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ABELIAN CHERN–SIMONS VORTICES AND HOLOMORPHIC BURGERS HIERARCHY

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We consider the Abelian Chern–Simons gauge field theory in 2+1 dimensions and its relation to the holomorphic Burgers hierarchy. We show that the relation between the complex potential and the complex gauge field as in incompressible and irrotational hydrodynamics has the meaning of the analytic Cole–Hopf transformation, linearizing the Burgers hierarchy and transforming it into the holomorphic Schrödinger hierarchy. The motion of planar vortices in Chern–Simons theory, which appear as pole singularities of the gauge field, then corresponds to the motion of zeros of the hierarchy. We use boost transformations of the complex Galilei group of the hierarchy to construct a rich set of exact solutions describing the integrable dynamics of planar vortices and vortex lattices in terms of generalized Kampe de Feriet and Hermite polynomials. We apply the results to the holomorphic reduction of the Ishimori model and the corresponding hierarchy, describing the dynamics of magnetic vortices and the corresponding lattices in terms of complexified Calogero–Moser models. We find corrections (in terms of Airy functions) to the two-vortex dynamics from the Moyal space–time noncommutativity.

Keywords: Chern–Simons gauge theory, Burgers hierarchy, noncommutative vortex, Ishimori model, holomorphic equation, Kampe de Feriet polynomial

1. Classical ferromagnetic model in continuous media

Volovik introduced a model of delocalized electrons in which the linear momentum density is restored using hydrodynamic variables, the density, and the normal velocity of the Fermi liquid, to solve the so-called momentum problem in planar ferromagnets [1]. Based on this, a simple model of a ferromagnetic fluid or a spin liquid, modifying the phenomenological Landau–Lifshitz equation, was proposed in [2]. In the model, the magnetic variable is described by the classical spin vector $\vec{S}=\vec{S}(x,y,t)$ valued on the two-dimensional sphere ($\vec{S}^2=1$), and the hydrodynamic variable is the velocity $\vec{v}(x,y,t)$ of the incompressible flow. For the particular anisotropic case of the space metric, we have the system

$$\begin{aligned}&\vec{S}_t+v_1\partial_1\vec{S}-v_2\partial_2\vec{S}=\vec{S}\times(\partial_1^2-\partial_2^2)\vec{S},\\&\partial_1v_2-\partial_2v_1=2\vec{S}(\partial_1\vec{S}\times\partial_2\vec{S}).\end{aligned}\tag{1}$$

The first equation is the Heisenberg model, where the time derivative $\partial/\partial t$ is replaced with the material derivative $D/Dt=\partial/\partial t+(\vec{v}\,\nabla)$, and the second equation is the relation between the hydrodynamic and spin variables called the Mermin–Ho relation [3]. It relates the flow vorticity to the topological charge density or the spin winding number. The following theorem [4] is applicable in this case.

Theorem. *For the flow constrained by the incompressibility condition*

$$\partial_1v_1+\partial_2v_2=0,\tag{2}$$

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the conservation law $\partial_t J_0 + \partial_2 J_2 - \partial_1 J_1 = 0$ *holds, where*

$$\begin{aligned}
J_0 &= (\partial_1 \vec{S})^2 + (\partial_2 \vec{S})^2, \\
J_1 &= -2\partial_1 \vec{S} \cdot \vec{S} \times (\partial_1^2 - \partial_2^2)\vec{S} + v_1 J_0 + 2\vec{S} \cdot (\partial_1 \vec{S} \times \partial_2^2 \vec{S} - \partial_1 \partial_2 \vec{S} \times \partial_2 \vec{S}), \\
J_2 &= 2\partial_2 \vec{S} \cdot \vec{S} \times (\partial_1^2 - \partial_2^2)\vec{S} + v_2 J_0 - 2\vec{S} \cdot (\partial_1^2 \vec{S} \times \partial_1 \partial_2 \vec{S} - \partial_1 \vec{S} \times \partial_2 \vec{S}).
\end{aligned} \tag{3}$$

By this theorem, for incompressible flow (2), functional (3) (the "energy" functional $E = \iint J_0 \, d^2x$) is conserved and bounded by the topological charge Q of a spin configuration: $E \geq 8\pi|Q|$ (Bogomol'nyi inequality). This inequality is saturated for spin configurations satisfying the first-order system

$$\partial_i \vec{S} \pm \epsilon_{ij} \vec{S} \times \partial_j \vec{S} = 0, \tag{4}$$

the Belavin–Polyakov self-duality equations. In contrast to the static case, the Bogomol'nyi inequalities hold for time-dependent fields in our case. The stereographic projections of the spin phase space are given by the formulas

$$S_+ = S_1 + iS_2 = \frac{2\zeta}{1+|\zeta|^2}, \qquad S_3 = \frac{1-|\zeta|^2}{1+|\zeta|^2}, \tag{5}$$

where ζ is a complex-valued function. In the complex derivatives $\partial_z = (\partial_1 - i\partial_2)/2$ and $\partial_{\bar{z}} = (\partial_1 + i\partial_2)/2$, self-duality equations (4) in the stereographic projection form become the respective analyticity or antianalyticity conditions $\zeta_{\bar{z}}(z,t) = 0$ or $\zeta_z(\bar{z},t) = 0$. It is easy to show by direct calculation that for incompressible flow (2), the holomorphic constraint $\zeta_{\bar{z}}(z,t) = 0$ is compatible with the time evolution $\partial\zeta_{\bar{z}}/\partial t = 0$.

2. Holomorphic reduction of the Ishimori model

From the theorem presented above, we see that incompressible flow admits the existence of a positive energy functional minimized by holomorphic reduction and that this reduction is compatible with the time evolution. This suggests solving the incompressibility conditions explicitly. We therefore consider topological magnet model (1) with the incompressibility condition solved in terms of a real function ψ, the stream function of the flow, $v_1 = \partial_2\psi$, $v_2 = -\partial_1\psi$. We then have the analytic reduction of the Ishimori model [5]

$$i\zeta_t - 2\psi_z\zeta_z + 2\zeta_{zz} - 4\frac{\zeta}{1+|\zeta|^2}\zeta_z^2 = 0, \tag{6}$$

$$\psi_{z\bar{z}} = -2\frac{\bar{\zeta}_{\bar{z}}\zeta_z}{(1+|\zeta|^2)^2}. \tag{7}$$

With the stream function

$$\psi = 2\log(1+|\zeta|^2), \tag{8}$$

Eq. (7) is satisfied automatically, and we obtain the holomorphic Schrödinger equation

$$i\zeta_t + 2\zeta_{zz} = 0 \tag{9}$$

from (6). Each zero of the function ζ in the complex plane z determines a magnetic vortex of the Ishimori model. The spin vector at the center of the vortex is $\vec{S} = (0,0,1)$, while $\vec{S} = (0,0,-1)$ at infinity. The motion of the zeros of Eq. (9) then determines the motion of the magnetic vortices in the plane. On the other hand, if we take the analytic function

$$f(z,t) = \frac{\Gamma}{2\pi i}\log\zeta(z,t) \tag{10}$$

as the complex potential of an effective flow [6], then each zero of the function ζ corresponds to a hydrodynamic vortex of the flow with the intensity Γ and to a simple-pole singularity of the complex velocity

$$u(\bar{z},t) = \bar{f}_{\bar{z}} = \frac{i\Gamma}{2\pi}(\log\bar{\zeta})_{\bar{z}}. \tag{11}$$

But the last relation has the meaning of the holomorphic Cole–Hopf transformation, according to which the complex velocity satisfies the holomorphic Burgers equation

$$iu_t + \frac{8\pi i}{\Gamma}uu_{\bar{z}} = 2u_{\bar{z}\bar{z}}. \tag{12}$$

Therefore, each magnetic vortex of the Ishimori model corresponds to a hydrodynamic vortex of the antiholomorphic Burgers equation. Moreover, relation (10) written in the form

$$\zeta = e^{2\pi i f/\Gamma} = e^{2\pi i(\phi+i\chi)/\Gamma} = \sqrt{\rho}\, e^{2\pi i\phi/\Gamma} \tag{13}$$

shows that the effective flow is just the Madelung representation for linear holomorphic Schrödinger equation (9), where the functions ϕ and χ are the respective velocity potential and stream function.

3. The N-vortex system

The system of N magnetic vortices is determined by N simple zeros

$$\zeta(z,t) = \prod_{k=1}^{N}(z - z_k(t)), \tag{14}$$

whose positions satisfy the system

$$\frac{dz_k}{dt} = \sum_{\substack{l=1\\ l\neq k}}^{N}\frac{4i}{z_k - z_l} \tag{15}$$

according to (9). In the one-dimensional space, this system was first considered in [7] (also see [8]) for moving poles of the Burgers equation determined by zeros of the heat equation. But complexification of the problem has several advantages. First, the problem of the roots of an algebraic equation of degree N, like the problem of the motion of the singularities of differential equations, is complete in the complex domain. In this case in contrast to one dimension, the pole dynamics in the plane becomes time reversible (see below) and can be interpreted as the vortex dynamics. Moreover, the (nonintegrable) generalization of system (15) to the case of three particles with different strengths was studied in [9] to explain the transition from regular to irregular motion as travel on a Riemann surface.

In Sec. 7, we show that the solution of this system is determined by N complex constants of motion; this is why the vortex dynamics in the Ishimori model is integrable. In fact, system (15) allows mapping to the complexified Calogero–Moser (type-I) N-particle problem [10], [11]. For this, we differentiate once and use the system again to obtain Newton's equations

$$\frac{d^2 z_k}{dt^2} = \sum_{\substack{l=1\\ l\neq k}}^{N}\frac{32}{(z_l - z_k)^3}. \tag{16}$$

These equations have the Hamiltonian form with the Hamiltonian function

$$H = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \sum_{k<l}^{N} \frac{16}{(z_k - z_l)^2} \tag{17}$$

and admit the Lax representation, whence follows the hierarchy of constants of motion in involution $I_k = \operatorname{tr} L^{k+1}$. Complexification of the classical Calogero–Moser model and holomorphic Hopf equation was recently considered in connection with the limit of an infinite number of particles, leading to quantum hydrodynamics and the quantum Benjamin–Ono equation [12]. On the other hand, the holomorphic version of the Burgers equation was considered in [13] to prove the existence and uniqueness of the nonlinear diffusion process for a system of Brownian particles with electrostatic repulsion in the case where the number of particles tends to infinity.

4. Integrable N-particle problem for N-vortex lattices

The function ζ of the form

$$\zeta(z,t) = \sin(z - z_k(t)) = (z - z_k(t)) \prod_{n=1}^{\infty}\left(1 - \frac{(z - z_k(t))^2}{n^2\pi^2}\right) \tag{18}$$

has a periodic infinite set of zeros and determines the vortex lattice. For (9), we first consider the system of N-vortex chain lattices periodic in x,

$$\zeta(z,t) = e^{-2iN^2 t} \prod_{k=1}^{N} \sin(z - z_k(t)), \tag{19}$$

such that the positions of the vortices satisfy the first-order system

$$\dot{z}_k = 2i \sum_{\substack{l=1 \\ l\neq k}}^{N} \cot(z_k - z_l). \tag{20}$$

Differentiating this system once with respect to time, we obtain the second-order equations of motion in the Newtonian form

$$\ddot{z}_k = 32 \sum_{\substack{l=1 \\ l\neq k}}^{N} \frac{\cot(z_k - z_l)}{\sin^2(z_k - z_l)} \tag{21}$$

with the Hamiltonian function of the Calogero–Moser type-II model [10]

$$H = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \sum_{k<l}^{N} \frac{16}{\sin^2(z_k - z_l)}. \tag{22}$$

For lattices periodic in y,

$$\zeta(z,t) = e^{2iN^2 t} \prod_{k=1}^{N} \sinh(z - z_k(t)), \tag{23}$$

we obtain the Calogero–Moser type-III Model

$$H = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \sum_{k<l}^{N} \frac{16}{\sinh^2(z_k - z_l)}. \tag{24}$$

5. Complex Galilei group and vortex generations

The complex Galilei group is generated by the algebra

$$[P_0, P_z] = 0, \qquad [P_0, K] = 4iP_z, \qquad [P_z, K] = -i, \tag{25}$$

where the respective energy and momentum operators are $P_0 = -i\partial_t$ and $P_z = -i\partial_z$ and the Galilean boost operator is $K = z + 4it\partial_z$. The Schrödinger operator $S = i\partial_t + 2\partial_z^2$ corresponds to the dispersion relation $P_0 = -2P_z^2$ and commutes with the Galilei group operators,

$$[P_0, S] = 0, \qquad [P_z, S] = 0, \qquad [K, S] = 0. \tag{26}$$

It is known from the theory of dynamical symmetry that if there exists an operator W such that

$$[S, W] = 0 \Rightarrow S(W\Phi) = W(S\Phi) = 0, \tag{27}$$

then it transforms a solution Φ of the Schrödinger equation into another solution $W\Phi$. This shows that the Galilei generators provide dynamical symmetries for the equation. Two of them are obvious: the time translation $P_0\colon e^{it_0P_0}\Phi(z,t) = \Phi(z, t+t_0)$ and the complex space translation $P_z\colon e^{it_0P_z}\Phi(z,t) = \Phi(z+z_0, t)$. The Galilean boost creates a new zero (new vortex in $\mathbb{C}$):

$$\Psi(z,t) = K\Phi(z,t) = (z + 4it\partial_z)\Phi(z,t). \tag{28}$$

Starting from the obvious solution $\Phi = 1$, we have the chain of n-vortex solutions, $K \cdot 1 = z = H_1(z, 2it)$, $K^2 \cdot 1 = z^2 + 4it = H_2(z, 2it)$, $K^3 \cdot 1 = z^3 + 12it = H_3(z, 2it)$, $\ldots$, $K^n \cdot 1 = H_n(z, 2it)$ in terms of the Kampe de Feriet polynomials [14]

$$H_n(z, it) = n! \sum_{k=0}^{[n/2]} \frac{(it)^k z^{n-2k}}{k!(n-2k)!}. \tag{29}$$

They satisfy the recursion relations

$$\begin{aligned} H_{n+1}(z, it) &= \left(z + 2it\frac{\partial}{\partial z}\right) H_n(z, it), \\ \frac{\partial}{\partial z} H_n(z, it) &= nH_{n-1}(z, it) \end{aligned} \tag{30}$$

and can be written in terms of the Hermite polynomials

$$H_n(z, 2it) = (-2it)^{n/2} H_n\left(\frac{z}{2\sqrt{-2it}}\right). \tag{31}$$

Let $w_n^{(k)}$ be the kth zero of the Hermite polynomial, $H_n(w_n^{(k)}) = 0$. Then the evolution of the corresponding vortex is given by

$$z_k(t) = 2w_n^{(k)}\sqrt{-2it}. \tag{32}$$

Under the time reflection $t \to -t$, the position of the vortex rotates through 90 degrees, $z_k \to z_k e^{i\pi/2}$. This transformation is also a symmetry of vortex equations (15). Using the formula

$$H_n(z, 2it) = \exp\left(it\frac{\partial^2}{\partial z^2}\right) z^n \tag{33}$$

and the superposition principle, we obtain the solution

$$\Phi(z,t)=\sum_{n=0}^{\infty}a_nH_n(z,2it)=\sum_{n=0}^{\infty}a_n\exp\left(2it\frac{\partial^2}{\partial z^2}\right)z^n=\exp\left(2it\frac{\partial^2}{\partial z^2}\right)\sum_{n=0}^{\infty}a_nz^n.$$

Hence, if $\chi(z)=\sum_{n=0}^{\infty}a_nz^n$ is an arbitrary analytic function, then $\Phi(z,t)=\exp(2it\,\partial^2/\partial z^2)\chi(z)$ is a solution determined by the integrals of motion $a_0,a_1,\dots$. Therefore, for a polynomial of degree n describing the evolution of n vortices, we have n complex integrals of motion.

The generating function of the Kampe de Feriet polynomials

$$\sum_{n=0}^{\infty}\frac{k^n}{n!}H_n(z,it)=e^{kz+ik^2t} \tag{34}$$

is also a solution of the plane-wave type. If we exponentiate the Galilean boost $e^{i\lambda K}=e^{i\lambda(z+4it\partial_z)}$, factor it by the Baker–Hausdorff formula $e^{A+B}=e^Be^Ae^{[A,B]/2}$ such that $e^{i\lambda K}=e^{i\lambda z+2i\lambda^2t}e^{-4\lambda t\partial_z}$, and apply it to a solution $\Phi(z,t)$, then we obtain

$$e^{i\lambda K}\Phi(z,t)=e^{i\lambda z+2i\lambda^2t}\Phi(z-4\lambda t,t), \tag{35}$$

the Galilean boost with the velocity 4λ, where the generating function of vortices (34) appears as the 1-cocycle.

Galilean boost (28) connecting two solutions of holomorphic Schrödinger equation (9) generates the auto-Bäcklund transformation

$$v=u+\frac{i\Gamma}{2\pi}\partial_{\bar z}\log\left(\bar z-\frac{8\pi t}{\Gamma}u\right) \tag{36}$$

between two solutions

$$u(\bar z,t)=\frac{i\Gamma}{2\pi}\frac{\bar\Phi_{\bar z}}{\bar\Phi},\qquad v(\bar z,t)=\frac{i\Gamma}{2\pi}\frac{\bar\Psi_{\bar z}}{\bar\Psi} \tag{37}$$

of antiholomorphic Burgers equation (12).

As an example, we consider the double-lattice solution

$$\zeta(z,t)=e^{-8it}\sin(z-z_1(t))\sin(z+z_1(t)), \tag{38}$$

where $\cos 2z_1=re^{8it}$ and r is a constant. Applying boost transformation (28), we obtain a solution describing the collision of a vortex with the double lattice,

$$\Psi(z,t)=\left(z+4it\frac{\partial}{\partial z}\right)\zeta(z,t). \tag{39}$$

Generalizing, we obtain N vortices interacting with M-vortex lattices,

$$\Psi(z,t)=e^{iMt}\left(z+4it\frac{\partial}{\partial z}\right)^N\prod_{k=1}^{M}\sin(z-z_k(t)), \tag{40}$$

where $z_1,\dots,z_k$ satisfy system (20).

6. Abelian Chern–Simons theory and the complex Burgers hierarchy

We now show how the antiholomorphic Burgers hierarchy appears in the Chern–Simons gauge field theory. The Chern–Simons functional is defined as

$$S(A) = \frac{\kappa}{4\pi}\int_{\mathcal{M}} A\wedge dA = \frac{\kappa}{4\pi}\int \varepsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda}\, d^3x, \tag{41}$$

where $\mathcal{M}$ is an oriented three-dimensional manifold, A is a $U(1)$ gauge connection, and κ is the coupling constant (the statistical parameter). In the canonical approach, $\mathcal{M} = \Sigma_2 \times \mathbb{R}$, where we interpret $\mathbb{R}$ as a time. Then $A_\mu = (A_0, A_i)$, $i = 1, 2$, where A_0 is the time component and the action takes the form

$$S = -\frac{\kappa}{4\pi}\int dt \int_\Sigma \epsilon^{ij}\left(A_i \frac{d}{dt}A_j - A_0 F_{ij}\right) d^2x. \tag{42}$$

In the first-order formalism, this implies that the Poisson bracket is

$$\{A_i(x), A_j(y)\} = \frac{4\pi}{\kappa}\epsilon_{ij}\delta(x-y) \tag{43}$$

and the Hamiltonian is $H = A_0\epsilon^{ij}F_{ij}$. The Hamiltonian is weakly vanishing ($H \approx 0$) because of the Chern–Simons Gauss law constraint

$$\partial_1 A_2 - \partial_2 A_1 = 0 \Leftrightarrow F_{ij} = 0. \tag{44}$$

Then the evolution is determined by the Lagrange multipliers A_0: $\partial_0 A_1 = \partial_1 A_0$ and $\partial_0 A_2 = \partial_2 A_0$. Because of the gauge invariance $A_\mu \to A_\mu + \partial_\mu\lambda$, we choose the Coulomb gauge condition to fix the gauge freedom: $\operatorname{div}\vec{A} = 0$. In addition, we have Chern–Simons Gauss law (44): $\operatorname{rot}\vec{A} = 0$. These two equations are identical to the incompressible and irrotational hydrodynamics. Solving the first equation in terms of the velocity potential φ: $A_k = \partial_k\varphi$, $k = 1, 2$, and the second equation in terms of the stream function ψ: $A_1 = \partial_2\psi$ and $A_2 = -\partial_1\psi$, we obtain the Cauchy–Riemann equations $\partial_1\varphi = \partial_2\psi$ and $\partial_2\varphi = -\partial_1\psi$. Hence, these two functions are harmonically conjugate, and the complex potential $f(z) = \varphi(x, y) + i\psi(x, y)$ is an analytic function of $z = x + iy$, $\partial f/\partial\bar{z} = 0$. The corresponding "complex gauge potential" $A = A_1 + iA_2 = \overline{f'(z)}$ is an antianalytic function. By analogy with hydrodynamics, the logarithmic singularities of the complex potential

$$f(z, t) = \frac{1}{2\pi i}\sum_{k=1}^{N}\Gamma_k \log(z - z_k(t)) \tag{45}$$

determine the poles of the complex gauge field

$$A = \frac{i}{2\pi}\sum_{k=1}^{N}\frac{\Gamma_k}{\bar{z} - \bar{z}_k(t)} \tag{46}$$

describing point vortices in the plane. Then the corresponding "statistical" magnetic field

$$B = \partial_1 A_2 - \partial_2 A_1 = -\Delta\psi = -\Delta \operatorname{Im} f(z), \tag{47}$$

where Δ is the Laplacian, determined by the stream function

$$\psi = -\frac{1}{2\pi}\sum_{k=1}^{N}\Gamma_k \log|z - z_k(t)| \tag{48}$$

is equal to

$$B = \frac{1}{2\pi}\sum_{k=1}^{N}\Gamma_k \Delta \log|z - z_k(t)| = \sum_{k=1}^{N}\Gamma_k \delta(\vec{r} - \vec{r}_k(t)). \tag{49}$$

The corresponding total magnetic flux is

$$\int_{\mathbb{R}^2}\int B\, d^2x = \sum_{k=1}^{N}\iint \Gamma_k \delta(\vec{r} - \vec{r}_k(t))\, d^2x = \Gamma_1 + \Gamma_2 + \dots + \Gamma_N. \tag{50}$$

Relation (49) can be interpreted as the Chern–Simons Gauss law

$$B = \frac{1}{\kappa}\bar{\psi}\psi = \frac{1}{\kappa}\rho \tag{51}$$

for point particles located at $\vec{r}_k(t)$ with the density

$$\rho = \sum_{k=1}^{N}\Gamma_k \delta(\vec{r} - \vec{r}_k(t)) \tag{52}$$

(with the masses $\Gamma_1, \Gamma_2, \dots, \Gamma_N$). Then magnetic fluxes are superimposed on particles and have the meaning of anyons. As a result, an integrable evolution of the complex gauge field singularities (vortices) would lead to the integrable evolution of anyons. Evolution of the antiholomorphic complex gauge potential is determined by the equation $\partial_0 A = 2\partial_{\bar{z}} A_0$, where the function A_0, as follows, is harmonic, $\Delta A_0 = 0$, and is given by $A_0 = [F_0(\bar{z}, t) + \overline{F}_0(z, t)]/2$. Then the evolution equation is

$$\partial_0 A = \partial_{\bar{z}} F_0. \tag{53}$$

Let

$$F_0 = \sum_{n=0}^{\infty} c_n F_0^{(n)}(\bar{z}, t), \tag{54}$$

where

$$F_0^{(n)}(\bar{z}, t) = (\partial_{\bar{z}} + A(\bar{z}, t))^n \cdot 1. \tag{55}$$

Then we have the antiholomorphic Burgers hierarchy

$$\partial_{t_n} A(\bar{z}, t) = \partial_{\bar{z}}[(\partial_{\bar{z}} + A(\bar{z}, t))^n \cdot 1] \tag{56}$$

for an arbitrary positive integer n. Using the recursion operator $R = \partial_{\bar{z}} + \partial_{\bar{z}} A \partial_{\bar{z}}^{-1}$, we write it in the form

$$\partial_{t_n} A = R^{n-1}\partial_{\bar{z}} A. \tag{57}$$

This hierarchy can be linearized by the antiholomorphic Cole–Hopf transformation for the complex gauge field

$$A = \frac{\bar{\Phi}_{\bar{z}}}{\bar{\Phi}} = (\log \bar{\Phi})_{\bar{z}} = \overline{(f(z, t))}_{\bar{z}} \tag{58}$$

in terms of the holomorphic Schrödinger (heat) hierarchy

$$\partial_{t_n} \Phi = \partial_z^n \Phi. \tag{59}$$

For $n = 2$, the second member of the hierarchy is just (9), and the zeros of this equation correspond to the magnetic vortices of the Ishimori model. The relation between Φ and complex potential f has the meaning of the Madelung representation for the hierarchy

$$\Phi(z,t) = e^{f(z,t)} = e^{\varphi + i\psi} = (e^{\varphi})e^{i\psi} = \sqrt{\rho}e^{i\psi}. \tag{60}$$

Therefore, the hierarchy of equations for f is the Madelung form of the holomorphic Schrödinger hierarchy

$$\partial_{t_n} f = (\partial_z + \partial_z f)^n \cdot 1 = e^{-f}\partial_z^n e^f \tag{61}$$

or

$$\partial_{t_n}(e^f) = \partial_z^n(e^f), \tag{62}$$

which is the potential Burgers hierarchy. We have the linear problem for the Burgers hierarchy

$$\Phi_z = \bar{A}\Phi, \qquad \Phi_{t_n} = \partial_z^n \Phi. \tag{63}$$

It can be written as the Abelian zero-curvature representation for the holomorphic Burgers hierarchy $\partial_{t_n} U - \partial_{\bar{z}} V_n = 0$, where $U = A$ and $V_n = (\partial_{\bar{z}} + A)^n \cdot 1$. For N vortices of equal strength,

$$\Phi(z,t) = e^f = \prod_{k=1}^{N}(z - z_k(t)), \tag{64}$$

the positions of the vortices correspond to the zeros of $\Phi(z,t)$. As a result, the vortex dynamics, leading to integrable anyon dynamics, is related to the motion of zeros satisfying vortex equations (15) in the case $n = 2$ and the equation

$$-\frac{dz_k(t_n)}{dt_n} = \mathrm{Res}\,|_{z=z_k}\left(\partial_z + \sum_{l=1}^{N}\frac{1}{z - z_l(t_n)}\right)^n \cdot 1, \quad k = 1,\ldots,N, \tag{65}$$

in the case of arbitrary n.

7. Galilei group hierarchy and vortex solutions

We now consider the complex Galilei group hierarchy

$$[P_0, P_z] = 0, \qquad [P_0, K_n] = i^n n P_z^{n-1}, \qquad [P_z, K_n] = -i, \tag{66}$$

where the hierarchy of boost transformations is generated by $K_n = z + nt\,\partial_z^{n-1}$, which commutes with the operator of the nth equation in the holomorphic Schrödinger hierarchy

$$S_n = \partial_t - \partial_z^n. \tag{67}$$

As a result, applying K_n to the solution Φ creates a solution with an additional vortex,

$$\Psi(z,t) = K_n\Phi(z,t) = (z + nt\,\partial_z^{n-1})\Phi(z,t). \tag{68}$$

In particular, we have $K_n \cdot 1 = z = H_1^{(n)}(z,t)$, $K_n^2 \cdot 1 = z^2 = H_2^{(n)}(z,t)$, $\ldots$, $K_n^{n-1} \cdot 1 = z^{n-1} = H_{n-1}^{(n)}(z,t)$, $K_n^n \cdot 1 = z^n + n!\, t = H_n^{(n)}(z,t)$, $\ldots$, $K_n^m \cdot 1 = H_m^{(n)}(z,t)$, where the generalized Kampe de Feriet polynomials [15]

$$H_m^{(n)}(z,t) = m! \sum_{k=0}^{[m/n]} \frac{t^k z^{m-nk}}{k!(m-nk)!} \tag{69}$$

satisfy holomorphic Schrödinger hierarchy (59),

$$\frac{\partial}{\partial t} H_m^{(n)}(z,t) = \partial_z^n H_m^{(n)}(z,t). \tag{70}$$

The generating function is given by

$$\sum_{m=0}^{\infty} \frac{k^m}{m!} H_m^{(n)}(z,t) = e^{kz + k^n t}. \tag{71}$$

From the operator representation

$$H_n^{(N)}(z,t) = \exp\left(t \frac{\partial^N}{\partial z^N}\right) z^n \Rightarrow \Phi(z,t) = \exp\left(t \frac{\partial^N}{\partial z^N}\right) \psi(z), \tag{72}$$

we obtain the solution of (59) in terms of an arbitrary analytic function ψ. The polynomials $H_m^{(N)}(z,t)$ are related to the generalized Hermite polynomials $H_m^{(N)}(x)$ [16] by

$$H_m^{(N)}(z,t) = t^{[m/N]} H_m^{(N)}\left(\frac{z}{\sqrt[N]{t}}\right). \tag{73}$$

Then the kth zero $w_n^{(N)k}$ of the generalized Hermite polynomial $H_n^{(N)}$ determines the evolution of the corresponding vortex,

$$H_n^{(N)}(w_n^{(N)k}) = 0 \Rightarrow z_k(t) = w_n^{(N)k} \sqrt[N]{t}. \tag{74}$$

The zeros are located on a circle (with a time-dependent radius) in the plane. As $t \to -t$, the vortex position rotates through the angle $z_k \to z_k e^{i\pi/N}$. Galilei boost hierarchy (68) provides the Bäcklund transformation for the nth member of antiholomorphic Burgers hierarchy (56),

$$v = u + \partial_z \log[z + Nt(\partial_z + u)^{N-1} \cdot 1]. \tag{75}$$

8. The negative Burgers hierarchy

The holomorphic Schrödinger hierarchy and the corresponding Burgers hierarchy can be analytically extended to negative values of N. Introducing the negative derivative (pseudodifferential) operator ∂_z^{-1} such that $\partial_z^{-m} z^n = z^{n+m}/\big((n+1)\cdots(n+m)\big)$, we obtain the hierarchy

$$\partial_{t_{-n}} \Phi = \partial_z^{-n} \Phi \tag{76}$$

or, differentiating n times, $\partial_{t_{-n}} \partial_z^n \Phi = \Phi$ in pure differential form. In terms of A defined by (58), we have the negative Burgers hierarchy

$$\partial_{t_{-n}} A = \partial_{\bar{z}} \frac{1 - \partial_{t_{-n}} A^n}{A^n}. \tag{77}$$

For $n = 1$, we have the equation $\partial_{t_{-1}}\Phi = \partial_z^{-1}\Phi$ or the Helmholz equation $\partial_{t_{-1}}\partial_z\Phi = \Phi$. The analytic continuation of the generalized Kampe de Feriet polynomials to $n = -1$ [15] is given by

$$H_M^{(-1)}(z,t) = M!\sum_{k=0}^{\infty}\frac{t^k z^{M+k}}{k!(M+k)!}. \tag{78}$$

Then

$$H_M^{(-1)}(z,t) = e^{t\partial_z^{-1}}H_M^{(-1)}(z,0), \qquad H_M^{(-1)}(z,0) = z^M. \tag{79}$$

Moreover, higher-order functions are generated by the "negative Galilean boost"

$$H_M^{(-1)}(z,t) = (z - t\partial_z^{-2})^M H_0^{(-1)}(z,t). \tag{80}$$

The functions $H_M^{(-1)}(z,t)$ are related to Bessel functions [15]. First, they are directly related to the Tricomi functions

$$C_M(zt) = \frac{z^{-M}}{M!}H_M^{(-1)}(z,t), \tag{81}$$

determined by the generating function

$$\sum_{M=-\infty}^{\infty}\lambda^M C_M(x) = e^{\lambda + x/\lambda}, \tag{82}$$

which is related to the Bessel functions by

$$J_M(x) = \left(\frac{x}{2}\right)^M C_M\left(-\frac{x^2}{4}\right). \tag{83}$$

Explicitly, we then have

$$H_M^{(-1)}(z,t) = M!\left(\frac{-z}{t}\right)^{M/2} J_M(2\sqrt{-zt}). \tag{84}$$

This yields the solution of the first negative flow (-1) Burgers equation $\partial_t A = \partial_{\bar z}(1 - \partial_t A)/A$ in the form

$$A = \frac{(H_M^{(-1)}(\bar z,t))_{\bar z}}{H_M^{(-1)}(\bar z,t)} = \frac{M}{2\bar z} + \sqrt{\frac{t}{-\bar z}}\frac{J_M'}{J_M} = \sqrt{\frac{t}{-\bar z}}\frac{J_{M-1}(2\sqrt{-\bar z t})}{J_M(2\sqrt{-\bar z t})}. \tag{85}$$

For an arbitrary member of the negative hierarchy, we have

$$\begin{aligned} &H_M^{(-N)}(z,t) = e^{t\partial_z^{-N}}H_M^{(-N)}(z,0), \\ &H_M^{(-N)}(z,0) = z^M, \end{aligned} \tag{86}$$

and

$$W_M^{(N)}(zt^{1/N}) = \frac{z^{-M}}{M!}H_M^{(-N)}(z,t), \tag{87}$$

where the Wright–Bessel functions $W_M^{(N)}(x)$ [15] are given by the generating function

$$\sum_{M=-\infty}^{\infty}\lambda^M W_M^{(N)}(x) = e^{\lambda + x/\lambda^N}. \tag{88}$$

9. Space–time noncommutativity

We now consider the influence of the space–time noncommutativity on the vortex dynamics. It is remarkable that the problem for two vortices can be solved explicitly. The noncommutative Burgers equation, its linearization by the Cole–Hopf transformation, and the two-soliton collision was considered in [17]. Here, we consider the holomorphic heat equation

$$\partial_t \Phi = \nu\, \partial_z^2 \Phi \tag{89}$$

with the two-vortex solution in the form

$$\Phi(z,t) = (z - z_1(t)) * (z - z_2(t)), \tag{90}$$

where the Moyal product is defined as

$$\begin{aligned} f(t,z) * g(t,z) &= e^{i\theta(\partial_t\partial_{z'} - \partial_{t'}\partial_z)} f(t,z) g(t',z')|_{z=z',\, t=t'} = \\ &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (\partial_t^{n-k}\partial_z^k f)(\partial_t^k \partial_z^{n-k} g). \end{aligned} \tag{91}$$

We then have the θ-deformed vortex equations

$$\begin{aligned} \dot z_1 &= \frac{-2\nu}{z_1 - z_2} - i\theta \frac{\ddot z_1 - \ddot z_2}{z_1 - z_2}, \\ \dot z_2 &= \frac{2\nu}{z_1 - z_2} + i\theta \frac{\ddot z_1 - \ddot z_2}{z_1 - z_2}. \end{aligned} \tag{92}$$

Adding them, we obtain the first integral of motion $z_1 + z_2 = C$ (the center of mass). Choosing the center of mass as the coordinate origin, we have $C = 0$ and $z_2 = -z_1$. Integrating the reduced equation for z_1 and substituting $z_1 = 2i\theta Y(t)$, we obtain the Ricatti equation

$$\dot Y + Y^2 = \frac{\nu}{2\theta^2}(t - t_0). \tag{93}$$

This can be linearized by substituting $Y = \dot\psi/\psi$, reducing it to the Airy equation

$$\ddot\psi = \frac{\nu}{2\theta^2}(t - t_0)\psi. \tag{94}$$

The solution is

$$z_1(t) = 2i\theta\beta \frac{\mathrm{Ai}'(\beta(t-t_0))}{\mathrm{Ai}(\beta(t-t_0))} = -i\sqrt{2\nu(t-t_0)}\, \frac{K_{2/3}\big(\sqrt{2\nu}(t-t_0)^{3/2}/(3\theta)\big)}{K_{1/3}\big(\sqrt{2\nu}(t-t_0)^{3/2}/(3\theta)\big)}, \tag{95}$$

where $\beta = (\nu/(2\theta^2))^{1/3}$ and K_n are modified Bessel functions of fractional order. This solution should be compared with the undeformed solution (32). The noncommutative corrections come from the ratio of two Bessel functions depending on θ. Using the asymptotic form of the Airy function, we obtain the correction in the form

$$z_1(t) = -z_2(t) \approx -i\sqrt{2\nu(t-t_0)} - \frac{i\theta}{2(t-t_0)} \tag{96}$$

as $t \to +\infty$. As can be easily seen, the correction is independent of the diffusion coefficient and is global.

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