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SOLITON RESONANCES FOR THE MKP-II

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Using the second flow (the derivative reaction–diffusion system) and the third one of the dissipative $SL(2,\mathbb{R})$ Kaup–Newell hierarchy, we show that the product of two functions satisfying those systems is a solution of the modified Kadomtsev–Petviashvili equation in 2+1 dimensions with negative dispersion (MKP-II). We construct Hirota's bilinear representations for both flows and combine them as the bilinear system for the MKP-II. Using this bilinear form, we find one- and two-soliton solutions for the MKP-II. For special values of the parameters, our solution shows resonance behavior with the creation of four virtual solitons. Our approach allows interpreting the resonance soliton as a composite object of two dissipative solitons in 1+1 dimensions.

Keywords: soliton resonance, dissipative soliton, modified Kadomtsev–Petviashvili equation, Hirota method, derivative reaction–diffusion system

1. Introduction

The gauge theory formulation of low-dimensional gravity models, such as the Jackiw–Teitelboim model [1], is based on the Cartan–Einstein vielbein or the moving-frame method. In terms of these variables in 1+1 dimensions, we deal with the so-called BF gauge theory and the zero-curvature equations of motion, providing a link with soliton equations [2]. But in these variables, represented as the "square root" of the pseudo-Riemannian metric, the soliton equations have a dissipative form; we therefore call them the dissipative solitons or dissipatons [2]. The dissipative version of the nonlinear Schrödinger equation with a rich resonance dynamics is a couple of nonlinear diffusion and antidiffusion equations, which we call the reaction–diffusion (RD) system [2], [3]. The dissipaton of that system is a pair of two real functions, one exponentially growing and the other decaying in space and time. But the product of these two functions has the perfect soliton form. As recently realized [4], if dissipatons of the RD system evolve with an additional time variable according to the next member after the RD system in the $SL(2,\mathbb{R})$ AKNS hierarchy with cubic dispersion, then this product can be considered a soliton of the (2+1)-dimensional Kadomtsev–Petviashvili equation with negative dispersion (KP-II). The resonance behavior of the KP-II solitons was thus found in terms of dissipatons of the (1+1)-dimensional models. Moreover, in this approach, the novel two-resonance soliton of the KP-II with four virtual solitons was constructed and interpreted as a degenerate four-soliton solution [4].

On the other hand, as previously shown [5], the dissipative version of the derivative nonlinear Schrödinger equations (DNLS) also admits dissipaton solutions with resonance interaction [6]. Moreover, these resonances show chirality properties, propagating in only one direction. In Sec. 2, following the strategy in [4], via the recursion operator of the Kaup–Newell (KN) hierarchy, we first construct the next dissipative system of the hierarchy with cubic dispersion. Using these two members of the $SL(2,\mathbb{R})$ KN hierarchy, we then show that the product of dissipaton functions is a solution of the modified KP-II (MKP-II). In

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Sec. 3, bilinearizing the two flows allows finding a bilinear form for the MKP-II. In Sec. 4, we consider chiral resonance dissipatons of the DRD system and their geometric meaning. In Sec. 5, we construct one- and two-soliton solutions of the MKP-II. We show that the soliton interaction has a resonance and that the chirality property imposes a restriction on the soliton collision angles. In conclusion, we discuss the main results in this paper.

2. MKP-II from the KN hierarchy

The KN hierarchy has the form [7]

$$\begin{pmatrix} q_{t_n} \\ r_{t_n} \end{pmatrix} = JL^n \begin{pmatrix} q \\ r \end{pmatrix}, \tag{2.1}$$

where the operator

$$J = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \tag{2.2}$$

is the first symplectic form,

$$L = \frac{1}{2}\begin{pmatrix} -\partial - r\partial^{-1}q\partial & -r\partial^{-1}r\partial \\ -q\partial^{-1}q\partial & \partial - q\partial^{-1}r\partial \end{pmatrix} \tag{2.3}$$

is the recursion operator of the hierarchy, and $\partial \equiv \partial/\partial x$. The second flow of the hierarchy is the system

$$q_{t_2} = \frac{1}{2}\big[q_{xx} + (q^2 r)_x\big], \tag{2.4a}$$

$$r_{t_2} = \frac{1}{2}\big[-r_{xx} + (r^2 q)_x\big], \tag{2.4b}$$

and the third one is

$$q_{t_3} = -\frac{1}{4}\left[q_{xx} + 3rqq_x + \frac{3}{2}(r^2q^2)q\right]_x, \tag{2.5a}$$

$$r_{t_3} = -\frac{1}{4}\left[r_{xx} - 3rqr_x + \frac{3}{2}(r^2q^2)r\right]_x. \tag{2.5b}$$

For the $SL(2,\mathbb{R})$ case of the KN hierarchy, we have the real time variables t_2 and t_3, here represented as $y \equiv t_2/2$ and $t \equiv -t_3/4$. In this case, the functions q and r are real, and we set

$$e^+ \equiv q, \qquad e^- \equiv -r. \tag{2.6}$$

We then have the DRD system [5]

$$e^+_y = e^+_{xx} - (e^+e^-e^+)_x, \tag{2.7a}$$

$$e^-_y = -e^-_{xx} - (e^+e^-e^-)_x \tag{2.7b}$$

and

$$e^+_t = e^+_{xxx} - 3(e^+e^-e^+_x)_x + \frac{3}{2}\big((e^+e^-)^2e^+\big)_x, \tag{2.8a}$$

$$e^-_t = e^-_{xxx} + 3(e^+e^-e^-_x)_x + \frac{3}{2}\big((e^+e^-)^2e^-\big)_x. \tag{2.8b}$$

We now consider the pair of functions of three variables $e^+(x,y,t)$ and $e^-(x,y,t)$ satisfying systems (2.7) and (2.8). These systems are compatible because they belong to the same hierarchy for different times. This compatibility can also be verified directly from the compatibility condition $e^{\pm}_{ty} = e^{\pm}_{yt}$ by using the respective conservation laws for Eqs. (2.7) and (2.8)

$$
\begin{aligned}
(e^+e^-)_y &= \left[(e^+_x e^- - e^+ e^-_x) - \frac{3}{2}(e^+e^-)^2\right]_x, \\
(e^+e^-)_t &= \left[(e^+e^-)_{xx} - 3(e^+_x e^-_x) + 3(e^+e^-)(e^+e^-_x - e^+_x e^-) + \frac{5}{2}(e^+e^-)^3\right]_x.
\end{aligned}
\tag{2.9}
$$

Proposition 1. *Let the functions $e^+(x,y,t)$ and $e^-(x,y,t)$ be simultaneous solutions of systems* (2.7) *and* (2.8). *Then the function $U(x,y,t) \equiv e^+e^-$ satisfies the MKP-II*

$$
\left(-4U_t + U_{xxx} - \frac{3}{2}U^2U_x - 3U_x\partial_x^{-1}U_y\right)_x = -3U_{yy} \tag{2.10}
$$

or, written in another form,

$$
-4U_t + U_{xxx} - \frac{3}{2}U^2U_x - 3U_xW = -3W_y, \tag{2.11a}
$$

$$
W_x = U_y \tag{2.11b}
$$

(*we obtain the second form from the first by introducing the auxiliary variable W according to Eq.* (2.11b) *and integrating over x*).

The proof is straightforward. From the definition of U and Eqs. (2.7) and (2.9), we have

$$
\begin{aligned}
U_y &= \left[(e^+_x e^- - e^+ e^-_x) - \frac{3}{2}U^2\right]_x, \\
U_{yy} &= \left[U_{xxx} - 4(e^+_x e^-_x)_x - 3U_x(e^+_x e^- - e^+ e^-_x) - U(e^+_x e^- - e^+e^-_x)_x - \frac{3}{2}(U^2)_y\right]_x.
\end{aligned}
\tag{2.12}
$$

On the other hand, Eqs. (2.8) and (2.9) give

$$
U_t = \left[U_{xx} - 3(e^+_x e^-_x) - 3U(e^+_x e^- - e^+e^-_x) + \frac{5}{2}U^3\right]_x, \tag{2.13}
$$

$$
U_{tx} = \left[U_{xx} - 3(e^+_x e^-_x) - 3U(e^+_x e^- - e^+e^-_x) + \frac{5}{2}U^3\right]_{xx}. \tag{2.14}
$$

We first combine Eqs. (2.12) and (2.14) to cancel the term $e^+_x e^-_x$ and then use Eq. (2.12) to eliminate $e^+_x e^- - e^+ e^-_x$ and its derivative according to

$$
(e^+_x e^- - e^+e^-_x)_x = U_y - \frac{3}{2}(U^2)_x \tag{2.15}
$$

and

$$
(e^+_x e^- - e^+e^-_x) = \partial_x^{-1}U_y - \frac{3}{2}U^2, \tag{2.16}
$$

obtained from (2.15) by integrating once. As a result, we obtain MKP-II (2.10).

3. Bilinear form for the second and third flows

We now construct a bilinear representation for systems (2.7) and (2.8) to find solutions of the MKP-II according to Proposition 1. In [5], we used the Hirota bilinear method to integrate DRD (2.7). We now apply the same method to Eq. (2.8) and the MKP-II. As noted in [5], the standard Hirota substitution as the ratio of two functions does not work directly for e^+ and e^- (this is also related to the complicated analytic structure of the DNLS [8]). To have the standard Hirota substitution, following [5], we first rewrite systems (2.7) and (2.8) in terms of the new functions Q^+ and Q^-,

$$e^+ = e^{+\int^x Q^+Q^-} Q^+, \qquad e^- = e^{-\int^x Q^+Q^-} Q^-, \tag{3.1}$$

and consequently have the systems

$$Q^+_y = Q^+_{xx} + Q^+Q^+Q^-_x - \frac{1}{2}(Q^+Q^-)^2Q^+, \tag{3.2a}$$

$$Q^-_y = -Q^-_{xx} + Q^-Q^-Q^+_x + \frac{1}{2}(Q^+Q^-)^2Q^-, \tag{3.2b}$$

and

$$Q^+_t = Q^+_{xxx} + 3Q^+_xQ^-_xQ^+ - \frac{3}{2}(Q^+Q^-)^2Q^+_x, \tag{3.3a}$$

$$Q^-_t = Q^-_{xxx} - 3Q^+_xQ^-_xQ^- - \frac{3}{2}(Q^+Q^-)^2Q^-_x. \tag{3.3b}$$

Then because

$$Q^+Q^- = e^+e^- = U, \tag{3.4}$$

systems (3.2) and (3.3) also provide a solution of the MKP-II, which we can formulate as follows.

Proposition 2. *Let the functions $Q^+(x,y,t)$ and $Q^-(x,y,t)$ be simultaneous solutions of systems* (3.2) *and* (3.3). *Then the function $U(x,y,t) \equiv Q^+Q^-$ satisfies MKP-II* (2.10) *or* (2.11).

To solve systems (3.2) and (3.3), we introduce four real functions g^+, g^-, f^+, and f^- according to the formulas

$$Q^+ = \frac{g^+}{f^+}, \qquad Q^- = \frac{g^-}{f^-}, \tag{3.5}$$

or, using Eqs. (3.1) and (3.4) for the original variables e^+ and e^-, we have the substitution

$$e^+ = \frac{g^+f^+}{(f^-)^2}, \qquad e^- = \frac{g^-f^-}{(f^+)^2}. \tag{3.6}$$

System (3.2) then bilinearizes in the form

$$(D_y \mp D_x^2)(g^\pm \cdot f^\pm) = 0, \tag{3.7a}$$

$$D_x^2(f^+ \cdot f^-) + \frac{1}{2}D_x(g^+ \cdot g^-) = 0, \tag{3.7b}$$

$$D_x(f^+ \cdot f^-) - \frac{1}{2}g^+g^- = 0. \tag{3.7c}$$

Similarly, for system (3.3), we have the next bilinear form

$$(D_t - D_x^3)(g^\pm \cdot f^\pm) = 0, \tag{3.8a}$$

$$D_x^2(f^+ \cdot f^-) + \frac{1}{2} D_x(g^+ \cdot g^-) = 0, \tag{3.8b}$$

$$D_x(f^+ \cdot f^-) - \frac{1}{2} g^+ g^- = 0. \tag{3.8c}$$

Comparing these two bilinear forms, we can see that the second and third equations in the systems have the same form. For a simultaneous solution of Eqs. (3.2) and (3.3), we therefore have the next bilinear system

$$(D_y \mp D_x^2)(g^\pm \cdot f^\pm) = 0, \tag{3.9a}$$

$$(D_t - D_x^3)(g^\pm \cdot f^\pm) = 0, \tag{3.9b}$$

$$D_x^2(f^+ \cdot f^-) + \frac{1}{2} D_x(g^+ \cdot g^-) = 0, \tag{3.9c}$$

$$D_x(f^+ \cdot f^-) - \frac{1}{2} g^+ g^- = 0. \tag{3.9d}$$

From the last equation, we obtain

$$U = e^+ e^- = Q^+ Q^- = \frac{g^+ g^-}{f^+ f^-} = 2\frac{D_x(f^+ \cdot f^-)}{f^+ f^-} = 2\frac{f_x^+ f^- - f^+ f_x^-}{f^+ f^-},$$

which provides the formula for the solution of the MKP-II

$$U = 2\left(\log \frac{f^+}{f^-}\right)_x. \tag{3.10}$$

4. Resonance solitons of the DRD system

We first consider DRD system (2.7) as an evolution equation with $y = x_0 = t$ interpreted as the time variable.

4.1. Chiral dissipaton solution. For the one-dissipaton solution, we have

$$g^\pm = e^{\eta^\pm}, \qquad f^\pm = 1 + e^{\phi^\pm} e^{\eta^+ + \eta^-}, \qquad e^{\phi^\pm} = \pm \frac{k^\mp}{2(k^+ + k^-)^2}, \tag{4.1}$$

where $\eta^\pm = k^\pm x \pm (k^\pm)^2 t + \eta_0^\pm$. Regularity requires that we choose the conditions $k^- > 0$ and $k^+ < 0$, and we hence have the dissipaton

$$Q^\pm = \frac{e^{\pm(\eta^+ - \eta^-)/2 - \alpha_\pm/2}}{2\cosh\left[(\eta^+ + \eta^- + \alpha_\pm)/2\right]} \tag{4.2}$$

with the solitonic density

$$e^+ e^- = Q^+ Q^- = \frac{2k^2}{\sqrt{v^2 - k^2}\,\cosh k(x - vt - x_0) + v}. \tag{4.3}$$

In the last equation, we introduce the dissipaton amplitude and velocity, $k = k^+ + k^-$ and $v = k^- - k^+$, in terms of which the above conditions mean that the dissipaton velocity is bounded from below by k. We note that in contrast to the dissipatons of the RD system, there is no critical value from above for the dissipaton velocity in our case.

For the mass and momentum densities, we have

$$\rho = e^+e^- = 2\partial_x \log \frac{f^+}{f^-}, \tag{4.4}$$

$$p = \frac{1}{2}\big(e^+\partial_x e^- - e^-\partial_x e^+ + (e^+e^-)^2\big) = \partial_x^2 \log f^+f^-, \tag{4.5}$$

which allows calculating the corresponding conserved quantities

$$M = \int_{-\infty}^{+\infty} \rho\,dx = 2\log\frac{f^+}{f^-}\bigg|_{-\infty}^{+\infty}, \qquad P = \int_{-\infty}^{+\infty} p\,dx = \partial_x \log f^+f^-\big|_{-\infty}^{+\infty}. \tag{4.6}$$

For the mass and momentum of a single dissipaton, we then obtain

$$M = \log\left(\frac{v+k}{v-k}\right)^2, \qquad P = k. \tag{4.7}$$

Because $|v| > |k|$, the mass M is positive. Rewriting the momentum in the canonical form $k = \mu v$, we find the effective mass $\mu = \tanh(M/4)$.

4.2. Geometric interpretation. The model has a geometric interpretation as a two-dimensional pseudo-Riemannian space–time with a constant scalar curvature $R = \Lambda < 0$ [2], [3]. This model is known as the Jackiw–Teitelboim gravity model [1]. It admits a gauge theory formulation as the BF theory. The gauge potentials are the Cartan–Einstein zweibein fields $e_\mu^\pm$, the metric is hence $g_{\mu\nu} = (e_\mu^+e_\nu^- + e_\nu^+e_\mu^-)/2$, and the spin connection is ω_μ, $\mu = 0, 1$. Then the equations of motion

$$D_\mu^\mp e_\nu^\pm = D_\nu^\mp e_\mu^\pm, \tag{4.8}$$

$$\partial_\mu\omega_\nu - \partial_\nu\omega_\mu = -\frac{\Lambda}{4}(e_\mu^+e_\nu^- - e_\nu^+e_\mu^-), \tag{4.9}$$

where $D_\mu^\pm = \partial_\mu \pm \omega_\mu$ have the respective meanings of the torsionless and the constant-curvature conditions. We fix the gauge freedom and the corresponding evolution by the conditions on the Lagrange multipliers

$$e_0^\pm = \pm\left(\partial_1 \mp \frac{\Lambda}{4} \mp e^+e^-\right)e^\pm \tag{4.10}$$

and spin connections

$$\omega_0 = \frac{\Lambda^2}{16} + \frac{\Lambda}{4}e^+e^-, \qquad \omega_1 = -\frac{\Lambda}{4}. \tag{4.11}$$

Identifying $(t, x) = (x_0, x_1)$, we then reduce system (4.8), (4.9) to DRD system (2.7). We note that in contrast to the RD [3], the scalar curvature in our case disappears from equations of motion but is still present in the linear problem. Moreover, it has the meaning of the squared spectral parameter $\Lambda = -8\lambda^2$.

The metric-tensor component g_{00} in terms of transformed variables (3.1) is given by

$$g_{00} = e_0^+ e_0^- = -\left(\partial_1 - \frac{\Lambda}{4}\right) Q^+ \left(\partial_1 + \frac{\Lambda}{4}\right) Q^-. \tag{4.12}$$

Calculating it for one dissipaton (4.2), we find that the event horizon $g_{00} = 0$ exists only for the bounded velocity $-k + (-\Lambda/2) < v < k + (-\Lambda/2)$ at the distance

$$x_{\mathrm{H}} - vt_{\mathrm{H}} - x_{0\mathrm{H}} = \frac{1}{k} \log \frac{4k^2(k - v - \Lambda/2)}{(k + v)(k + v + \Lambda/2)}. \tag{4.13}$$

In contrast to the RD dissipaton with two symmetric event horizons reflecting two directions of motion, we now have only one directional motion, and we call the corresponding single event horizon the *chiral event horizon*.

5. Resonance solitons of the MKP-II

We now consider a solution of system (3.9) giving a (2+1)-dimensional solution of the MKP-II. For a one-soliton solution, we have

$$g^\pm = e^{\eta_1^\pm}, \qquad f^\pm = 1 + e^{\phi_{11}^\pm} e^{\eta_1^+ + \eta_1^-}, \qquad e^{\phi_{11}^\pm} = \pm \frac{k_1^\mp}{2(k_1^+ + k_1^-)^2}, \tag{5.1}$$

where $\eta_1^\pm = k_1^\pm x \pm (k_1^\pm)^2 y + (k^\pm)^3 t + \eta_0^\pm$. The regularity condition requires $k_1^+ \le 0$ and $k_1^- \ge 0$. We then have

$$U(x, y, t) = \frac{2k^2}{\sqrt{p^2 - k^2}\, \cosh k\big(x - py + (k^2 + 3p^2)t/4 - a_0\big) + p}, \tag{5.2}$$

where $k = k_1^+ + k_1^-$, $p = k_1^- - k_1^+ > 0$, and the parameter $p^2 > k^2$, bounded from below, is positive $p > 0$. The geometric meaning of this parameter is $p^{-1} = \tan\alpha$, where α is the slope of the soliton line with respect to the x axis. Because of the condition $p > 0$, the direction of this line is restricted between $0 < \alpha < \pi/2$ (this is the space analogue of the chirality property of the dissipaton in 1+1 dimensions for the DNLS [5], where it propagates in only one direction). The soliton velocity is the two-dimensional vector $\mathbf{v} = (\omega, -\omega/p)$, where $\omega = (k^2 + 3p^2)/4$, directed at the angle γ to the soliton line, where $\cos\gamma = 1 - 1/p^2$. When $p = 1$, the soliton velocity is orthogonal to the soliton line.

For a two-soliton solution, we have

$$g^\pm = e^{\eta_1^\pm} + e^{\eta_2^\pm} + \alpha_1^\pm e^{\eta_2^+ + \eta_2^- + \eta_1^\pm} + \alpha_2^\pm e^{\eta_1^+ + \eta_1^- + \eta_2^\pm},$$

$$f^\pm = 1 + \sum_{i,j=1}^{2} e^{\phi_{ij}^\pm} e^{\eta_i^+ + \eta_j^-} + \beta^\pm e^{\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^-},$$

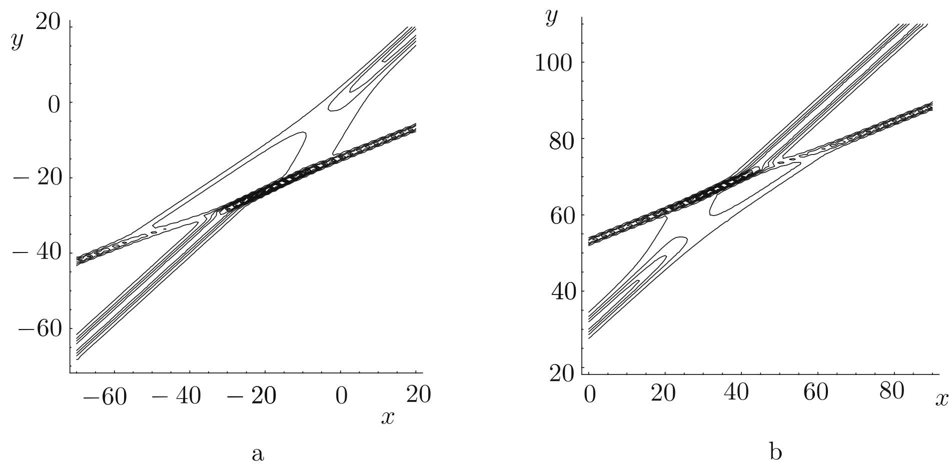

Fig. 1

where $\eta_i^{\pm} = k_i^{\pm} x \pm (k_i^{\pm})^2 y + (k_i^{\pm})^3 t + \eta_{i0}^{\pm}$, $k_{ij}^{nm} \equiv (k_i^n + k_j^m)$ and

$$\alpha_1^{\pm} = \pm\frac{1}{2}\frac{k_2^{\mp}(k_1^{\pm} - k_2^{\pm})^2}{(k_{22}^{+-})^2(k_{12}^{\pm\mp})^2}, \qquad \alpha_2^{\pm} = \pm\frac{1}{2}\frac{k_1^{\mp}(k_1^{\pm} - k_2^{\pm})^2}{(k_{11}^{+-})^2(k_{21}^{\pm\mp})^2},$$

$$\beta^{\pm} = \frac{(k_1^{+} - k_2^{+})^2(k_1^{-} - k_2^{-})^2}{4(k_{11}^{+-}k_{12}^{+-}k_{21}^{+-}k_{22}^{+-})^2}k_1^{\mp}k_2^{\mp},$$

$$e^{\phi_{ii}^{\pm}} = \pm\frac{k_i^{\mp}}{2(k_{ii}^{+-})^2}, \qquad e^{\phi_{ij}^{+}} = \frac{k_j^{-}}{2(k_{ij}^{+-})^2}, \qquad e^{\phi_{ij}^{-}} = -\frac{k_i^{+}}{2(k_{ij}^{+-})^2}.$$

The regularity conditions are now the same as for one soliton: $k_i^{+} \leq 0$ and $k_i^{-} \geq 0$. This solution then describes a collision of two solitons propagating in the plane and creating resonance states for some parameter values. The time evolution of these states is shown in Fig. 1.

6. Conclusion

We have constructed virtual soliton resonance solutions for the MKP-II in terms of dissipatons of (1+1)-dimensional equations as the DRD system and its next higher member in the $SL(2,\mathbb{R})$ KN hierarchy. The difference from the KP-II resonance is the additional restrictions on soliton angles due to the regularity conditions.

After this paper was finished, Konopelchenko, at the workshop “Nonlinear Physics: Theory and Experiment III,” directed our attention to the relations between the MKP equation and (1+1)-dimensional models by symmetry reduction of (2+1)-dimensional models [9], [10]. But in [9], only a relation between the MKP and the Burgers hierarchy was established, while [10] related the MKP to the DNLS in the Nakamura–Chen form but not in the KN form. Moreover, there are no results concerning resonance solitons in those papers.

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