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RINGS WHOSE MODULES ARE WEAKLY SUPPLEMENTED ARE PERFECT. APPLICATIONS TO CERTAIN RING EXTENSIONS

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Abstract

In this note we show that a ring R is left perfect if and only if every left R-module is weakly supplemented if and only if R is semilocal and the radical of the countably infinite free left R-module has a weak supplement.

H. Bass characterized in [1] those ring R whose left R-modules have projective covers and termed them *left perfect rings*. He characterized them as those semilocal rings which have a left t-nilpotent Jacobson radical $\mathrm{Jac}(R)$. Bass' *semiperfect rings* are those whose finitely generated left (or right) R-modules have projective covers and can be characterized as those semilocal rings which have the property that idempotents lift modulo $\mathrm{Jac}(R)$. Kasch and Mares transferred in [5] the notions of perfect and semiperfect rings to modules and characterized semiperfect modules by a lattice-theoretical condition as follows: a module M is called *supplemented* if for any submodule N of M there exists a submodule L of M minimal with respect to $M = N + L$. The left perfect rings are then shown to be exactly those rings whose left R-modules are supplemented while the semiperfect rings are those whose finitely generated left R-modules are supplemented. Equivalently it is enough for a ring R to be semiperfect if the left (or right) R-module R is supplemented. Recall that a submodule N of a module M is called *small*, denoted by $N \ll M$, if $N + L \neq M$ for all proper submodules L of M. Weakening the "supplemented" condition, one calls a module *weakly supplemented* if for every submodule N

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of M there exists a submodule L of M with $N + L = M$ and $N \cap L \ll M$. In this case L is called a *weak supplement* of N in M. The semilocal rings R are precisely those rings whose finitely generated left (or right) R-modules are weakly supplemented. Again it is enough that R is weakly supplemented as left (or right) R-module. Semilocal rings which are not semiperfect are examples of weakly supplemented modules which are not supplemented. In this note we prove that if R is semilocal and the radical of the countably infinite free left R-module has a weak supplement, then R has to be left perfect, i.e. every left R-module is supplemented.

Throughout this note all rings are associative with unit and modules are considered to be unital. An ideal I of a ring R is called *left t-nilpotent* if for any family $\{a_i\}_{i\in\mathsf{N}}$ of elements of I there exists $n > 0$ such that $a_1a_2\ldots a_n = 0$. A ring R is left perfect if and only if it is semilocal and $\mathrm{Jac}(R)$ is left t-nilpotent. Recall that an infinite family $\{A_\lambda \mid \lambda \in \Lambda\}$ of left ideals of R is called *left vanishing* if given any sequence $a_1, a_2, \ldots,$ with $a_i \in A_{\lambda_i}$ and $\lambda_i \neq \lambda_j$ for all $i \neq j$, there exists a number $n \geq 1$ for which $a_1a_2a_3\ldots a_n = 0$. It follows from Ware and Zelmanowitz [9, Theorem 1] that, if F is a free left R-module with $F = R^{(\Lambda)} := \bigoplus_{\lambda\in\Lambda} R_\lambda$ and $f \in \mathrm{Jac}(\mathrm{End}(F))$, then the family $\{\pi_\lambda(\mathrm{Im}(f))\}_{\lambda\in\Lambda}$ of left ideals of R is left vanishing. (Here, for each λ in the index set Λ, $R_\lambda = R$ and $\pi_\lambda : F \to R_\lambda$ is the natural projection, while $\mathrm{End}(F)$ is the endomorphism ring of F.) Using this result we can prove our main result:

THEOREM 1. *The following statements are equivalent for a ring R:*
(a) *Every left R-module is weakly supplemented;*
(b) *$R^{(\mathsf{N})}$ is weakly supplemented as left R-module;*
(c) *R is semilocal and* $\mathrm{Rad}({}_RR^{(\mathsf{N})})$ *has a weak supplement in* ${}_RR^{(\mathsf{N})}$*;*
(d) *R is left perfect.*

PROOF. (d) $\Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow$ (c) is clear and we just need to show (c) $\Rightarrow$ (d). Set $F = R^{(\mathsf{N})}$ and denote $J = \mathrm{Jac}(R)$. Suppose that R is semilocal, then $JF = \mathrm{Rad}(F)$ by [6, Proposition 2.24]. Let L be a weak supplement of JF in F, i.e. $JF + L = F$ and $JF \cap L \ll F$. Then, for any $i \in \mathsf{N}$, taking $\pi_i : F \to R$ to be the projection map, we have $R = \pi_i(JF + L) = J + \pi_i(L) = \pi_i(L)$ and so there exists $x_i \in L$ such that $\pi_i(x_i) = 1$. Let $\{a_i\}_{i\in\mathsf{N}}$ be any family of elements of J. Then $a_ix_i \in JL \subseteq JF \cap L \ll F$ and $\pi_i(a_ix_i) = a_i$ for any $i \in \mathsf{N}$. Define $f \in \mathrm{End}(F)$ by $f(z) = \sum_{i\in\mathsf{N}} z_ia_ix_i$ for all $z = (z_i)_{i\in\mathsf{N}}$. Since $\mathrm{Im}(f) \ll F$, it follows from Ware and Zelmanowitz [9, Lemma 1] that $f \in \mathrm{Jac}(\mathrm{End}(F))$ and so, by [9, Theorem 1], that $\{\pi_i(\mathrm{Im}(f))\}_{i\in\mathsf{N}}$ is left vanishing. Thus there exists $n > 0$ such that

$$a_1a_2\ldots a_n = \pi_1(a_1x_1)\pi_2(a_2x_2)\ldots\pi_n(a_nx_n) = 0.$$

This shows that $\mathrm{Jac}(R)$ is left t-nilpotent and hence R is left perfect.

Let $\sigma[M]$ denote the Wisbauer category of a module M, i.e. the full subcategory of R-Mod consisting of submodules of quotients of direct sums of copies of M. A module M is called a self-generator if any of its submodules is an image of a direct sum of copies of M.

COROLLARY 2. *Let M be a finitely generated, self-projective, self-generator. Then every module in $\sigma[M]$ is weakly supplemented if and only if* $\mathrm{End}(M)$ *is left perfect.*

PROOF. By [10, 18.3] M is projective in $\sigma[M]$ and by [10, 18.5] M is a generator in $\sigma[M]$. Hence by [10, 46.2] the functor $\mathrm{Hom}(M, -)$ is a Morita equivalence between $\sigma[M]$ and $\mathrm{End}(M)$-Mod. Thus every module in $\sigma[M]$ is weakly supplemented if and only if every left $\mathrm{End}(M)$-module is weakly supplemented, which holds if and only if $\mathrm{End}(M)$ is left perfect by the Theorem.

Recall that a left R-module M is called *semi-projective* if for any endomorphism $f \in S = \mathrm{End}(M)$ we have $Sf = \mathrm{Hom}(M, \mathrm{Im}(f))$. The module M is called *π-projective* if for any submodules N, L of M with $M = N + L$ we have $S = \mathrm{Hom}(M, N) + \mathrm{Hom}(M, L)$ (see, [2]).

PROPOSITION 3. *Suppose M is a semi-projective and π-projective R-module. Then $S/\mathrm{Jac}(S)$ is regular if and only if $\mathrm{Im}(f)$ has a weak supplement in M for each $f \in S$.*

PROOF. ($\Rightarrow$) Let $f \in S$. By hypothesis there is a $g \in S$ such that $f - fgf \in J(S)$. We have $\mathrm{Im}(f) + \mathrm{Im}(1 - fg) = M$. It is easy to see that $\mathrm{Im}(f) \cap \mathrm{Im}(1 - fg) \subseteq \mathrm{Im}(f - fgf)$, but since $f - fgf \in \mathrm{Jac}(S)$ we have $\mathrm{Im}(f - fgf) \ll M$ by [2, 4.28(3)]. Hence $\mathrm{Im}(1 - fg)$ is a weak supplement of $\mathrm{Im}(f)$ in M.

($\Leftarrow$) Let $f \in S$ and K be a weak supplement of $\mathrm{Im}(f)$ in M. Since M is semi-projective and π-projective we have $S = \mathrm{Hom}(M, \mathrm{Im}(f)) + \mathrm{Hom}(M, K) = Sf + \mathrm{Hom}(M, K)$. Since $Sf \cap \mathrm{Hom}(M, K) = \mathrm{Hom}(M, \mathrm{Im}(f) \cap K)$ and $\mathrm{Im}(f) \cap K \ll M$, we get $Sf \cap \mathrm{Hom}(M, K) \subseteq \mathrm{Jac}(S)$. Thus Sf has a weak supplement for all f, which implies that $S/\mathrm{Jac}(S)$ is von Neumann regular by [7, 3.18].

The last proposition generalizes [7, 3.18]. Also as a consequence we conclude that the endomorphism ring of a semi-projective, π-projective weakly supplemented module is regular modulo its Jacobson radical.

1. Applications to certain ring extensions

We exploit the existence of a Morita equivalence in the context of various algebraic structures to apply our theorem.

1.1. Azumaya Algebras

Let k be a commutative ring and A be a central k-algebra, i.e. $Z(A) = k$, and $A^e = A \otimes_k A^{op}$ be its enveloping algebra. Recall that A is called an *Azumaya algebra* if the multiplication map $A^e \to A$ splits as A-bimodule map.

COROLLARY 4. *Let A be an Azumaya algebra. Then any A-bimodule is weakly supplemented (as A-bimodule) if and only if k is a perfect ring. In this case A is a left and right perfect ring.*

PROOF. By [11, 28.1] A is a projective generator in the category of A-bimodules, which is the module category over A^e. A is also finitely generated and projective over k and $\mathrm{Hom}(_{,}A^e)(A, -)$ gives an equivalence between A^e-Mod and k-Mod. Hence by Corollary 2 every A-bimodule is weakly supplemented if and only if $\mathrm{End}_{A^e}(A) \simeq Z(A) = k$ is perfect. Note that the radical $\mathrm{Rad}(_{A^e}A)$ of A as A-bimodule is the intersection of maximal ideals of A, its Brown-McCoy radical. Since any Azumaya algebra is a PI-algebra any primitive ideal is also maximal, i.e. $\mathrm{Jac}(A) = \mathrm{Rad}(_{A^e}A)$. If furthermore A is a weakly supplemented A-bimodule, then $A/\mathrm{Rad}(_{A^e}A) = A/\mathrm{Jac}(A)$ is a semisimple artinian A-bimodule and hence semisimple artinian since A is PI. Thus A is a semilocal ring. On the other hand, since A is a projective A^e-module, it is a direct summand of A^e as A-bimodule. Thus $\mathrm{Rad}(_{A^e}A) \subseteq \mathrm{Jac}(A^e)$. Since k and A^e are Morita equivalent, A^e is left and right perfect and hence $\mathrm{Jac}(A^e)$ and therefore also $\mathrm{Rad}(_{A^e}A) = \mathrm{Jac}(A)$ is left and right t-nilpotent. Hence A is left and right t-nilpotent, i.e. left and right perfect.

1.2. Graded modules

Let k be a commutative ring and G a group. A G-graded k-algebra A is an algebra over k with decomposition $A = \bigoplus_{g \in G} A_g$ into additive subgroups such that $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$. Note that A_e, with e the neutral element of G, is a subring of A. A left A-module M is called G-graded, if $M = \bigoplus_{g \in G} M_g$ and $A_g M_h \subseteq M_{gh}$. Since the partially ordered set of graded submodules of a graded module is a modular lattice, it makes sense to talk about weakly supplemented graded modules.

A G-graded algebra A is called *strongly graded* if $A_g A_h = A_{gh}$:

COROLLARY 5. *Let G be a finite group and A a strongly G-graded algebra. Then every G-graded left A-module is weakly supplemented if and only if A_e is a left perfect ring.*

PROOF. If A is strongly graded, then the category of G-graded left A-modules is Morita equivalent to the category of right A_e-modules by [3, 2.12 and 2.2]. Hence all graded A-modules are weakly supplemented iff all right A_e-modules are weakly supplemented iff A_e is right perfect by Corollary 2.

1.3. Hopf-Galois extensions

The reader is referred to S. Montgomery's book [8] for all Hopf-theoretical notions. Let H be a k-Hopf algebra and A a right H-comodule algebra, i.e. an algebra in the category of right H-comodules. Let $\rho : A \to A \otimes H$ denote the coaction of H on A. The subring of coinvariants is $B = A^{coH} = \{a \in A \mid \rho(a) = a \otimes 1\}$. The extension $B \subseteq A$ is called an H-Hopf-Galois extension if the following map is an isomorphism:

$$\beta : A \otimes_B A \to A \otimes_k H \qquad \text{with} \quad \beta(a \otimes a') = a\rho(a')$$

A theorem of Ulbrich says that a G-graded k-algebra A is strongly graded if and only if $A_e \subseteq A$ is a $k[G]$-Hopf Galois extension (see [8, Chapter 8]). If k is a field and H finite dimensional then [8, 8.3.3] says that $B \subseteq A$ is H-Galois if and only if the category of (A, H)-bimodules, i.e. the category of left A-modules which are also right H-comodules, is Morita equivalent to the category of right B-modules. Under these conditions, A is a progenerator in the category of (A, H)-bimodules whose endomorphism ring is isomorphic to B. Applying again Corollary 2 we have:

COROLLARY 6. *Let $B \subseteq A$ be an H-Hopf-Galois extension with H a finite dimensional Hopf algebra over a field k. Then any (A, H)-bimodule is weakly supplemented if and only if B is right perfect.*

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