvee_{i\in I} x_i$ there exists a finite subset F of I such that $c \leq \bigvee_{i\in F} x_i$. A lattice L is said to be compact if 1 is compact and compactly generated (or algebraic) if each of its elements is a join of compact elements (see [10]). If $a < b$ and $a \leq c < b$ implies $c = a$, then we say that a is covered by b (or b covers a). If 0 is covered by a for some element a of L, then a is called an atom (see [12]). A lattice L is called semiatomic if 1 is a join of atoms in L (see [3]).

Proposition 2.13 — Let L be an E-complemented lattice and a be an element of L different from 0, 1. If the quotient sublattice $1/a$ is complemented, then there are elements b_1, b_2 in L such that b_1 is a complement of b_2, $b_1/0$ is complemented, $a \trianglelefteq b_2/0$ and b_2/a is complemented.

If L is compactly generated, then the converse holds.

PROOF : There exists b_1 in L such that $b_1 \wedge a = 0$ and $b_1 \vee a \trianglelefteq L$. Since $1/a$ is complemented, there is a complement b_2 of $b_1 \vee a$ in $1/a$. So $1 = (b_1 \vee a) \vee b_2 = b_1 \vee b_2$ and $0 = b_1 \wedge a = b_1 \wedge [(b_1 \vee a) \wedge b_2] = b_1 \wedge [(b_1 \wedge b_2) \vee a] = (b_1 \wedge b_2) \vee (b_1 \wedge a) = b_1 \wedge b_2$. Furthermore b_2/a and $b_1/0 = b_1/(b_1 \wedge a) \cong (b_1 \vee a)/a$ are complemented by Lemma 2.11. Since $b_1 \vee a \trianglelefteq L$, $a = (b_1 \wedge b_2) \vee a = (b_1 \vee a) \wedge b_2 \trianglelefteq b_2/0$ by Lemma 2.12.

Now suppose that L is compactly generated and there are elements b_1, b_2 satisfying the conditions. Sublattices $(b_1 \vee a)/a$ and b_2/a are compactly generated by [3, Exercise 2.7 and Exercise 2.9 (iii)]. Since $(b_1 \vee a)/a \cong b_1/(b_1 \wedge a) = b_1/0$, it is complemented. Compactly generated complemented lattices $(b_1 \vee a)/a$ and b_2/a are semiatomic by [3, Theorem 6.8]. Then $1/a = (b_1 \vee b_2)/a = (b_1 \vee a)/a \ \vee \ b_2/a$ is semiatomic and since L is compactly generated, $1/a$ is compactly generated. Therefore $1/a$ is complemented by [3, Theorem 6.8]. □

Lemma 2.14 — Let L be an E-complemented lattice and a be an element of L different from 0, 1. The quotient sublattice $1/a$ is complemented if and only if for every element b of L, there exists an element c in L such that $b \vee c = 1$ and $b \wedge c \leq a$.

PROOF : $(\Rightarrow)$ Let b be an element of L. Since $b \vee a$ is in $1/a$, it has a complement c in $1/a$. Then $(b \wedge c) \vee a = (b \vee a) \wedge c = a$, therefore $b \wedge c \leq a$ and $b \vee c = (b \vee a) \vee c = 1$.

$(\Leftarrow)$ Let $b \in 1/a$. There is an element c of L with $b \vee c = 1$ and $b \wedge c \leq a$. Then $b \vee (c \vee a) = b \vee c = 1$ and $b \wedge (c \vee a) = (b \wedge c) \vee a = a$, that is $c \vee a$ is a complement of b in $1/a$. So $1/a$ is complemented. □

Recall that the meet of all maximal elements (different from 1) in L is called the radical of L (see [10]), denoted by $\mathrm{rad}(L)$. If $a \ll L$ and m is a maximal element in L, then $m \vee a \neq 1$, therefore $m \vee a = m$ and so $a \leq m$. It means that the radical of L contains all small elements of L (see also [10, Proposition 6]). A lattice L is said to be semilocal if the quotient sublattice $1/\mathrm{rad}(L)$ is complemented (cf. [4, 17.1]).

***Theorem* 2.15** — *If L is an E-complemented weakly supplemented lattice, then it is semilocal and there are elements b_1, b_2 in L such that b_1 is a complement of b_2 with $b_1/0$ complemented and $\mathrm{rad}(L) \trianglelefteq b_2/0$.*

PROOF : Since L is weakly supplemented, for every element b of L there exists an element c of L such that $b \vee c = 1$ and $b \wedge c \ll L$, therefore $b \wedge c \leq \mathrm{rad}(L)$. Then the sufficient condition of Lemma 2.14 is satisfied if take $a = \mathrm{rad}(L)$. Therefore $1/\mathrm{rad}(L)$ is complemented. Then the rest statements of the theorem follows from Proposition 2.13. □

Corollary 2.16 — Let L be an E-complemented lattice with small radical. Then L is weakly supplemented if and only if it is semilocal.

PROOF : $(\Rightarrow)$ By Theorem 2.15.

$(\Leftarrow)$ Assume L is semilocal, i.e. $1/\mathrm{rad}(L)$ is complemented. By Lemma 2.14, for every element a of L there is an element b in L such that $a \vee b = 1$ and $a \wedge b \leq \mathrm{rad}(L) \ll L$. So b is a weak supplement of a in L. □

Corollary 2.17 — An E-complemented lattice L with zero radical is weakly supplemented if and only if it is complemented.

Remark 2.18 : Since pseudo-complemented lattices are E-complemented the last five statements are true for pseudo-complemented lattices as well.

3. Cofinitely Weak Supplemented Lattices

For compactly generated lattices, without loss of generality, weak supplements of cofinite elements can be regarded as compact elements:

Lemma 3.1 — (cf. [1, Lemma 2.1]) Let L be a compactly generated lattice and a be a cofinite element of L. If b is a weak supplement of a in L, then a has a weak supplement c in L such that $c \leq b$ and c is compact.

PROOF : Since L is compactly generated, $b = \bigvee_{i \in I} c_i$ where each c_i is compact.

Then

$$1 = a \vee b = a \vee (\bigvee_{i \in I} c_i) = \bigvee_{i \in I} (a \vee c_i).$$

Since $1/a$ is compact, $1 = \bigvee_{i \in F} (a \vee c_i) = a \vee (\bigvee_{i \in F} c_i)$ for some finite subset F of I. But $c = \bigvee_{i \in F} c_i$ is compact by [3, Proposition 2.1]. Clearly c is a weak supplement of a. □

The following example shows that Lemma 3.1 need not be true for lattices that are not compactly generated.

Example 3.2 — Let $L = \{(x, 0) \mid x \in [0, 1]\} \cup \{(0, y) \mid y \in [0, 1]\} \subseteq \mathbb{R}^2$ and define the order $\preceq$ on L as follows. $(a, b) \preceq (c, d)$ if either $b = d = 0$ and $a \leq c$; or $a = c = 0$ and $b \leq d$; or $b = c = 0$ and $a \leq d$. One can easily verify that L is a complete modular lattice with the largest element $(0, 1)$ and the smallest element $(0, 0)$. Since the quotient sublattice $(0, 1)/(1, 0)$ is simple, it is compact. So $(1, 0)$ is a cofinite element of L. Let a be a real number with $0 < a < 1$. Clearly $(0, a) \vee (1, 0) = (0, 1)$ and $(0, a) \wedge (1, 0) = (a, 0)$ is small in L, so $(0, a)$ is a weak supplement of $(1, 0)$ in L. On the other hand, there is no compact element in L except for $(0, 0)$, therefore there is no compact weak supplement (b, c) of $(1, 0)$ with $(b, c) \preceq (0, a)$.

Proposition 3.3 — If L is a *cws*-lattice, then for every element a of L, $1/a$ is also a *cws*-sublattice of L.

PROOF : Let b be a cofinite element of $1/a$. Then $1/b$ is a compact sublattice in $1/a$, so $1/b$ is a compact quotient sublattice in L. This means that b is a cofinite element of L. Since L is a *cws*-lattice, b has a weak supplement x in L, i.e. $x \vee b = 1$ and $x \wedge b \ll L$. Since $x \wedge b \ll L$, $(x \vee a) \wedge b = (x \wedge b) \vee a \ll (1 \vee a)/a = 1/a$ by Lemma 2.2(1). So $x \vee a$ is a weak supplement of b in $1/a$. □

We are going to prove that L is *cws*-lattice if and only if every maximal element of L has a weak supplement. This result was proved for *cws*-modules in [1]. The proof of the following lemma for modules ([1, Lemma 2.15]) is valid for lattices as well.

Lemma 3.4 — Let a and b be elements of L such that b is a weak supplement of a maximal element m of L. If $a \vee b$ has a weak supplement in L, then a has a weak supplement in L.

Let Γ be the set of all elements b of L such that b is a weak supplement of some maximal element of L and let $\mathrm{cws}(L)$ denote the join of all elements of Γ.

***Theorem* 3.5** — *A lattice L is a cws-lattice if and only if every maximal element of L has a weak supplement.*

PROOF : $(\Rightarrow)$ Clear since every maximal element is cofinite.
$(\Leftarrow)$ Observing that every nonzero compact lattice has a maximal element by [3, Lemma 2.4], this part of the proof is analogous to the proof of Theorem 2.16 in [1]. □

Using this theorem we prove that an arbitrary join of *cws*-lattices is a *cws*-lattice (see [1, Proposition 2.12]).

***Theorem* 3.6** — *Let $\{a_i/0\}_{i\in I}$ be a collection of cws-sublattices of L with $1 = \bigvee_{i\in I} a_i$. Then L is a cws-lattice.*

PROOF : Let m be any maximal element of L. If $a_i \leq m$ for all $i \in I$, then $1 = \bigvee_{i\in I} a_i \leq m$ which is a contradiction. So there exists a $j \in I$ such that $a_j \nleq m$. Then $1 = a_j \vee m$. Since $a_j/(a_j \wedge m) \cong (a_j \vee m)/m = 1/m$, the element $a_j \wedge m$ is maximal in $a_j/0$. By hypothesis there is a weak supplement c of $a_j \wedge m$ in $a_j/0$, i.e. $(a_j \wedge m) \vee c = a_j$ and $a_j \wedge m \wedge c \ll a_j/0$. If $c \leq m$ then $a_j = (a_j \wedge m) \vee c \leq m$, a contradiction. So $c \nleq m$. Therefore $1 = m \vee c$ and $m \wedge c = a_j \wedge m \wedge c \ll L$ by Lemma 2.2(3). Thus c is a weak supplement of m in L. By Theorem 3.5, L is a *cws*-lattice. □

Theorem 3.5 is also used in the proof of the following theorem which in its turn gives a new result for modules.

***Theorem* 3.7** — *If $a/0$ is a cws-sublattice of L and $1/a$ has no maximal element, then L is also a cws-lattice.*

PROOF : Let b be a maximal element of L. If $a \leq b$ then b is a maximal element of $1/a$, but $1/a$ has no maximal element. So $a \nleq b$, therefore $a \vee b = 1$ and $a/(a \wedge b) \cong (a \vee b)/b = 1/b$. Since b is a maximal element of L, $a \wedge b$ is a maximal and therefore a cofinite element of $a/0$. Then there is a weak supplement c of $a \wedge b$ in $a/0$, that is $(a \wedge b) \vee c = a$ and $(a \wedge b) \wedge c \ll a/0$. Since c is in $a/0$, $c \wedge b = c \wedge (a \wedge b) \ll L$. $c \vee b = c \vee (a \wedge b) \vee b = a \vee b = 1$. So c is a weak supplement of b in L. By Theorem 3.5, L is a *cws*-lattice. □

Lemma 3.8 — Let L be a compactly generated lattice and a be a cofinite element of L. If a has a weak supplement b in L and for every compact element c with $c \leq b$, $\mathrm{rad}(c/0) = c \wedge \mathrm{rad}(L)$, then a has a compact supplement in L.

PROOF : Since a is cofinite, $1/a$ is compact. So by Lemma 3.1, a has a compact weak supplement c with $c \leq b$, i.e. $1 = a \vee c$ and $a \wedge c \ll L$. Then $a \wedge c \leq \mathrm{rad}(L)$. So $a \wedge c \leq c \wedge \mathrm{rad}(L) = \mathrm{rad}(c/0)$. Since c is compact, $\mathrm{rad}(c/0) \ll c/0$ by [10, Proposition 9 (iii)]. Thus $a \wedge c \ll c/0$. Hence c is a supplement of a in L. □

Using Lemma 3.8 one can easily modify the proofs of [1, Theorem 2.19] and [1, Corollary 2.20] to prove Theorem 3.9 and Corollary 3.10.

***Theorem* 3.9** — *Let L be a compactly generated lattice such that for every compact element c of L, $\mathrm{rad}(c/0) = c \wedge \mathrm{rad}(L)$. Then L is cofinitely weak supplemented if and only if L is cofinitely supplemented.*

Corollary 3.10 — Let L be a compact lattice such that for every compact element c of L, $\mathrm{rad}(c/0) = c \wedge \mathrm{rad}(L)$. Then L is weakly supplemented if and only if L is supplemented. Furthermore in this case every compact element of L is a supplement.

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