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Integrable hierarchies and information measures

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Abstract

In this paper we investigate integrable models from the perspective of information theory, exhibiting various connections. We begin by showing that compressible hydrodynamics for a one-dimensional isentropic fluid, with an appropriately motivated information theoretic extension, is described by a general nonlinear Schrödinger (NLS) equation. Depending on the choice of the enthalpy function, one obtains the cubic NLS or other modified NLS equations that have applications in various fields. Next, by considering the integrable hierarchy associated with the NLS model, we propose higher order information measures which include the Fisher measure as their first member. The lowest members of the hierarchy are shown to be included in the expansion of a regularized Kullback–Leibler measure while, on the other hand, a suitable combination of the NLS hierarchy leads to a Wootters type measure related to a NLS equation with a relativistic dispersion relation. Finally, through our approach, we are led to construct integrable relativistic NLS equations.

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1. Introduction

Integrable equations are fascinating not just because of their soliton solutions and the connections they make among different areas of mathematics, but also because they do describe real physical systems in some limit. An example is the cubic nonlinear Schrödinger equation,

$$\mathrm{i}\psi_t + \psi_{xx} + 2\kappa^2|\psi|^2\psi = 0 \tag{1}$$

which is of relevance in quantum optics, condensed matter physics and other areas. The basic equation (1) can be modified while still preserving integrability, for example by adding to the right-hand side of (1) a term proportional to

$$Q = s\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}, \tag{2}$$

where $\rho = |\psi|^2$. Through a change of variables one can actually absorb that extra term and regain the form (1) at the expense of redefined parameters [1]. However this is possible only if parameter $s < 1$, whereas if $s > 1$ one ends up with a reaction-diffusion equation [1]. Such Q augmented NLS equations have appeared in plasma physics [2], where they describe transmission of uni-axial waves in a cold collisionless plasma subject to a transverse magnetic field.

The reason for using the symbol Q is because such a term, often referred to as a 'quantum potential', appeared first in alternate ways of writing the usual linear Schrödinger equation of quantum mechanics [4–6]. Consider the one-dimensional time-dependent Schrödinger equation (we set the mass $m = 1$),

$$\mathrm{i}\hbar\psi_t + \frac{\hbar^2}{2}\psi_{xx} - U(x)\psi = 0. \tag{3}$$

Then substituting into this equation the Madelung representation of the wavefunction

$$\psi = \sqrt{\rho}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S} \tag{4}$$

decomposes it into two real equations,

$$S_t + \frac{1}{2}(S_x)^2 + U - \frac{\hbar^2}{2}\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}} = 0, \tag{5}$$

$$\rho_t + (\rho S_x)_x = 0. \tag{6}$$

The first equation may be viewed as a generalization of the usual Hamilton–Jacobi equation by the term with explicit $\hbar$ dependence, the quantum potential, encoding the quantum aspects of the theory. The second equation is the continuity equation expressing the conservation of probability.

Several attempts have been made to motivate the form of Q and thus obtain a derivation of Schrödinger's equation from classical dynamics [7]. Here we adopt an information theoretic perspective similar to that used in statistical mechanics and which is usually referred to as the 'maximum entropy method' [8]. The idea is that if one has a system that has to be described probabilistically then, lacking any information of the detailed microscopic dynamics, one should choose the probability distribution with minimum bias. This is achieved by maximizing an appropriate measure of uncertainty (entropy), such as the Gibbs–Shannon measure used in classical statistical mechanics.

In order to proceed with an information theoretic interpretation of (5) and (6), it is useful to approach those equations through a variational principle [9]: one minimizes the action

$$\Phi = \int \rho\left[S_t + \frac{1}{2}(S_x)^2 + U\right]\mathrm{d}x\,\mathrm{d}t + \frac{\hbar^2}{8}I_F \tag{7}$$

with respect to the field variables ρ and S. The positive quantity

$$I_F \equiv \int \mathrm{d}x\,\mathrm{d}t\rho\left(\frac{\rho_x}{\rho}\right)^2 \tag{8}$$

resembles the 'Fisher information' measure used in statistics [10, 11]. Since a broader probability distribution $\rho(x)$ represents a greater uncertainty in x, I_F may be thought of as an inverse uncertainty measure.

Equations (7) and (8) were used in [9] to interpret Schrödinger's equation as follows: first one notes that without the term I_F, varying equation (7) gives rise to the Hamilton–Jacobi equation describing a classical *ensemble*. The probability, $\rho(x)$ appears in this context because one supposes that there is uncertainty in our knowledge of the initial position of the particle. One then adopts the principle of maximum uncertainty [8] to constrain the probability distribution $\rho(x)$ characterizing the ensemble: we would like to be as unbiased as possible in its choice, consistent with our lack of information. That constraint is implemented in (7) by minimizing I_F when varying the classical action: $\hbar^2/8$ is the Lagrange multiplier.

It remains to explain why I_F is chosen as the information measure in the above quantum mechanical context as opposed to say the Gibbs–Shannon measure. In information theory and statistical mechanics the Gibbs–Shannon measure is the simplest possibility that satisfies certain axioms that are deemed necessary in those contexts [8]. Similarly one can derive the Fisher measure as the relevant quantity that satisfies axioms relevant for classical ensemble dynamics and hence appropriate for use in deriving Schrödinger's equation [12].

In this paper we would like to apply the above information theoretic reasoning to motivate the NLS (1) and its various extensions. In the next section we first review the derivation of the action for a classical compressible fluid in one dimension. Then in section 3 we use information theoretic arguments to modify the action and so arrive at a general nonlinear Schrödinger equation. In section 4 we employ an expansion of the enthalpy function to obtain specific examples of the nonlinear Schrödinger equation. In section 5 we consider the hierarchy associated with the NLS equation and use that to define a hierarchy of higher derivative information measures, relating the information hierarchy to other information measures in the literature. In section 6 we use the information measures to construct NLS equations with relativistic dispersion relations. Our conclusion is in section 7 while in the appendices we discuss some details of the integrability of the NLS hierarchy; while much of that material is known, we present some novel representations (such as using operator q-numbers and operator q-derivatives) and then apply the general procedures to construct the new relativistic integrable equations, associated with higher derivative information measures, in section 6.

2. Compressible fluid in one dimension

In this paper we focus on a specific physical model, hydrodynamics, to illustrate our approach, though we believe that much of it can be generalized to other contexts.

The Euler and continuity equations for a one-dimensional compressible fluid are

$$v_t + vv_x + \frac{1}{\rho}P_x = 0, \tag{9}$$

$$\rho_t + (\rho v)_x = 0, \tag{10}$$

where the hydrodynamical variables corresponding to the density of fluid, velocity of fluid and pressure have been denoted by $\rho(x,t)$, $v(x,t)$ and $P(x,t)$, respectively. In addition, one has the thermodynamic equation [13]

$$\frac{d\S}{dt} = \frac{\partial \S}{\partial t} + v\S_x = 0 \tag{11}$$

expressing the conservation of entropy, §, if one assumes the absence of the heat exchange between parts of the medium. To complete the description of the dynamics one also needs an equation of state

$$P = P(\S, \rho) \tag{12}$$

whose concrete form depends on the properties of the fluid. For example, an ideal gas has

$$P = e^{\S/c_V}\rho^\gamma, \tag{13}$$

where $\gamma = c_P/c_V$ (the Poisson adiabate) is the ratio of specific heat capacities at constant pressure and volume, respectively. We will consider an *isentropic* fluid which has a spacetime constant § so that (11) is automatically satisfied. For such barotropic processes the pressure becomes a function of density only,

$$P = P(\rho). \tag{14}$$

It is convenient to introduce the enthalpy function defined through the relation

$$\frac{\partial}{\partial x}\mathcal{E}(\rho) = \frac{1}{\rho}\frac{\partial}{\partial x}P(\rho) \tag{15}$$

which implies, for $\rho_x \neq 0$,

$$\mathcal{E}'(\rho) = \frac{1}{\rho}P'(\rho) \tag{16}$$

or

$$\mathcal{E}(\rho) = \int_{\rho_0}^{\rho}\frac{\mathrm{d}P}{\rho}. \tag{17}$$

Then the system of equations (9) and (10) becomes

$$v_t + v v_x + (\mathcal{E}(\rho))_x = 0, \tag{18}$$

$$\rho_t + (\rho v)_x = 0. \tag{19}$$

This system may be written in the Lagrangian form by first introducing the velocity potential

$$v(x,t) = S_x(x,t) \tag{20}$$

and then integrating the first equation once and introducing the enthalpy potential

$$\mathcal{E}(\rho) = \frac{\mathrm{d}V(\rho)}{\mathrm{d}\rho} \tag{21}$$

to obtain

$$S_t + \frac{(S_x)^2}{2} + \frac{\mathrm{d}V(\rho)}{\mathrm{d}\rho} = 0, \tag{22}$$

$$\rho_t + (\rho S_x)_x = 0. \tag{23}$$

The action for this system is

$$A = \int\left(\rho S_t + \frac{\rho(S_x)^2}{2} + V(\rho)\right)\mathrm{d}x\,\mathrm{d}t \tag{24}$$

and equations (22) and (23) appear by varying this functional with respect to ρ and S. The resemblance of (24) to the classical part of (7) will be the starting point for the extension in the next section.

We note, for later use below, that when the enthalpy vanishes, $\mathcal{E} = 0$, the fluid equations (22) and (23) are invariant under a scaling, $\rho \to \alpha\rho$. That is, with $\mathcal{E} = 0$, the magnitude of the density does not matter, only its variation. When $\mathcal{E}$ is not zero, the equations to be derived later become generalized nonlinear Schrödinger equations, with a sensitivity to the magnitude of ρ.

3. Information-theoretical extension of compressible fluid dynamics

The action (24) gives the classical equations of motion for the fluid. Since the density ρ informs us about the likelihood of finding the microscopic fluid elements at a certain region of spacetime, it plays a role analogous to the probability density in quantum mechanics. Thus from the density we may form an information measure I that quantifies our knowledge of the microstates and we may demand, as in the previous section, that the equations of motion follow from (24) but constrained such that our uncertainty (information) is maximized (minimized). This will lead to modified hydrodynamics equations that depend on the form of information measure chosen in the procedure. Now, the density $\rho(x)$ is positive definite and if it is uniform it tells us that the underlying particles of the fluid could be anywhere: we have no information (maximum uncertainty). If the density is peaked somewhere, we know that a fluid particle is more likely to be there, that is we have gained information. Thus we require that our scalar information functional $I[\rho]$ have the property that it is positive definite and $I \to 0$ as $\rho \to$ a constant.

We prefer local equations, and so we may write I as an integral over a density function $J(\rho)$,

$$I = \int \mathrm{d}x\, \mathrm{d}t \rho J(\rho). \tag{25}$$

Next we assume the density to be slowly varying and so do a derivative expansion,

$$J(\rho) = J_0(\rho) + \rho' J_1(\rho) + \rho'' J_{21}(\rho) + (\rho')^2 J_{22}(\rho) + \text{higher derivative terms}, \tag{26}$$

where $J_0(\rho)$, $J_1(\rho)$, ... do not contain any derivatives. We assume that when (26) is used in (25) the integrals are convergent term by term.

We also impose the strong condition that the information measure, I, does not break the invariance of the $\mathcal{E} = 0$ equations of motion (22) and (23) under the scaling of $\rho \to \alpha\rho$. That is, although $\mathcal{E}$ will generally break that invariance, we demand that the terms in the modified equations of motion that come from I do not do so: the information measure is chosen to be neutral to the magnitude of ρ but measures only local variations. So here we see the first difference between the contributions of our I and $\mathcal{E}$: I is insensitive to the size of ρ.

In order to achieve our goal, we need to demand that $J(\rho)$ in (25) is scale invariant (note we already factored out a ρ in the integral form of I). In that way the equation of motion terms that come from varying I will be scale invariant. This is satisfied if (26) has the form

$$J(\rho) = a_0 + a_1 \times (\rho'/\rho) + a_{21} \times (\rho''/\rho) + a_{22} \times (\rho'/\rho)^2 + \text{higher derivative terms}, \tag{27}$$

where $a_k, a_{kl}, \ldots$ for $k, l = 1, 2, \ldots,$ are constants. Using this in the integral that defines I, equation (25), and dropping constants and total derivatives, only the a_{22} term survives to leading order and it gives precisely the Fisher information measure!

Recall that we still need to demand positivity of our information measure: that fixes the Lagrange multiplier to be positive if we are minimizing the information. Fortunately, the Fisher measure already satisfies the other required property, that it vanishes as $\rho \to$ a constant.

Note that in the fluid problem we have in general the boundary condition $\rho \to \rho_0$ as $|x| \to \infty$, so that for normalization of the probability we have

$$\int (\rho(x,t) - \rho_0)\, \mathrm{d}x = 1. \tag{28}$$

With the choice of the Fisher information measure

$$I_F = \int \frac{(\rho_x)^2}{\rho}\, \mathrm{d}x = 4 \int (\sqrt{\rho})_x (\sqrt{\rho})_x\, \mathrm{d}x \tag{29}$$

as motivated above, the extension of the normalization condition (28) from the usual case in quantum mechanics does not modify the convergence properties of I. Thus we have the variational functional

$$A + \frac{\lambda^2}{8} I_F, \tag{30}$$

where λ is a Lagrange multiplier and the equations of motion that follow are

$$S_t + \frac{(S_x)^2}{2} + \frac{\mathrm{d}V(\rho)}{\mathrm{d}\rho} - \frac{\lambda^2}{2}\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}} = 0, \tag{31}$$

$$\rho_t + (\rho S_x)_x = 0. \tag{32}$$

These equations may be combined into one complex equation through the inverse Madelung transformation to give the following general nonlinear Schrödinger equation,

$$\mathrm{i}\lambda\psi_t + \frac{\lambda^2}{2}\psi_{xx} - \mathcal{E}(|\psi|^2)\psi = 0. \tag{33}$$

In summary, although both the enthalpy function, $\mathcal{E}$, and the information functional, I, will contribute ρ-dependent terms to the equations of motion, their structure and origin is in general different. Using the lowest order information measure with the properties described above we get a generalized NLS equation (33) that depends on the form of $\mathcal{E}$. Using more generalized information measures will give further extensions of the NLS equations.

We remark also that unlike the quantum mechanics case [12], the deduction of the Fisher measure above did not use the separability condition: rather here we assumed a convergent derivative expansion of the information density. See also [14] for similar arguments used in the relativistic case.

4. Weak nonlinearity and nonlinear Schrödinger equation

In this section we study the simplest form of the function $\mathcal{E}$ in (33) that will give rise to integrable systems. If $\mathcal{E}$ as a function of $\rho = |\psi|^2$ is analytic then

$$\mathcal{E}(|\psi|^2) = \mathcal{E}_0 + \mathcal{E}_1|\psi|^2 + \mathcal{E}_2|\psi|^4 + \cdots + \mathcal{E}_n|\psi|^{2n} + \cdots. \tag{34}$$

This equation implies that the pressure according to (15) is also an analytic function of the form

$$P(\rho) = \mathcal{E}_0 + \frac{1}{2}\mathcal{E}_1\rho^2 + \frac{2}{3}\mathcal{E}_2\rho^3 + \cdots \frac{n}{n+1}\mathcal{E}_n\rho^{n+1} + \cdots. \tag{35}$$

At the lowest order of nonlinearity we get the nonlinear Schrödinger equation (NLS) with cubic nonlinearity

$$\mathrm{i}\lambda\psi_t + \frac{\lambda^2}{2}\psi_{xx} - (\mathcal{E}_0 + \mathcal{E}_1|\psi|^2)\psi = 0. \tag{36}$$

This model is integrable for both signs of $\mathcal{E}_1$. For $\mathcal{E}_1 > 0$ it is defocusing (e.g. repulsive Bose gas) and nontrivial soliton solutions exist only with nontrivial boundary conditions, so in this case $\mathcal{E}_0 \neq 0$. For $\mathcal{E}_1 < 0$ we have the focusing case (e.g. attractive Bose gas) for which soliton solutions exist for vanishing boundary conditions; so in this case we can put $\mathcal{E}_0 = 0$. Both cases have applications in nonlinear optics describing pulse propagating in nonlinear media. Consider the second case again: by rescaling space and time variables $t' = t/\lambda$, $x' = \sqrt{2}x/\lambda$ and the coupling constant $\mathcal{E}_1 = -2\kappa^2$ we may rewrite it in the form (we now skip all upper scripts)

$$\mathrm{i}\psi_t + \psi_{xx} + 2\kappa^2|\psi|^2\psi = 0. \tag{37}$$

As is well known [26], integrability of NLS (37) is connected with existence of the flat non-Abelian connections

$$J_1 = \begin{pmatrix} -\frac{i}{2}p & -\kappa^2\bar\psi \\ \psi & \frac{i}{2}p \end{pmatrix}, \qquad J_0 = \begin{pmatrix} -\frac{i}{2}p^2 + i\kappa^2|\psi|^2 & -\kappa^2(p\bar\psi - i\bar\psi_x) \\ p\psi + i\psi_x & \frac{i}{2}p^2 - i\kappa^2|\psi|^2 \end{pmatrix}, \tag{38}$$

where constant p is the spectral parameter, so that the zero curvature condition

$$\partial_t J_1 - \partial_x J_0 + [J_0, J_1] = 0 \tag{39}$$

is equivalent to (37). It implies the linear system

$$\frac{\partial}{\partial x}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\frac{i}{2}p & -\kappa^2\bar\psi \\ \psi & \frac{i}{2}p \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = J_1\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{40}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\frac{i}{2}p^2 + i\kappa^2|\psi|^2 & -\kappa^2(p\bar\psi - i\bar\psi_x) \\ p\psi + i\psi_x & \frac{i}{2}p^2 - i\kappa^2|\psi|^2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = J_0\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{41}$$

The linear problem (40) and (41) is called the Zakharov–Shabat problem and it can be solved by the inverse scattering method [26].

The 'semiclassical' or dispersionless limit of equation (37) was studied in [15] in relation to shock wave propagation in nonlinear optics. The wave form of this semiclassical limit is a NLS equation perturbed by a quantum potential [16]

$$i\psi_t + \psi_{xx} + 2\kappa^2|\psi|^2\psi = \frac{|\psi|_{xx}}{|\psi|}\psi. \tag{42}$$

Then we can conclude that inclusion of information characteristics in the form of the Fisher measure, produces NLS (37) from dispersionless NLS (42) and corresponding solitons of the first one from the shock waves of the second one [16].

5. Integrable NLS hierarchy and higher derivative information measures

It is well known that one can construct a hierarchy of higher order differential equations that are related to the cubic NLS (37) and its complex conjugate, which are still integrable [17],

$$i\sigma_3\begin{pmatrix} \psi \\ \bar\psi \end{pmatrix}_{t_N} = \mathcal{R}^N\begin{pmatrix} \psi \\ \bar\psi \end{pmatrix} \tag{43}$$

where t_N, $N = 1, 2, 3, \ldots$, is an infinite time hierarchy and $\mathcal{R}$ is the matrix integro-differential operator—the recursion operator of the NLS hierarchy—

$$\mathcal{R} = i\sigma_3\begin{pmatrix} \partial_x + 2\kappa^2\psi\int^x\bar\psi & -2\kappa^2\psi\int^x\psi \\ -2\kappa^2\bar\psi\int^x\bar\psi & \partial_x + 2\kappa^2\bar\psi\int^x\psi \end{pmatrix} \tag{44}$$

and σ_3 is the Pauli matrix; see the appendix for more details.

The first few members of the hierarchy $N = 1, 2, 3, 4$, are

$$\psi_{t_1} = \psi_x, \tag{45}$$

$$i\psi_{t_2} + \psi_{xx} + 2\kappa^2|\psi|^2\psi = 0, \tag{46}$$

$$\psi_{t_3} + \psi_{xxx} + 6\kappa^2|\psi|^2\psi_x = 0, \tag{47}$$

$$i\psi_{t_4} = \psi_{xxxx} + 2\kappa^2\left(2|\psi_x|^2\psi + 4|\psi|^2\psi_{xx} + \bar\psi_{xx}\psi^2 + 3\bar\psi\psi_x^2\right) + 6\kappa^4|\psi|^4\psi. \tag{48}$$

In the linear approximation, when $\kappa = 0$, the recursion operator is just the momentum operator

$$\mathcal{R}_0 = i\sigma_3\frac{\partial}{\partial x} \tag{49}$$

and the NLS hierarchy (43) becomes the linear Schrödinger hierarchy

$$\mathrm{i}\psi_{t_n} = \mathrm{i}^n \partial_x^n \psi. \tag{50}$$

Written in the Madelung representation it produces the complex Burgers hierarchy so that this representation plays the role of the complex Cole–Hopf transformation [18].

5.1. *Fourth order flow*

Let us look more explicitly at the fourth order flow for which the Hamiltonian is

$$H = \int \left[\bar{\psi}_{xx}\psi_{xx} - 8\kappa^2\bar{\psi}_x\psi_x\bar{\psi}\psi - \kappa^2\left(\bar{\psi}_x^2\psi^2 + \bar{\psi}^2\psi_x^2\right) + 2\kappa^4|\psi|^6\right]\mathrm{d}x. \tag{51}$$

In the Madelung representation

$$\psi = \sqrt{\rho}\,\mathrm{e}^{\mathrm{i}S} = \mathrm{e}^{R+\mathrm{i}S}, \tag{52}$$

this becomes

$$H = \int \left[\frac{\rho_{xx}^2}{4\rho} - \frac{\rho_{xx}\rho_x^2}{4\rho^2} + \frac{\rho_x^4}{16\rho^3}\right. \tag{53}$$

$$\left. + \rho S_{xx}^2 + \rho S_x^4 + 2\rho_x S_x S_{xx} - 2\kappa^2\left(\frac{5}{4}\rho_x^2 + 3\rho^2 S_x^2\right) + 2\kappa^4\rho^3\right]\mathrm{d}x. \tag{54}$$

In fact for configurations with $S = \text{const}$ we have only contributions from the first three terms which can be combined into

$$H = \int (\sqrt{\rho})_{xx}(\sqrt{\rho})_{xx}\,\mathrm{d}x. \tag{55}$$

This may be considered as a higher order analogue of the Fisher information measure I_F (29).

5.2. *Hierarchy of information measures*

Generalizing, the above linearized Schrödinger hierarchy suggests, after the substitution $S = \text{const}$, the even order information measure hierarchy

$$I_2 = \int (\sqrt{\rho})_x(\sqrt{\rho})_x\,\mathrm{d}x, \tag{56}$$

$$I_4 = \int (\sqrt{\rho})_{xx}(\sqrt{\rho})_{xx}\,\mathrm{d}x, \tag{57}$$

and in general,

$$I_{2n} = \int (\sqrt{\rho})_{x\ldots x}(\sqrt{\rho})_{x\ldots x}\,\mathrm{d}x. \tag{58}$$

Here all odd members vanish because their integrands are total derivatives.

We will use the above information hierarchy in section 6 to construct relativistic NLS equations and exhibit links between different information measures known in the literature.

5.3. Soliton information measures

As an illustration, we compute the above information measures for the one soliton solution of the NLS equation. The measure is

$$\rho_\nu(x) = \frac{\nu}{2\cosh^2 \nu x} \tag{59}$$

which satisfies

$$\int_{-\infty}^{\infty} \rho_\nu(x)\,\mathrm{d}x = 1 \tag{60}$$

and is characterized by a real parameter ν so that

$$\lim_{\nu\to\infty} \rho_\nu(x) = \delta(x). \tag{61}$$

Then information measures (55), (56), ... for this distribution apart from numerical constants are simply

$$I_2 = \nu^2, \qquad I_4 = \nu^4, \ldots, \qquad I_{2\nu} = \nu^{2n}, \ldots. \tag{62}$$

5.4. Relation to Kullback–Liebler measure

The Gibbs–Shannon entropy

$$I_{\mathrm{GS}} = -\int \rho(x) \ln \rho(x)\,\mathrm{d}x \tag{63}$$

may be generalized to the Kullback–Leibler information [11]

$$I_{\mathrm{KL}}(p, r) = -\int \rho(x) \ln \frac{\rho(x)}{r(x)} \mathrm{d}x, \tag{64}$$

where $r(x)$ is a reference probability distribution. If one chooses the reference distribution to be the same as $\rho(x)$ but with infinitesimally shifted arguments, that is $r(x) = \rho(x + \Delta x)$, then to lowest order,

$$I_{\mathrm{KL}}(\rho(x), \rho(x + \Delta(x)) = \frac{-(\Delta x)^2}{2} I_F(\rho(x)) + O(\Delta x)^3, \tag{65}$$

that is, the Fisher measure is recovered as the lowest order term in the expansion.

One may further generalize the Kullback–Liebler information by introducing a parameter $0 < \eta < 1$, as used for example in [19],

$$M \equiv \int \rho(x) \ln \frac{\rho(x)}{(1-\eta)\rho(x) + \eta\rho(x+\eta L)} \mathrm{d}x. \tag{66}$$

This form is nonsingular even if the density vanishes at any point. For $L \ll 1$ we have the expansion

$$M = L^2 \frac{\eta^4}{2} \int \frac{\rho_x^2}{\rho} \mathrm{d}x - L^3 \left(\frac{\eta^6}{3} - \frac{\eta^5}{4}\right) \int \frac{\rho_x^3}{\rho^2} \mathrm{d}x$$

$$- L^4 \left[\frac{\eta^6}{24} \int \frac{\rho_{xx}^2}{\rho} \mathrm{d}x + \left(\frac{\eta^7}{3} - \frac{\eta^6}{9} - \frac{\eta^8}{4}\right) \int \frac{\rho_x^4}{\rho^3} \mathrm{d}x\right] + O(L^5)$$

where a number of surface terms have been dropped after integration by parts. Let us look at the symmetrized measure

$$M(+L) + M(-L) = L^2\eta^4 \int \frac{\rho_x^2}{\rho}\,\mathrm{d}x - L^4\left[\frac{\eta^6}{12}\int \frac{\rho_{xx}^2}{\rho}\,\mathrm{d}x + 2\left(\frac{\eta^7}{3} - \frac{\eta^6}{9} - \frac{\eta^8}{4}\right)\int \frac{\rho_x^4}{\rho^3}\,\mathrm{d}x\right] + O(L^6),$$

where as before, the lowest order term, proportional to L^2, is the Fisher measure. By choosing the parameter η to satisfy

$$\eta^2 - \frac{4}{3}\eta + \frac{3}{8} = 0, \tag{67}$$

or $\eta = (2 \pm \sqrt{5/8})/3$ we can rewrite the next, $O(L^4)$ term, as the higher derivative information measure given by equation (54).

Thus the first two members of the information hierachy I_{2n} we proposed in section 5 are contained in the Kullback–Leibler information.

6. Relativistic NLS equations

Now let us consider other ways of combining and using the higher derivative information measures. Take a Hamiltonian of the form

$$H = c_2 I_2 + c_4 I_4 + \cdots + c_{2n} I_{2n} + \cdots \tag{68}$$

where the constant coefficients c_i depend on the context.

6.1. Semi-relativistic NLS

For example, for low momenta one may as usual expand the relativistic dispersion relation $E = \sqrt{m^2c^4 + p^2c^2}$ to obtain

$$E = mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \cdots. \tag{69}$$

This may be used to construct a 'semi-relativistic' Schrödinger equation as a formal power series

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi = mc^2\left(1 - \frac{\hbar^2}{2m^2c^2}\frac{\partial^2}{\partial x^2} - \frac{\hbar^4}{8m^4c^4}\frac{\partial^4}{\partial x^4} + \cdots\right)\psi \tag{70}$$

$$\equiv \hat{h}\psi \tag{71}$$

where apart from a constant, the average of $\hat{h}$ for real ψ is precisely (68) for a particular choice of coefficients. Combining two complex conjugate equations together we have

$$\mathrm{i}\sigma_3\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix}_t = mc^2\sqrt{1 + \frac{1}{m^2c^2}\left(\mathrm{i}\sigma_3\frac{\partial}{\partial x}\right)^2}\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix}. \tag{72}$$

In fact, following the general procedure described in appendix A.4.2 one may proceed further: by replacing the derivative operator $\mathcal{R}_0 = \mathrm{i}\sigma_3\frac{\partial}{\partial x}$ or momenta with the full recursion operator $\mathcal{R}$ (44), one obtains an *integrable* relativistic nonlinear Schrödinger equation

$$\mathrm{i}\sigma_3\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix}_t = mc^2\sqrt{1 + \frac{1}{m^2c^2}\mathcal{R}^2}\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix}, \tag{73}$$

where the square root operator has meaning of the formal power series so that

$$\mathrm{i}\sigma_3\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix}_t = mc^2\left(1 + \frac{1}{2m^2c^2}\mathcal{R}^2 - \frac{1}{8m^4c^4}\mathcal{R}^4 + \frac{1}{16m^6c^6}\mathcal{R}^6 \pm \cdots\right)\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix}. \tag{74}$$

6.1.1. The linear problem. Applying the general result of appendix A.4.3 to the above relativistic dispersion, we have the next linear problem for equation (73)

$$\frac{\partial}{\partial x}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\frac{\mathrm{i}}{2}p & -\kappa^2\bar{\psi} \\ \psi & \frac{\mathrm{i}}{2}p \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{75}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \mathrm{i}A & -\kappa^2\bar{C} \\ C & -\mathrm{i}A \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{76}$$

where

$$\begin{pmatrix} C \\ \bar{C} \end{pmatrix} = \frac{\sqrt{m^2c^4 + \mathcal{R}^2c^2} - \sqrt{m^2c^4 + p^2c^2}}{\mathcal{R} - p}\begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} \tag{77}$$

$$A = -\frac{1}{2}\sqrt{m^2c^4 + p^2c^2} - \mathrm{i}\kappa^2\left(\int^x \bar{\psi}, -\int^x \psi\right)\frac{\sqrt{m^2c^4 + \mathcal{R}^2c^2} - \sqrt{m^2c^4 + p^2c^2}}{\mathcal{R} - p}\begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} \tag{78}$$

and the spectral parameter p has meaning of the classical momentum.

We note that relativistic versions of the Schrödinger equation have been considered in different contexts, for example to study relativistic quarks in nuclei [20] and gravitational collapse of a boson star [21]. A nonlinear version has appeared as the semi-relativistic Hartree–Fock equation [22]. But none of those models is known to be integrable. By contrast the model (73), where the square root is considered as a formal power series (matrix pseudo-differential operator), is an integrable nonlinear Schrödinger equation with relativistic dispersion:

$$\mathrm{i}\psi_t = mc^2\sqrt{1 - \frac{1}{m^2c^2}\frac{\partial^2}{\partial x^2}}\psi + F(\psi), \tag{79}$$

where the nonlinearity expanded in $1/c^2$ is the infinite sum

$$\begin{aligned} F(\psi) = {}& \frac{1}{2m}[-2\kappa^2|\psi|^2\psi] \\ & - \frac{1}{8m^3c^2}\left[2\kappa^2\left(2|\psi_x|^2\psi + 4|\psi|^2\psi_{xx} + \bar{\psi}_{xx}\psi^2 + 3\bar{\psi}\psi_x^2\right) + 6\kappa^4|\psi|^4\psi\right] \\ & + O\left(\frac{1}{c^4}\right). \end{aligned} \tag{80}$$

What is amazing is that if we expand also the dispersion part in $1/c^2$, then at every order of $1/c^2$ we get an integrable system. It means that we have integrable relativistic corrections to the NLS equation at any order. And the Fisher information has appeared here as the nonrelativistic approximation of the relativistic information measure hierarchy.

6.2. Relativistic quantum mechanics and Wootters measure

Another way of constructing a relativistic model that includes higher derivative information measures is to use rapidity variables for the relativistic dispersion relation,

$$E = mc^2\cosh\chi, \qquad p = mc\sinh\chi. \tag{81}$$

This gives the relativistic model with Hamiltonian [23]

$$H = mc^2\int\bar{\psi}\cosh\left(\mathrm{i}\lambda\frac{\partial}{\partial x}\right)\psi\,\mathrm{d}x, \tag{82}$$

where $\lambda = \hbar/(mc)$ is the Compton wave length of the relativistic particle. Expanding in powers of λ,

$$\cosh\left(\mathrm{i}\lambda\frac{\partial}{\partial x}\right) = 1 + \frac{1}{2!}\left(\mathrm{i}\lambda\frac{\partial}{\partial x}\right)^2 + \frac{1}{4!}\left(\mathrm{i}\lambda\frac{\partial}{\partial x}\right)^4 + \cdots \tag{83}$$

we again have a member of the information hierarchy (68). The Fisher measure then corresponds to the nonrelativistic approximation of order $O(\lambda^2)$. The relativistic quantum mechanics in one space dimension then is described by the Schrödinger equation, which we rewrite for couple of complex conjugate equations as

$$\mathrm{i}\sigma_3\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix}_t = mc^2\cosh\left(\mathrm{i}\lambda\sigma_3\frac{\partial}{\partial x}\right)\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix}. \tag{84}$$

Following integrable nonlinearization procedure described in appendix A.4.2 we have nonlinear relativistic quantum mechanical (NRQM) wave equation

$$\mathrm{i}\sigma_3\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix}_t = mc^2\cosh(\lambda\mathcal{R})\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix}. \tag{85}$$

6.2.1. Linear problem for NRQM. The linear problem for this equation is given by the Zakharov–Shabat problem (75) for the space part, and (76) for the time part, where coefficient functions are

$$\begin{pmatrix}C\\ \bar{C}\end{pmatrix} = mc^2\frac{\cosh(\lambda\mathcal{R}) - \cosh(\lambda p)}{\mathcal{R} - p}\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix} \tag{86}$$

and

$$A = -\frac{1}{2}mc^2\cosh(\lambda p) - \mathrm{i}\kappa^2 mc^2\left(\int^x \bar{\psi}, -\int^x \psi\right)\frac{\cosh(\lambda\mathcal{R}) - \cosh(\lambda p)}{\mathcal{R} - p}\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix}. \tag{87}$$

6.2.2. Wootters measure. Finally we note that the above free Hamiltonian may be represented as a finite difference operator

$$H = \frac{mc^2}{2}\int \bar{\psi}\left(\mathrm{e}^{L\frac{\partial}{\partial x}} + \mathrm{e}^{-L\frac{\partial}{\partial x}}\right)\psi\,\mathrm{d}x = \frac{mc^2}{2}\int(\bar{\psi}(x)\psi(x+L) + \bar{\psi}(x)\psi(x-L))\,\mathrm{d}x. \tag{88}$$

The dispersive part of this hierarchy for $S = \text{const}$ gives a Wootters type [24] measure

$$I_W = \int(\sqrt{\rho(x)}\sqrt{\rho(x+L)} + \sqrt{\rho(x)}\sqrt{\rho(x-L)})\,\mathrm{d}x. \tag{89}$$

7. Summary

We have shown how information theory arguments can be used to motivate the general nonlinear Schrödinger equation in the context of hydrodynamics. This then led us to study different information measures.

We noted that the integrable hierarchy of linear and nonlinear Schrödinger equations, in their Madelung form, naturally suggest a hierarchy of information measures of which the Fisher measure represents the first member. The lowest members of the information hierarchy were shown to be included in the expansion of a regularized Kullback–Leibler measure.

We also showed how to construct integrable semi-relativistic nonlinear Schrödinger equations using various combinations of the information measures. These classes of equations, which are distinct from those obtained in [14] and references therein, might be useful in analyzing relativistic corrections to solitons, Bose–Einstein condensates or other condensed matter systems with effective equations of relativistic form.

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Appendix

A.1. NLS hierarchy

Consider the Zakharov–Shabat linear problem (40)

$$\frac{\partial}{\partial x}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\frac{\mathrm{i}}{2}p & -\kappa^2\bar{\psi} \\ \psi & \frac{\mathrm{i}}{2}p \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = J_1\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{A.1}$$

for the space evolution, and the generalized problem

$$\frac{\partial}{\partial t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \mathrm{i}A & -\kappa^2\bar{C} \\ C & -\mathrm{i}A \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = J_0\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{A.2}$$

for the time evolution, where the real $A(x, t, p)$ and complex $C(x, t, p)$ functions are determined by the zero-curvature condition (39) and subject to

$$\partial_t\psi = \partial_x C + 2\mathrm{i}A\psi + \mathrm{i}pC \tag{A.3}$$

$$\partial_t\bar{\psi} = \partial_x\bar{C} - 2\mathrm{i}A\psi - \mathrm{i}p\bar{C} \tag{A.4}$$

$$\partial_x A = \mathrm{i}\kappa^2(\bar{C}\psi - C\bar{\psi}). \tag{A.5}$$

Substitution [17]:

$$A_N = \sum_{n=0}^{N} A^{(n)}\left(-\frac{p}{2}\right)^n, \qquad C_N = \sum_{n=0}^{N} C^{(n)}\left(-\frac{p}{2}\right)^n \tag{A.6}$$

to (A.3) gives the evolution equation

$$\partial_{t_N}\psi = \partial_x C^{(0)} + 2\mathrm{i}A^{(0)}\psi \tag{A.7}$$

and $C^{(N)} = 0$, $A^{(N)} = a_N = \text{const}$. In our further consideration we fix this constant so that $a_N = (-2)^{N-1}$. Then we have the recurrence relations

$$C^{(n)} = \frac{1}{2\mathrm{i}}\partial_x C^{(n+1)} + A^{(n+1)}\psi \tag{A.8}$$

$$\partial_x A^{(n)} = \mathrm{i}\kappa^2(\bar{C}^{(n)}\psi - C^{(n)}\bar{\psi}), \tag{A.9}$$

where $n = 0, 1, 2, \ldots, N-1$. Integrating the last equation one has

$$A^{(n)} = -\mathrm{i}\kappa^2\int^x(\bar{\psi}C^{(n)} - \psi\bar{C}^{(n)}). \tag{A.10}$$

A.2. Recursion operator

Substituting (A.10) into (A.8) and into its complex conjugate, we find the recursion formula

$$\begin{pmatrix} C^{(n)} \\ \bar{C}^{(n)} \end{pmatrix} = -\frac{1}{2}\mathcal{R}\begin{pmatrix} C^{(n+1)} \\ \bar{C}^{(n+1)} \end{pmatrix} \tag{A.11}$$

which starts from

$$\begin{pmatrix} C^{(N-1)} \\ \bar{C}^{(N-1)} \end{pmatrix} = a_N \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}, \tag{A.12}$$

where $\mathcal{R}$ is the matrix integro-differential operator—the recursion operator of the NLS hierarchy—

$$\mathcal{R} = \mathrm{i}\sigma_3 \begin{pmatrix} \partial_x + 2\kappa^2\psi \int^x \bar{\psi} & -2\kappa^2\psi \int^x \psi \\ -2\kappa^2\bar{\psi} \int^x \bar{\psi} & \partial_x + 2\kappa^2\bar{\psi} \int^x \psi \end{pmatrix} \tag{A.13}$$

and σ_3—the Pauli matrix. Repeating k-times it gives

$$\begin{pmatrix} C^{(N-k)} \\ \bar{C}^{(N-k)} \end{pmatrix} = \frac{a_N}{(-2)^{k-1}} \mathcal{R}^{k-1} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}. \tag{A.14}$$

For $k = N$ steps we obtain

$$\begin{pmatrix} C^{(0)} \\ \bar{C}^{(0)} \end{pmatrix} = \frac{a_N}{(-2)^{N-1}} \mathcal{R}^{N-1} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}. \tag{A.15}$$

Using (A.9) for $n = 0$, the evolution equation (A.7) can be rewritten by the recursion operator (44) as

$$\mathrm{i}\sigma_3 \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}_{t_N} = \mathcal{R} \begin{pmatrix} C^{(0)} \\ \bar{C}^{(0)} \end{pmatrix}. \tag{A.16}$$

Hence substituting (A.15) and fixing $a_N = (-2)^{N-1}$ we obtain equation (43)

$$\mathrm{i}\sigma_3 \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}_{t_N} = \mathcal{R}^N \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}, \tag{A.17}$$

where t_N, $N = 1, 2, 3, \ldots,$ is an infinite time hierarchy.

A.3. Linear problem for the NLS hierarchy

Every equation of the hierarchy (A.17) is integrable. The linear problem for the Nth equation is given by the Zakharov–Shabat problem (A.1) for the space part and

$$\frac{\partial}{\partial t_N} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \mathrm{i} \ A_N & -\kappa^2 \bar{C}_N \\ C_N & -\mathrm{i}A_N \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = J_{0_N} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{A.18}$$

for the time part. Coefficient functions C_N and A_N can be found by substituting (A.14) into (A.6) so that

$$\begin{pmatrix} C_N \\ \bar{C}_N \end{pmatrix} = \sum_{k=0}^{N} \left(-\frac{p}{2}\right)^{N-k} \begin{pmatrix} C^{(N-k)} \\ \bar{C}^{(N-k)} \end{pmatrix} = \sum_{k=1}^{N} \left(-\frac{p}{2}\right)^{N-k} \begin{pmatrix} C^{(N-k)} \\ \bar{C}^{(N-k)} \end{pmatrix} \tag{A.19}$$

or

$$\begin{pmatrix} C_N \\ \bar{C}_N \end{pmatrix} = \sum_{k=1}^{N} p^{N-k} \mathcal{R}^{k-1} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = (p^{N-1} + p^{N-2}\mathcal{R} + \cdots + \mathcal{R}^{N-1}) \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}. \tag{A.20}$$

To write this expression in a compact form, by analogy with q-calculus it is convenient to introduce notation of the q-number operator

$$1 + q + q^2 + \cdots + q^{N-1} \equiv [N]_q, \tag{A.21}$$

where q is a linear operator. Hence with operator $q \equiv \mathcal{R}/p$ we have the finite Laurent form in the spectral parameter p,

$$1 + \frac{\mathcal{R}}{p} + \left(\frac{\mathcal{R}}{p}\right)^2 + \cdots + \left(\frac{\mathcal{R}}{p}\right)^{N-1} \equiv [N]_{\mathcal{R}/p}. \tag{A.22}$$

Then we have shortly

$$\begin{pmatrix} C_N \\ \bar{C}_N \end{pmatrix} = p^{N-1}[N]_{\mathcal{R}/p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}. \tag{A.23}$$

In a similar way

$$A_N = -\frac{p^N}{2} - \mathrm{i}\kappa^2 \left(\int^x \bar{\psi}, -\int^x \psi\right) \begin{pmatrix} C_N \\ \bar{C}_N \end{pmatrix} \tag{A.24}$$

and using (A.23)

$$A_N = -\frac{p^N}{2} - \mathrm{i}\kappa^2 p^{N-1} \left(\int^x \bar{\psi}, -\int^x \psi\right) [N]_{\mathcal{R}/p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}. \tag{A.25}$$

Equations (A.18), (A.23) and (A.25) give the time part of the linear problem (the Lax representation) for the *N*th flow of NLS hierarchy (A.17).

A.4. General NLS hierarchy equation

For the time t determined by the formal series

$$\partial_t = \sum_{N=0}^{\infty} E_N \partial_{t_N}, \tag{A.26}$$

where E_N are arbitrary constants, the general NLS hierarchy equation is [27]

$$\mathrm{i}\sigma_3 \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}_t = (E_0 + E_1\mathcal{R} + \cdots + E_N\mathcal{R}^N + \cdots) \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}. \tag{A.27}$$

A.4.1. Linear problem. Integrability of this equation is associated with the Zakharov–Shabat problem (A.1) and the time evolution

$$J_0 = \sum_{N=0}^{\infty} E_N J_{0_N} = \begin{pmatrix} \mathrm{i}A & -\kappa^2\bar{C} \\ C & -\mathrm{i}A \end{pmatrix}, \tag{A.28}$$

where

$$\begin{pmatrix} C \\ \bar{C} \end{pmatrix} = \sum_{N=0}^{\infty} E_N \begin{pmatrix} C_N \\ \bar{C}_N \end{pmatrix} = \sum_{N=1}^{\infty} E_N p^{N-1}[N]_{\mathcal{R}/p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}. \tag{A.29}$$

In the last equation we have used that for $N = 0$, $C_0 = 0$. Then we have

$$A = \sum_{N=0}^{\infty} E_N A_N = -\frac{1}{2}\sum_{N=0}^{\infty} E_N p^N - \mathrm{i}\kappa^2 \left(\int^x \bar{\psi}, -\int^x \psi\right) \begin{pmatrix} C \\ \bar{C} \end{pmatrix}. \tag{A.30}$$

A.4.2. Integrable nonlinearization. The above equation (A.27) gives integrable nonlinear extension of a linear Schrödinger equation with general analytic dispersion. Let one considers the classical particle system with the energy–momentum relation

$$E = E(p) = E_0 + E_1 p + E_2 p^2 + \cdots. \tag{A.31}$$

Then the corresponding time-dependent Schrödinger wave equation is

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\psi = H\left(-\mathrm{i}\hbar\frac{\partial}{\partial x}\right)\psi, \tag{A.32}$$

where the Hamiltonian operator results from the standard substitution for momentum $p \to -\mathrm{i}\hbar\frac{\partial}{\partial x}$ in the dispersion (A.31). Equation (A.32) together with its complex conjugate can be rewritten as

$$\mathrm{i}\hbar\sigma_3\frac{\partial}{\partial t}\begin{pmatrix}\psi\\ \bar\psi\end{pmatrix} = H\left(-\mathrm{i}\hbar\sigma_3\frac{\partial}{\partial x}\right)\begin{pmatrix}\psi\\ \bar\psi\end{pmatrix}. \tag{A.33}$$

The momentum operator here is just the recursion operator (49) in the linear approximation $\mathcal{R}_0 = \mathrm{i}\sigma_3\frac{\partial}{\partial x}$. Hence (A.33) can be rewritten as the linear Schrödinger equation with arbitrary analytic dispersion

$$\mathrm{i}\hbar\sigma_3\frac{\partial}{\partial t}\begin{pmatrix}\psi\\ \bar\psi\end{pmatrix} = H(\mathcal{R}_0)\begin{pmatrix}\psi\\ \bar\psi\end{pmatrix} = \left(E_0 + E_1\mathcal{R}_0 + E_2\mathcal{R}_0^2\cdots\right)\begin{pmatrix}\psi\\ \bar\psi\end{pmatrix}. \tag{A.34}$$

Then the nonlinear integrable extension of this equation appears as (A.27), which corresponds to the replacement $\mathcal{R}_0 \to \mathcal{R}$, $(\hbar = 1)$, so that

$$\mathrm{i}\sigma_3\begin{pmatrix}\psi\\ \bar\psi\end{pmatrix}_t = H(\mathcal{R})\begin{pmatrix}\psi\\ \bar\psi\end{pmatrix}. \tag{A.35}$$

From this point of view the standard substitution for classical momentum $p \to -\mathrm{i}\hbar\frac{\partial}{\partial x}$ or equivalently $p \to -\mathrm{i}\hbar\sigma_3\frac{\partial}{\partial x} = \mathcal{R}_0$ for the equation in spinor form, gives quantization in the form of the linear Schrödinger equation. While substitution $p \to \mathcal{R}$ gives 'nonlinear quantization' and the nonlinear Schrödinger hierarchy equation.

A.4.3. The Lax representation. The related Lax representation for equation (A.35) is given by (A.29) and (A.30). Using the definition of the q-derivative

$$D_q^{(\zeta)} f(\zeta) = \frac{f(q\zeta) - f(\zeta)}{(q-1)\zeta} \tag{A.36}$$

for operator $q = \mathcal{R}/p$ we have the relation

$$D_{\mathcal{R}/p}^{(p)} p^N = [N]_{\mathcal{R}/p}\, p^{N-1}. \tag{A.37}$$

Then equation (A.29) can be rewritten as

$$\begin{pmatrix}C\\ \bar C\end{pmatrix} = \sum_{N=1}^{\infty} E_N p^{N-1}[N]_{\mathcal{R}/p}\begin{pmatrix}\psi\\ \bar\psi\end{pmatrix} = \sum_{N=1}^{\infty} E_N D_{\mathcal{R}/p}^{(p)} p^N\begin{pmatrix}\psi\\ \bar\psi\end{pmatrix} \tag{A.38}$$

or using linearity of (A.36) and dispersion (A.31)

$$\begin{pmatrix}C\\ \bar C\end{pmatrix} = D_{\mathcal{R}/p}^{(p)}\sum_{N=0}^{\infty} E_N p^N\begin{pmatrix}\psi\\ \bar\psi\end{pmatrix} = D_{\mathcal{R}/p}^{(p)} E(p)\begin{pmatrix}\psi\\ \bar\psi\end{pmatrix}. \tag{A.39}$$

Using definition (A.36) it gives simple formula

$$\begin{pmatrix} C \\ \bar{C} \end{pmatrix} = \frac{E(\mathcal{R}) - E(p)}{\mathcal{R} - p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}, \tag{A.40}$$

where

$$\frac{E(\mathcal{R}) - E(p)}{\mathcal{R} - p} = E_1 + E_2(\mathcal{R} + p) + E_3(\mathcal{R}^2 + \mathcal{R}p + p^2) + \cdots. \tag{A.41}$$

Then for A we obtain

$$A = -\frac{1}{2}E(p) - \mathrm{i}\kappa^2 \left(\int^x \bar{\psi}, -\int^x \psi \right) \frac{E(\mathcal{R}) - E(p)}{\mathcal{R} - p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}. \tag{A.42}$$

Equations (A.40) and (A.42) give the Lax representation of the general integrable NLS hierarchy model (A.35). It is worth noting here that special form of the dispersion $E = E(p)$ is fixed by physical problem. In section 6 we discuss the relativistic form of this dispersion and corresponding semi-relativistic NLS equation.

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