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DEGENERATE FOUR-VIRTUAL-SOLITON RESONANCE FOR THE KP-II

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We propose a method for solving the (2+1)-dimensional Kadomtsev–Petviashvili equation with negative dispersion (KP-II) using the second and third members of the disipative version of the AKNS hierarchy. We show that dissipative solitons (dissipatons) of those members yield the planar solitons of the KP-II. From the Hirota bilinear form of the $SL(2,\mathbb{R})$ AKNS flows, we formulate a new bilinear representation for the KP-II, by which we construct one- and two-soliton solutions and study the resonance character of their mutual interactions. Using our bilinear form, for the first time, we create a four-virtual-soliton resonance solution of the KP-II, and we show that it can be obtained as a reduction of a four-soliton solution in the Hirota–Satsuma bilinear form for the KP-II.

Keywords: dissipative soliton, Ablowitz–Kaup–Newell–Segur hierarchy, Kadomtsev–Petviashvili equation, Hirota method, soliton resonance, reaction–diffusion system

1. Introduction

A dissipative version of the AKNS hierarchy [1] was recently considered in connection with (1+1)-dimensional (lineal) gravity models [2]. It was found that the second flow described by the dissipative version of the nonlinear Schrödinger (NLS) equation, the so-called reaction–diffusion system, admits new soliton-type solutions called *dissipatons*. Dissipatons have exponentially growing and decaying amplitudes with a perfect soliton shape for their bilinear product and a resonance interaction behavior.

In the present paper, we study resonance dissipative solitons (dissipatons) in the AKNS hierarchy and show that they yield the planar solitons of the (2+1)-dimensional Kadomtsev–Petviashvili equation with negative dispersion (KP-II). Our approach is based on a method for generating solutions of the (2+1)-dimensional KP equation: we show that if a simultaneous solution of the second and third flows of the AKNS hierarchy is considered, then the product e^+e^- satisfies the KP-II (see the proposition in Sec. 4). Using these results, we construct a new bilinear representation of the KP-II with one- and two-soliton solutions. We show that our two-soliton solution corresponds to the degenerate four-soliton solution in the standard Hirota form of the KP and displays a four-virtual-soliton resonance.

2. The $SL(2,\mathbb{R})$ AKNS hierarchy

The dissipative $SL(2,\mathbb{R})$ AKNS hierarchy of evolution equations

$$\frac{1}{2}\sigma_3 \begin{pmatrix} e^+ \\ e^- \end{pmatrix}_{t_N} = \Re^{N+1} \begin{pmatrix} e^+ \\ e^- \end{pmatrix}, \tag{1}$$

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where $N = 0, 1, 2, \ldots$, $\Lambda < 0$, is generated by the recursion operator $\Re$,

$$\Re = \begin{pmatrix} \partial_x - \frac{\Lambda}{4} e^+ \int^x e^- & -\frac{\Lambda}{4} e^+ \int^x e^+ \\ -\frac{\Lambda}{4} e^- \int^x e^- & \partial_x + \frac{\Lambda}{4} e^- \int^x e^+ \end{pmatrix}. \tag{2}$$

The second and third members of AKNS hierarchy are then

$$\begin{cases} e^+_{t_1} = e^+_{xx} + \frac{\Lambda}{4} e^+ e^- e^+, \\ -e^-_{t_1} = e^-_{xx} + \frac{\Lambda}{4} e^+ e^- e^- \end{cases} \tag{3}$$

and

$$\begin{cases} e^+_{t_2} = e^+_{xxx} + \frac{3\Lambda}{4} e^+ e^- e^+_x, \\ e^-_{t_2} = e^-_{xxx} + \frac{3\Lambda}{4} e^+ e^- e^-_x. \end{cases} \tag{4}$$

System (3), the dissipative version of the NLS equation, is called the reaction–diffusion (RD) system [2]. It is connected with the gauge theoretical formulation of (1+1)-dimensional gravity, constant-curvature surfaces in pseudo-Euclidean space [2], and the NLS soliton problem in the quantum potential [2], [3].

3. Resonance dissipatons in the AKNS hierarchy

3.1. Dissipatons of the RD system. The second member of the AKNS hierarchy, RD system (3), by the substitution

$$e^{\pm} = \sqrt{\frac{8}{-\Lambda}} \frac{G^{\pm}(x,t)}{F(x,t)} \tag{5}$$

admits the Hirota bilinear representation, $t \equiv t_1$,

$$(\pm D_t - D_x^2)(G^{\pm} \cdot F) = 0, \qquad D_x^2(F \cdot F) = -2G^+G^-. \tag{6}$$

Any solution of this system then determines a solution of RD system (3). The simplest solution of bilinear system (6) has the form [3]

$$G^{\pm} = \pm e^{\eta_1^{\pm}}, \qquad F = 1 + \frac{e^{(\eta_1^+ + \eta_1^-)}}{(k_1^+ + k_1^-)^2}, \tag{7}$$

where $\eta_1^{\pm} = k_1^{\pm} x \pm (k_1^{\pm})^2 t + \eta_1^{\pm(0)}$. This solution determines a soliton-type solution of the RD system with exponentially growing and decaying amplitudes, called the dissipaton [2]. But for the product e^+e^-, we have the perfect one-soliton shape

$$e^+e^- = \frac{8k^2}{\Lambda \cosh^2\big[k(x - vt - x_0)\big]} \tag{8}$$

with the amplitude $k = (k_1^+ + k_1^-)/2$, propagating with the velocity $v = -(k_1^+ - k_1^-)$, where the initial position is $x_0 = -\log(k_1^+ + k_1^-)^2 + \eta_1^{+(0)} + \eta_1^{-(0)}$.

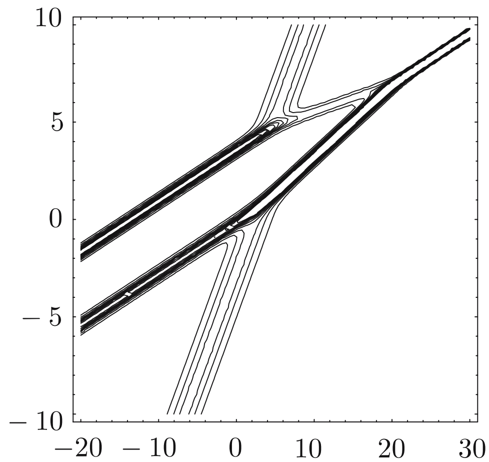

Fig. 1

The RD system has a geometric interpretation in the language of constant-curvature surfaces [2]. It follows that when $e^{\pm}$ satisfy RD equations (3), the Riemannian metric describes a two-dimensional pseudo-Riemannian space–time with the constant curvature Λ: $R = g^{\mu\nu}R_{\mu\nu} = \Lambda$. If we calculate the metric for one-dissipaton solution (7), it shows a singularity (sign change) at $\tanh k(x-vt) = \pm v/2k$. This singularity (called the causal singularity) has a physical interpretation in terms of black-hole physics and relates to the resonance properties of solitons. In fact, constructing a two-dissipaton solution, we find that it describes a collision of two dissipatons creating a resonance (metastable) bound state [3].

3.2. Dissipatons for the third flow. For the third flow of the AKNS hierarchy, we have cubic dispersion system (4). The bilinear representation of this system for the functions $e^{\pm}(x,t)$ in terms of three real functions $G^{\pm}$ and F, as in (5), is

$$(D_t + D_x^3)(G^{\pm}\cdot F) = 0, \qquad D_x^2(F\cdot F) = -2G^+G^-. \tag{9}$$

From the last equation, we have the expression for the product

$$U = e^+e^- = \frac{8}{-\Lambda}\frac{G^+G^-}{F^2} = \frac{4}{\Lambda}\frac{D_x^2(F\cdot F)}{F^2} = \frac{8}{\Lambda}\frac{\partial^2}{\partial x^2}\log F. \tag{10}$$

The simplest solution of this system,

$$G^{\pm} = \pm e^{\eta_1^{\pm}}, \qquad F = 1 + \frac{e^{(\eta_1^+ + \eta_1^-)}}{(k_1^+ + k_1^-)^2}, \tag{11}$$

where $\eta_1^{\pm} = k_1^{\pm}x - (k_1^{\pm})^3 t + \eta_1^{\pm(0)}$, defines a one-dissipaton solution of system (4),

$$e^{\pm} = \pm\sqrt{\frac{8}{-\Lambda}}\,\frac{|k_{11}^{+-}|}{2}\,\frac{e^{\pm(\eta_1^+ - \eta_1^-)/2}}{\cosh[(k_1^+ + k_1^-)(x - vt - x_0)/2]}, \tag{12}$$

where $v = {k_1^+}^2 - k_1^+k_1^- + {k_1^-}^2$, $x_0 = \left(\eta_1^{+(0)} + \eta_1^{-(0)}\right)/k_1^+k_1^-$, and $\phi_{11} = -2\log k_{11}^{+-}$. The resonance interaction from three to two dissipatons for system (4) is illustrated in Fig. 1.

System (4) admits the symmetric reduction $e^+ = e^- = u$, leading to the MKdV equation

$$u_{t_2} = u_{xxx} + \frac{3\Lambda}{4} u^2 u_x. \tag{13}$$

Under this reduction, $k_1^+ = k_1^- \equiv k$, and dissipaton (12) becomes a one-soliton solution of the MKdV equation,

$$e^+ = e^- = u(x,t) = \sqrt{\frac{8}{-\Lambda}} \frac{|k|}{\cosh k(x - k^2 t - x_0)}. \tag{14}$$

We can thus see that the dissipaton is a more general object reducible to the real soliton of the MKdV equation. Similarly, the two-dissipaton solution of system (4) under the reduction $k_1^+ = k_1^-$, $k_2^+ = k_2^-$ is reducible to the two-soliton solution of the MKdV equation. The natural problem is to find an evolution equation for the dissipaton product e^+e^-. As we show below, it is the KP-II.

4. The KP-II resonance solitons

4.1. The KP-II and the AKNS hierarchy. The AKNS hierarchy also allows developing a new method for finding a solution of the (2+1)-dimensional KP equation. Depending on the sign of the dispersion, two types of the KP equations are known. The minus sign in the right-hand side of the KP corresponds to the case of negative dispersion and is called the KP-II. To relate the KP-II to the AKNS hierarchy, we consider the pair of functions $e^+(x,y,t)$ and $e^-(x,y,t)$ satisfying the second and third members of the dissipative AKNS hierarchy. Here, we rename the time variables $t_1 \equiv y$ and $t_2 \equiv t$. Respectively differentiating Eqs. (3) and (4) with respect to t and y, we can see that they are compatible.

Proposition. *Let the functions $e^+(x,y,t)$ and $e^-(x,y,t)$ be simultaneous solutions of Eqs.* (3) *and* (4)*. Then the function $U(x,y,t) \equiv e^+e^-$ satisfies the KP-II*

$$\left(4U_t + \frac{3\Lambda}{4}(U^2)_x + U_{xxx}\right)_x = -3U_{yy}. \tag{15}$$

Proof. We take the derivative of U with respect to y and use Eq. (3). Hence, $U_y = (e_x^+ e^- - e_x^- e^+)_x$,

$$U_{yy} = \left(e_{xxx}^+ e^- + e_{xxx}^- e^+ - (e_x^+ e_x^-)_x\right) + \frac{\Lambda}{2} U_x U. \tag{16}$$

Similarly for U_t, we have

$$U_t = -\left(e_{xxx}^+ e^- + \frac{3\Lambda}{4} U e^- e_x^- + e_{xxx}^- e^+ + \frac{3\Lambda}{4} U e_x^- e^+\right), \tag{17}$$

$$U_{xt} = -\left(e_{xxx}^+ e^- + e_{xxx}^- e^+ + \frac{3\Lambda}{4} U U_x\right)_x. \tag{18}$$

Combining these formulas,

$$4U_{xt} + 3U_{yy} = \left[-e_{xxx}^+ e^- - e_{xxx}^- e^+ - \frac{3\Lambda}{2} U U_x - 3(e_x^+ e_x^-)_x\right]_x, \tag{19}$$

and using $U_{xxx} = e_{xxx}^+ e^- + e_{xxx}^- e^+ + 3e_{xx}^+ e_x^- + 3e_x^+ e_{xx}^-$, we obtain KP-II (15).[1]

[1] As Konopelchenko recently mentioned to us, similar results are also known in the literature as symmetry reductions of the KP (see, e.g., [4]).

4.2. Bilinear representation of the KP-II by AKNS flows. Using bilinear representations for systems (3) and (4) and the proposition, we can find a bilinear representation for the KP-II. The bilinear form is given by (6) for RD system (3) and by Eqs. (9) for the third flow, system (4).

We now consider $G^{\pm}$ and F as functions of three variables, $G^{\pm} = G^{\pm}(x, y, t)$ and $F = F(x, y, t)$, and require that these functions be a simultaneous solution of bilinear systems (6) and (9). Because the second equation in both systems is the same, it suffices to consider the next bilinear system

$$\begin{cases} (\pm D_y - D_x^2)(G^{\pm} \cdot F) = 0, \\ (D_t + D_x^3)(G^{\pm} \cdot F) = 0, \\ D_x^2(F \cdot F) = -2G^+G^-. \end{cases} \tag{20}$$

According to the proposition, any solution of this system then generates a solution of the KP-II. From the last equation, we can derive U directly in terms of only function F,

$$U = e^+e^- = \frac{8}{-\Lambda}\frac{G^+G^-}{F^2} = \frac{4}{\Lambda}\frac{D_x^2(F \cdot F)}{F^2} = \frac{8}{\Lambda}\frac{\partial^2}{\partial x^2}\log F. \tag{21}$$

The simplest solution of this system,

$$G^{\pm} = \pm e^{\eta_1^{\pm}}, \qquad F = 1 + \frac{e^{(\eta_1^+ + \eta_1^-)}}{(k_1^+ + k_1^-)^2}, \tag{22}$$

where $\eta_1^{\pm} = k_1^{\pm}x \pm (k_1^{\pm})^2 y - (k_1^{\pm})^3 t + \eta_1^{\pm(0)}$, defines a one-soliton solution of the KP-II according to Eq. (21),

$$U = \frac{2(k_1^+ + k_1^-)^2}{\Lambda\cosh^2\big[\big((k_1^+ + k_1^-)x + ({k_1^+}^2 - {k_1^-}^2)y - ({k_1^+}^3 + {k_1^-}^3)t + \gamma\big)/2\big]}, \tag{23}$$

where $\gamma = -\log(k_1^+ + k_1^-)^2 + \eta_1^{+(0)} + \eta_1^{-(0)}$. This soliton is a planar wave barrier traveling in an arbitrary direction and is called the planar soliton.

4.3. Two-soliton solution. Continuing Hirota's expansion, we find a two-soliton solution in the form

$$G^{\pm} = \pm(e^{\eta_1^{\pm}} + e^{\eta_2^{\pm}} + \alpha_1^{\pm}e^{\eta_1^+ + \eta_1^- + \eta_2^{\pm}} + \alpha_2^{\pm}e^{\eta_2^+ + \eta_2^- + \eta_1^{\pm}}), \tag{24}$$

$$F = 1 + \frac{e^{\eta_1^+ + \eta_1^-}}{(k_{11}^{+-})^2} + \frac{e^{\eta_1^+ + \eta_2^-}}{\big(k_{12}^{+-}\big)^2} + \frac{e^{\eta_2^+ + \eta_1^-}}{(k_{21}^{+-})^2} + \frac{e^{\eta_2^+ + \eta_2^-}}{(k_{22}^{+-})^2} + \beta e^{\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^-}, \tag{25}$$

where $\eta_i^{\pm} = k_i^{\pm}x \pm (k_i^{\pm})^2 y - (k_i^{\pm})^3 t + \eta_i^{\pm(0)}$ and $k_{ij}^{ab} = k_i^a + k_j^b$, $i, j = 1, 2$, $a, b = +, -$,

$$\alpha_1^{\pm} = \frac{(k_1^{\pm} - k_2^{\pm})^2}{(k_{11}^{+-}k_{21}^{\pm\mp})^2}, \qquad \alpha_2^{\pm} = \frac{(k_1^{\pm} - k_2^{\pm})^2}{(k_{22}^{+-}k_{12}^{\pm\mp})^2}, \qquad \beta = \frac{(k_1^+ - k_2^+)^2(k_1^- - k_2^-)^2}{(k_{11}^{+-}k_{12}^{+-}k_{21}^{+-}k_{22}^{+-})^2}.$$

It provides a two-soliton solution of the KP-II regular everywhere according to Eq. (21).

4.4. Degenerate four-soliton solution. Another bilinear form in terms of only the function F is known for the KP-II [5],

$$(D_xD_t + D_x^4 + D_y^2)(F \cdot F) = 0. \tag{26}$$

It is therefore natural to compare the soliton solutions of our bilinear equations (20) with the ones given by this equation. To solve Eq. (26), we consider $F = 1 + \varepsilon F_1 + \varepsilon^2 F_2 + \dots$. The solution $F_1 = e^{\eta_1}$, where $\eta_1 = k_1 x + \Omega_1 y + \omega_1 t + \eta_1^0$, the dispersion is $k_1\omega_1 + k_1^4 + \Omega_1^2 = 0$, and $F_n = 0$, $n = 2, 3, \dots$, under identifications $k_1 = k_1^+ + k_1^-$, $\Omega_1 = \sqrt{3}\,(k_1^{+2} - k_1^{-2})$, and $\omega_1 = -4(k_1^{+3} + k_1^{-3})$ and with the rescaling $4t \to t$ and $\sqrt{3}\,y \to y$ determines a one-soliton solution of KP-II (15). We realize that it coincides with our one-soliton solution (23). But the two-soliton solution of Eq. (26) [6] does not correspond to our two-soliton solution (24), (25). The appearance of four different terms $e^{\eta_i^\pm + \eta_k^\pm}$ in Eq. (25) suggests that our two-soliton solution should correspond to some degenerate case of the four-soliton solution of Eq. (26).[2] To construct a four-soliton solution, we first find the solutions of bilinear equations (26),

$$F_1 = e^{\eta_1}, \qquad F_2 = e^{\eta_2}, \qquad F_4 = e^{\eta_3}, \tag{27}$$

where $\eta_i = k_i x + \Omega_i y + \omega_i t + \eta_i^0$, $i = 1, 2, 3$, the dispersion relations are

$$k_i\omega_i + k_i^4 + \Omega_i^2 = 0, \tag{28}$$

and

$$F_3 = \alpha_{12} e^{\eta_1 + \eta_2}, \qquad F_5 = \alpha_{13} e^{\eta_1 + \eta_3}, \qquad F_6 = \alpha_{23} e^{\eta_1 + \eta_3}, \tag{29}$$

where

$$\alpha_{ij} = -\frac{(k_i - k_j)(\omega_i - \omega_j) + (k_i - k_j)^4 + (\Omega_i - \Omega_j)^2}{(k_i + k_j)(\omega_i + \omega_j) + (k_i + k_j)^4 + (\Omega_i + \Omega_j)^2}, \quad i, j = 1, 2, 3. \tag{30}$$

We then parameterize our solution in the form

$$\begin{aligned}
k_1 &= k_1^+ + k_1^-, & \omega_1 &= -4(k_1^{+3} + k_1^{-3}), & \Omega_1 &= \sqrt{3}\,(k_1^{+2} - k_1^{-2}),\\
k_2 &= k_2^+ + k_2^-, & \omega_2 &= -4(k_2^{+3} + k_2^{-3}), & \Omega_2 &= \sqrt{3}\,(k_2^{+2} - k_2^{-2}),\\
k_3 &= k_1^+ + k_2^-, & \omega_3 &= -4(k_1^{+3} + k_2^{-3}), & \Omega_3 &= \sqrt{3}\,(k_1^{+2} - k_2^{-2}),\\
k_4 &= k_2^+ + k_1^-, & \omega_4 &= -4(k_2^{+3} + k_1^{-3}), & \Omega_4 &= \sqrt{3}\,(k_2^{+2} + k_1^{-2}),
\end{aligned} \tag{31}$$

satisfying dispersion relations (28). Substituting these parameterizations in the above solutions, we find that

$$\alpha_{13} = 0 \quad \Rightarrow \quad F_5 = 0, \qquad \alpha_{23} = 0 \quad \Rightarrow \quad F_6 = 0. \tag{32}$$

Continuing Hirota's expansion with the solution $F_7 = e^{\eta_4}$, where $\eta_4 = k_4 x + \Omega_4 y + \omega_4 t + \eta_4^0$, we find that $F_8 = \alpha_{14} e^{\eta_1 + \eta_4}$, where

$$\alpha_{14} = -\frac{(k_1 - k_4)(\omega_1 - \omega_4) + (k_1 - k_4)^4 + (\Omega_1 - \Omega_4)^2}{(k_1 + k_4)(\omega_1 + \omega_4) + (k_1 + k_4)^4 + (\Omega_1 + \Omega_4)^2}, \tag{33}$$

and after parameterization (31), it also vanishes:

$$\alpha_{14} = 0 \quad \Rightarrow \quad F_8 = 0. \tag{34}$$

[2] One of the authors (O. K. P.) thanks Professor J. Hietarinta for this suggestion.

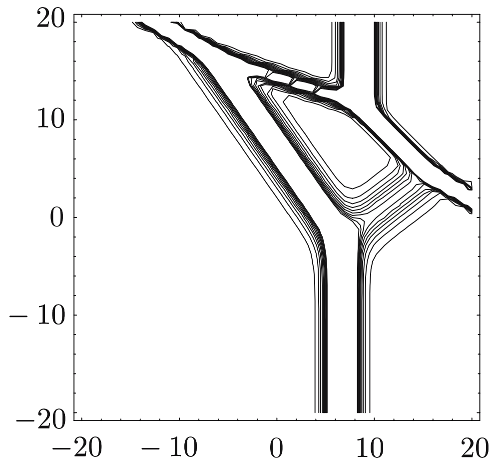

Fig. 2

The next solution $F_9 = \alpha_{24} e^{\eta_2+\eta_4}$, where

$$\alpha_{24} = -\frac{(k_2 - k_4)(\omega_2 - \omega_4) + (k_2 - k_4)^4 + (\Omega_2 - \Omega_4)^2}{(k_2 + k_4)(\omega_2 + \omega_4) + (k_2 + k_4)^4 + (\Omega_2 + \Omega_4)^2}, \tag{35}$$

is also zero:

$$\alpha_{24} = 0 \quad \Rightarrow \quad F_9 = 0. \tag{36}$$

We then have $F_{10} = 0$ and $F_{11} = \alpha_{34} e^{\eta_3+\eta_4}$, where

$$\alpha_{34} = -\frac{(k_3 - k_4)(\omega_3 - \omega_4) + (k_3 - k_4)^4 + (\Omega_3 - \Omega_4)^2}{(k_3 + k_4)(\omega_3 + \omega_4) + (k_3 + k_4)^4 + (\Omega_3 + \Omega_4)^2}. \tag{37}$$

When we check it for higher-order terms, we find that $F_{12} = F_{13} = \cdots = 0$. Therefore, we have a degenerate four-soliton solution of Eqs. (26) in the form

$$F = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + e^{\eta_4} + \alpha_{12} e^{\eta_1+\eta_2} + \alpha_{34} e^{\eta_3+\eta_4}. \tag{38}$$

Comparing this solution with the one in Eq. (25) and taking into account that $\eta_1 + \eta_2 = \eta_3 + \eta_4$ according to parameterization (31), we find that they coincide. The above consideration shows that our two-soliton solution of the KP-II can be obtained by reducing the four-soliton solution in canonical Hirota form (26). Moreover, it allows finding a new four-virtual-soliton resonance for the KP-II.

4.5. Resonance interaction of planar solitons. Choosing different values of parameters for our two-soliton solution, we find the resonance character of the soliton–soliton interaction. For the parameter choice $k_1^+ = 2$, $k_1^- = 1$, $k_2^+ = 1$, and $k_2^- = 0.3$ and a zero value of the position-shift constants, we obtain a two-soliton solution moving in the plane with a constant velocity with the creation of the four so-called virtual solitons, i.e., solitons without asymptotic states at infinity (Fig. 2).

The resonance character of the interactions of our planar solitons is related to the resonance nature of the dissipatons considered in Sec. 3. It has also been reported for several systems, but the four-virtual-soliton resonance does not seem to have been obtained for the KP-II [7] prior to our work. At the workshop "Nonlinear Physics: Theory and Experiment III," we realized that Biondini and Kodama, using Sato's theory, also very recently constructed resonance solitons for the KP-II [8]. A comparison shows that our bilinear constraint plays the same role as the Toda lattice in their paper.

5. Conclusions

The idea to use a couple of equations from the AKNS hierarchy to generate a solution of the KP can also be applied to multidimensional systems with a zero-curvature structure, such as the Chern–Simons gauge theory. Our three-dimensional zero-curvature representation of the KP-II gives a flat non-Abelian connection for $SL(2,\mathbb{R})$ and corresponds to a sector of the three-dimensional gravity theory. We recently showed that an idea similar to the one presented here can also be applied to the Kaup–Newell hierarchy. In this case, combining the second and third flows of the dissipative version of the derivative NLS equation, we found resonance soliton dynamics for a modified KP-II [9].

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