IOP PUBLISHING — JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 452001 (9pp) doi:10.1088/1751-8113/41/45/452001

FAST TRACK COMMUNICATION

Soliton resonances in a generalized nonlinear Schrödinger equation

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Received 19 August 2008
Published 14 October 2008
Online at stacks.iop.org/JPhysA/41/452001

Abstract

It is shown that a generalized nonlinear Schrödinger equation proposed by Malomed and Stenflo admits, for a specific range of parameters, resonant soliton interaction. The equation is transformed to the 'resonant' nonlinear Schrödinger equation, as originally introduced to describe black holes in a Madelung fluid and recently derived in the context of uniaxial wave propagation in a cold collisionless plasma. A Hirota bilinear representation is obtained and soliton solutions are thereby derived. The one-soliton solution interpretation in terms of a black hole in two-dimensional spacetime is given. For the two-soliton solution, resonant interactions of several kinds are found. The addition of a quantum potential term is considered and the reduction is obtained to the resonant NLS equation.

PACS numbers: 02.30.Ik, 02.30.Jr, 04.60.Kz, 05.45.−a, 42.81.Dp, 52.35.Sb

(Some figures in this article are in colour only in the electronic version)

1. Malomed–Stenflo NLS and RNLS connections

In a search for generalizations of the nonlinear Schrödinger equation which admit Hamiltonian form, Malomed and Stenflo [1] derived the equation

$$\mathrm{i}u_t + u_{xx} + 2p|u|^2u = \left(\bar{c}\frac{u_x^2}{u^2} + c\frac{\bar{u}_x^2}{\bar{u}^2} - 2c\frac{\bar{u}_{xx}}{\bar{u}} - 2c\frac{\bar{u}_x u_x}{\bar{u}u}\right)u \tag{1}$$

with the Hamiltonian density

$$\mathcal{H} = |u_x|^2 - p|u|^4 + c\frac{u}{\bar{u}}\bar{u}_x^2 + \bar{c}\frac{\bar{u}}{u}u_x^2 \tag{2}$$

and the complex parameter $c = c_1 + \mathrm{i}c_2$. As was shown by Natterman [2], under the restriction of this parameter to the open disc $|c| < \frac{1}{2}$, equation (1) can be transformed into the NLS equation and, accordingly, is integrable (see also Auberson and Sabatier [3] for real c). Here, it will be shown that (1) is integrable for all values of the complex parameter c, and that, in a specific range of the parameters, it admits resonance solitons.

If we set $u = \mathrm{e}^{R+\mathrm{i}S}$ then (1) yields

$$-S_t - (1-2c_1)S_x^2 + 2p\,\mathrm{e}^{2R} + 2c_2 S_{xx} + (1+2c_1)\big(R_{xx} + R_x^2\big) = 0, \tag{3}$$

$$R_t + (1-2c_1)(S_{xx} + 2R_x S_x) + 2c_2 R_{xx} + 4c_2 R_x^2 = 0 \tag{4}$$

and it is readily seen that the linear transformation

$$S = \hat{S} + \frac{2c_2}{2c_1 - 1}\hat{R}, \qquad R = \hat{R}, \qquad \hat{t} = (2c_1 - 1)t \tag{5}$$

transforms this system into the Madelung form

$$\hat{S}_{\hat{t}} - \hat{S}_x^2 - \frac{2p}{2c_1 - 1}\mathrm{e}^{2\hat{R}} - \frac{4|c|^2 - 1}{(2c_1 - 1)^2}\big(\hat{R}_{xx} + \hat{R}_x^2\big) = 0 \tag{6}$$

$$-\hat{R}_{\hat{t}} + (\hat{S}_{xx} + 2\hat{R}_x\hat{S}_x) = 0. \tag{7}$$

Introduction of the new wavefunction

$$\psi = \mathrm{e}^{\hat{R} - \mathrm{i}\hat{S}} \tag{8}$$

produces the resonant NLS (RNLS) equation of Pashaev and Lee [4],

$$\mathrm{i}\psi_{\hat{t}} + \psi_{xx} - \frac{2p}{2c_1 - 1}|\psi|^2\psi = s\frac{|\psi|_{xx}}{|\psi|}\psi, \tag{9}$$

where

$$s = 1 + \frac{4|c|^2 - 1}{(2c_1 - 1)^2}. \tag{10}$$

2. RNLS reductions

2.1. Undercritical case

If $s < 1$ so that $|c| < \frac{1}{2}$, then on rescaling time and the phase of the wavefunction according to

$$\hat{t} = \frac{\tilde{t}}{\sqrt{1-s}}, \qquad \hat{S}(x,t) = \sqrt{1-s}\,\tilde{S}(x,\tilde{t}), \qquad \hat{R}(x,t) = \tilde{R}(x,\tilde{t}), \tag{11}$$

where

$$\sqrt{1-s} = \frac{1 - 4|c|^2}{(1 - 2c_1)^2}, \tag{12}$$

then we retrieve the usual NLS equation

$$\mathrm{i}\tilde{\psi}_{\tilde{t}} + \tilde{\psi}_{xx} + 2p\frac{1 - 2c_1}{1 - 4|c|^2}|\tilde{\psi}|^2\tilde{\psi} = 0 \tag{13}$$

in $\tilde{\psi} = \mathrm{e}^{\tilde{R} - \mathrm{i}\tilde{S}}$. This, as in [2], establishes that when $|c| < \frac{1}{2}$ the Malomed–Stefflo equation (1) may be transformed into the standard NLS equation.

2.2. Critical case

If $s = 1$ so that $|c| = \frac{1}{2}$, then on the circle $c_1^2 + c_2^2 = \frac{1}{4}$ equation (9) becomes dispersionless and the resultant NLS equation can be linearized.

2.3. Special case

In the special case when $c_1 = \frac{1}{2}$ and c_2 is an arbitrary real number, the system (3)–(4) reduces [2] to the heat equation

$$-S_t + 2c_2 S_{xx} + 2p\rho + 2\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}} = 0 \tag{14}$$

with density and quantum potential-type sources, together with the heat equation

$$\rho_t + 2c_2\rho_{xx} = 0. \tag{15}$$

for the density $\rho = |u|^2 = \mathrm{e}^{2R}$.

2.4. Overcritical (resonant) case

If $s > 1$, so that $|c| > \frac{1}{2}$ then except on the vertical line $c = \frac{1}{2} + \mathrm{i}c_2$, the RNLS equation cannot be reduced to the NLS form. However, the rescaling

$$\hat{t} = \frac{\tilde{t}}{\sqrt{s-1}}, \qquad \hat{S}(x,t) = \sqrt{s-1}\,\tilde{S}(x,\tilde{t}), \qquad \hat{R}(x,t) = \tilde{R}(x,\tilde{t}), \tag{16}$$

where

$$\sqrt{s-1} = \frac{\sqrt{4|c|^2 - 1}}{|2c_1 - 1|} \tag{17}$$

and the introduction of the two real functions E^+, E^- according to

$$E^+ = \mathrm{e}^{\tilde{R}+\tilde{S}}, \qquad E^- = -\mathrm{e}^{\tilde{R}-\tilde{S}} \tag{18}$$

produces the coupled system

$$-E^+_{\tilde{t}} + E^+_{xx} + 2p\frac{2c_1 - 1}{4|c|^2 - 1}E^+E^-E^+ = 0, \tag{19}$$

$$E^-_{\tilde{t}} + E^-_{xx} + 2p\frac{2c_1 - 1}{4|c|^2 - 1}E^+E^-E^- = 0. \tag{20}$$

2.5. Bilinear representation of the resonant case

The system (19) and (20) can be bilinearized in terms of three real functions G^+, G^- and F where

$$E^+ = \sqrt{\frac{4|c|^2 - 1}{|p(2c_1 - 1)|}}\frac{G^+}{F}, \qquad E^- = \sqrt{\frac{4|c|^2 - 1}{|p(2c_1 - 1)|}}\frac{G^-}{F} \tag{21}$$

satisfy the system

$$\left(+D_{\tilde{t}} - D_x^2\right)(G^+ \cdot F) = 0, \tag{22}$$

$$\left(-D_{\tilde{t}} - D_x^2\right)(G^- \cdot F) = 0, \tag{23}$$

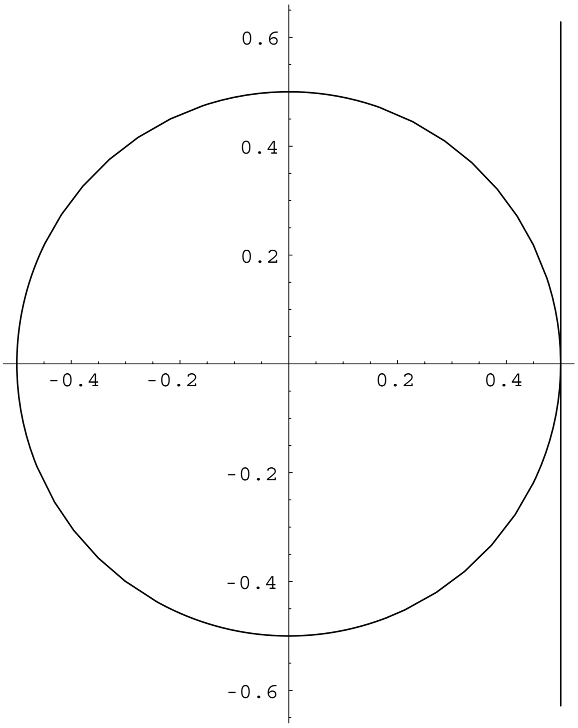

Figure 1. The complex c plane. The region inside the circle $|c| < \frac{1}{2}$ corresponds to the NLS. The circle $|c| = \frac{1}{2}$ is associated with the dispersionless limit of the NLS. Points along the vertical line $x = \frac{1}{2}$ correspond to linear diffusion reductions. The region $|c| > \frac{1}{2}$ corresponds to the resonant case. The right half-plane with $c_1 > \frac{1}{2}$ admits nonsingular solutions for the coupling constant $p < 0$, and for the left half-plane $c_1 < \frac{1}{2}$ for $p > 0$.

$$D_x^2(F \cdot F) = 2\kappa^2 G^+ G^-, \tag{24}$$

where the latter equation shows that

$$-|u|^2 = E^+E^- = \kappa^2 \frac{4|c|^2 - 1}{|p(2c_1 - 1)|} (\ln F)_{xx}, \tag{25}$$

where $\kappa^2 = \text{sign}\, p((2c_1 - 1)) = \pm 1$.

In the focusing case $p > 0$, for $c_1 > \frac{1}{2}$ we have $\kappa^2 = 1$ while for $c_1 < \frac{1}{2}$ we have $\kappa^2 = -1$ (see figure 1).

In the defocusing case $p < 0$, for $c_1 > \frac{1}{2}$ we have $\kappa^2 = -1$ while for $c_1 < \frac{1}{2}$ we have $\kappa^2 = +1$ (see figure 1).

It is noted that the solution u of the Malomed–Stenflo equation (1) may be written explicitly in a bilinear form as

$$u(x,t) = \left[\frac{4|c|^2 - 1}{|p(2c_1 - 1)|} \frac{1}{F^2} \left(\frac{G^+}{-G^-} \right)^{\mathrm{i}\frac{\sqrt{4|c|^2-1}}{2|2c_1-1|}} \right]^{\frac{2c-1}{2(2c_1-1)}}, \tag{26}$$

where $G^{\pm}(x, \tilde{t}) = G^{\pm}(x, \sqrt{4|c|^2 - 1}t)$, $F(x, \tilde{t}) = F(x, \sqrt{4|c|^2 - 1}t)$, $c = c_1 + \mathrm{i}c_2$.

2.6. Single-soliton solution

For the one-soliton solution we have

$$G^{\pm} = \pm\, \mathrm{e}^{\eta_1^{\pm}}, \qquad F = 1 - \kappa^2\, \mathrm{e}^{\eta_1^{+}+\eta_1^{-}+\phi_{11}}, \qquad \mathrm{e}^{\phi_{11}} = \frac{1}{\left(k_1^{+}+k_1^{-}\right)^2}, \tag{27}$$

where $\eta_1^{\pm} = k_1^{\pm}\, x \pm \left(k_1^{\pm}\right)^2 \tilde{t} + \eta_1^{\pm(0)}$, and $k_1^{\pm}$, $\eta_1^{\pm(0)}$ are arbitrary real constants. This solution is regular only if $\kappa^2 < 0$, which corresponds to the cases $p > 0, c_1 < \frac{1}{2}$ or $p < 0, c_1 > \frac{1}{2}$, when $\kappa^2 = -1$ (see figure 1). Here, we focus on this case. From the preceding we have

$$\mathrm{e}^{\hat{R}} = \sqrt{\frac{4|c|^2-1}{|p(2c_1-1)|}}\, \frac{\left|k_1^{+}+k_1^{-}\right|}{2\cosh\frac{\eta_1^{+}+\eta_1^{-}+\phi_{11}}{2}}, \qquad \hat{S} = \frac{\sqrt{4|c|^2-1}}{|2c_1-1|}\, \frac{\eta_1^{+}-\eta_1^{-}}{2}. \tag{28}$$

Denoting $v \equiv \left(k_1^{-}-k_1^{+}\right)\sqrt{4|c|^2-1}$, $k \equiv \left(k_1^{-}+k_1^{+}\right)/2$ and using $\tilde{t} = \pm\sqrt{4|c|^2-1}\,t$ we obtain a single-soliton solution of the model (1) in the form

$$u(x,t) = \sqrt{\frac{4|c|^2-1}{|p(2c_1-1)|}}\, \frac{|k|\, \mathrm{e}^{\mathrm{i}\Phi(x,t)}}{\cosh k(x-vt-x_0)}, \tag{29}$$

where

$$\Phi = \frac{1}{|2c_1-1|}\left[-\frac{vx}{2} + \left[(4|c|^2-1)k^2 + \frac{v^2}{4}\right]t\right] - \frac{2c_2}{2c_1-1}\ln[\cosh k(x-vt-x_0)] + \phi_0. \tag{30}$$

2.7. Hyperbolic metrics and black hole interpretation

Substitution of the Madelung form $u = \mathrm{e}^{R+\mathrm{i}S}$ into the Hamiltonian density (2) yields

$$\mathcal{H} = \left[(1+2c_1)R_x^2 + (1-2c_1)S_x^2 + 4c_2 R_x S_x\right]\mathrm{e}^{2R} - p\,\mathrm{e}^{4R}. \tag{31}$$

The dispersion is positive definite if $|c| < \frac{1}{2}$ and indefinite when $|c| > \frac{1}{2}$. In the present resonant case, the dispersion is of indefinite sign. Thus in terms of (5)

$$\mathcal{H} = \left[\left(\frac{4|c|^2-1}{2c_1-1}\right)\hat{R}_x^2 + (1-2c_1)\hat{S}_x^2\right]\mathrm{e}^{2\hat{R}} - p\,\mathrm{e}^{4\hat{R}} \tag{32}$$

whence, when $|c| > \frac{1}{2}$ the dispersion is indefinite and it changes sign at points in the spacetime where

$$\hat{R}_x = \pm\frac{1-2c_1}{\sqrt{4|c|^2-1}}\hat{S}_x. \tag{33}$$

For the one-soliton solution (29) this gives

$$\tanh k(x-vt-x_0) = \pm\frac{v}{2k}, \tag{34}$$

a solution of which exists if $|v| < 2|k|$. As in [4, 5], we can construct a two-dimensional pseudo-Riemannian metric for (19), (20) and the RNLS, namely

$$\mathrm{d}l^2 = \left[(4|c|^2-1)\hat{R}_x^2 - (2c_1-1)^2\hat{S}_x^2\right]\mathrm{e}^{2\hat{R}}\,\mathrm{d}t^2 - 2\hat{S}_x|2c_1-1|\,\mathrm{e}^{2\hat{R}}\,\mathrm{d}x\,\mathrm{d}t - \mathrm{e}^{2\hat{R}}\,\mathrm{d}x^2 \tag{35}$$

so that evolution according to equation (1) implies the two-dimensional spacetime with the constant scalar curvature

$$R = 8p\frac{2c_1-1}{4|c|^2-1}. \tag{36}$$

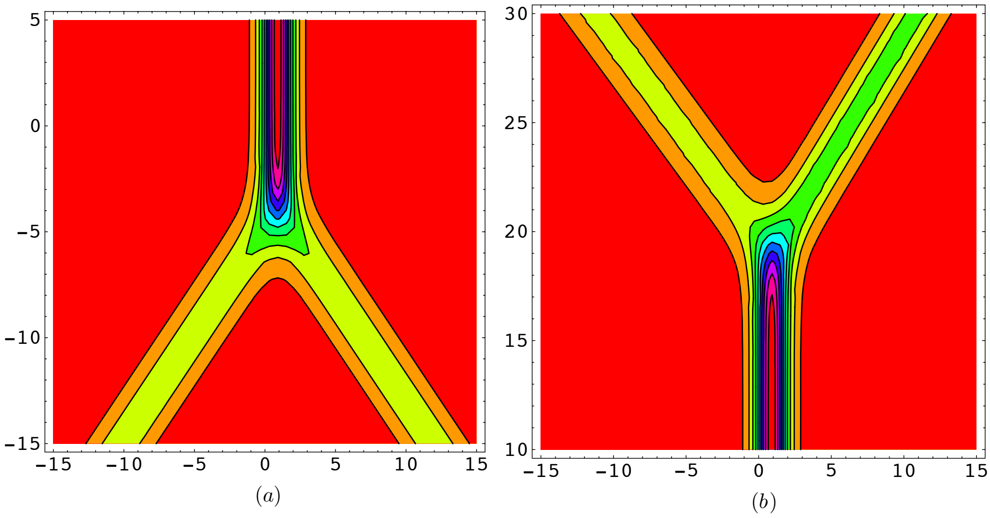

Figure 2. Fusion and fission of two solitons (*a*) fusion of two solitons (*b*) fission of two solitons.

With our choice of parameters, namely $c_1 > \frac{1}{2}$, $p < 0$ or $c_1 < \frac{1}{2}$, $p > 0$, R is negative-valued. The time component of the metric is the dispersion term ϵ_0 for the energy

$$g_{00} = \left[(4|c|^2 - 1)\hat{R}_x^2 - (2c_1 - 1)^2 \hat{S}_x^2\right] \mathrm{e}^{2\hat{R}} = (2c_1 - 1)\epsilon_0. \tag{37}$$

Points where g_{00} vanishes correspond to the event horizon of a black hole. For the one-soliton solution this corresponds to condition (34). Solitons of the equation (1) moving with the velocity $|v| < 2|k|$ correspond to black holes with event horizon dependent on the velocity of the soliton.

2.8. Two-soliton solution

The Hirota bilinear representation (22)–(24) admits two-soliton solutions with

$$G^{\pm} = \pm\left(\mathrm{e}^{\eta_1^{\pm}} + \mathrm{e}^{\eta_2^{\pm}} + \alpha_1^{\pm}\,\mathrm{e}^{\eta_1^{+}+\eta_1^{-}+\eta_2^{\pm}} + \alpha_2^{\pm}\,\mathrm{e}^{\eta_2^{+}+\eta_2^{-}+\eta_1^{\pm}}\right), \tag{38}$$

$$F = 1 + \frac{\mathrm{e}^{\eta_1^{+}+\eta_1^{-}}}{\left(k_{11}^{+-}\right)^2} + \frac{\mathrm{e}^{\eta_1^{+}+\eta_2^{-}}}{\left(k_{12}^{+-}\right)^2} + \frac{\mathrm{e}^{\eta_2^{+}+\eta_1^{-}}}{\left(k_{21}^{+-}\right)^2} + \frac{\mathrm{e}^{\eta_2^{+}+\eta_2^{-}}}{\left(k_{22}^{+-}\right)^2} + \beta\,\mathrm{e}^{\eta_1^{+}+\eta_1^{-}+\eta_2^{+}+\eta_2^{-}}, \tag{39}$$

where $\eta_i^{\pm} = k_i^{\pm}x \pm \left(k_i^{\pm}\right)^2\tilde{t} + \eta_i^{\pm(0)}$, $k_{ij}^{ab} = k_i^a + k_j^b$, $(i, j = 1, 2)$, $(a, b = +-)$,

$$\alpha_1^{\pm} = \frac{\left(k_1^{\pm} - k_2^{\pm}\right)^2}{\left(k_{11}^{+-}k_{21}^{\pm\mp}\right)^2}, \qquad \alpha_2^{\pm} = \frac{\left(k_1^{\pm} - k_2^{\pm}\right)^2}{\left(k_{22}^{+-}k_{12}^{\pm\mp}\right)^2}, \qquad \beta = \frac{\left(k_1^{+} - k_2^{+}\right)^2\left(k_1^{-} - k_2^{-}\right)^2}{\left(k_{11}^{+-}k_{12}^{+-}k_{21}^{+-}k_{22}^{+-}\right)^2}. \tag{40}$$

2.9. Resonance interaction of solitons

In figure 2, fusion and fission of two solitons is shown for the parameter values $k_1^{+} = 0.1$, $k_1^{-} = 1$, $k_2^{+} = 1$, $k_2^{-} = 0$ and large phase shift. The horizontal and vertical axes represent space x and time t coordinates, respectively.

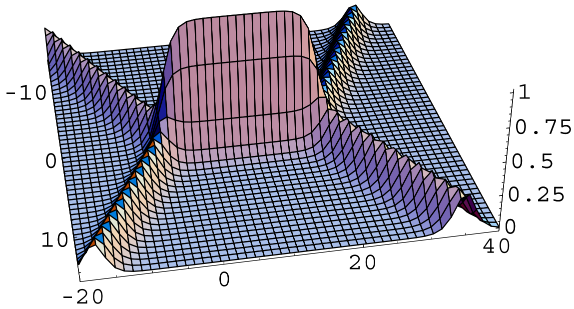

Figure 3. Two-soliton resonant state.

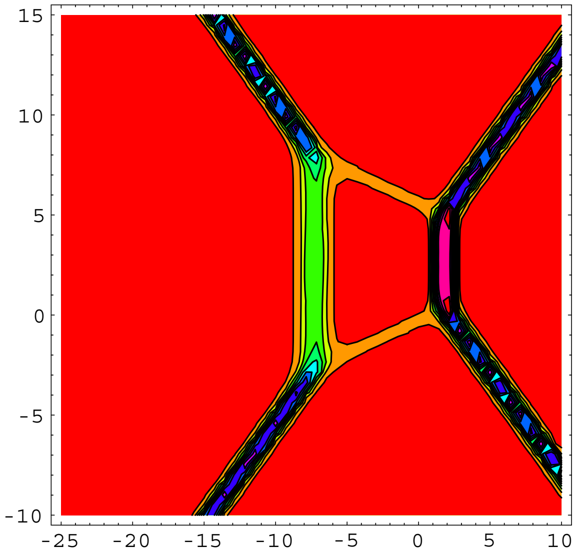

Figure 4. Four-soliton resonance scattering.

In figure 3, the creation of soliton resonance with a finite lifetime is shown. The parameters in this case are the same as above, except for the phase shift $d = 15$.

In figure 4, four virtual soliton resonance scattering is shown for $k_1^+ = 2$, $k_1^- = 1$, $k_2^+ = 1$, $k_2^- = 2$ and $d = 16$.

3. Nontrivial boundary conditions

In the application of the RNLS model to the propagation of solitonic magnetoacoustic waves in [6] the required asymptotic behavior is $|\psi|^2 = \rho \to 1$ at infinity. In this case, we can derive a one-soliton solution of (1) with

$$|u|^2(x,t) = 1 + \frac{v^2 - 4p(1-2c_1)}{4p(1-2c_1)}\operatorname{sech}^2\left[\frac{\sqrt{v^2-4p(1-2c_1)}}{2\sqrt{4|c|^2-1}}(x+vt+x_0)\right] \tag{41}$$

and the phase

$$S(x,t) = S_0 + 2pt + \frac{c_2}{2c_1 - 1}\ln|u|^2(x,t) \tag{42}$$

$$+\frac{\sqrt{4|c|^2-1}}{2|2c_1-1|}\ln\frac{v+\sqrt{v^2-4p(1-2c_1)}\tanh\left[\frac{\sqrt{v^2-4p(1-2c_1)}}{2\sqrt{4|c|^2-1}}(x+vt+x_0)\right]}{v-\sqrt{v^2-4p(1-2c_1)}\tanh\left[\frac{\sqrt{v^2-4p(1-2c_1)}}{2\sqrt{4|c|^2-1}}(x+vt+x_0)\right]}. \tag{43}$$

It is seen that the velocity of this soliton is bounded below with $|v| > 2|p(1-2c_1)|$. This contrasts with the case of the defocusing NLS equation where the dark soliton velocity is bounded above. Moreover if the soliton of the defocusing NLS is a hole-like (bubble) excitation with $\rho = |u|^2 < 1$, for the Malomed–Stenflo equation this has $\rho = |u|^2 > 1$. It is noted that the two-soliton solution can be constructed alternatively via a Backlund–Darboux transformation [6]. Solutions of the RNLS equation with nontrivial boundary conditions have been investigated by Lee and Pashaev in [7]. These results may be carried over '*mutatis mutandis*' to the Malomed–Stenflo equation (1).

4. Conclusion

It has been established that the generalized nonlinear Schrödinger equation (1) introduced in [1], for a specific range of parameters, admits resonant soliton interaction. Indeed, a natural integrable extension of this equation is suggested, namely

$$\mathrm{i}u_t + u_{xx} + 2p|u|^2u = \left(\bar{c}\frac{u_x^2}{u^2} + c\frac{\bar{u}_x^2}{\bar{u}^2} - 2c\frac{\bar{u}_{xx}}{\bar{u}} - 2c\frac{\bar{u}_x u_x}{\bar{u}u}\right)u + 4\nu\frac{|u|_{xx}}{|u|}u \tag{44}$$

corresponding to the addition of a 'quantum potential' term with strength ν. This extension can be motivated in an information theory context to reflect uncertainty conditions in the measurement process and described by the Fisher measure [8]. The generalized NLS equation (44) is Hamiltonian with

$$\mathcal{H} = |u_x|^2 - p|u|^4 + c\frac{u}{\bar{u}}\bar{u}_x^2 + \bar{c}\frac{\bar{u}}{u}u_x^2 - 4\nu(|u|_x)^2. \tag{45}$$

Following the same procedure as that for (1), reduction may be made to the RNLS form (9) but now with the parameter

$$s = 1 + \frac{4|c|^2 - 1 - 4\nu(2c_1 - 1)}{(2c_1 - 1)^2}. \tag{46}$$

The reductions of the extended model equation (44) then depend on both the complex parameter $c = c_1 + \mathrm{i}c_2$ and the real quantum potential strength ν. In geometrical terms, the circle $|c| = \frac{1}{2}$ in figure 1 is modified by the presence of the additional parameter ν to become

$$(c_1 - \nu)^2 + c_2^2 = \left(\nu - \tfrac{1}{2}\right)^2. \tag{47}$$

The region inside this circle corresponds to the NLS reduction, while the outside corresponds to the resonant NLS case. It is noted that when $\nu = \frac{1}{2}$, the disc shrinks to a point and no reduction to the classical NLS is possible. In this case

$$s = 1 + \frac{(2c_1 - 1)^2 + 4c_2^2}{(2c_1 - 1)^2} \tag{48}$$

whence $s > 1$ and the model equation (44) is necessarily of resonant type.

Acknowledgments

This work was partially supported by Izmir Institute of Technology, Turkey, Grant BAP 25 2008 and the Institute of Mathematics, Academia Sinica, Taipei, Taiwan.

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