ON A RECENT GENERALIZATION OF SEMIPERFECT RINGS

ENGIN BÜYÜKAŞIK AND CHRISTIAN LOMP

ABSTRACT. It follows from a recent paper by Ding and Wang that any ring which is generalized supplemented as left module over itself is semiperfect. The purpose of this note is to show that Ding and Wang's claim is not true and that the class of generalized supplemented rings lies properly between the class of semilocal and semiperfect rings. Moreover we rectify their "theorem" by introducing a wider notion of local submodules.

1. INTRODUCTION

H. Bass characterized in [3] those rings R whose left R-modules have projective covers and termed them *left perfect rings*. He characterized them as those semilocal rings which have a left t-nilpotent Jacobson radical $\mathrm{Jac}(R)$. Bass's *semiperfect rings* are those whose finitely generated left (or right) R-modules have projective covers and can be characterized as those semilocal rings which have the property that idempotents lift modulo $\mathrm{Jac}(R)$.

Kasch and Mares transferred in [9] the notions of perfect and semiperfect rings to modules and characterized semiperfect modules by a lattice-theoretical condition as follows. A module M is called *supplemented* if for any submodule N of M there exists a submodule L of M minimal with respect to $M = N + L$. The left perfect rings are then shown to be exactly those rings whose left R-modules are supplemented while the semiperfect rings are those whose finitely generated left R-modules are supplemented. Equivalently it is enough for a ring R to be semiperfect if the left (or right) R-module R is supplemented. Recall that a submodule N of a module M is called *small*, denoted by $N \ll M$, if $N + L \neq M$ for all proper submodules L of M. Weakening the "supplemented" condition one calls a module *weakly supplemented* if for every submodule N of M there exists a submodule L of M with $N + L = M$ and $N \cap L \ll M$. The semilocal rings R are precisely those rings whose finitely generated left (or right) R-modules are weakly supplemented. Again it is enough that R is weakly supplemented as left (or right) R-module. There are many semilocal rings which are not semiperfect, e.g. the localization of the integers at two distinct primes.

Recently another notion of "supplement" submodule has emerged called *Rad-supplement*. A submodule N of a module M has a *generalized supplement* or *Rad-supplement* L in M if $N + L = M$ and $N \cap L \subseteq \mathrm{Rad}(L)$ (see [11]). Here $\mathrm{Rad}(L)$ denotes the *radical* of L, namely the intersection of all maximal submodules of L or, if L has no such submodules, simply L itself. If every submodule of M has a Rad-supplement, then M is called *Rad-supplemented*. Note that Rad-supplements L of M are also called *coneat submodules* in the literature and can be characterized by the fact that any module with zero radical is injective with respect to the inclusion $L \subseteq M$ (see [5, 10.14] or [2]). Recall that a submodule N of a module M is called *cofinite* if M/N is finitely generated. In our terminology, Ding and Wang claimed that the following is true:

Theorem (Ding-Wang [11, Theorem 2.11]). For any left R-module M the following conditions are equivalent:

(a) every submodule of M is contained in a maximal submodule and every cofinite submodule of M has a Rad-supplement in M.

(b) M has a small radical and can be written as an irredundant sum of local submodules.

Here a module L is called *local* if $L/\mathrm{Rad}(L)$ is simple and $\mathrm{Rad}(L)$ is small in L.

Since it is well-known that finite sums of supplemented modules are supplemented (see [13, 41.2]), Ding and Wang's claim implies that a finitely generated module M is supplemented if and only if it is Rad-supplemented; in particular for $M = R$, R is semiperfect if and only if R is Rad-supplemented as left (or right) R-module. The purpose of this note is to show that Ding and Wang's claim is not true and that the class of Rad-supplemented rings lies properly between the class of semilocal and semiperfect rings. Moreover we rectify the above theorem by introducing a different "local" condition for modules.

Throughout the paper, R is an associative ring with identity and all modules are unital left R-modules.

2. Examples

It is clear from the introduction that one has the following implications of conditions on submodules of a module with a small radical:

$$supplements \Rightarrow Rad\text{-}supplements \Rightarrow weak\ supplements,$$

and this implies that any semiperfect ring is Rad-supplemented and any Rad-supplemented ring is semilocal.

2.1. **Semilocal rings which are not Rad-supplemented.** Since finitely generated modules have small radical it is clear that finitely generated Rad-supplements are supplements. Thus any left noetherian, left Rad-supplemented ring is semiperfect and any left noetherian semilocal non-semiperfect ring would give an example of a semilocal ring which is not Rad-supplemented. It is well-known that the localization of the integers by two distinct primes p and q, namely

$$R = \left\{ \frac{a}{b} \in \mathbb{Q} \mid b \text{ is neither divisible by } p \text{ nor by } q \right\},$$

is a semilocal noetherian ring which is not semiperfect and hence gives such example.

2.2. **Rad-supplemented rings which are not semiperfect.**
While a noetherian ring is Rad-supplemented if and only it is semiperfect, we now show that a ring with idempotent Jacobson radical is semilocal if and only if it is Rad-supplemented.

Proposition 2.1. *Let R be a ring with idempotent Jacobson radical J. Then R is semilocal if and only if R is Rad-supplemented as left or right R-module.*

Proof. Let R be semilocal with idempotent Jacobson radical. Let I be a left ideal I of R. If $I \subseteq J$, then I is small and R is a Rad-supplement of I. Hence suppose that $I \not\subseteq J$. Then $(I + J)/J$ is a direct summand in the semisimple ring R/J and there exists a left ideal K of R containing J such that $I + K = R$ and $I \cap K \subseteq J$. Since $J \subseteq K$ and J is idempotent, we have $J = J^2 \subseteq JK = \mathrm{Rad}(K)$. Thus K is a Rad-supplement of I in R.
The converse is clear. $\qquad\square$

Hence any indecomposable non-local semilocal ring with idempotent Jacobson radical is an example of a Rad-supplemented ring which is not semiperfect. To construct such a ring we will look at the endomorphism rings of uniserial modules. Recall that a unital ring R is called a *nearly simple uniserial domain* if it is uniserial as left and right R-module, i.e. its lattices of left, resp. right, ideals are linearly ordered, such that $\mathrm{Jac}(R)$ is the unique nonzero two-sided ideal of R. Let R be any nearly simple uniserial domain and $r \in \mathrm{Jac}(R)$ and let $S = \mathrm{End}(R/rR)$ be the endomorphism ring of the uniserial cyclic right R-module R/rR. Then S is semilocal by a result of Herbera and Shamsuddin (see [7, 4.16]). Moreover Puninski showed in [10, 6.7] that S has exactly three nonzero proper two-sided ideals, namely the two maximal ideals

$$I = \{f \in S \mid f \text{ is not injective}\} \quad \text{and} \quad K = \{g \in S \mid g \text{ is not surjective}\},$$

and the Jacobson radical $\mathrm{Jac}(S) = I \cap K$ which is idempotent. Note that since R/rR is indecomposable, S also has no non-trivial idempotents. Moreover S is a prime ring. Hence S is an indecomposable, non-local semilocal ring with idempotent Jacobson radical.

Our example depends on the existence of nearly simple uniserial domains. Such rings were first constructed by Dubrovin in 1980 (see [6]), as follows. Let

$$G = \{f : \mathbb{Q} \to \mathbb{Q} \mid f(t) = at + b \text{ for } a, b \in \mathbb{Q} \text{ and } a > 0\}$$

be the group of affine linear functions on the field of rational numbers $\mathbb{Q}$. Choose any irrational number $\epsilon \in \mathbb{R}$ and set

$$P = \{f \in G \mid \epsilon \leq f(\epsilon)\} \quad \text{and} \quad P^+ = \{f \in G \mid \epsilon < f(\epsilon)\}.$$

Note that P, resp. P^+, defines a left order on G. Take an arbitrary field F and consider the semigroup group ring $F[P]$ in which the right ideal $M = \sum_{g \in P^+} gF[P]$ is maximal. The set $F[P] \setminus M$ is a left and right Ore set and the corresponding localization R is a nearly simple uniserial domain (see [4, 6.5]). Thus taking any nonzero element $r \in R$, $S = \mathrm{End}(R/rR)$ is a Rad-supplemented ring which is not semiperfect.

3. Cofinitely Rad-supplemented modules

We say that the module M is *cofinitely Rad-supplemented* if any cofinite submodule of M has a Rad-supplement. Note that every submodule of a finitely generated module is cofinite, and hence a finitely generated module is Rad-supplemented if and only if it is cofinitely Rad-supplemented. Ding and Wang's theorem tried to describe cofinitely Rad-supplemented modules with small radical as sums of local modules. In order to correct their theorem we introduce a different module-theoretic local condition.

3.1. **w-local modules.** We say that a module is *w-local* if it has a unique maximal submodule. It is clear that a module is w-local if and only if its radical is maximal. The question whether any projective w-local module must be local was first studied by R. Ware in [12, p. 250]. There he proved that if R is commutative or left noetherian or if idempotents lift modulo $\mathrm{Jac}(R)$, then any projective w-local module must be local (see [12, 4.9–11]). The first example of a w-local non-local projective module was given by Gerasimov and Sakhaev in [8].

In general, unless the module has small radical, a w-local module need not be local as we now show by taking the abelian group $M = \mathbb{Q} \oplus \mathbb{Z}/p\mathbb{Z}$ for any prime p. Clearly $J = \mathbb{Q} \oplus 0$ is maximal in M. We show that $J = \mathrm{Rad}(M)$. Indeed, if N is another maximal submodule of M, then $M = N + J$ and thus $J/(N \cap J) \simeq M/N$ is simple, which is impossible since $\mathbb{Q}$ has no simple factors. Thus $J = \mathbb{Q} \oplus 0$ is the unique maximal submodule of (the non-local module) M.

The same construction is possible for any ring R with a nonzero left R-module Q with $\mathrm{Rad}(Q) = Q$. Then taking any simple left R-module E one concludes that $Q \oplus E$ is w-local module which is not local. In other words, if every w-local module over a ring R is local, then $\mathrm{Rad}(Q) \neq Q$ for all non-zero left R-modules Q, i.e. R is a *left max ring*. Conversely, any left module over a left max ring has a small radical, hence any w-local module is local. We have just proved:

Lemma 3.1. *A ring R is a left max ring if and only if every w-local left R-module is local.*

3.2. **Rad-supplements of maximal submodules.** While local (or hollow) modules are supplemented, w-local modules might not be.

Lemma 3.2. *Every w-local module M is cofinitely Rad-supplemented.*

Proof. Let U be a cofinite submodule of M. Since M/U is finitely generated, U is contained in a maximal submodule of M and so $U \subseteq \mathrm{Rad}\,M$. Now M is a Rad-supplement of U in M, because $U + M = M$ and $U \cap M = U \subseteq \mathrm{Rad}\,M$. $\qquad\square$

The gap in Ding and Wang's theorem is that Rad-supplements of maximal submodules need not be local. However they are always w-local, as we now prove.

Lemma 3.3. *Any Rad-supplement of a maximal submodule is w-local.*

Proof. Let K be a maximal submodule of a module M and L be a Rad-supplement of K in M. Then $K + L = M$ and $K \cap L \subseteq \mathrm{Rad}\,L$. Since $M/K \cong L/K \cap L$ is simple, $\mathrm{Rad}\,L$ is a maximal submodule of L. Thus L is a *w*-local module. $\qquad\square$

Let R be again a nearly simple uniserial domain, $0 \neq r \in \mathrm{Jac}(R)$ and $S = \mathrm{End}(R/rR)$. Then S has two maximal two-sided ideals I and K such that $I \cap K = J = J^2$ where $J = \mathrm{Jac}(S)$. Since $J \subset K$ we have $J = J^2 \subseteq JK \subseteq J$. Thus $I \cap K = J = JK = \mathrm{Rad}(K)$. Analogously one shows $I \cap K = \mathrm{Rad}(I)$. Hence I and K are mutual Rad-supplements. By [10, p. 239] K is a cyclic uniserial left ideal, while the left ideal I is not finitely generated. Hence K is a supplement of I. Moreover K is a supplement of any maximal left ideal M of S containing I, because $K + M \supseteq K + I = S$ and $K \cap M \subseteq K$ is small in K since K is uniserial. Hence K is a local submodule. On the other hand I is not a supplement of K since otherwise K and I would be mutual supplements and hence direct summands which is impossible.

3.3. Closure properties of cofinitely Rad-supplemented modules. We establish some general closure properties of cofinitely Rad-supplemented modules. To begin with, we prove the following.

Lemma 3.4. *Let $N \subseteq M$ be modules such that N is cofinitely Rad-supplemented. If U is a cofinite submodule of M such that $U + N$ has a Rad-supplement in M, then U also has a Rad-supplement in M.*

Proof. Let K be a Rad-supplement of $U + N$ in M. Since $N/[N \cap (U + K)] \cong M/(U + K)$ is finitely generated, $N \cap (U + K)$ is a cofinite submodule of N. Let L be a Rad-supplement of $N \cap (U + K)$ in N, i.e. $L + (N \cap (U + K)) = N$ and $L \cap (U + K) \subseteq \mathrm{Rad}\, L$. Then we have

$$M = U + N + K = U + K + L + (N \cap (U + K)) = U + K + L$$

and

$$U \cap (K + L) \subseteq (K \cap (U + L)) + (L \cap (U + K)) \subseteq \mathrm{Rad}\, K + \mathrm{Rad}\, L \subseteq \mathrm{Rad}(K + L).$$

Hence $K + L$ is a Rad-supplement of U in M. $\qquad\square$

With this last Lemma at hand we can now prove:

Theorem 3.5. *The class of cofinitely Rad-supplemented modules is closed under arbitrary direct sums and homomorphic images.*

Proof. Let $\mathcal{C}$ denote the class of cofinitely Rad-supplemented left R-modules over a fixed ring R. We will first show that $\mathcal{C}$ is closed under factor modules. First of all, it is clear that $\mathcal{C}$ is closed under isomorphisms, since being Rad-supplemented is a lattice-theoretical notion. Take any $M \in \mathcal{C}$ and $f : M \to X$, where X is a left R-module. We may assume that $f(M)$ is of the form M/N for some submodule N of M. Let K/N be a cofinite submodule of M/N. Then K is a cofinite submodule of M, so that K has a Rad-supplement L in M. That is, $K + L = M$ and $K \cap L \subseteq \mathrm{Rad}(L)$. Then we have $K/N + (L + N)/N = M/N$ and

$$K \cap (L + N)/N = (K \cap L + N)/N \subseteq (\mathrm{Rad}\, L + N)/N \subseteq \mathrm{Rad}((L + N)/N).$$

Therefore $(L + N)/N$ is a Rad-supplement of K/N in M/N. Hence $M/N \in \mathcal{C}$.

Let $M = \bigoplus_{i \in I} M_i$, with $M_i \in \mathcal{C}$ and U a cofinite submodule of M. Then there is a finite subset $F = \{i_1, i_2, \ldots, i_k\} \subseteq I$ such that $M = U + \left(\bigoplus_{n=1}^{k} M_{i_n} \right)$. Since $(U + \bigoplus_{n=2}^{k} M_{i_n})$ is

cofinite and $M = M_{i_1} + (U + \bigoplus_{n=2}^{k} M_{i_n})$ has trivially a Rad-supplement in M, by Lemma 3.4 $U + \bigoplus_{n=2}^{k} M_{i_n}$ has a Rad-supplement in M. By repeated use of Lemma 3.4 $k-1$ times we get a Rad-supplement for U in M. Hence M is a cofinitely Rad-supplemented. $\square$

It follows from Theorem 3.5 that a ring R is left Rad-supplemented if and only if every left R-module is cofinitely Rad-supplemented.

Now we give an example showing that there are cofinitely Rad-supplemented modules, which are not Rad-supplemented. Let $N \subseteq M$ be left R-modules. If M/N has no maximal submodule, then any proper Rad-supplement L of N in M also has no maximal submodules, i.e. $L = \mathrm{Rad}(L)$, because if L is a Rad-supplement of N in M and K a maximal submodule of L, then

$$M/(N+K) \simeq L/((N \cap L) + K) = L/K \neq 0$$

is a nonzero simple module, a contradiction.

Next let R be a discrete valuation ring with quotient field K. Then $F/T \cong K = \mathrm{Rad}(K)$ as R-modules, for a free R-module F and $T \subseteq F$. Since R is hereditary, any submodule of F is projective and has a proper radical; thus T cannot have a Rad-supplement in F by the preceding remark, i.e. F is not Rad-supplemented. On the other hand, since R is local and hence Rad-supplemented, F is cofinitely Rad-supplemented by Theorem 3.5.

Call a module M *radical-full* if $M = \mathrm{Rad}(M)$. Obviously radical-full modules are Rad-supplemented.

Lemma 3.6. *Any extension of a radical-full module by a cofinitely Rad-supplemented module is cofinitely Rad-supplemented.*

Proof. Let N be a cofinitely Rad-supplemented submodule of M such that M/N is radical-full. For any cofinite $U \subseteq M$, $(U+N)/N$ is a cofinite submodule of M/N and hence $U + N = M$. Hence, by Lemma 3.4, U has a Rad-supplement in M. $\square$

3.4. **Characterization of Rad-supplemented modules with small radical.** Recall that Ding and Wang's theorem expresses a cofinitely Rad-supplemented module with small radical as an irredundant sum of local modules, which is not possible in general as shown above. Here we show that any cofinitely Rad-supplemented module can be written as the sum of w-local modules instead. The following is a cofinitely Rad-supplemented analogue of a result in [1] by Alizade, Bilhan and Smith for cofinitely supplemented modules.

Theorem 3.7. *The following statements are equivalent for a module M.*

(a) *M is cofinitely Rad-supplemented;*
(b) *every maximal submodule of M has a Rad-supplement in M;*
(c) *$M/wLoc(M)$ has no maximal submodules, where $wLoc(M)$ is the sum of all w-local submodules of M;*
(d) *$M/cgs(M)$ has no maximal submodules, where $cgs(M)$ is the sum of all cofinitely Rad-supplemented submodules of M.*

Proof. $(a) \Rightarrow (b)$ Clear.

$(b) \Rightarrow (c)$ Suppose $wLoc(M) \subseteq K$ for some maximal submodule K of M. Let N be a Rad-supplement of K in M, i.e. $K + N = M$ and $K \cap N \subseteq \operatorname{Rad} N$. Then, by Lemma 3.3, N is a w-local submodule of M and so $N \subseteq wLoc(M) \subseteq K$, a contradiction. Thus $M/wLoc(M)$ has no maximal submodules.

$(c) \Rightarrow (d)$ Suppose K is a maximal submodule of M with $cgs(M) \subseteq K$. Since $M/wLoc(M)$ has no maximal submodules, we have $K + wLoc(M) = M$. So $K + L = M$ for some w-local submodule L of M. By Lemma 3.2, L is cofinitely Rad-supplemented and so $L \subseteq cgs(M) \subseteq K$, a contradiction. Hence $M/cgs(M)$ has no maximal submodules.

$(d) \Rightarrow (a)$ By Theorem 3.5, $cgs(M)$ is cofinitely Rad-supplemented and so, by Lemma 3.6, M is cofinitely Rad-supplemented. $\square$

For finitely generated modules M, e.g. $M = R$, we can rephrase the theorem as follows:

Corollary 3.8. *The following statements are equivalent for a finitely generated module M.*

   (a) *M is Rad-supplemented;*
   (b) *every maximal submodule of M has a Rad-supplement in M;*
   (c) *M is the sum of finitely many w-local submodules.*

4. Concluding remarks

While a ring is semiperfect (i.e. supplemented) if and only if it is a sum of local submodules, we showed that a ring is left Rad-supplemented if and only if it is a sum of w-local submodules. Moreover the class of Rad-supplemented rings lies strictly between the semilocal and the semiperfect rings. In particular any example of a left Rad-supplemented ring which is not semiperfect, must contain a w-local left ideal which is not local (cyclic).

References

[1] R. Alizade, G. Bilhan, and P. F. Smith, *Modules whose maximal submodules have supplements*, Comm. Algebra **29** (2001), 2389–2405.

[2] K. Al-Takhman, C. Lomp, and R. Wisbauer, *τ-complemented and τ-supplemented modules*, Algebra Discrete Math. **3** (2006), 1–15.

[3] H. Bass, *Finitistic dimension and a homological generalization of semiprimary rings.*, Trans. Amer. Math. Soc. **95** (1960), 466–488.

[4] C. Bessenrodt, H. H. Brungs, and G.Törner, *Right chain rings. Part 1*, Schriftenreihe des Fachbereiches Mathematik, Universität Duisburg (1990), no. 181.

[5] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, *Lifting Modules. Supplements and Projectivity in Module Theory*, Frontiers in Mathematics, Birkhäuser, Basel, 2006.

[6] N.I. Dubrovin, *Chain domains.*, Mosc. Univ. Math. Bull. **35** (1980), no. 2, 56-60.

[7] A.Facchini, *Module Theory. Endomorphism rings and direct sum decompositions in some classes of modules*, Progress in Mathematics, Birkhäuser, Basel, 1998.

[8] V. N. Gerasimov and I. I. Sakhaev, *A counterexample to two conjectures on projective and flat modules*, (Russian), Sibirsk. Mat. Zh. **25** (1984), 31–35.

[9] F. Kasch and E. A. Mares, *Eine Kennzeichnung semi-perfekter Moduln*, Nagoya Math. J. **27** (1966), 525–529.

[10] G. Puninski, *Projective modules over the endomorphism ring of a biuniform module*, J. Pure Appl. Algebra **188** (2004), 227–246.

[11] Y. Wang and N. Ding, *Generalized supplemented modules.*, Taiwanese J. Math **10** (2006), 1589–1601.

[12] R. Ware, *Endomorphism rings of projective modules*, Trans. Amer. Math. Soc. **155** (1971), 233–256.
[13] R. Wisbauer, *Foundations of Modules and Rings*, Gordon and Breach, 1991.

Izmir Institute of Technology, Department of Mathematics, 35430, Urla, Izmir, Turkey
  E-mail address: enginbuyukasik@iyte.edu.tr

Departamento de Matemâtica Pura da Faculdade de Ciências da Universidade do Porto, R.Campo Alegre 687, 4169-007 Porto, Portugal
  E-mail address: clomp@fc.up.pt