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The Cauchy problem for the planar spin-liquid model

Nai-Heng Chang[1] **and Oktay K Pashaev**[2,3]

[1] Department of Applied Mathematics, National University of Kaohsiung, Kaohsiung, Taiwan 811, Republic of China
[2] Department of Mathematics, Izmir Institute of Technology, Urla-Izmir 35437, Turkey

E-mail: nhchang@nuk.edu.tw and oktaypashaev@iyte.edu.tr

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Abstract
In this paper, we study the Cauchy problem of a two-dimensional model for a moving ferromagnetic continuum and prove global existence and uniqueness of solutions. In addition, equivalence to the coupled system of nonlinear Schrödinger equations interacting with a Chern–Simons gauge field is established.

Mathematics Subject Classification: 35Q55, 35A05, 58J47

1. Introduction

The Heisenberg model for an isotropic ferromagnetic spin system, with classical spin $u \in \mathbb{S}^2 \subset \mathbb{R}^3$, is given by [17, 28]

$$u_t = u \wedge \Delta u, \qquad x \in \mathbb{R}^m, \quad m = 1, 2, 3, \tag{1.1}$$

where Δ is the Laplacian on $\mathbb{R}^m$, and $\wedge$ is the wedge product on $\mathbb{R}^3$. This model has been investigated by many mathematicians and physicists [5,7,8,10,11,28]. In one space dimension, it is integrable and gauge equivalent to the focusing cubic Schrödinger equation [7]. In higher space dimensions, it is not integrable in general and the only known result regarding the global well-posedness of the Cauchy problem was established for small solutions under radial symmetry assumption [5].

Recently, a modification of the Heisenberg model (1.1) has been proposed by Volovik [32] for restoration of the correct linear momentum density of the ferromagnets. Using a simple model of delocalized electrons, by analogy with superfluid motion at $T = 0$ in $\mathrm{He}^3 - \mathrm{A}$, he introduced the normal velocity v of the fermionic liquid as an additional hydrodynamical variable, describing the background fermionic vacuum. This magnetic fluid or spin-liquid is characterized by the local magnetization field $u(x, t) \in \mathbb{S}^2$, subject to the modified Heisenberg

[3] Also at: Joint Institute for Nuclear Research, Dubna (Moscow) 141980, Russian Federation.

model in the moving frame with velocity $v(x, t)$, and supplied with the continuity equation $\rho_t + \nabla \cdot (\rho v) = 0$ for density $\rho(x, t)$ [21]. For incompressible flow $\rho_t = 0$ the last equation simplifies to $\nabla \cdot v = 0$ and allows one to exclude ρ from consideration. Moreover, for planar magnetic systems the existence of topologically nontrivial vortex configurations requires the fluid to be rotational with nonvanishing vorticity function.

In the present work, we consider a two-dimensional incompressible 'spin fluid' system

$$\begin{aligned} &u_t + v_1 u_x + v_2 u_y = u \wedge (u_{xx} + u_{yy}), \\ &v_{2x} - v_{1y} = 2u \cdot (u_x \wedge u_y), \\ &\nabla \cdot v = 0, \end{aligned} \tag{1.2}$$

where the velocity of the ferromagnetic continuum is given by $v = (v_1, v_2)$. The constraint

$$v_{2x} - v_{1y} = 2u \cdot (u_x \wedge u_y) \tag{1.3}$$

requiring that the fluid vorticity is proportional to the magnetic topological current is known in the theory of superfluid He^3 as the Mermin–Ho relation [19].

The model (1.2) was studied by Martina *et al* in [23]. They found a Hamiltonian formulation, a symmetry algebra of the Kac–Moody type with a loop-algebra structure, and the conformal invariance property, as well as many nonlinear excitations (helicon, roton and meron solutions). It can be also considered as a special case of the modified Ishimori model [23]

$$\begin{aligned} &u_t + u_x \phi_y - \beta^2 u_y \phi_x = u \wedge (u_{xx} + \alpha^2 u_{yy}), \\ &\phi_{xx} + \alpha^2 \beta^2 \phi_{yy} = -2\alpha^2 \beta^2 u \cdot (u_x \wedge u_y), \\ &\alpha^2 = \pm 1, \qquad \beta^2 = \pm 1. \end{aligned} \tag{1.4}$$

If we let $\phi_y = v_1$, $\phi_x = -v_2$, (1.4) with $\beta^2 = 1, \alpha^2 = 1$ corresponds to (1.2). For the case $\beta^2 = -1$, (1.4) describes the Ishimori model [14]

$$\begin{aligned} &u_t + \phi_y u_x + \phi_x u_y = u \wedge (u_{xx} + \alpha^2 u_{yy}), \\ &\phi_{xx} - \alpha^2 \phi_{yy} = 2\alpha^2 u \cdot (u_x \wedge u_y), \qquad \alpha^2 = \pm 1. \end{aligned} \tag{1.5}$$

The Ishimori model, proposed in 1984, is an integrable (2 + 1) dimensional topological spin field model, which has been studied in many theoretical frameworks [12, 13, 15, 29], and its integrability was established by the inverse scattering method [3, 16, 18]. Furthermore, there is a gauge relation between the Ishimori model and the Davey–Stewartson equations [18].

Concerning the results on the Cauchy problem for the Ishimori model, the case $\alpha^2 = -1$ was studied by Soyeur [29] and he obtained local well-posedness and global existence of solutions for small data in an appropriate Sobolev space. Hayashi and Saut [13] were the first to study the case $\alpha^2 = 1$. They considered the problem in a class of analytic functions, which allowed them to obtain local and global existence for small analytic data. Also, Hayashi [12] removed the analyticity assumptions in [13] by establishing the local well-posedness of the initial value problem for the case $\alpha^2 = 1$ with small data in the weighted Sobolev space. The smallness assumptions were removed recently by Kenig *et al* [15]. The type of nonlinearities in (1.2) are closely related to the nonderivative nonlinearities appearing in Schrödinger maps. Local in time estimates for these types of terms can be found in the work of Nahmod, Stefanov and Uhlenbeck [25, 26]. Here we are interested however in global in time estimates for small data.

In this paper, we study the global well-posedness of the Cauchy problem for (1.2). The generalized Hasimoto transform and estimates for Schrödinger equations were used to study the Cauchy problem for (1.1) in [5]. We utilize their method to treat (1.2) and obtain similar results; that is, we prove that small energy implies global existence and uniqueness, and that small energy implies regularity for smooth initial data. Moreover, there are some new features in this work:

- There is no symmetry assumption here. In [5], global existence was only proved for equivariant solutions of the Heisenberg model (1.1). For the spin-liquid model (1.2), the velocity constraint (1.3) allows us to prove global existence of solutions without symmetry assumption (see [27]).
- By applying the generalized Hasimoto transform, we obtain a different kind of nonlocal term which requires new estimates.
- We establish equivalence between the two systems, which was not done for the Heisenberg model (1.1) in [5].

The main results are as follows:

Theorem 1.1. *For solutions $u \in C(\mathbb{R}, H^2(\mathbb{R}^2))$, the system (1.2) is equivalent to the following coupled nonlinear Schrödinger equations*

$$q_{1t} + \mathrm{i}a_0 q_1 = \mathrm{i}\Delta q_1 + \mathrm{i}(a_1^2 + a_2^2)q_1 + (a_{1x} - a_{2y})q_1 + (a_{2x} + a_{1y})q_2, \tag{1.6}$$

$$q_{2t} + \mathrm{i}a_0 q_2 = \mathrm{i}\Delta q_2 + \mathrm{i}(a_1^2 + a_2^2)q_2 - (a_{1x} - a_{2y})q_2 + (a_{2x} + a_{1y})q_1, \tag{1.7}$$

$$p = \mathrm{i}q_{1x} + \mathrm{i}q_{2y} + a_1 q_1 + a_2 q_2, \tag{1.8}$$

$$\mathrm{div} \cdot a = a_{1x} + a_{2y} = 0, \tag{1.9}$$

$$a_{1y} - a_{2x} = \mathrm{Im}(\bar{q}_1 q_2), \tag{1.10}$$

$$a_{1t} - a_{0x} = \mathrm{Im}(\bar{q}_1 p), \tag{1.11}$$

$$a_{2t} - a_{0y} = \mathrm{Im}(\bar{q}_2 p), \tag{1.12}$$

$$q_{1y} + \mathrm{i}a_2 q_1 = q_{2x} + \mathrm{i}a_1 q_2. \tag{1.13}$$

Remark 1.1. The system (1.6)–(1.13) seems over-determined, however, we will show in section 2 that (1.11)–(1.13) can be viewed as the compatibility conditions for the system (1.6)–(1.10). In physical language this system represents the pair of nonlinear Schrödinger equations for the matter field interacting with an Abelian Chern–Simons gauge field. Then (1.9) has the meaning of the Coulomb gauge condition, while (1.10) is the Gauss law of Chern–Simons dynamics. It shows that the gauge field a is determined by the matter density for q_1 and q_2.

Theorem 1.2. *Given initial data $u_0(x) \in H^2(\mathbb{R}^2)$, there exists an $\epsilon_0 > 0$ such that if $\|\nabla u_0\|_{L^2(\mathbb{R}^2)} \leqslant \epsilon_0$, the Cauchy problem of (1.2) admits a unique global solution (u, v_1, v_2) such that*

$$u \in C(\mathbb{R}, H^2(\mathbb{R}^2)) \cap L^4_{\mathrm{loc}}(\mathbb{R}, W^{2,4}(\mathbb{R}^2)),$$
$$\nabla v_1, \nabla v_2 \in L^4(\mathbb{R}, L^{4/3}(\mathbb{R}^2)) \cap L^2(\mathbb{R}, L^2(\mathbb{R}^2)).$$

Remark 1.2. Throughout the paper, u is a unit vector and we frequently use the notation $u \in C(\mathbb{R}, H^2(\mathbb{R}^2))$ or $L^4(\mathbb{R}, W^{2,4}(\mathbb{R}^2))$ which means that $\nabla u \in C(\mathbb{R}, H^1(\mathbb{R}^2))$ or $L^4(\mathbb{R}, W^{1,4}(\mathbb{R}^2))$.

The paper is organized as follows: in section 2, we prove that the Cauchy problem of the system (1.6)–(1.13) has a unique global solution in an appropriate space. In section 3, we utilize the generalized Hasimoto transform and the Frobenius theorem to show that (1.2) is actually equivalent to the system (1.6)–(1.13). Therefore, we conclude the global existence and uniqueness of solutions for the Cauchy problem of (1.2).

2. Global existence of solutions to the system (1.6)–(1.13)

In this section, we prove the global existence and uniqueness of smooth solutions for the Cauchy problem of the system (1.6)–(1.13). First, we need the following estimate [4].

Lemma 2.1. *In two space dimensions, solutions to the linear equation*

$$q_t = \mathrm{i}(\Delta q + h),$$
$$q(0) = q_0,$$

satisfy the following space–time estimate

$$\|q\|_{L^\infty([0,T],L^2(\mathbb{R}^2))} + \|q\|_{L^4([0,T],L^4(\mathbb{R}^2))} \leqslant C\|q_0\|_{L^2(\mathbb{R}^2)} + C\|h\|_{L^{4/3}([0,T],L^{4/3}(\mathbb{R}^2))} \tag{2.1}$$

for any $T > 0$.

We introduce some notation. Let $\mathcal{C} \overset{\text{def}}{=} [0,T]\times\mathbb{R}^2$ $(T < \infty)$. For $1 \leqslant p, q \leqslant \infty$, we define

$$L^pL^q(\mathcal{C}) \overset{\text{def}}{=} L^p([0,T],L^q(\mathbb{R}^2)), \qquad p \neq q,$$
$$L^p(\mathcal{C}) \overset{\text{def}}{=} L^p([0,T],L^p(\mathbb{R}^2)), \qquad p = q.$$

For $q = (q_1, q_2)$, we define the norms of q by

$$\begin{aligned}\|q\|_T &= \sum_{i=1}^{2}(\|q_i\|_{L^\infty L^2(\mathcal{C})} + \|q_i\|_{L^4(\mathcal{C})}),\\ \|q\|_\infty &= \sum_{i=1}^{2}(\|q_i\|_{L^\infty(\mathbb{R},L^2(\mathbb{R}^2))} + \|q_i\|_{L^4(\mathbb{R},L^4(\mathbb{R}^2))}).\end{aligned} \tag{2.2}$$

To show global existence of smooth solutions for the system (1.6)–(1.13), we proceed in two steps. First, we establish the following result for an auxiliary system.

Proposition 2.1. *Given initial data q_{10}, $q_{20} \in L^2(\mathbb{R}^2)$, there exists an $\epsilon_0 > 0$ such that if $\|q_{10}\|_{L^2(\mathbb{R}^2)} + \|q_{20}\|_{L^2(\mathbb{R}^2)} \leqslant \epsilon_0$, then the Cauchy problem*

$$\begin{aligned}
&q_{1t} + \mathrm{i}a_0q_1 = \mathrm{i}\Delta q_1 + \mathrm{i}(a_1^2 + a_2^2)q_1 + (a_{1x} - a_{2y})q_1 + (a_{2x} + a_{1y})q_2,\\
&q_{2t} + \mathrm{i}a_0q_2 = \mathrm{i}\Delta q_2 + \mathrm{i}(a_1^2 + a_2^2)q_2 - (a_{1x} - a_{2y})q_2 + (a_{2x} + a_{1y})q_1,\\
&\Delta a_1 = \partial_y \operatorname{Im}(\bar{q}_1q_2),\\
&\Delta a_2 = \partial_x \operatorname{Im}(\bar{q}_2q_1),\\
&\Delta a_0 = \tfrac{1}{2}\partial_x^2|q_2|^2 + \tfrac{1}{2}\partial_y^2|q_1|^2 - \tfrac{1}{2}\partial_x^2|q_1|^2 - \tfrac{1}{2}\partial_y^2|q_2|^2 - 2\partial_x\partial_y \operatorname{Re}(q_1\bar{q}_2)\\
&\qquad +2\partial_x \operatorname{Im}(a_2q_1\bar{q}_2) + 2\partial_y \operatorname{Im}(a_1\bar{q}_1q_2),\\
&q_1(x,0) = q_{10}(x),\\
&q_2(x,0) = q_{20}(x)
\end{aligned} \tag{2.3}$$

has a unique global solution $(q_1, q_2, a_0, a_1, a_2)$ such that

$$q_1, q_2 \in C(\mathbb{R}, L^2(\mathbb{R}^2)) \cap L^4(\mathbb{R}, L^4(\mathbb{R}^2)),$$
$$\nabla a_1, \nabla a_2 \in L^4(\mathbb{R}, L^{4/3}(\mathbb{R}^2)) \cap L^2(\mathbb{R}, L^2(\mathbb{R}^2)).$$

Proof. By Duhamel's principle we rewrite the equations for q_1 and q_2 in terms of the integral equations

$$\begin{aligned} q_1(t) &= U(t)q_{10} + \int_0^t U(t-s)N_1(q_1, q_2)\,\mathrm{d}s, \\ q_2(t) &= U(t)q_{20} + \int_0^t U(t-s)N_2(q_1, q_2)\,\mathrm{d}s, \end{aligned} \tag{2.4}$$

where $U(t)$ denotes the solution operator of the linear Schrödinger equation in two space dimensions, and N_1 and N_2 consist of all the nonlinear terms in the equations for q_1 and q_2, respectively. The basic idea of the proof is to use a contraction type argument in the space $L^4(\mathcal{C}) \cap L^\infty L^2(\mathcal{C})$ such that (2.4) has a fixed point. We begin by expressing a_0, a_1 and a_2 in terms of q_1 and q_2 by solving the Poisson-like equations:

$$\begin{aligned} a_1 &= N * [\partial_y \operatorname{Im}(\bar{q}_1 q_2)] \stackrel{\text{def}}{=} \Delta^{-1}[\partial_y \operatorname{Im}(\bar{q}_1 q_2)], \\ a_2 &= N * [\partial_x \operatorname{Im}(\bar{q}_2 q_1)] \stackrel{\text{def}}{=} \Delta^{-1}[\partial_y \operatorname{Im}(\bar{q}_2 q_1)], \\ a_0 &= N * [-\tfrac{1}{2}\partial_x^2|q_1|^2 + \tfrac{1}{2}\partial_x^2|q_2|^2 + \tfrac{1}{2}\partial_y^2|q_1|^2 - \tfrac{1}{2}\partial_y^2|q_2|^2 - 2\partial_x\partial_y \operatorname{Re}(q_1\bar{q}_2) \\ &\quad +2\partial_x \operatorname{Im}(a_2 q_1 \bar{q}_2) + 2\partial_y \operatorname{Im}(a_1 \bar{q}_1 q_2)] \\ &\stackrel{\text{def}}{=} \Delta^{-1}[-\tfrac{1}{2}\partial_x^2|q_1|^2 + \tfrac{1}{2}\partial_x^2|q_2|^2 + \tfrac{1}{2}\partial_y^2|q_1|^2 - \tfrac{1}{2}\partial_y^2|q_2|^2 - 2\partial_x\partial_y \operatorname{Re}(q_1\bar{q}_2) \\ &\quad +2\partial_x \operatorname{Im}(a_2 q_1 \bar{q}_2) + 2\partial_y \operatorname{Im}(a_1 \bar{q}_1 q_2)], \end{aligned} \tag{2.5}$$

where N stands for the fundamental solution $(1/2\pi)\log|x|$ in two space dimensions. From (2.5),

$$\Delta a_{1y} = \partial_y^2 \operatorname{Im}(\bar{q}_1 q_2).$$

By setting

$$a_{1y} = E(\operatorname{Im}(\bar{q}_1 q_2)),$$

where the singular integral operator E is defined in Fourier variables by

$$\widehat{E(f)} = \frac{\xi_2^2}{\xi_1^2 + \xi_2^2}\hat{f},$$

the Calderon–Zygmund theorem implies that E is a bounded operator in $L^p(\mathbb{R}^2)$, $1 < p < \infty$ [6,9]. Hence there exists a constant $C = C_p > 0$ such that

$$\|a_{1y}\|_{L^p(\mathbb{R}^2)} \leqslant C_p \|\operatorname{Im}(\bar{q}_1 q_2)\|_{L^p(\mathbb{R}^2)}. \tag{2.6}$$

In particular, if $q_i \in C(\mathbb{R}, L^2(\mathbb{R}^2)) \cap L^4(\mathbb{R}, L^4(\mathbb{R}^2))$, then

$$\|a_{1y}\|_{L^{4/3}(\mathbb{R}^2)} \leqslant C\|q_1\|_{L^4(\mathbb{R}^2)}\|q_2\|_{L^2(\mathbb{R}^2)}. \tag{2.7}$$

Similarly, it holds that

$$\|a_{1x}\|_{L^{4/3}(\mathbb{R}^2)} \leqslant C\|q_1\|_{L^4(\mathbb{R}^2)}\|q_2\|_{L^2(\mathbb{R}^2)}. \tag{2.8}$$

By the Gagliardo–Nirenberg–Sobolev inequality, we have

$$\|a_1\|_{L^4(\mathbb{R}^2)} \leqslant C\|\nabla a_1\|_{L^{4/3}(\mathbb{R}^2)} \leqslant C\|q_1\|_{L^2(\mathbb{R}^2)}\|q_2\|_{L^4(\mathbb{R}^2)}, \tag{2.9}$$

which yields the space–time norm of a_1

$$\|a_1\|_{L^4(\mathcal{C})} \leqslant C\|q_1\|_{L^\infty L^2(\mathcal{C})}\|q_2\|_{L^4(\mathcal{C})} \leqslant C\|q\|_T^2. \tag{2.10}$$

In a similar fashion we can show that

$$\|a_2\|_{L^4(\mathcal{C})} \leqslant C\|q\|_T^2. \tag{2.11}$$

For $q_1, q_2 \in L^4(\mathbb{R}, L^4) \cap L^\infty(\mathbb{R}, L^2(\mathbb{R}^2))$, a_0 can be estimated in the following manner:

- The first term of a_0 in (2.5) can be estimated by

$$\|\Delta^{-1}(\tfrac{1}{2}\partial_x^2|q_1|^2)\|_{L^2(\mathbb{R}^2)} \leqslant C\|q_1\|^2_{L^4(\mathbb{R}^2)}, \tag{2.12}$$

  which implies that the bound on the space–time norm is

$$\|\Delta^{-1}(\tfrac{1}{2}\partial_x^2|q_1|^2)\|_{L^2(\mathcal{C})} \leqslant C\|q_1\|^2_{L^4(\mathcal{C})} \leqslant C\|q\|^2_T. \tag{2.13}$$

  The estimate (2.13) also holds for $\Delta^{-1}(\frac{1}{2}\partial_y^2|q_2|^2)$, $\Delta^{-1}(\frac{1}{2}\partial_x^2|q_2|^2)$, $\Delta^{-1}(\frac{1}{2}\partial_y^2|q_1|^2)$ and $\Delta^{-1}(\frac{1}{2}\partial_x\partial_y \operatorname{Re}(q_1\bar{q_2}))$.
- The remaining two terms can be estimated by

$$\begin{aligned}
&\|\Delta^{-1}2\partial \operatorname{Im}(a_2q_1\bar{q_2})\|_{L^4(\mathbb{R}^2)} \\
&\quad \leqslant C\|\Delta^{-1}\partial^2 \operatorname{Im}(a_2q_1\bar{q_2})\|_{L^{4/3}(\mathbb{R}^2)}, \qquad \text{(by Sobolev)} \\
&\quad \leqslant C\|\operatorname{Im}(a_2q_1\bar{q_2})\|_{L^{4/3}(\mathbb{R}^2)}, \qquad \text{(by the Calderon–Zygmund theorem)} \\
&\quad \leqslant C\|a_2\|_{L^4(\mathbb{R}^2)}\|q_1\|_{L^4(\mathbb{R}^2)}\|q_2\|_{L^4(\mathbb{R}^2)}, \\
&\quad \leqslant C\|q_1\|_{L^2(\mathbb{R}^2)}\|q_2\|^2_{L^4(\mathbb{R}^2)}\|q_1\|_{L^4(\mathbb{R}^2)} \qquad \text{[by (2.9)]}.
\end{aligned} \tag{2.14}$$

After obtaining estimates for a_0, a_1 and a_2 in terms of q_1 and q_2, we are able to get *a priori* estimates for q_1 and q_2 in (2.4). By using lemma 2.1 we have

$$\begin{aligned}
\|q_1\|_{L^\infty L^2(\mathcal{C})} + \|q_1\|_{L^4(\mathcal{C})} &\leqslant C\|q_{10}\|_{L^2(\mathbb{R}^2)} + C\|N_1(q_1, q_2)\|_{L^{4/3}(\mathcal{C})}, \\
\|q_2\|_{L^\infty L^2(\mathcal{C})} + \|q_2\|_{L^4(\mathcal{C})} &\leqslant C\|q_{20}\|_{L^2(\mathbb{R}^2)} + C\|N_2(q_1, q_2)\|_{L^{4/3}(\mathcal{C})},
\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}
\|N_1(q_1, q_2)\|_{L^{4/3}(\mathcal{C})} &\leqslant \|(a_1^2 + a_2^2)q_1\|_{L^{4/3}(\mathcal{C})} + \|(a_{1x} - a_{2y})q_1\|_{L^{4/3}(\mathcal{C})} \\
&\quad + \|(a_{2x} + a_{1y})q_2\|_{L^{4/3}(\mathcal{C})} + \|a_0q_1\|_{L^{4/3}(\mathcal{C})} \\
&\overset{\text{def}}{=} \|\mathcal{R}_1\|_{L^{4/3}(\mathcal{C})} + \|\mathcal{R}_2\|_{L^{4/3}(\mathcal{C})} + \|\mathcal{R}_3\|_{L^{4/3}(\mathcal{C})} + \|\mathcal{R}_4\|_{L^{4/3}(\mathcal{C})}, \\
\|N_2(q_1, q_2)\|_{L^{4/3}(\mathcal{C})} &\leqslant \|(a_1^2 + a_2^2)q_2\|_{L^{4/3}(\mathcal{C})} + \|(a_{1x} - a_{2y})q_2\|_{L^{4/3}(\mathcal{C})} \\
&\quad + \|(a_{2x} + a_{1y})q_1\|_{L^{4/3}(\mathcal{C})} + \|a_0q_2\|_{L^{4/3}(\mathcal{C})}.
\end{aligned} \tag{2.16}$$

As in the following, we show how to get the estimate on $N_1(q_1, q_2)$. A similar argument also works for $N_2(q_1, q_2)$.

- From (2.10) and (2.11) the space norm of $\mathcal{R}_1$ can be bounded by

$$\begin{aligned}
\|\mathcal{R}_1\|_{L^{4/3}(\mathbb{R}^2)} &\leqslant C(\|a_1\|^2_{L^4(\mathbb{R}^2)} + \|a_2\|^2_{L^4(\mathbb{R}^2)})\|q_1\|_{L^4(\mathbb{R}^2)} \\
&\leqslant C\|q_1\|^2_{L^2(\mathbb{R}^2)}\|q_2\|^2_{L^4(\mathbb{R}^2)}\|q_1\|_{L^4(\mathbb{R}^2)}.
\end{aligned}$$

  Then the space–time norm of $\mathcal{R}_1$ can be estimated by

$$\|\mathcal{R}_1\|_{L^{4/3}(\mathcal{C})} \leqslant C\|q_1\|^2_{L^\infty L^2(\mathcal{C})}\|q_2\|^2_{L^4(\mathcal{C})}\|q_1\|_{L^4(\mathcal{C})} \leqslant \|q\|^5_T.$$

- We can estimate the space norms of $\mathcal{R}_2$ and $\mathcal{R}_3$ by

$$\|\partial_i a_j q_k\|_{L^{4/3}(\mathbb{R}^2)} \leqslant C\|\partial_i a_j\|_{L^2(\mathbb{R}^2)}\|q_k\|_{L^4(\mathbb{R}^2)} \leqslant C\|q_k\|^3_{L^4(\mathbb{R}^2)}. \tag{2.17}$$

  This gives the bound on the space–time norm

$$\|\partial_i a_j q_k\|_{L^{4/3}(\mathcal{C})} \leqslant C\|q_k\|^3_{L^4(\mathcal{C})} \leqslant C\|q\|^3_T, \tag{2.18}$$

  which implies that

$$\begin{aligned}
\|\mathcal{R}_2\|_{L^{4/3}(\mathcal{C})} &\leqslant C\|q\|^3_T, \\
\|\mathcal{R}_3\|_{L^{4/3}(\mathcal{C})} &\leqslant C\|q\|^3_T.
\end{aligned} \tag{2.19}$$

- From (2.12) and (2.14) the space norm of $\mathcal{R}_4$ can be computed by

$$\|\mathcal{R}_4\|_{L^{4/3}(\mathbb{R}^2)} \leqslant C\|q_1\|^3_{L^4(\mathbb{R}^2)} + C\|q_1\|^2_{L^2(\mathbb{R}^2)}\|q_2\|^2_{L^4(\mathbb{R}^2)}\|q_1\|_{L^4(\mathbb{R}^2)}.$$

Then the space–time norm of $\|\mathcal{R}_4\|_{L^{4/3}(\mathcal{C})}$ is bounded by

$$\begin{aligned}\|\mathcal{R}_4\|_{L^{4/3}(\mathcal{C})} &\leqslant C\|q_1\|^3_{L^4(\mathcal{C})} + C\|q_1\|^2_{L^\infty L^2(\mathcal{C})}\|q_2\|^2_{L^4(\mathcal{C})}\|q_1\|_{L^4(\mathcal{C})}\\ &\leqslant C\|q\|^3_T + C\|q\|^5_T.\end{aligned}$$

From (2.15) q_2 has the same bound as q_1, thus we have

$$\|q\|_T \leqslant C\|q_0\|_{L^2(\mathbb{R}^2)} + C\|q\|^3_T + C\|q\|^5_T. \tag{2.20}$$

Therefore, if $\|q_0\|_{L^2(\mathbb{R}^2)}$ is small enough, the above inequality (2.20) leads to a global bound

$$\|q\|_\infty \leqslant C\|q_0\|_{L^2(\mathbb{R}^2)}. \tag{2.21}$$

This *a priori* bound coupled with a contraction mapping argument is sufficient to prove existence and uniqueness of a fixed point of (2.4) in the space $C(\mathbb{R}, L^2(\mathbb{R}^2)) \cap L^4(\mathbb{R}, L^4(\mathbb{R}^2))$ which solves (2.3). □

Similarly, one can obtain H^1-regularity of the solution (q_1, q_2) and higher regularity for smooth data.

Corollary 2.1. *Under the additional assumption that q_{10} and $q_{20} \in H^1(\mathbb{R}^2)$, the above solution (q_1, q_2) is in the space*

$$\begin{aligned}&q_1, q_2 \in C(\mathbb{R}, H^1(\mathbb{R}^2)) \cap L^4_{\mathrm{loc}}(\mathbb{R}, W^{1,4}(\mathbb{R}^2)),\\ &\nabla a_1, \nabla a_2 \in L^4(\mathbb{R}, W^{1,4/3}) \cap L^2(\mathbb{R}, H^1).\end{aligned}$$

Proof. Let $w_i = \partial q_i$ be any spatial derivative, we only show how to estimate w_1 and the result follows similarly for w_2.

$$\begin{aligned}w_{1t} - \mathrm{i}\Delta w_1 &= \mathrm{i}(a_1^2 + a_2^2)w_1 - \mathrm{i}a_0 w_1 + (a_{1x} - a_{2y})w_1 + (a_{2x} + a_{1y})w_2\\ &\quad + \mathrm{i}\partial(a_1^2 + a_2^2)q_1 - \mathrm{i}\partial a_0 q_1 + \partial(a_{1x} - a_{2y})q_1 + \partial(a_{2x} + a_{1y})q_2\\ &\overset{\text{def}}{=} f(w_1, w_2).\end{aligned} \tag{2.22}$$

Applying (2.1) to (2.22), one obtains

$$\|w_1\|_{L^\infty L^2(\mathcal{D})} + \|w_1\|_{L^4(\mathcal{D})} \leqslant C\|w_1(t_1)\|_{L^2(\mathbb{R}^2)} + C\|f(w_1, w_2)\|_{L^{4/3}(\mathcal{D})}, \tag{2.23}$$

where $\mathcal{D} = [t_1, t_2] \times \mathbb{R}^2$. To get *a priori* estimates on $f(w_1, w_2)$, we proceed in the following manner:

- First,

$$\begin{aligned}\|(a_1^2 + a_2^2)w_1\|_{L^{4/3}(\mathcal{D})} &\leqslant C\|(a_1^2 + a_2^2)\|_{L^2(\mathcal{D})}\|w_1\|_{L^4(\mathcal{D})}\\ &\leqslant C\|q_i\|^2_{L^\infty L^2(\mathcal{D})}\|q_j\|^2_{L^4(\mathcal{D})}\|w_1\|_{L^4(\mathcal{D})} \qquad \text{[by (2.10)]}\\ &\leqslant C\|q\|^2_{L^\infty L^2(\mathcal{D})}\|q\|^2_{L^4(\mathcal{D})}\|w\|_{L^4(\mathcal{D})},\end{aligned} \tag{2.24}$$

here

$$\|w\|_{L^4(\mathcal{D})} = \|w_1\|_{L^4(\mathcal{D})} + \|w_2\|_{L^4(\mathcal{D})}.$$

- Second, from (2.12) and (2.14) we have

$$\|a_0 w_1\|_{L^{4/3}(\mathbb{R}^2)} \leqslant C\|q_1\|^2_{L^4(\mathbb{R}^2)}\|w_1\|_{L^4(\mathbb{R}^2)} + C\|q_1\|_{L^2(\mathbb{R}^2)}\|q_2\|^2_{L^4(\mathbb{R}^2)}\|q_1\|_{L^4(\mathbb{R}^2)}\|w_1\|_{L^2(\mathbb{R}^2)}, \tag{2.25}$$

then the space–time norm of $a_0 w_1$ can be bounded by

$$\|a_0 w_1\|_{L^{4/3}(\mathcal{D})} \leqslant C\|q\|^2_{L^4(\mathcal{D})}\|w_1\|_{L^4(\mathcal{D})} + C\|q\|_{L^\infty L^2(\mathcal{D})}\|q\|^3_{L^4(\mathcal{D})}\|w_1\|_{L^\infty L^2(\mathcal{D})}.$$

- We can estimate the third and fourth terms by

$$\begin{aligned} \|w_i \partial a_j\|_{L^{4/3}(\mathbb{R}^2)} &\leqslant C\|w_i\|_{L^4(\mathbb{R}^2)}\|\partial a_j\|_{L^2(\mathbb{R}^2)} \\ &\leqslant C\|w_i\|_{L^4(\mathbb{R}^2)}\|q\|^2_{L^4(\mathbb{R}^2)} \qquad \text{[by (2.6)]}. \end{aligned} \tag{2.26}$$

The above estimate implies that

$$\|w_i \partial a_j\|_{L^{4/3}(\mathcal{D})} \leqslant C\|w\|_{L^4(\mathcal{D})}\|q\|^2_{L^4(\mathcal{D})}.$$

- The space norm of $\partial(a_1^2 + a_2^2)q_1$ can be estimated by

$$\|\partial(a_1^2 + a_2^2)q_1\|_{L^{4/3}(\mathbb{R}^2)} \leqslant C\|a_1\nabla a_1 + a_2\nabla a_2\|_{L^4(\mathbb{R}^2)}\|q_1\|_{L^2(\mathbb{R}^2)},$$

$$\begin{aligned} \|a_1\nabla a_1\|_{L^4(\mathbb{R}^2)} &\leqslant C(\|a_1\Delta a_1\|_{L^{4/3}(\mathbb{R}^2)} + \|\nabla a_1 \nabla a_1\|_{L^{4/3}(\mathbb{R}^2)}) \\ &\leqslant C(\|a_1\partial_y \operatorname{Im}(\bar{q}_1 q_2)\|_{L^{4/3}(\mathbb{R}^2)} + \|\nabla a_1\|_{L^2(\mathbb{R}^2)}\|\nabla a_1\|_{L^4(\mathbb{R}^2)}) \\ &\leqslant C\|\nabla q_k\|_{L^4(\mathbb{R}^2)}\|q_k\|_{L^4(\mathbb{R}^2)}\|a_1\|_{L^4(\mathbb{R}^2)} + C\|q_k\|^2_{L^4(\mathbb{R}^2)}\|\Delta a_1\|_{L^{4/3}(\mathbb{R}^2)} \\ &\leqslant C\|\nabla q_k\|_{L^4(\mathbb{R}^2)}\|q_k\|_{L^4(\mathbb{R}^2)}\|a_1\|_{L^4(\mathbb{R}^2)} \\ &\quad + C\|q_k\|^2_{L^4(\mathbb{R}^2)}\|\nabla q_k\|_{L^4(\mathbb{R}^2)}\|q_k\|_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{2.27}$$

Thus we have

$$\begin{aligned} \|\partial(a_1^2 + a_2^2)q_1\|_{L^{4/3}(\mathbb{R}^2)} &\leqslant C(\|\nabla q_k\|_{L^4(\mathbb{R}^2)}\|q_k\|_{L^4(\mathbb{R}^2)}\|a_1\|_{L^4(\mathbb{R}^2)} \\ &\quad + \|q_k\|^2_{L^4(\mathbb{R}^2)}\|\nabla q_k\|_{L^4(\mathbb{R}^2)}\|q_k\|_{L^2(\mathbb{R}^2)})\|q_1\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Therefore, the space–time norm of $\partial(a_1^2 + a_2^2)q_1$ is bounded by

$$\|\partial(a_1^2 + a_2^2)q_1\|_{L^{4/3}(\mathcal{D})} \leqslant C\|q\|^2_{L^\infty L^2(\mathcal{D})}\|q\|^2_{L^4(\mathcal{D})}\|w\|_{L^4(\mathcal{D})}. \tag{2.28}$$

- The space norm of $\partial a_{1y} q_2$ can be estimated by

$$\begin{aligned} \|\partial a_{1y} q_2\|_{L^{4/3}(\mathbb{R}^2)} &\leqslant C\|\partial a_{1y}\|_{L^2(\mathbb{R}^2)}\|q_2\|_{L^4(\mathbb{R}^2)} \\ &\leqslant C\|\Delta a_1\|_{L^2(\mathbb{R}^2)}\|q_2\|_{L^4(\mathbb{R}^2)} \\ &\leqslant C\|\partial_y \operatorname{Im}(\bar{q}_1 q_2)\|_{L^2(\mathbb{R}^2)}\|q_2\|_{L^4(\mathbb{R}^2)} \qquad \text{[by (2.3)]} \\ &\leqslant C\|w\|_{L^4(\mathbb{R}^2)}\|q\|^2_{L^4(\mathbb{R}^2)} \end{aligned}$$

and this implies

$$\begin{aligned} \|\partial(a_{1x} - a_{2y})q_1\|_{L^{4/3}(\mathbb{R}^2)} &\leqslant C\|w\|_{L^4(\mathbb{R}^2)}\|q\|^2_{L^4(\mathbb{R}^2)}, \\ \|\partial(a_{2x} + a_{1y})q_2\|_{L^{4/3}(\mathbb{R}^2)} &\leqslant C\|w\|_{L^4(\mathbb{R}^2)}\|q\|^2_{L^4(\mathbb{R}^2)}. \end{aligned}$$

Therefore, the space–time norms can be bounded by

$$\begin{aligned} \|\partial(a_{1x} - a_{2y})q_1\|_{L^{4/3}(\mathcal{D})} &\leqslant C\|w\|_{L^4(\mathcal{D})}\|q\|^2_{L^4(\mathcal{D})}, \\ \|\partial(a_{2x} + a_{1y})q_2\|_{L^{4/3}(\mathcal{D})} &\leqslant C\|w\|_{L^4(\mathcal{D})}\|q\|^2_{L^4(\mathcal{D})}. \end{aligned} \tag{2.29}$$

- From (2.3) we have

$$\begin{aligned} \Delta\partial a_0 = {}& \tfrac{1}{2}\partial_x^2\partial|q_2|^2 + \tfrac{1}{2}\partial_y^2\partial|q_1|^2 - \tfrac{1}{2}\partial_x^2\partial|q_1|^2 - \tfrac{1}{2}\partial_y^2\partial|q_2|^2 - 2\partial_x\partial_y\partial \operatorname{Re}(q_1\bar{q}_2) \\ & + 2\partial_x\partial \operatorname{Im}(a_2 q_1 \bar{q}_2) + 2\partial_y \partial \operatorname{Im}(a_1 \bar{q}_1 q_2), \end{aligned}$$

then the space norm of $\partial a_0 q_1$ can be computed by

$$\begin{aligned} \|q_1\partial a_0\|_{L^{4/3}(\mathbb{R}^2)} &\leqslant C\|\partial q \cdot q\|_{L^2(\mathbb{R}^2)}\|q_1\|_{L^4(\mathbb{R}^2)} + C\|a_i q_j q_k\|_{L^4(\mathbb{R}^2)}\|q_1\|_{L^2(\mathbb{R}^2)} \\ &\leqslant C\|w\|_{L^4(\mathbb{R}^2)}\|q\|^2_{L^4(\mathbb{R}^2)} + C\|a_i q_j q_k\|_{L^4(\mathbb{R}^2)}\|q\|_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{2.30}$$

Furthermore,

$$\begin{aligned}\|a_i q_j q_k\|_{L^4(\mathbb{R}^2)} &\leqslant C\|\partial(a_i q_j q_k)\|_{L^{4/3}(\mathbb{R}^2)}\\ &\leqslant C\|\partial a_i q_j q_k\|_{L^{4/3}(\mathbb{R}^2)} + C\|a_i w_j q_k\|_{L^{4/3}(\mathbb{R}^2)}\\ &\leqslant C\|\partial a_i\|_{L^4(\mathbb{R}^2)}\|q_j\|_{L^4(\mathbb{R}^2)}\|q_k\|_{L^4(\mathbb{R}^2)}\\ &\quad + C\|a_i\|_{L^4(\mathbb{R}^2)}\|w_j\|_{L^4(\mathbb{R}^2)}\|q_k\|_{L^4(\mathbb{R}^2)}\\ &\leqslant C\|q_i q_j\|_{L^4(\mathbb{R}^2)}\|q_j\|_{L^4(\mathbb{R}^2)}\|q_k\|_{L^4(\mathbb{R}^2)}\\ &\quad + C\|a_i\|_{L^4(\mathbb{R}^2)}\|w_j\|_{L^4(\mathbb{R}^2)}\|q_k\|_{L^4(\mathbb{R}^2)}\\ &\leqslant C\|\partial(q_i q_j)\|_{L^{4/3}(\mathbb{R}^2)}\|q_j\|_{L^4(\mathbb{R}^2)}\|q_k\|_{L^4(\mathbb{R}^2)}\\ &\quad + C\|q_i\|_{L^2(\mathbb{R}^2)}\|q_j\|_{L^4(\mathbb{R}^2)}\|q_k\|_{L^4(\mathbb{R}^2)}\|w_j\|_{L^4(\mathbb{R}^2)}\\ &\leqslant C\|q_i\|_{L^2(\mathbb{R}^2)}\|q_j\|_{L^4(\mathbb{R}^2)}\|q_k\|_{L^4(\mathbb{R}^2)}\|w_i\|_{L^4(\mathbb{R}^2)}\\ &\leqslant C\|q\|^2_{L^4(\mathbb{R}^2)}\|q\|_{L^2(\mathbb{R}^2)}\|w\|_{L^4(\mathbb{R}^2)}.\end{aligned} \tag{2.31}$$

Thus we derive that

$$\|q_1\partial a_0\|_{L^{4/3}(\mathbb{R}^2)} \leqslant C\|w\|_{L^4(\mathbb{R}^2)}\|q\|_{L^4(\mathbb{R}^2)} + C\|q\|^2_{L^4(\mathbb{R}^2)}\|q\|^2_{L^2(\mathbb{R}^2)}\|w\|_{L^4(\mathbb{R}^2)},$$

which leads to the space–time estimate

$$\|\partial a_0 q_1\|_{L^{4/3}(\mathcal{D})} \leqslant C\|q\|^2_{L^4(\mathcal{D})}\|w\|_{L^4(\mathcal{D})} + C\|q\|^2_{L^\infty L^2(\mathcal{D})}\|q\|^2_{L^4(\mathcal{D})}\|w\|_{L^4(\mathcal{D})}. \tag{2.32}$$

Combining all the estimates (2.24)–(2.32) on $f(w_1, w_2)$ we arrive at

$$\|w_1\|_{L^\infty L^2(\mathcal{D})} + \|w_1\|_{L^4(\mathcal{D})} \leqslant C\|w_1(t_1)\|_{L^2(\mathbb{R}^2)} + C(\|q\|^2_{L^4(\mathcal{D})} + \|q\|^2_{L^\infty L^2(\mathcal{D})}\|q\|^2_{L^4(\mathcal{D})})\|w\|_{L^4(\mathcal{D})}. \tag{2.33}$$

Similarly, one has

$$\|w_2\|_{L^\infty L^2(\mathcal{D})} + \|w_2\|_{L^4(\mathcal{D})} \leqslant C\|w_2(t_1)\|_{L^2(\mathbb{R}^2)} + C(\|q\|^2_{L^4(\mathcal{D})} + \|q\|^2_{L^\infty L^2(\mathcal{D})}\|q\|^2_{L^4(\mathcal{D})})\|w\|_{L^4(\mathcal{D})}. \tag{2.34}$$

Then it holds that

$$\|w\|_{L^\infty L^2(\mathcal{D})} + \|w\|_{L^4(\mathcal{D})} \leqslant C\|w(t_1)\|_{L^2(\mathbb{R}^2)} + C(\|q\|^2_{L^4(\mathcal{D})} + \|q\|^2_{L^\infty L^2(\mathcal{D})}\|q\|^2_{L^4(\mathcal{D})})\|w\|_{L^4(\mathcal{D})}. \tag{2.35}$$

Moreover, from (2.21) the space–time norm of q is bounded by

$$\|q\|_{L^4(\mathbb{R}^3)} + \|q\|_{L^\infty(\mathbb{R}, L^2(\mathbb{R}^2))} \leqslant C\|q_0\|_{L^2(\mathbb{R}^2)}. \tag{2.36}$$

Therefore, we conclude that

$$\|\partial q\|_{L^\infty([0,T], L^2(\mathbb{R}^2))} + \|\partial q\|_{L^4([0,T], L^4(\mathbb{R}^2)} \leqslant C_*(T), \tag{2.37}$$

where C_* is a bounded function depending on the norm of $w(0) \in L^2(\mathbb{R}^2)$ and $q \in L^4(\mathbb{R}^3)$. Thus $q_1, q_2 \in C(\mathbb{R}, H^1(\mathbb{R}^2)) \cap L^4_{\mathrm{loc}}(\mathbb{R}, W^{1,4}(\mathbb{R}^2))$. □

Corollary 2.2. *If the initial data are smooth, (2.3) has a unique global smooth solution.*

Proof. The proof is by estimates on the derivatives of higher order. □

To establish global existence and uniqueness of solutions for the system (1.6)–(1.13), the idea is to show that the solution constructed for (2.3) also solves (1.6)–(1.13). We have the following theorem.

Theorem 2.1. *Given smooth initial data q_{10} and $q_{20} \in H^1(\mathbb{R}^2)$, there exists an $\epsilon_0 > 0$ such that if $\|q_{10}\|_{L^2(\mathbb{R}^2)} + \|q_{20}\|_{L^2(\mathbb{R}^2)} \leqslant \epsilon_0$, then the Cauchy problem for the system (1.6)–(1.13) has a unique global smooth solution $(q_1, q_2, a_0, a_1, a_2)$.*

Proof. Let $(q_1, q_2, a_0, a_1, a_2)$ be the unique global smooth solution of (2.3). To show that it also solves the system (1.6)–(1.13) we only need to verify that (1.9)–(1.13) hold. Equations (1.9) and (1.10) can be verified easily as follows:

- $\mathrm{div}\cdot a = \Delta^{-1}[\partial_x\partial_y \,\mathrm{Im}(\bar{q}_1 q_2) + \partial_y\partial_x \,\mathrm{Im}(\bar{q}_2 q_1)] = 0.$
- $\partial_y a_1 - \partial_x a_2 = \Delta^{-1}[\partial_y^2 \,\mathrm{Im}(\bar{q}_1 q_2) - \partial_x^2 \,\mathrm{Im}(\bar{q}_2 q_1)] = \mathrm{Im}(\bar{q}_1 q_2).$

The verifications of (1.11)–(1.13) can be done simultaneously by deriving an equation for $Y = q_{1y} + \mathrm{i}a_2 q_1 - q_{2x} - \mathrm{i}a_1 q_2$. We differentiate equations (1.6) and (1.7) with respect to y and x, respectively, and obtain

$$\begin{aligned}(q_{1y})_t + \mathrm{i}a_{0y}q_1 + \mathrm{i}a_0 q_{1y} &= \mathrm{i}\Delta q_{1y} + \mathrm{i}(a_1^2 + a_2^2)q_{1y} + \mathrm{i}(a_1^2 + a_2^2)_y q_1 \\ &\quad + (a_{1x} - a_{2y})q_{1y} + (a_{1x} - a_{2y})_y q_1 + (a_{2x} + a_{1y})q_{2y} + (a_{2x} + a_{1y})_y q_2, \end{aligned} \tag{2.38}$$

$$\begin{aligned}(q_{2x})_t + \mathrm{i}a_{0x}q_2 + \mathrm{i}a_0 q_{2x} &= \mathrm{i}\Delta q_{2x} + \mathrm{i}(a_1^2 + a_2^2)_x q_2 + \mathrm{i}(a_1^2 + a_2^2)q_{2x} \\ &\quad - (a_{1x} - a_{2y})_x q_2 - (a_{1x} - a_{2y})q_{2x} + (a_{2x} + a_{1y})q_{1x} + (a_{2x} + a_{1y})_x q_1. \end{aligned} \tag{2.39}$$

By subtracting (2.39) from (2.38) and substituting $Y = q_{1y} + \mathrm{i}a_2 q_1 - q_{2x} - \mathrm{i}a_1 q_2$ into the equation, one obtains

$$\begin{aligned}&(Y - \mathrm{i}a_2 q_1 + \mathrm{i}a_1 q_2)_t + \mathrm{i}a_0(Y - \mathrm{i}a_2 q_1 + \mathrm{i}a_1 q_2) + \mathrm{i}a_{0y}q_1 - \mathrm{i}a_{0x}q_2 \\ &\quad = \mathrm{i}\Delta(Y - \mathrm{i}a_2 q_1 + \mathrm{i}a_1 q_2) + \mathrm{i}(a_1^2 + a_2^2)(Y - \mathrm{i}a_2 q_1 + \mathrm{i}a_1 q_2)q_2\Delta a_1 \\ &\qquad - q_1\Delta a_2 + 2\mathrm{i}(a_1 a_{1y} + a_2 a_{2y})q_1 - 2\mathrm{i}(a_1 a_{1x} + a_2 a_{2x})q_2 \\ &\qquad - 2a_{2y}q_{1y} + 2a_{1x}q_{2x} + (a_{2x} + a_{1y})(q_{2y} - q_{1x}). \end{aligned}$$

After some calculation the above equation can be simplified as

$$\begin{aligned}&Y_t + \mathrm{i}[a_{0y} - a_{2t} + \mathrm{Im}(\bar{q}_2 p)]q_1 + \mathrm{i}[a_{1t} - a_{0x} - \mathrm{Im}(\bar{q}_1 p)]q_2 + \mathrm{i}a_0 Y \\ &\quad = \mathrm{i}\Delta Y + \mathrm{i}(a_1^2 + a_2^2)Y + \mathrm{i}\,\mathrm{Im}(\bar{q}_2 p)q_1 - \mathrm{i}\,\mathrm{Im}(\bar{q}_1 p)q_2 \\ &\qquad + \mathrm{i}\,\mathrm{Im}(\bar{q}_1 q_2)(a_1 q_1 + a_2 q_2) + (a_{2x} + a_{1y})(q_{2y} - q_{1x}) \\ &\qquad + 2a_{1x}q_{2x} - 2a_{2y}q_{1y}. \end{aligned}$$

Further computation gives us

$$Y_t = \mathrm{i}\Delta Y - \mathrm{i}a_0 Y + \mathrm{i}(a_1^2 + a_2^2)Y - \mathrm{i}[a_{0y} - a_{2t} + \mathrm{Im}(\bar{q}_2 p)]q_1 - \mathrm{i}[a_{1t} - a_{0x} - \mathrm{Im}(\bar{q}_1 p)]q_2. \tag{2.40}$$

In order to write (2.40) as an equation in terms of Y, let us assume that we have proved the following claim:

Claim 2.1. *Let $(q_1, q_2, a_0, a_1, a_2)$ be the smooth solution of (2.3), then the following holds*

$$a_{1t} - a_{0x} - \mathrm{Im}(\bar{q}_1 p) = \Delta^{-1}[\partial_x^2 \,\mathrm{Re}(\bar{q}_2 Y) - 2\partial_x\partial_y \,\mathrm{Re}(\bar{q}_1 Y) - \partial_y^2 \,\mathrm{Re}(\bar{q}_2 Y)], \tag{2.41}$$

$$a_{0y} - a_{2t} - \mathrm{Im}(\bar{q}_2 p) = \Delta^{-1}[\partial_y^2 \,\mathrm{Re}(\bar{q}_1 Y) - 2\partial_x\partial_y \,\mathrm{Re}(\bar{q}_2 Y) - \partial_x^2 \,\mathrm{Re}(\bar{q}_1 Y)]. \tag{2.42}$$

□

Using the above claim (2.40) becomes

$$\begin{aligned}Y_t &= \mathrm{i}\Delta Y - \mathrm{i}a_0 Y + \mathrm{i}(a_1^2 + a_2^2)Y + \mathrm{i}[\partial_x^2\Delta^{-1}\,\mathrm{Re}(\bar{q}_1 Y) + 2\partial_x\partial_y\Delta^{-1}\,\mathrm{Re}(\bar{q}_2 Y) \\ &\quad - \partial_y^2\Delta^{-1}\,\mathrm{Re}(\bar{q}_1 Y)]q_1 + \mathrm{i}[\partial_y^2\Delta^{-1}\,\mathrm{Re}(\bar{q}_2 Y) + 2\partial_x\partial_y\Delta^{-1}\,\mathrm{Re}(\bar{q}_1 Y) - \partial_x^2\Delta^{-1}\,\mathrm{Re}(\bar{q}_2 Y)]q_2. \end{aligned} \tag{2.43}$$

Multiplying (2.43) by $\bar{Y}$ and integrating over space, one obtains

$$\frac{d}{dt}\int |Y|^2\,dx = \int i[\partial_x^2\Delta^{-1}\operatorname{Re}(\bar{q}_1Y) + 2\partial_x\partial_y\Delta^{-1}\operatorname{Re}(\bar{q}_2Y) - \partial_y^2\Delta^{-1}\operatorname{Re}(\bar{q}_1Y)] \times (q_1\bar{Y} - \bar{q}_1Y)\,dx + \int i[\partial_y^2\Delta^{-1}\operatorname{Re}(\bar{q}_2Y) + 2\partial_x\partial_y\Delta^{-1}\operatorname{Re}(\bar{q}_1Y) - \partial_x^2\Delta^{-1}\operatorname{Re}(\bar{q}_2Y)](q_2\bar{Y} - \bar{q}_2Y)\,dx.$$

It follows that

$$\begin{aligned}\frac{d}{dt}\|Y\|^2_{L^2(\mathbb{R}^2)} &\leqslant C(\|q_2Y\|_{L^2(\mathbb{R}^2)} + \|q_1Y\|_{L^2(\mathbb{R}^2)})^2 \\ &\leqslant C(\|q_1\|_{L^\infty(\mathbb{R}^2)} + \|q_2\|_{L^\infty(\mathbb{R}^2)})^2\|Y\|^2_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{2.44}$$

From (2.37) and corollary 2.2, we have that for any $T > 0$,

$$\|q_1\|_{L^\infty(\mathbb{R}^2)} + \|q_2\|_{L^\infty(\mathbb{R}^2)} \leqslant C(T)$$

for $0 \leqslant t \leqslant T$. Thus (2.44) becomes

$$\frac{d}{dt}\int |Y|^2\,dx \leqslant C\|Y\|^2_{L^2(\mathbb{R}^2)},$$

which implies that $Y = 0$ for $0 \leqslant t \leqslant T$ if $Y = 0$ initially. Therefore, (1.13) is verified. (1.11) and (1.12) follow from claim 2.1. We conclude that $(q_1, q_2, a_0, a_1, a_2)$ solves the system (1.6)–(1.13). □

To complete the proof of the theorem, it remains to prove the claim.

Proof of claim 2.1. From (2.3),

$$\begin{aligned}a_{1t} - a_{0x} - \operatorname{Im}(\bar{q}_1p) &= \Delta^{-1}[\partial_y\partial_t\operatorname{Im}(\bar{q}_1q_2)] - \Delta^{-1}\partial_x[\partial_x\operatorname{Im}(\bar{p}q_1) + \partial_y\operatorname{Im}(\bar{p}q_2)] \\ &\quad - \operatorname{Im}(\bar{q}_1p) + \Delta^{-1}\partial_x[\partial_x\operatorname{Im}(\bar{p}q_1) + \partial_y\operatorname{Im}(\bar{P}q_2)] \\ &\quad - \Delta^{-1}\partial_x[\tfrac{1}{2}\partial_x^2|q_2|^2 + \tfrac{1}{2}\partial_y^2|q_1|^2 - \tfrac{1}{2}\partial_x^2|q_1|^2 - \tfrac{1}{2}\partial_y^2|q_2|^2 \\ &\quad - 2\partial_x\partial_y\operatorname{Re}(q_1\bar{q}_2) + 2\partial_x\operatorname{Im}(a_2q_1\bar{q}_2) + 2\partial_y\operatorname{Im}(a_1\bar{q}_1q_2)] \\ &= \mathrm{I} + \mathrm{II}.\end{aligned} \tag{2.45}$$

We compute

$$\begin{aligned}\mathrm{I} &= \Delta^{-1}[\partial_y\partial_t\operatorname{Im}(\bar{q}_1q_2)] - \Delta^{-1}\partial_x[\partial_x\operatorname{Im}(\bar{p}q_1) + \partial_y\operatorname{Im}(\bar{p}q_2)] - \operatorname{Im}(\bar{q}_1p) \\ &= \Delta^{-1}[\partial_y\partial_t\operatorname{Im}(\bar{q}_1q_2)] - \Delta^{-1}[\partial_x^2\operatorname{Im}(\bar{p}q_1)] - \Delta^{-1}[\partial_y^2\operatorname{Im}(\bar{p}q_1)] \\ &\quad + \Delta^{-1}[\partial_y^2\operatorname{Im}(\bar{p}q_1)] - \Delta^{-1}[\partial_x\partial_y\operatorname{Im}(\bar{p}q_2)] - \operatorname{Im}(\bar{q}_1p) \\ &= \Delta^{-1}\{\partial_y[\partial_t\operatorname{Im}(\bar{q}_1q_2) + \partial_y\operatorname{Im}(\bar{p}q_1) - \partial_x\operatorname{Im}(\bar{p}q_2)]\} \\ &= \Delta^{-1}\{\partial_y[\operatorname{Im}(\bar{q}_{1t}q_2 + \bar{q}_1q_{2t} + \bar{p}_yq_1 + \bar{p}q_{1y} - \bar{p}_xq_2 - \bar{p}q_{2x})]\}.\end{aligned} \tag{2.46}$$

By using the equations for q_1 and q_2 in (2.3) we obtain

$$\begin{aligned}\operatorname{Im}(\bar{q}_{1t}q_2) &= -\operatorname{Re}[q_2\Delta\bar{q}_1 + (a_1^2 + a_2^2)\bar{q}_1q_2 - a_0\bar{q}_1q_2)] + \operatorname{Im}[(a_{1x} - a_{2y})\bar{q}_1q_2], \\ \operatorname{Im}(\bar{q}_1q_{2t}) &= \operatorname{Re}[\bar{q}_1\Delta q_2 + (a_1^2 + a_2^2)\bar{q}_1q_2 - a_0\bar{q}_1q_2] - \operatorname{Im}[(a_{1x} - a_{2y})\bar{q}_1q_2].\end{aligned}$$

Therefore, we have

$$\operatorname{Im}(\bar{q}_{1t}q_2 + \bar{q}_1q_{2t}) = \operatorname{Re}(\bar{q}_1\Delta q_2 - q_2\Delta\bar{q}_1). \tag{2.47}$$

From (1.8) we have

$$\bar{p}_y = -i\bar{q}_{1xy} - i\bar{q}_{2yy} + a_{1y}\bar{q}_1 + a_1\bar{q}_{1y} + a_{2y}\bar{q}_2 + a_2\bar{q}_{2y}$$

and this implies that

$$\mathrm{Im}(\bar{p}_y q_1) = -\mathrm{Re}(q_1\bar{q}_{1xy}) - \mathrm{Re}(q_1\bar{q}_{2yy}) + \mathrm{Im}(a_1 q_1\bar{q}_{1y}) + \mathrm{Im}(a_{2y}q_1\bar{q}_2) + \mathrm{Im}(a_2 q_1\bar{q}_{2y}). \tag{2.48}$$

Since

$$Y = q_{1y} + \mathrm{i}a_2 q_1 - q_{2x} - \mathrm{i}a_1 q_2,$$

it follows that

$$q_{1xy} = Y_x - \mathrm{i}a_{2x}q_1 - \mathrm{i}a_2 q_{1x} + q_{2xx} + \mathrm{i}a_{1x}q_2 + \mathrm{i}a_1 q_{2x}.$$

Hence one gets

$$\mathrm{Re}(q_1\bar{q}_{1xy}) = \mathrm{Re}(q_1\bar{Y}_x) - \mathrm{Im}(a_2 q_1\bar{q}_{1x}) + \mathrm{Re}(q_1\bar{q}_{2xx}) + \mathrm{Im}(a_{1x}q_1\bar{q}_2) + \mathrm{Im}(a_1 q_1\bar{q}_{2x}).$$

Therefore, we have

$$\begin{aligned}\mathrm{Im}(\bar{p}_y q_1) = &-\mathrm{Re}(q_1\bar{Y}_x) + \mathrm{Im}(a_2 q_1\bar{q}_{1x}) - \mathrm{Re}(q_1\bar{q}_{2xx}) - \mathrm{Im}(a_{1x}q_1\bar{q}_2) - \mathrm{Im}(a_1 q_1\bar{q}_{2x})\\ &-\mathrm{Re}(q_1\bar{q}_{2yy}) + \mathrm{Im}(a_1 q_1\bar{q}_{1y}) + \mathrm{Im}(a_{2y}q_1\bar{q}_2) + \mathrm{Im}(a_2 q_1\bar{q}_{2y})\end{aligned} \tag{2.49}$$

and

$$\mathrm{Im}(\bar{p}q_{1y}) = -\mathrm{Re}(\bar{q}_{1x}q_{1y}) - \mathrm{Re}(q_{1y}\bar{q}_{2y}) + \mathrm{Im}(a_1 q_1 q_{1y}) + \mathrm{Im}(a_2\bar{q}_2 q_{1y}). \tag{2.50}$$

Moreover,

$$\bar{p}_x = -\mathrm{i}\bar{q}_{1xx} - \mathrm{i}\bar{q}_{2xy} + a_{1x}\bar{q}_1 + a_1\bar{q}_{1x} + a_{2x}\bar{q}_2 + a_2\bar{q}_{2x}.$$

By using the expression for q_{2x}

$$q_{2x} = q_{1y} + \mathrm{i}a_2 q_1 - \mathrm{i}a_1 q_2 - Y,$$

we obtain

$$\bar{q}_{2xy} = \bar{q}_{1yy} - \mathrm{i}a_{2y}\bar{q}_1 - \mathrm{i}a_2\bar{q}_{1y} + \mathrm{i}a_{1y}\bar{q}_2 + \mathrm{i}a_1\bar{q}_{2y} - \bar{Y}_y.$$

Then one may compute

$$\begin{aligned}\bar{p}_x q_2 = &-\mathrm{i}q_2\bar{q}_{1xx} - \mathrm{i}\bar{q}_{1yy}q_2 - a_{2y}\bar{q}_1 q_2 - a_2 q_2\bar{q}_{1y} + a_{1y}|q_2|^2 + a_1 q_2\bar{q}_{2y} + \mathrm{i}\bar{Y}_y q_2\\ &+ a_{1x}\bar{q}_1 q_2 + a_1\bar{q}_{1x}q_2 + a_{2x}|q_2|^2 + a_2\bar{q}_{2x}q_2,\end{aligned}$$

which implies

$$\mathrm{Im}(\bar{p}_x q_2) = -\mathrm{Re}(q_2\Delta\bar{q}_1) - \mathrm{Im}(a_{2y}\bar{q}_1 q_2) - \mathrm{Im}(a_2 q_2\bar{q}_{1y}) + \mathrm{Im}(a_1 q_2\bar{q}_{2y}) + \mathrm{Re}(\bar{Y}_y q_2). \tag{2.51}$$

From the expression for p, we have

$$\mathrm{Im}(\bar{p}q_{2x}) = -\mathrm{Re}(\bar{q}_{1x}q_{2x}) - \mathrm{Re}(q_{2x}\bar{q}_{2y}) + \mathrm{Im}(a_1\bar{q}_1 q_{2x}) + \mathrm{Im}(a_2\bar{q}_2 q_{2x}). \tag{2.52}$$

Combining (2.48)–(2.52) we get

$$\begin{aligned}\mathrm{I} &= \Delta^{-1}\{\partial_y[\mathrm{Im}(\bar{q}_{1t}q_2 + \bar{q}_1 q_{2t} + \bar{p}_y q_1 + \bar{p}q_{1y} - \bar{p}_x q_2 - \bar{p}q_{2x})]\}\\ &= \Delta^{-1}\{\partial_y[\mathrm{Re}(\bar{q}_1\Delta q_2) - \mathrm{Re}(q_2\Delta\bar{q}_1) - \mathrm{Re}(q_1\bar{Y}_x) + \mathrm{Im}(a_2 q_1\bar{q}_{1x}) - \mathrm{Im}(a_{1x}q_1\bar{q}_2)\\ &\quad -\mathrm{Im}(a_1 q_1\bar{q}_{2x}) - \mathrm{Re}(q_1\Delta\bar{q}_2) + \mathrm{Im}(a_1 q_1\bar{q}_{1y}) + \mathrm{Im}(a_{2y}q_1\bar{q}_2) + \mathrm{Im}(a_2 q_1\bar{q}_{2y})\\ &\quad -\mathrm{Re}(\bar{q}_{1x}q_{1y}) - \mathrm{Re}(q_{1y}\bar{q}_{2y}) + \mathrm{Im}(a_1 q_1 q_{1y}) + \mathrm{Im}(a_2\bar{q}_2 q_{1y}) + \mathrm{Re}(q_2\Delta\bar{q}_1) + \mathrm{Im}(a_{2y}\bar{q}_1 q_2)\\ &\quad +\mathrm{Im}(a_2 q_2\bar{q}_{1y}) - \mathrm{Im}(a_1 q_2\bar{q}_{2y}) - \mathrm{Re}(\bar{Y}_y q_2) + \mathrm{Re}(\bar{q}_{1x}q_{2x}) + \mathrm{Re}(q_{2x}\bar{q}_{2y}) - \mathrm{Im}(a_1\bar{q}_1 q_{2x})\\ &\quad -\mathrm{Im}(a_2\bar{q}_2 q_{2x}) - \mathrm{Im}(a_{1x}\bar{q}_1 q_2) - \mathrm{Im}(a_1\bar{q}_{1x}q_2) - \mathrm{Im}(a_2 q_2\bar{q}_{2x})]\}\\ &= \Delta^{-1}\{\partial_y[-\mathrm{Re}(q_1\bar{Y}_x) + \mathrm{Im}(a_2 q_1\bar{q}_{1x}) + \mathrm{Im}(a_2 q_1\bar{q}_{2y}) - \mathrm{Re}(\bar{q}_{1x}q_{1y}) - \mathrm{Re}(q_{1y}\bar{q}_{2y})\\ &\quad -\mathrm{Im}(a_1 q_2\bar{q}_{2y}) - \mathrm{Re}(\bar{Y}_y q_2) + \mathrm{Re}(\bar{q}_{1x}q_{2x}) + \mathrm{Re}(q_{2x}\bar{q}_{2y}) - \mathrm{Im}(a_1 q_2\bar{q}_{1x})]\}\\ &= \Delta^{-1}\{\partial_y[-\mathrm{Re}(q_1\bar{Y}_x) - \mathrm{Re}(\bar{Y}_y q_2) - \mathrm{Re}(\mathrm{i}a_2 q_1\bar{q}_{1x} + q_{1y}\bar{q}_{1x} - q_{2x}\bar{q}_{1x} - \mathrm{i}a_1 q_2\bar{q}_{1x})\\ &\quad -\mathrm{Re}(\mathrm{i}a_2 q_1\bar{q}_{2y} + q_{1y}\bar{q}_{2y} - \mathrm{i}a_1 q_2\bar{q}_{2y} - q_{2x}\bar{q}_{2y})]\}\\ &= \Delta^{-1}\{\partial_y[-\mathrm{Re}(q_1\bar{Y}_x) - \mathrm{Re}(q_2\bar{Y}_y) - \mathrm{Re}(\bar{q}_{2y}Y) - \mathrm{Re}(Y\bar{q}_{1x})]\}.\end{aligned} \tag{2.53}$$

On the other hand,

$$\mathrm{II} = \Delta^{-1}\{\partial_x[\partial_x \operatorname{Im}(\bar{p}q_1) + \partial_y \operatorname{Im}(\bar{p}q_2)]\} - \Delta^{-1}a_{0x}. \tag{2.54}$$

From the expressions for $\operatorname{Im}(\bar{p}q_1)$ and $\operatorname{Im}(\bar{p}q_2)$

$$\begin{aligned}\operatorname{Im}(\bar{p}q_1) &= -\operatorname{Re}(q_1\bar{q}_{1x}) - \operatorname{Re}(q_1\bar{q}_{2y}) + \operatorname{Im}(a_2q_1\bar{q}_2),\\ \operatorname{Im}(\bar{p}q_2) &= -\operatorname{Re}(q_2\bar{q}_{1x}) - \operatorname{Re}(q_2\bar{q}_{2y}) + \operatorname{Im}(a_1\bar{q}_1q_2),\end{aligned}$$

(2.54) can be written as

$$\begin{aligned}\mathrm{II} = {}& \Delta^{-1}\{\partial_x[-\partial_x \operatorname{Re}(q_1\bar{q}_{1x}) - \partial_x \operatorname{Re}(q_1\bar{q}_{2y}) + \partial_x \operatorname{Im}(a_2q_1\bar{q}_2)\\ &-\partial_y \operatorname{Re}(q_2\bar{q}_{1x}) - \partial_y \operatorname{Re}(q_2\bar{q}_{2y}) + \partial_y \operatorname{Im}(a_1\bar{q}_1q_2) - \tfrac{1}{2}\partial_x^2|q_2|^2 - \tfrac{1}{2}\partial_y^2|q_1|^2 + \tfrac{1}{2}\partial_x^2|q_1|^2\\ &+\tfrac{1}{2}\partial_y^2|q_2|^2 + 2\partial_x\partial_y \operatorname{Re}(q_1\bar{q}_2) - 2\partial_x \operatorname{Im}(a_2q_1\bar{q}_2) - 2\partial_y \operatorname{Im}(a_1\bar{q}_1q_2)]\}\\ ={}& \Delta^{-1}\{\partial_x[-\partial_x \operatorname{Re}(q_1\bar{q}_{1x}) - \partial_x\partial_y \operatorname{Re}(q_1\bar{q}_2) + \partial_x \operatorname{Re}(q_{1y}\bar{q}_2) + \partial_x \operatorname{Im}(a_2q_1\bar{q}_2)\\ &-\partial_y\partial_x \operatorname{Re}(q_2\bar{q}_1) + \partial_y \operatorname{Re}(q_{2x}\bar{q}_1) - \partial_y \operatorname{Re}(q_2\bar{q}_{2y}) + \partial_y \operatorname{Im}(a_1\bar{q}_1q_2) - \tfrac{1}{2}\partial_x^2|q_2|^2 - \tfrac{1}{2}\partial_y^2|q_1|^2\\ &+\tfrac{1}{2}\partial_x^2|q_1|^2 + \tfrac{1}{2}\partial_y^2|q_2|^2 + 2\partial_x\partial_y \operatorname{Re}(q_1\bar{q}_2) - 2\partial_x \operatorname{Im}(a_2q_1\bar{q}_2) - 2\partial_y \operatorname{Im}(a_1\bar{q}_1q_2)]\}\\ ={}& \Delta^{-1}\{\partial_x[\partial_x \operatorname{Re}(q_{1y}\bar{q}_2) + \partial_x \operatorname{Im}(a_2q_1\bar{q}_2) + \partial_y \operatorname{Re}(q_{2x}\bar{q}_1) + \partial_y \operatorname{Im}(a_1\bar{q}_1q_2)\\ &-\tfrac{1}{2}\partial_x^2|q_2|^2 - \tfrac{1}{2}\partial_y^2|q_1|^2 - 2\partial_x \operatorname{Im}(a_2q_1\bar{q}_2) - 2\partial_y \operatorname{Im}(a_1\bar{q}_1q_2)]\}\\ ={}& \Delta^{-1}\{\partial_x[\partial_x \operatorname{Re}(q_{1y}\bar{q}_2) - \partial_x \operatorname{Im}(a_2q_1\bar{q}_2) + \partial_y \operatorname{Re}(q_{2x}\bar{q}_1)\\ &-\partial_y \operatorname{Im}(a_1\bar{q}_1q_2) - \tfrac{1}{2}\partial_x^2|q_2|^2 - \tfrac{1}{2}\partial_y^2|q_1|^2]\}.\end{aligned} \tag{2.55}$$

By using the expression for Y

$$Y = q_{1y} + \mathrm{i}a_2q_1 - q_{2x} - \mathrm{i}a_1q_2$$

we can write II as

$$\begin{aligned}\mathrm{II} = {}& \Delta^{-1}\{\partial_x[\partial_x \operatorname{Re}(Y\bar{q}_2) + \partial_x \operatorname{Im}(a_2q_1\bar{q}_2) + \partial_x \operatorname{Re}(q_{2x}\bar{q}_2) - \partial_x \operatorname{Im}(a_2q_1\bar{q}_2)\\ &+ \partial_y \operatorname{Re}(q_{2x}\bar{q}_1) - \partial_y \operatorname{Im}(a_1\bar{q}_1q_2) - \tfrac{1}{2}\partial_x^2|q_2|^2 - \tfrac{1}{2}\partial_y^2|q_1|^2]\}\\ ={}& \Delta^{-1}\{\partial_x[\partial_x \operatorname{Re}(Y\bar{q}_2) + \partial_y \operatorname{Re}(\bar{q}_1q_{1y}) + \partial_y \operatorname{Im}(a_1\bar{q}_1q_2) - \partial_y \operatorname{Re}(\bar{q}_1Y)\\ &- \partial_y \operatorname{Im}(a_1\bar{q}_1q_2) - \tfrac{1}{2}\partial_y^2|q_1|^2]\}\\ ={}& \Delta^{-1}\{\partial_x[\partial_x \operatorname{Re}(Y\bar{q}_2) - \partial_y \operatorname{Re}(\bar{q}_1Y)]\}.\end{aligned} \tag{2.56}$$

Therefore, we obtain

$$\begin{aligned}a_{1t} - a_{0x} - \operatorname{Im}(\bar{q}_1p) &= \mathrm{I} + \mathrm{II}\\ &= \Delta^{-1}\{\partial_y[-\operatorname{Re}(q_1\bar{Y}_x) - \operatorname{Re}(q_2\bar{Y}_y) - \operatorname{Re}(\bar{q}_{2y}Y) - \operatorname{Re}(Y\bar{q}_{1x})]\}\\ &\quad + \Delta^{-1}\{\partial_x[\partial_x \operatorname{Re}(Y\bar{q}_2) - \partial_y \operatorname{Re}(Y\bar{q}_1)]\}\\ &= \Delta^{-1}[\partial_x^2 \operatorname{Re}(Y\bar{q}_2)] - 2\Delta^{-1}[\partial_x\partial_y \operatorname{Re}(Y\bar{q}_1)] - \Delta^{-1}[\partial_y^2 \operatorname{Re}(Y\bar{q}_2)].\end{aligned} \tag{2.57}$$

In a similar fashion, $a_{0y} - a_{2t} + \operatorname{Im}(\bar{q}_2p)$ can be computed by

$$\begin{aligned}a_{0y} - a_{2t} + \operatorname{Im}(\bar{q}_2p) &= \Delta^{-1}\{\partial_y[\partial_x \operatorname{Im}(\bar{p}q_1) + \partial_y \operatorname{Im}(\bar{p}q_2)]\} - \Delta^{-1}\{\partial_t[\partial_x \operatorname{Im}(\bar{q}_2q_1)]\}\\ &\quad +\operatorname{Im}(\bar{q}_2p) + \Delta^{-1}\{\partial_y[\tfrac{1}{2}\partial_x^2|q_2|^2 + \tfrac{1}{2}\partial_y^2|q_1|^2 - \tfrac{1}{2}\partial_x^2|q_1|^2\\ &\quad -\tfrac{1}{2}\partial_y^2|q_2|^2 - 2\partial_x\partial_y \operatorname{Re}(q_1\bar{q}_2) + 2\partial_x \operatorname{Im}(a_2q_1\bar{q}_2)\\ &\quad +2\partial_y \operatorname{Im}(a_1\bar{q}_1q_2)]\} - \Delta^{-1}\{\partial_y[\partial_x \operatorname{Im}(\bar{p}q_1) + \partial_y \operatorname{Im}(\bar{p}q_2)]\}\\ &\stackrel{\text{def}}{=} A + B.\end{aligned} \tag{2.58}$$

$$A = \Delta^{-1}\{\partial_y[\partial_x \operatorname{Im}(\bar{p}q_1) + \partial_y \operatorname{Im}(\bar{p}q_2)]\} - \Delta^{-1}\{\partial_t[\partial_x \operatorname{Im}(\bar{q}_2 q_1)]\} + \operatorname{Im}(p\bar{q}_2)$$
$$= \Delta^{-1}\{[\partial_x\partial_y \operatorname{Im}(\bar{p}q_1) - \partial_t\partial_x \operatorname{Im}(q_1\bar{q}_2) - \partial_x^2 \operatorname{Im}(\bar{p}q_2)]\}$$
$$= \partial_x\Delta^{-1}\{\operatorname{Im}[\partial_y(q_1\bar{p}) - \partial_t(q_1\bar{q}_2) - \partial_x(q_2\bar{p})]\}$$
$$= -\partial_x\Delta^{-1}\{\operatorname{Im}[\mathrm{i}q_1\bar{Y}_x + \mathrm{i}q_2\bar{Y}_y + \mathrm{i}\bar{q}_{2y}Y + \mathrm{i}\bar{q}_{1x}Y]\}$$
$$= -\partial_x^2\Delta^{-1}\operatorname{Re}(\bar{q}_1 Y) - \partial_x\partial_y\Delta^{-1}\operatorname{Re}(\bar{q}_2 Y). \tag{2.59}$$

On the other hand,

$$B = \Delta^{-1}\{\partial_y[\partial_x \operatorname{Re}(q_2\bar{q}_{2x}) + \partial_y \operatorname{Re}(q_1\bar{q}_{1y}) - \partial_x \operatorname{Re}(q_1\bar{q}_{1x}) - \partial_y \operatorname{Re}(q_2\bar{q}_{2y}) - 2\partial_x\partial_y \operatorname{Re}(q_1\bar{q}_2)$$
$$+ 2\partial_x \operatorname{Im}(a_2 q_1\bar{q}_2) + 2\partial_y \operatorname{Im}(a_1\bar{q}_1 q_2) - \partial_x \operatorname{Im}(\bar{p}q_1) - \partial_y \operatorname{Im}(\bar{p}q_2)]\}. \tag{2.60}$$

Since

$$p = \mathrm{i}q_{1x} + \mathrm{i}q_{2y} + a_1 q_1 + a_2 q_2,$$

then we have

$$\begin{aligned} \operatorname{Im}(\bar{p}q_1) &= -\operatorname{Re}(\bar{q}_{1x}q_1) - \operatorname{Re}(\bar{q}_{2y}q_1) + \operatorname{Im}(a_2 q_1\bar{q}_2),\\ \operatorname{Im}(\bar{p}q_2) &= -\operatorname{Re}(\bar{q}_{1x}q_2) - \operatorname{Re}(\bar{q}_{2y}q_2) + \operatorname{Im}(a_1\bar{q}_1 q_2). \end{aligned} \tag{2.61}$$

Using (2.61), B can be written as

$$B = \Delta^{-1}\{\partial_y[\partial_x \operatorname{Re}(q_2\bar{q}_{2x}) + \partial_y \operatorname{Re}(q_1\bar{q}_{1y}) - \partial_x \operatorname{Re}(q_1\bar{q}_{1x}) - \partial_y \operatorname{Re}(q_2\bar{q}_{2y}) - 2\partial_x\partial_y \operatorname{Re}(q_1\bar{q}_2)$$
$$+2\partial_x \operatorname{Im}(a_2 q_1\bar{q}_2) + 2\partial_y \operatorname{Im}(a_1\bar{q}_1 q_2) + \partial_x \operatorname{Re}(\bar{q}_{1x}q_1) + \partial_x \operatorname{Re}(\bar{q}_{2y}q_1)$$
$$-\partial_x \operatorname{Im}(a_2 q_1\bar{q}_2) + \partial_y \operatorname{Re}(\bar{q}_{1x}q_2) + \partial_y \operatorname{Re}(\bar{q}_{2y}q_2) - \partial_y \operatorname{Im}(a_1\bar{q}_1 q_2)]\}$$
$$= \Delta^{-1}\{\partial_y[\partial_x \operatorname{Re}(q_2\bar{q}_{2x}) + \partial_y \operatorname{Re}(q_1\bar{q}_{1y}) - 2\partial_x\partial_y \operatorname{Re}(q_1\bar{q}_2) + \partial_y \operatorname{Re}(q_2\bar{q}_{1x})$$
$$+\partial_x \operatorname{Im}(a_2 q_1\bar{q}_2) + \partial_y \operatorname{Im}(a_1\bar{q}_1 q_2) + \partial_x \operatorname{Re}(q_1\bar{q}_{2y})]\}$$
$$= \Delta^{-1}\{\partial_y[\partial_x \operatorname{Re}(q_2\bar{q}_{2x}) + \partial_y \operatorname{Re}(q_1\bar{q}_{1y}) - 2\partial_x\partial_y \operatorname{Re}(q_1\bar{q}_2) + \partial_x \operatorname{Im}(a_2 q_1\bar{q}_2)$$
$$+\partial_y \operatorname{Im}(a_1\bar{q}_1 q_2) + \partial_x\partial_y \operatorname{Re}(q_1\bar{q}_2) - \partial_x \operatorname{Re}(q_{1y}\bar{q}_2) + \partial_x\partial_y \operatorname{Re}(q_2\bar{q}_1) - \partial_y \operatorname{Re}(q_{2x}\bar{q}_1)]\}$$
$$= \Delta^{-1}\{\partial_y\{\partial_x \operatorname{Re}[\bar{q}_2(q_{2x} - \mathrm{i}a_2 q_1 - q_{1y} + \mathrm{i}a_1 q_2)]\}$$
$$+\partial_y\{\operatorname{Re}[\bar{q}_1(q_{1y} - \mathrm{i}a_1 q_2 - q_{2x} + \mathrm{i}a_2 q_1)]\}\}$$
$$= \Delta^{-1}\{\partial_y[\partial_x \operatorname{Re}(-Y\bar{q}_2) + \partial_y \operatorname{Re}(Y\bar{q}_1)]\}. \tag{2.62}$$

Finally, we obtain

$$a_{0y} - a_{2t} + \operatorname{Im}(\bar{q}_2 p) = -\Delta^{-1}[\partial_x^2 \operatorname{Re}(\bar{q}_1 Y) + \partial_x\partial_y \operatorname{Re}(\bar{q}_2 Y) + \partial_x\partial_y \operatorname{Re}(\bar{q}_2 Y) - \partial_y^2 \operatorname{Re}(\bar{q}_1 Y)]$$
$$= -\Delta^{-1}[\partial_x^2 \operatorname{Re}(\bar{q}_1 Y) + 2\partial_x\partial_y \operatorname{Re}(\bar{q}_2 Y) - \partial_y^2 \operatorname{Re}(\bar{q}_1 Y)]. \tag{2.63}$$

Thus the proof of the claim is complete. □

We have shown that the system (1.6)–(1.13) has a unique global smooth solution. In order to solve the Cauchy problem for (1.2), we need to show the equivalence of (1.2) and (1.6)–(1.13) which will be done in the next section.

3. The equivalence of (1.2) and (2.3)

In this section, we establish the generalized Hasimoto transform and the reverse procedure to prove the equivalence of the systems (1.2) and (1.6)–(1.13) for smooth solutions. For simplicity, we adopt the notation in [5] and rewrite (1.2) as

$$\begin{aligned} u_t + v_1 u_x + v_2 u_y &= J(D_x u_x + D_y u_y),\\ v_{2x} - v_{1y} &= 2\langle J u_x, u_y\rangle_g, \end{aligned} \tag{3.1}$$

where u is a map from $\mathbb{R}^2 \times \mathbb{R}$ to a compact Riemann surface N with a metric g and a complex structure J, and D_k denotes the covariant derivative on $u^{-1}TN$ induced by the metric g; i.e. for any $V(x,t) \in T_{u(x,t)}N$,

$$D^k V^a = \delta^{kl} D_j V^a = \delta^{kl}\left(\frac{\partial V^a}{\partial x^k} + \Gamma^a_{bc}\frac{\partial u^b}{\partial x^k} V^c\right),$$

where Γ^a_{bc} denotes the Christoffel symbol of the metric g. We will restrict ourselves to the case where $N = \mathbb{S}^2$ (1.2), then the covariant derivative D can be written explicitly as

$$D_x = \partial_x + \langle u_x, \cdot\rangle u,$$
$$D_y = \partial_y + \langle u_y, \cdot\rangle u,$$
$$D_t = \partial_t + \langle u_t, \cdot\rangle u.$$

As in [5], to show equivalence between (1.2) and (1.6)–(1.13), let (u, v_1, v_2) be the smooth solution of (1.2) with initial data $u_0(x)$ and $\{e, Je\}$ be an orthonormal frame on $u^{-1}TN$. Since $\{e, Je\}$ is an orthonormal frame, we have

$$\begin{aligned} D_x e &= a_1 Je, \\ D_y e &= a_2 Je \end{aligned} \tag{3.2}$$

or equivalently,

$$\begin{aligned} \partial_x e + \langle u_x, e\rangle u &= a_1 Je, \\ \partial_y e + \langle u_y, e\rangle u &= a_2 Je, \end{aligned} \tag{3.3}$$

where a_1 and a_2 are two real-valued functions corresponding to the gauge condition. With the following choice of gauge

$$a_1 = -\frac{v_1}{2}, \qquad a_2 = -\frac{v_2}{2}$$

(1.3) implies that the compatibility condition of (3.3) is satisfied. Therefore, the linear system (3.3) has a global solution e. Similarly, we have

$$D_t e = J a_0 e$$

for some real-valued function a_0. In this frame, the coordinates of $\partial_x u$, $\partial_y u$ and $\partial_t u$ are given by three complex-valued functions q_1, q_2 and p such that

$$\begin{aligned} \partial_x u &= q_1 e, \\ \partial_y u &= q_2 e, \\ \partial_t u &= pe. \end{aligned} \tag{3.4}$$

Under this setting, we have the first part of theorem 1.1.

Theorem 3.1. $(q_1, q_2, p, a_0, a_1, a_2)$ *satisfy the coupled nonlinear Schrödinger equations (1.6)–(1.13).*

Proof. The functions q_1, q_2, p and a_0 are related as follows.

- From the compatibility conditions:

$$\begin{aligned} D_x u_t &= D_t u_x, \\ D_y u_t &= D_t u_y, \\ D_x u_y &= D_y u_x, \end{aligned} \tag{3.5}$$

  one has

$$p_x + \mathrm{i}a_1 p = q_{1t} + \mathrm{i}a_0 q_1, \tag{3.6}$$

$$p_y + \mathrm{i}a_2 p = q_{2t} + \mathrm{i}a_0 q_2, \tag{3.7}$$

$$q_{1y} + \mathrm{i}a_2 q_1 = q_{2x} + \mathrm{i}a_1 q_2. \tag{3.8}$$

- From the curvature identities:

$$D_x D_t e = D_t D_x e + R(u_t, u_x)e,$$
$$D_y D_t e = D_t D_y e + R(u_t, u_y)e,$$
$$D_x D_y e = D_y D_x e + R(u_y, u_x)e,$$

we have

$$\partial_t a_1 - \partial_x a_0 = \mathrm{Im}(\bar{q}_1 p), \tag{3.9}$$
$$\partial_t a_2 - \partial_y a_0 = \mathrm{Im}(\bar{q}_2 p), \tag{3.10}$$
$$\partial_y a_1 - \partial_x a_2 = \mathrm{Im}(\bar{q}_1 q_2) \tag{3.11}$$

by using the fact that the Gaussian curvature $= 1$ for $\mathbb{S}^2$.

- From the spin-liquid model (1.2), we have

$$p = \mathrm{i}q_{1x} + \mathrm{i}q_{2y} + a_1 q_1 + a_2 q_2. \tag{3.12}$$

Substituting the above expression into (3.6) and using (3.8), we get an equation for q_1

$$q_{1t} + \mathrm{i}a_0 q_1 = \mathrm{i}q_{1xx} + \mathrm{i}q_{1yy} + \mathrm{i}(a_1^2 + a_2^2)q_1 + (a_{1x} - a_{2y})q_1 + (a_{1y} + a_{2x})q_2.$$

Similarly, an equation for q_2 can be obtained from (3.7), (3.8) and (3.12)

$$q_{2t} + \mathrm{i}a_0 q_2 = \mathrm{i}q_{2xx} + \mathrm{i}q_{2yy} + \mathrm{i}(a_1^2 + a_2^2)q_2 - (a_{1x} - a_{2y})q_2 + (a_{2x} + a_{1y})q_1.$$

Thus we show how to construct q from u in theorem 1.1. □

Remark 3.1. The generalized Hasimoto transform can also be applied to the Ishimori model (1.5) to derive the Davey–Stewartson equations. We will not repeat the procedure here (see [27]).

Let $(q_1, q_2, p, a_0, a_1, a_2)$ be the smooth solution to the coupled Schrödinger equations (1.6)–(1.13) with initial data (q_{10}, q_{20}). To construct solutions of (1.2) from (1.6)–(1.13), the idea is to solve (u, e) given by the following over-determined system:

$$\begin{aligned} u_x &= q_1 e, \\ u_y &= q_2 e, \\ u_t &= pe, \\ D_x e &= a_1 J e, \\ D_y e &= a_2 J e, \\ D_t e &= a_0 J e, \end{aligned} \tag{3.13}$$

where D_k denotes the Levi-Civita connection. It is equivalent to treat the system

$$\begin{aligned} \partial_x u &= q_1^{\mathrm{Re}} e + q_1^{\mathrm{Im}} u \wedge e \stackrel{\mathrm{def}}{=} f_1(x, y, t), \\ \partial_x e &= a_1 u \wedge e - q_1^{\mathrm{Re}} u \stackrel{\mathrm{def}}{=} g_1(x, y, t), \\ \partial_y u &= q_2^{\mathrm{Re}} e + q_2^{\mathrm{Im}} u \wedge e \stackrel{\mathrm{def}}{=} f_2(x, y, t), \\ \partial_y e &= a_2 u \wedge e - q_2^{\mathrm{Re}} u \stackrel{\mathrm{def}}{=} g_2(x, y, t), \\ \partial_t u &= p^{\mathrm{Re}} e + p^{\mathrm{Im}} u \wedge e \stackrel{\mathrm{def}}{=} f_3(x, y, t), \\ \partial_t e &= a_0 u \wedge e - p^{\mathrm{Re}} u \stackrel{\mathrm{def}}{=} g_3(x, y, t) \end{aligned} \tag{3.14}$$

with the conditions

$$\begin{aligned} u \cdot e &= 0, \\ |u| &= 1, \\ |e| &= 1, \end{aligned} \tag{3.15}$$

where $q_1^{\rm Re}$ and $q_2^{\rm Im}$ denote the real and imaginary parts of q_1 and q_2, respectively. To solve (3.14), first we need the following proposition.

Proposition 3.1. *The compatibility conditions for (3.14) are satisfied.*

$$\partial_y f_1 = \partial_x f_2, \tag{3.16}$$

$$\partial_t f_1 = \partial_x f_3, \tag{3.17}$$

$$\partial_t f_2 = \partial_y f_3, \tag{3.18}$$

$$\partial_y g_1 = \partial_x g_2, \tag{3.19}$$

$$\partial_x g_3 = \partial_t g_1, \tag{3.20}$$

$$\partial_y g_3 = \partial_t g_2. \tag{3.21}$$

Proof. To verify (3.16), we compute the following:

$$\begin{aligned}
\text{LHS} &= \partial_y(q_1^{\rm Re} e + q_1^{\rm Im} u\wedge e)\\
&= q_{1y}^{\rm Re} e + q_1^{\rm Re}\partial_y e + q_{1y}^{\rm Im} u\wedge e + q_1^{\rm Im}\partial_y(u\wedge e)\\
&= q_{1y}^{\rm Re} e + q_{1y}^{\rm Im} u\wedge e + q_1^{\rm Re}(a_2 u\wedge e - q_2^{\rm Re} u) + q_1^{\rm Im}(\partial_y u\wedge e) + q_1^{\rm Im}(u\wedge\partial_y e)\\
&= q_{1y}^{\rm Re} e + (q_{1y}^{\rm Im} + a_2 q_1^{\rm Re}) u\wedge e - q_1^{\rm Re} q_2^{\rm Re} u + q_1^{\rm Im}(q_2^{\rm Re} e + q_2^{\rm Im} u\wedge e)\wedge e\\
&\quad + q_1^{\rm Im} u\wedge(a_2 u\wedge e - q_2^{\rm Re} u)\\
&= q_{1y}^{\rm Re} e + (q_{1y}^{\rm Im} + a_2 q_1^{\rm Re}) u\wedge e - q_1^{\rm Re} q_2^{\rm Re} u - q_1^{\rm Im} q_2^{\rm Im} u - a_2 q_1^{\rm Im} e\\
&= (q_{1y}^{\rm Re} - a_2 q_1^{\rm Im}) e + (q_{1y}^{\rm Im} + a_2 q_1^{\rm Re}) u\wedge e - (q_1^{\rm Re} q_2^{\rm Re} + q_1^{\rm Im} q_2^{\rm Im}) u.
\end{aligned} \tag{3.22}$$

On the other hand, we have

$$\begin{aligned}
\text{RHS} &= \partial_x q_2^{\rm Re} e + q_2^{\rm Re}\partial_x e + q_{2x}^{\rm Im} u\wedge e + q_2^{\rm Im}(\partial_x u\wedge e) + q_2^{\rm Im}(u\wedge\partial_x e)\\
&= \partial_x q_2^{\rm Re} e + q_2^{\rm Re}(a_1 u\wedge e - q_1^{\rm Re} u) + q_2^{\rm Im}(q_1^{\rm Re} e + q_1^{\rm Im} u\wedge e)\wedge e\\
&\quad + q_{2x}^{\rm Im} u\wedge e + q_2^{\rm Im} u\wedge(a_1 u\wedge e - q_1^{\rm Re} u)\\
&= \partial_x q_2^{\rm Re} e + (a_1 q_2^{\rm Re} + q_{2x}^{\rm Im}) u\wedge e - q_1^{\rm Re} q_2^{\rm Re} u - q_1^{\rm Im} q_2^{\rm Im} u - a_1 q_2^{\rm Im} e\\
&= (\partial_x q_2^{\rm Re} - a_1 q_2^{\rm Im}) e + (a_1 q_2^{\rm Re} + q_{2x}^{\rm Im}) u\wedge e - (q_1^{\rm Re} q_2^{\rm Re} + q_1^{\rm Im} q_2^{\rm Im}) u.
\end{aligned} \tag{3.23}$$

In order to prove that LHS = RHS, we need to show the following:

$$\begin{aligned}
(q_{1y}^{\rm Re} - a_2 q_1^{\rm Im}) &= (q_{2x}^{\rm Re} - a_1 q_2^{\rm Im}),\\
(q_{1y}^{\rm Im} + a_2 q_1^{\rm Re}) &= (q_{2x}^{\rm Im} + a_1 q_2^{\rm Re}).
\end{aligned} \tag{3.24}$$

From theorem 2.1 we have shown that

$$q_{1y} + {\rm i} a_2 q_1 = q_{2x} + {\rm i} a_1 q_2. \tag{3.25}$$

By splitting q_1 and q_2 into real and imaginary parts we obtain

$$q_{1y}^{\rm Re} + {\rm i} q_{1y}^{\rm Im} + {\rm i} a_2(q_1^{\rm Re} + {\rm i} q_1^{\rm Im}) = q_{2x}^{\rm Re} + {\rm i} q_{2x}^{\rm Im} + {\rm i} a_1(q_2^{\rm Re} + {\rm i} q_2^{\rm Im}). \tag{3.26}$$

Therefore, (3.26) implies (3.24) and (3.16).

The verifications for (3.17) and (3.18) are similar, so it suffices to show (3.18). The left-hand side of (3.18) can be computed by

$$\begin{aligned}
\partial_t f_2 &= \partial_t(q_2^{\rm Re} e + q_2^{\rm Im} u\wedge e)\\
&= q_{2t}^{\rm Re} e + q_2^{\rm Re}\partial_t e + q_{2t}^{\rm Im} u\wedge e + q_2^{\rm Im} u_t\wedge e + q_2^{\rm Im} u\wedge\partial_t e\\
&= q_{2t}^{\rm Re} e + q_2^{\rm Re}(a_0 u\wedge e - p^{\rm Re} u) + q_{2t}^{\rm Im} u\wedge e + q_2^{\rm Im}(p^{\rm Re} e + p^{\rm Im} u\wedge e)\wedge e\\
&\quad + q_2^{\rm Im} u\wedge(a_0 u\wedge e - p^{\rm Re} u)\\
&= q_{2t}^{\rm Re} e + a_0 q_2^{\rm Re} u\wedge e - q_2^{\rm Re} p^{\rm Re} u + q_{2t}^{\rm Im} u\wedge e + q_2^{\rm Im} p^{\rm Re} e - q_2^{\rm Im} p^{\rm Im} u - a_0 q_2^{\rm Im} e\\
&= (q_{2t}^{\rm Re} - a_0 q_2^{\rm Im}) e - (q_2^{\rm Re} p^{\rm Re} + q_2^{\rm Im} p^{\rm Im}) u + (a_0 q_2^{\rm Re} + q_{2t}^{\rm Im}) u\wedge e.
\end{aligned} \tag{3.27}$$

The right-hand side can be computed by

$$
\begin{aligned}
\partial_y f_3 &= \partial_y(p^{\mathrm{Re}} e + p^{\mathrm{Im}} u \wedge e) \\
&= p_y^{\mathrm{Re}} e + p^{\mathrm{Re}} \partial_y e + p_y^{\mathrm{Im}} u \wedge e + p^{\mathrm{Im}} \partial_y (u \wedge e) \\
&= p_y^{\mathrm{Re}} e + p^{\mathrm{Re}} (a_2 u \wedge e - q_2^{\mathrm{Re}} u) + p_y^{\mathrm{Im}} u \wedge e + p^{\mathrm{Im}} (u_y \wedge e) + p^{\mathrm{Im}} u \wedge (\partial_y e) \\
&= p_y^{\mathrm{Re}} e + a_2 p^{\mathrm{Re}} u \wedge e - p^{\mathrm{Re}} q_2^{\mathrm{Re}} u + p_y^{\mathrm{Im}} u \wedge e + p^{\mathrm{Im}} (q_2^{\mathrm{Re}} e + q_2^{\mathrm{Im}} u \wedge e) \wedge e \\
&\quad + p^{\mathrm{Im}} u \wedge (a_2 u \wedge e - q_2^{\mathrm{Re}} u) \\
&= p_y^{\mathrm{Re}} e + a_2 p^{\mathrm{Re}} u \wedge e - p^{\mathrm{Re}} q_2^{\mathrm{Re}} u + p_y^{\mathrm{Im}} u \wedge e - p^{\mathrm{Im}} q_2^{\mathrm{Im}} u - a_2 p^{\mathrm{Im}} e \\
&= (p_y^{\mathrm{Re}} - a_2 p^{\mathrm{Im}}) e - (p^{\mathrm{Re}} q_2^{\mathrm{Re}} + p^{\mathrm{Im}} q_2^{\mathrm{Im}}) u + (a_2 p^{\mathrm{Re}} + p_y^{\mathrm{Im}}) u \wedge e.
\end{aligned}
\tag{3.28}
$$

To show (3.18), it remains to check the following:

$$
\begin{aligned}
q_{2t}^{\mathrm{Re}} - a_0 q_2^{\mathrm{Im}} &= p_y^{\mathrm{Re}} - a_2 p^{\mathrm{Im}}, \\
a_2 p^{\mathrm{Re}} + p_y^{\mathrm{Im}} &= a_0 q_2^{\mathrm{Re}} + q_{2t}^{\mathrm{Im}}.
\end{aligned}
\tag{3.29}
$$

From (1.8) we can write

$$
\begin{aligned}
p_y &= \mathrm{i} q_{1xy} + \mathrm{i} q_{2yy} + a_{1y} q_1 + a_1 q_{1y} + a_{2y} q_2 + a_2 q_{2y} \\
&= \mathrm{i}(q_{2xx} + \mathrm{i} a_{1x} q_2 + \mathrm{i} a_1 q_{2x} - \mathrm{i} a_{2x} q_1 - \mathrm{i} a_2 q_{1x}) + \mathrm{i} q_{2yy} + a_{1y} q_1 + a_1 q_{1y} + a_{2y} q_2 + a_2 q_{2y} \\
&= \mathrm{i} \Delta q_2 + (a_{2y} - a_{1x}) q_2 + (a_{1y} + a_{2x}) q_1 - a_1 q_{2x} + a_2 q_{1x} + a_1 q_{1y} + a_2 q_{2y}.
\end{aligned}
\tag{3.30}
$$

Then we have

$$
\begin{aligned}
p_y^{\mathrm{Re}} - a_2 p^{\mathrm{Im}} = &- \Delta q_2^{\mathrm{Im}} + (a_{2y} - a_{1x}) q_2^{\mathrm{Re}} + (a_{1y} + a_{2x}) q_1^{\mathrm{Re}} \\
&- a_1 q_{2x}^{\mathrm{Re}} + a_2 q_{1x}^{\mathrm{Re}} + a_1 q_{1y}^{\mathrm{Re}} + a_2 q_{2y}^{\mathrm{Re}} - a_2 (q_{1x}^{\mathrm{Re}} + q_{2y}^{\mathrm{Re}} + a_1 q_1^{\mathrm{Im}} + a_2 q_2^{\mathrm{Im}}).
\end{aligned}
\tag{3.31}
$$

To simplify (3.31), we notice from (1.13) that

$$
\begin{aligned}
q_{1y}^{\mathrm{Re}} - a_2 q_1^{\mathrm{Im}} &= q_{2x}^{\mathrm{Re}} - a_1 q_2^{\mathrm{Im}}, \\
q_{1y}^{\mathrm{Im}} + a_2 q_1^{\mathrm{Re}} &= q_{2x}^{\mathrm{Im}} + a_1 q_2^{\mathrm{Re}}, \\
a_1 q_{1y}^{\mathrm{Re}} &= a_1 a_2 q_1^{\mathrm{Im}} + a_1 q_{2x}^{\mathrm{Re}} - a_1^2 q_2^{\mathrm{Im}}.
\end{aligned}
\tag{3.32}
$$

Hence $p_y^{\mathrm{Re}} - a_2 p^{\mathrm{Im}}$ can be written as

$$
\begin{aligned}
p_y^{\mathrm{Re}} - a_2 p^{\mathrm{Im}} = &- \Delta q_2^{\mathrm{Im}} + (a_{2y} - a_{1x}) q_2^{\mathrm{Re}} + (a_{1y} + a_{2x}) q_1^{\mathrm{Re}} - a_1 q_{2x}^{\mathrm{Re}} \\
&+ a_2 q_{1x}^{\mathrm{Re}} + a_1 a_2 q_1^{\mathrm{Im}} + a_1 q_{2x}^{\mathrm{Re}} - a_1^2 q_2^{\mathrm{Im}} - a_2 q_{1x}^{\mathrm{Re}} - a_1 a_2 q_1^{\mathrm{Im}} - a_2^2 q_2^{\mathrm{Im}} \\
= &- \Delta q_2^{\mathrm{Im}} + (a_{2y} - a_{1x}) q_2^{\mathrm{Re}} + (a_{1y} + a_{2x}) q_1^{\mathrm{Re}} - (a_1^2 + a_2^2) q_2^{\mathrm{Im}}.
\end{aligned}
\tag{3.33}
$$

From (1.7), we have

$$
\begin{aligned}
q_{2t}^{\mathrm{Re}} - a_0 q_2^{\mathrm{Im}} &= [\mathrm{i} \Delta q_2 + \mathrm{i}(a_1^2 + a_2^2) q_2 - (a_{1x} - a_{2y}) q_2 + a_{2x} + a_{1y}]^{\mathrm{Re}} \\
&= -\Delta q_2^{\mathrm{Im}} - (a_1^2 + a_2^2) q_2^{\mathrm{Im}} - (a_{1x} - a_{2y}) q_2^{\mathrm{Re}} + (a_{2x} + a_{1y}) q_1^{\mathrm{Re}}, \\
q_{2t}^{\mathrm{Im}} + a_0 q_2^{\mathrm{Re}} &= [\mathrm{i} \Delta q_2 + \mathrm{i}(a_1^2 + a_2^2) q_2 - (a_{1x} - a_{2y}) q_2 + a_{2x} + a_{1y}]^{\mathrm{Im}} \\
&= \Delta q_2^{\mathrm{Re}} + (a_1^2 + a_2^2) q_2^{\mathrm{Re}} - (a_{1x} - a_{2y}) q_2^{\mathrm{Im}} + (a_{2x} + a_{1y}) q_1^{\mathrm{Im}}.
\end{aligned}
\tag{3.34}
$$

Combining (3.33) and (3.34) we have verified the first equation in (3.29). To verify the second, we take the imaginary part of p_y in (3.30) and obtain

$$
p_y^{\mathrm{Im}} = \Delta q_2^{\mathrm{Re}} + (a_{2y} - a_{1x}) q_2^{\mathrm{Im}} + (a_{1y} + a_{2x}) q_1^{\mathrm{Im}} - a_1 q_{2x}^{\mathrm{Im}} + a_2 q_{1x}^{\mathrm{Im}} + a_1 q_{1y}^{\mathrm{Im}} + a_2 q_{2y}^{\mathrm{Im}}.
\tag{3.35}
$$

Then $p_y^{\mathrm{Im}} + a_2 p^{\mathrm{Re}}$ can be written as

$$\begin{aligned} p_y^{\mathrm{Im}} + a_2 p^{\mathrm{Re}} &= \Delta q_2^{\mathrm{Re}} + (a_{2y} - a_{1x}) q_2^{\mathrm{Im}} + (a_{1y} + a_{2x}) q_1^{\mathrm{Im}} - a_1 q_{2x}^{\mathrm{Im}} + a_2 q_{1x}^{\mathrm{Im}} \\ &\quad + a_1 q_{1y}^{\mathrm{Im}} + a_2 q_{2y}^{\mathrm{Im}} + a_2(-q_{1x}^{\mathrm{Im}} - q_{2y}^{\mathrm{Im}} + a_1 q_1^{\mathrm{Re}} + a_2 q_2^{\mathrm{Re}}) \\ &= \Delta q_2^{\mathrm{Re}} + (a_{2y} - a_{1x}) q_2^{\mathrm{Im}} + (a_{1y} + a_{2x}) q_1^{\mathrm{Im}} - a_1 q_{2x}^{\mathrm{Im}} + a_2 q_{1x}^{\mathrm{Im}} \\ &\quad + a_1 q_{1y}^{\mathrm{Im}} + a_2 q_{2y}^{\mathrm{Im}} - a_2 q_{1x}^{\mathrm{Im}} - a_2 q_{2y}^{\mathrm{Im}} + a_1 a_2 q_1^{\mathrm{Re}} + a_2^2 q_2^{\mathrm{Re}}. \end{aligned} \tag{3.36}$$

By (1.13), (3.36) becomes

$$\begin{aligned} p_y^{\mathrm{Im}} + a_2 p^{\mathrm{Re}} &= \Delta q_2^{\mathrm{Re}} + (a_{2y} - a_{1x}) q_2^{\mathrm{Im}} + (a_{1y} + a_{2x}) q_1^{\mathrm{Im}} - a_1 q_{2x}^{\mathrm{Im}} \\ &\quad + a_1 q_{2x}^{\mathrm{Im}} + a_1^2 q_2^{\mathrm{Re}} - a_1 a_2 q_1^{\mathrm{Re}} + a_1 a_2 q_1^{\mathrm{Re}} + a_2^2 q_2^{\mathrm{Re}} \\ &= \Delta q_2^{\mathrm{Re}} + (a_{2y} - a_{1x}) q_2^{\mathrm{Im}} + (a_{1y} + a_{2x}) q_1^{\mathrm{Im}} + (a_1^2 + a_2^2) q_2^{\mathrm{Re}} \\ &= q_{2t}^{\mathrm{Im}} + a_0 q_2^{\mathrm{Re}}. \end{aligned} \tag{3.37}$$

Therefore, (3.18) holds. To verify (3.19), we compute the left-hand side by

$$\begin{aligned} \mathrm{LHS} &= \partial_y (a_1 u \wedge e - q_1^{\mathrm{Re}} u) \\ &= a_{1y} u \wedge e + a_1 u_y \wedge e + a_1 u \wedge e_y - q_{1y}^{\mathrm{Re}} u - q_1^{\mathrm{Re}} u_y \\ &= a_{1y} u \wedge e + a_1 (q_2^{\mathrm{Re}} e + q_2^{\mathrm{Im}} u \wedge e) \wedge e + a_1 u \wedge (a_2 u \wedge e - q_2^{\mathrm{Re}} u) \\ &\quad - q_{1y}^{\mathrm{Re}} u - q_1^{\mathrm{Re}} (q_2^{\mathrm{Re}} e + q_2^{\mathrm{Im}} u \wedge e) \\ &= a_{1y} u \wedge e - a_1 q_2^{\mathrm{Im}} u - a_1 a_2 e - q_{1y}^{\mathrm{Re}} u - q_1^{\mathrm{Re}} q_2^{\mathrm{Re}} e - q_1^{\mathrm{Re}} q_2^{\mathrm{Im}} u \wedge e \\ &= (a_{1y} - q_1^{\mathrm{Re}} q_2^{\mathrm{Im}}) u \wedge e - (a_1 q_2^{\mathrm{Im}} + q_{1y}^{\mathrm{Re}}) u - (a_1 a_2 + q_1^{\mathrm{Re}} q_2^{\mathrm{Re}}) e. \end{aligned} \tag{3.38}$$

On the other hand, the right-hand side can be computed as follows:

$$\begin{aligned} \mathrm{RHS} &= \partial_x (a_2 u \wedge e - q_2^{\mathrm{Re}} u) \\ &= a_{2x} u \wedge e + a_2 u_x \wedge e + a_2 u \wedge e_x - q_{2x}^{\mathrm{Re}} u - q_2^{\mathrm{Re}} u_x \\ &= a_{2x} u \wedge e + a_2 (q_1^{\mathrm{Re}} e + q_1^{\mathrm{Im}} u \wedge e) \wedge e + a_2 u \wedge (a_1 u \wedge e - q_1^{\mathrm{Re}} u) \\ &\quad - q_{2x}^{\mathrm{Re}} u - q_2^{\mathrm{Re}} (q_1^{\mathrm{Re}} e + q_1^{\mathrm{Im}} u \wedge e) \\ &= a_{2x} u \wedge e - a_2 q_1^{\mathrm{Im}} u - a_1 a_2 e - q_{2x}^{\mathrm{Re}} u - q_1^{\mathrm{Re}} q_2^{\mathrm{Re}} e - q_1^{\mathrm{Im}} q_2^{\mathrm{Re}} u \wedge e \\ &= (a_{2x} - q_1^{\mathrm{Im}} q_2^{\mathrm{Re}}) u \wedge e - (a_2 q_1^{\mathrm{Im}} + q_{2x}^{\mathrm{Re}}) u - (a_1 a_2 + q_1^{\mathrm{Re}} q_2^{\mathrm{Re}}) e. \end{aligned} \tag{3.39}$$

Thus it remains to show that

$$\begin{aligned} a_{1y} - q_1^{\mathrm{Re}} q_2^{\mathrm{Im}} &= a_{2x} - q_1^{\mathrm{Im}} q_2^{\mathrm{Re}}, \\ a_1 q_2^{\mathrm{Im}} + q_{1y}^{\mathrm{Re}} &= a_2 q_1^{\mathrm{Im}} + q_{2x}^{\mathrm{Re}}. \end{aligned} \tag{3.40}$$

From (1.10) we have

$$a_{1y} - a_{2x} = \mathrm{Im}(\bar{q}_1 q_2) = -q_1^{\mathrm{Im}} q_2^{\mathrm{Re}} + q_1^{\mathrm{Re}} q_2^{\mathrm{Im}},$$

then the first equality in (3.40) is proved. To verify the second identity, we notice from (1.13) that

$$q_{1y} + \mathrm{i} a_2 q_1 = q_{2x} + \mathrm{i} a_1 q_2.$$

By splitting it into real and imaginary parts, we get

$$q_{1y}^{\mathrm{Re}} + \mathrm{i} q_{1y}^{\mathrm{Im}} + \mathrm{i} a_2 q_1^{\mathrm{Re}} - a_2 q_1^{\mathrm{Im}} = q_{2x}^{\mathrm{Re}} + \mathrm{i} q_{2x}^{\mathrm{Im}} + \mathrm{i} a_1 q_2^{\mathrm{Re}} - a_1 q_2^{\mathrm{Im}}.$$

Therefore, we obtain

$$\begin{aligned} q_{1y}^{\mathrm{Re}} - a_2 q_1^{\mathrm{Im}} &= q_{2x}^{\mathrm{Re}} - a_1 q_2^{\mathrm{Im}}, \\ q_{1y}^{\mathrm{Im}} + a_2 q_1^{\mathrm{Re}} &= q_{2x}^{\mathrm{Im}} + a_1 q_2^{\mathrm{Re}}, \end{aligned}$$

which imply (3.40). To show (3.20) we proceed as follows:

$$\begin{aligned}
\partial_x g_3 &= \partial_x(a_0 u \wedge e - p^{\mathrm{Re}} u) \\
&= \partial_x a_0 u \wedge e + a_0 u_x \wedge e + a_0 u \wedge e_x - p_x^{\mathrm{Re}} u - p^{\mathrm{Re}} u_x \\
&= \partial_x a_0 u \wedge e + a_0(q_1^{\mathrm{Re}} e + q_1^{\mathrm{Im}} u \wedge e) \wedge e - p_x^{\mathrm{Re}} u \\
&\quad + a_0 u \wedge (a_1 u \wedge e - q_1^{\mathrm{Re}} u) - p^{\mathrm{Re}}(q_1^{\mathrm{Re}} e + q_1^{\mathrm{Im}} u \wedge e) \\
&= \partial_x a_0 u \wedge e - a_0 q_1^{\mathrm{Im}} u - a_0 a_1 e - p_x^{\mathrm{Re}} u - p^{\mathrm{Re}} q_1^{\mathrm{Re}} e - p^{\mathrm{Re}} q_1^{\mathrm{Im}} u \wedge e \\
&= (\partial_x a_0 - p^{\mathrm{Re}} q_1^{\mathrm{Im}}) u \wedge e - (a_0 q_1^{\mathrm{Im}} + p_x^{\mathrm{Re}}) u - (a_0 a_1 + p^{\mathrm{Re}} q_1^{\mathrm{Re}}).
\end{aligned} \tag{3.41}$$

On the other hand,

$$\begin{aligned}
\partial_t g_1 &= \partial_t(a_1 u \wedge e - q_1^{\mathrm{Re}} u) \\
&= a_{1t} u \wedge e + a_1(p^{\mathrm{Re}} e + p^{\mathrm{Im}} u \wedge e) \wedge e - q_{1t}^{\mathrm{Re}} u \\
&\quad + a_1 u \wedge (a_0 u \wedge e - p^{\mathrm{Re}} u) - q_1^{\mathrm{Re}}(p^{\mathrm{Re}} e + p^{\mathrm{Im}} u \wedge e) \\
&= a_{1t} u \wedge e - a_1 p^{\mathrm{Im}} u - a_0 a_1 e - q_{1t}^{\mathrm{Re}} u - q_1^{\mathrm{Re}} p^{\mathrm{Re}} e - q_1^{\mathrm{Re}} p^{\mathrm{Im}} u \wedge e \\
&= (a_{1t} - q_1^{\mathrm{Re}} p^{\mathrm{Im}}) u \wedge e - (a_1 p^{\mathrm{Im}} + q_{1t}^{\mathrm{Re}}) u - (a_0 a_1 + q_1^{\mathrm{Re}} p^{\mathrm{Re}}) e.
\end{aligned} \tag{3.42}$$

To prove that (3.41) and (3.42) are equal, we have to verify the following identities:

$$\begin{aligned}
& a_{1t} - q_1^{\mathrm{Re}} p^{\mathrm{Im}} = a_{0x} - p^{\mathrm{Re}} q_1^{\mathrm{Im}}, \\
& a_0 q_1^{\mathrm{Im}} + p_x^{\mathrm{Re}} = a_1 p^{\mathrm{Im}} + q_{1t}^{\mathrm{Re}}.
\end{aligned} \tag{3.43}$$

From (1.11) we can write

$$\begin{aligned}
a_{1t} - a_{0x} &= \mathrm{Im}(\bar{q}_1 p) \\
&= \mathrm{Im}[(q_1^{\mathrm{Re}} - \mathrm{i} q_1^{\mathrm{Im}})(p^{\mathrm{Re}} + \mathrm{i} p^{\mathrm{Im}})] \\
&= q_1^{\mathrm{Re}} p^{\mathrm{Im}} - q_1^{\mathrm{Im}} p^{\mathrm{Re}},
\end{aligned}$$

thus the first equality in (3.43) holds. To prove the second, we substitute the expressions for p into the equation and get

$$\begin{aligned}
& q_{1t}^{\mathrm{Re}} + a_1(\mathrm{i} q_{1xx} + \mathrm{i} q_{2y} + a_1 q_1 + a_2 q_2)^{\mathrm{Im}} - a_0 q_1^{\mathrm{Im}} - (\mathrm{i} q_{1x} + \mathrm{i} q_{2y} + a_1 q_1 + a_2 q_2)_x^{\mathrm{Re}} \\
&\quad = q_{1t}^{\mathrm{Re}} + a_1 q_{1x}^{\mathrm{Re}} + a_1 q_{2y}^{\mathrm{Re}} + a_1^2 q_1^{\mathrm{Im}} + a_1 a_2 q_2^{\mathrm{Im}} - a_0 q_1^{\mathrm{Im}} \\
&\qquad - (\mathrm{i} q_{1xx} + \mathrm{i} q_{2xy} + a_{1x} q_1 + a_1 q_{1x} + a_{2x} q_2 + a_2 q_{2x})^{\mathrm{Re}} \\
&\quad = q_{1t}^{\mathrm{Re}} + a_1 q_{1x}^{\mathrm{Re}} + a_1 q_{2y}^{\mathrm{Re}} + a_1^2 q_1^{\mathrm{Im}} + a_1 a_2 q_2^{\mathrm{Im}} - a_0 q_1^{\mathrm{Im}} \\
&\qquad + q_{1xx}^{\mathrm{Im}} + q_{2xy}^{\mathrm{Im}} - a_{1x} q_1^{\mathrm{Re}} - a_1 q_{1x}^{\mathrm{Re}} - a_{2x} q_2^{\mathrm{Re}} - a_2 q_{2x}^{\mathrm{Re}}.
\end{aligned} \tag{3.44}$$

From (1.13), one has

$$\begin{aligned}
q_{2xy} &= q_{1yy} + \mathrm{i} a_{2y} q_1 + \mathrm{i} a_2 q_{1y} - \mathrm{i} a_{1y} q_2 - \mathrm{i} a_1 q_{2y}, \\
q_{2xy}^{\mathrm{Im}} &= q_{1yy}^{\mathrm{Im}} + a_{2y} q_1^{\mathrm{Re}} + a_2 q_{1y}^{\mathrm{Re}} - a_{1y} q_2^{\mathrm{Re}} - a_1 q_{2y}^{\mathrm{Re}}.
\end{aligned}$$

Then (3.44) can be rewritten as

$$\begin{aligned}
& q_{1t}^{\mathrm{Re}} + a_1 q_{1x}^{\mathrm{Re}} + a_1 q_{2y}^{\mathrm{Re}} + a_1^2 q_1^{\mathrm{Im}} + a_1 a_2 q_2^{\mathrm{Im}} - a_0 q_1^{\mathrm{Im}} + q_{1xx}^{\mathrm{Im}} + q_{1yy}^{\mathrm{Im}} + a_{2y} q_1^{\mathrm{Re}} + a_2 q_{1y}^{\mathrm{Re}} \\
&\quad - a_{1y} q_2^{\mathrm{Re}} - a_1 q_{2y}^{\mathrm{Re}} - a_{1x} q_1^{\mathrm{Re}} - a_1 q_{1x}^{\mathrm{Re}} - a_{2x} q_2^{\mathrm{Re}} - a_2 q_{2x}^{\mathrm{Re}} \\
&\qquad = q_{1t}^{\mathrm{Re}} - a_0 q_1^{\mathrm{Im}} + \Delta q_1^{\mathrm{Im}} - (a_{1x} - a_{2y}) q_1^{\mathrm{Re}} - (a_{2x} + a_{1y}) q_2^{\mathrm{Re}} \\
&\qquad\quad + a_1^2 q_1^{\mathrm{Im}} + a_1 a_2 q_2^{\mathrm{Im}} + a_2(\mathrm{i} a_1 q_2 - \mathrm{i} a_2 q_1)^{\mathrm{Re}} \\
&\qquad = q_{1t}^{\mathrm{Re}} - a_0 q_1^{\mathrm{Im}} + \Delta q_1^{\mathrm{Im}} - (a_{1x} - a_{2y}) q_1^{\mathrm{Re}} - (a_{2x} + a_{1y}) q_2^{\mathrm{Re}} \\
&\qquad\quad + (a_1^2 + a_2^2) q_1^{\mathrm{Im}} = 0
\end{aligned} \tag{3.45}$$

by using the real part of equation (1.6). Therefore (3.20) is valid and (3.21) follows from a similar argument. This completes the proof. □

Proposition 3.2. *The system (3.14) with initial data*

$$u(0,0,0) = u*,$$
$$e(0,0,0) = e*,$$

where $u*$ *and* e *are fixed unit vectors on* $u*TN$*, has a unique smooth solution* $u(x, y, t)$ *in a neighbourhood of* $(0, 0, 0)$.

Proof. We basically implement the proof in [30] (p 254). We want to construct $u(x, y, t)$ and $e(x, y, t)$ satisfying (3.14). The first step is to solve the following ordinary differential equation (ODE)

$$\begin{aligned} u_1'(x) &= f_1(x,0,0),\\ e_1'(x) &= g_1(x,0,0),\\ u_1(0) &= u^*,\\ e_1(0) &= e^*. \end{aligned} \tag{3.46}$$

Clearly (3.46) admits a unique solution (u_1, e_1) defined for $|x| < \epsilon_1$ for some $\epsilon_1 > 0$. By taking the inner product with u_1 and e_1 in the first two equations in (3.46), one has

$$\begin{aligned} (|u_1|^2(x))' &= 2q_1^{\mathrm{Re}}\langle e_1, u_1\rangle,\\ (|e_1|^2(x))' &= -2q_2^{\mathrm{Re}}\langle e_1, u_1\rangle,\\ \langle u_1, e_1\rangle' &= q_1^{\mathrm{Re}}(|e_1|^2 - |u_1|^2). \end{aligned}$$

This ODE system for $|u_1|^2$, $|e_1|^2$ and $\langle e_1, u_1\rangle$ has a unique solution $|u_1|^2 = 1$, $|e_1|^2 = 1$, $\langle e_1, u_1\rangle = 0$ with the initial data

$$\begin{aligned} |u_1|^2(0) &= |u^*| = 1,\\ |e_1|^2(0) &= |e^*| = 1,\\ \langle e_1, u_1\rangle(0) &= 0. \end{aligned}$$

We define

$$\begin{aligned} u(x,0,0) &= u_1(x),\\ e(x,0,0) &= e_1(x), \end{aligned}$$

then $\forall |x| \leqslant \epsilon_1$ it holds that

$$\begin{aligned} \partial_x u(x,0,0) &= f_1(x,0,0),\\ \partial_x e(x,0,0) &= g_1(x,0,0),\\ |u(x,0,0)| &= 1,\\ |e(x,0,0)| &= 1,\\ \langle u(x,0,0), e(x,0,0)\rangle &= 0. \end{aligned}$$

Now for each fixed x with $|x| < \epsilon_1$, we consider the equations

$$\begin{aligned} u_2'(y) &= f_2(x,y,0),\\ e_2'(y) &= g_2(x,y,0),\\ u_2(0) &= u(x,0,0),\\ e_2(0) &= e(x,0,0). \end{aligned}$$

This system has a unique local solution (u_2, e_2). If ϵ_1 is small enough, then $u_2(y)$ and $e_2(y)$ are defined for $|y| < \epsilon_2$ for some $\epsilon_2 > 0$. Similarly, we can show that

$$\begin{aligned}
&|u_2| = 1,\\
&|e_2| = 1,\\
&\langle e_2, u_2\rangle = 0.
\end{aligned}$$

Then we define $u(x, y, 0) = u_2(y)$ and $e(x, y, 0) = e_2(y)$ for $|x| < \epsilon_1$ and $|y| < \epsilon_2$ which satisfy

$$\begin{aligned}
&\partial_y u(x, y, 0) = f_2(x, y, 0),\\
&\partial_y e(x, y, 0) = g_2(x, y, 0),\\
&|u(x, y, 0)| = 1,\\
&|e(x, y, 0)| = 1,\\
&\langle u(x, y, 0), e(x, y, 0)\rangle = 0.
\end{aligned}$$

We need to verify that for each fixed x with $|x| < \epsilon_1$, $u(x, y, 0)$ and $e(x, y, 0)$ also satisfy the equations

$$\begin{aligned}
&\partial_x u(x, y, 0) = f_1(x, y, 0),\\
&\partial_x e(x, y, 0) = g_1(x, y, 0)
\end{aligned} \tag{3.47}$$

for $|y| < \epsilon_2$. Let

$$\begin{aligned}
&G_1(y) = \partial_x u(x, y, 0) - f_1(x, y, 0),\\
&G_2(y) = \partial_x e(x, y, 0) - g_1(x, y, 0)
\end{aligned}$$

and clearly one has

$$\begin{aligned}
&G_1(0) = 0,\\
&G_2(0) = 0.
\end{aligned}$$

We compute the derivatives of G_1 and G_2 by

$$\begin{aligned}
\partial_y G_1(y) &= \partial_y(\partial_x u(x, y, 0) - f_1(x, y, 0))\\
&= \partial_x \partial_y u(x, y, 0) - \partial_y f_1(x, y, 0)\\
&= \partial_x f_2 - \partial_y f_1\\
&= 0 \qquad \text{[from (3.16)]},\\
\partial_y G_2(y) &= \partial_y \partial_x e(x, y, 0) - \partial_y g_1(x, y, 0)\\
&= \partial_x \partial_y e(x, y, 0) - \partial_y g_1(x, y, 0)\\
&= \partial_x g_2(x, y, 0) - \partial_y g_1(x, y, 0)\\
&= 0 \qquad \text{[from (3.19)]}.
\end{aligned}$$

Therefore, $G_1(y) = G_2(y) = 0$, which implies that (3.47) is satisfied. For $|x| < \epsilon_1$, $|y| < \epsilon_2$, the following system

$$\begin{aligned}
&u_3'(t) = f_3(x, y, t),\\
&e_3'(t) = g_3(x, y, y),\\
&u_3(0) = u(x, y, 0),\\
&e_3(0) = e(x, y, 0)
\end{aligned} \tag{3.48}$$

admits a unique local solution defined for $t < \epsilon_3$ for some $\epsilon_3 > 0$. Similarly, it can be easily verified that $|u_3| = 1$, $|e_3| = 1$ and $\langle e_3, u_3\rangle = 0$. For $|x| < \epsilon_1$, $|y| < \epsilon_2$, $t < \epsilon_3$, we let

$$\begin{aligned} u(x, y, t) &= u_3(t), \\ e(x, y, t) &= e_3(t). \end{aligned}$$

It remains to prove that

$$\begin{aligned} \partial_x u(x, y, t) &= f_1(x, y, t), \\ \partial_x e(x, y, t) &= g_1(x, y, t), \\ \partial_y u(x, y, t) &= f_2(x, y, t), \\ \partial_y e(x, y, t) &= g_2(x, y, t). \end{aligned} \tag{3.49}$$

For $|x| < \epsilon_1$, $|y| < \epsilon_2$ and $t < \epsilon_3$, we define $F_i(t)$ for $t < \epsilon$ as follows:

$$\begin{aligned} F_1(t) &= \partial_x u(x, y, t) - f_1(x, y, t), \\ F_2(t) &= \partial_x e(x, y, t) - g_1(x, y, t), \\ F_3(t) &= \partial_y u(x, y, t) - f_2(x, y, t), \\ F_4(t) &= \partial_y e(x, y, t) - g_2(x, y, t). \end{aligned}$$

Then we have

$$F_i(0) = 0, \qquad 1 \leqslant i \leqslant 4.$$

From proposition 3.2, we also have

$$F_i'(t) = 0, \qquad 1 \leqslant i \leqslant 4, \quad t < \epsilon_3.$$

Thus $F_i(t) = 0$ and we have constructed $u(x, y, t)$ and $e(x, y, t)$ in a neighbourhood of (0, 0, 0) which solve (3.13). □

After constructing a unique local smooth solution $u(x, y, t)$ and $e(x, y, t)$ of (3.13), since we have global bounds on u and e, we can extend the local solution to obtain a global solution. Thus we are ready to prove the following theorem.

Theorem 3.2. *The smooth solution of (3.13) solves the system (1.2).*

Proof. Let (u, e) be the solution constructed in proposition 3.2. To show that u and e solve (1.2), we compute the following:

$$\begin{aligned} &D_x(u_t + v_1u_x + v_2u_y - JD_xu_x - JD_yu_y) \\ &\quad= D_tu_x + D_x(v_1q_1e + v_2q_2e - JD_xq_1e - JD_yq_2e) \\ &\quad= D_t(q_1e) + v_{1x}q_1e + v_1q_{1x}e + v_1q_1D_xe + v_{2x}q_2e + v_2D_xq_2e + v_2q_2D_xe \\ &\qquad -JD_x(q_{1x}e + q_1D_xe) - JD_x(q_{2y}e + q_2D_ye) \\ &\quad= q_{1t}e + q_1D_te + v_{1x}q_1e + v_1q_{1x}e + v_1q_1a_1Je + v_{2x}q_2e + v_2q_{2x}e + v_2q_2a_1Je \\ &\qquad -Jq_{1xx}e + a_1q_{1x}e - Jq_{1x}a_1Je - Jq_1a_{1x}Je - Jq_1a_1JD_xe - Jq_{2xy}e \\ &\qquad -Jq_{2y}Ja_1e - Jq_{2x}Ja_2e + q_2a_{2x}e + a_2q_2Ja_1e \\ &\quad= q_{1t}e + a_0q_1Je - 2a_{1x}q_1e - 2a_1q_{1x}e - 2a_1^2q_1Je - 2a_{2x}q_2e - 2a_2q_{2x}e \\ &\qquad -2a_1a_2q_2Je - Jq_{1xx}e + a_1q_{1x}e + a_1q_{1x}e + a_{1x}q_1e + a_1^2q_1Je + a_1q_{2y}e \\ &\qquad +a_2q_{2x}e + a_{2x}q_2e + a_1a_2q_2Je - Jq_{2xy}e \\ &\quad= q_{1t}e + a_0q_1Je - a_{1x}q_1e - a_1^2q_1Je - a_{2x}q_2e - a_2q_{2x}e \\ &\qquad -a_1a_2q_2Je - Jq_{1xx}e + a_1q_{2y}e - Jq_{2xy}e. \end{aligned} \tag{3.50}$$

By (1.13) we have

$$q_{2xy} = q_{1yy} + \mathrm{i}a_{2y}q_1 + \mathrm{i}a_2q_{1y} - \mathrm{i}a_{1y}q_2 - \mathrm{i}a_1q_{2y},$$

then it follows that

$$q_{2xy}Je = q_{1yy}Je - a_{2y}q_1e - a_2q_{1y}e + a_{1y}q_2e + a_1q_{2y}e.$$

Therefore, (3.50) becomes

$$\begin{aligned}
&D_x(u_t + v_1u_x + v_2u_y - JD_xu_x - JD_yu_y)\\
&\quad = q_{1t}e + a_0q_1Je - a_{1x}q_1e - a_1^2q_1Je - a_{2x}q_2e - a_2q_{2x}e - a_1a_2q_2Je\\
&\qquad + a_1q_{2y}e - q_{1yy}Je + a_{2y}q_1e + a_2q_{1y}e - a_{1y}q_2e - a_1q_{2y}e\\
&\quad = q_{1t}e + a_0q_1Je - a_{1x}q_1e - a_1^2q_1Je - a_{2x}q_2e - a_2q_{2x}e - a_1a_2q_2Je\\
&\qquad - q_{1yy}Je + a_{2y}q_1e - a_{1y}q_2e - q_{1xx}Je + a_2(q_{2x} + a_1q_2 - a_2q_1J)e\\
&\quad = q_{1t}e + a_0q_1Je - a_{1x}q_1e - a_1^2q_1Je - a_{2x}q_2e - a_2q_{2x}e - a_1a_2q_2Je\\
&\qquad - q_{1xx}Je - q_{1yy}Je + a_{2y}q_1e - a_{1y}q_2e + a_2q_{2x}e + a_1a_2q_2Je - a_2^2q_1Je\\
&\quad = [q_{1t} + \mathrm{i}a_0q_1 - (a_{1x} - a_{2y})q_1 - (a_{2x} + a_{1y})q_2 - \mathrm{i}\Delta q_1 - \mathrm{i}(a_1^2 + a_2^2)q_1]e\\
&\quad = 0 \qquad [(\text{from } 1.6)]. \qquad (3.51)
\end{aligned}$$

Following a similar procedure, we also obtain

$$D_y(u_t + v_1u_x + v_2u_y - JD_xu_x - JD_yu_y) = 0.$$

Thus we deduce that

$$u_t + v_1u_x + v_2u_y - JD_xu_x - JD_yu_y = C, \qquad (3.52)$$

where C is a constant vector in $\mathbb{R}^3$. In the frame we constructed, (3.52) can be written as

$$\begin{aligned}
&u_t + v_1u_x + v_2u_y - JD_xu_x - JD_yu_y\\
&\quad = pe - 2a_1q_1e - 2a_2q_2e - JD_x(q_1e) - JD_y(q_2e)\\
&\quad = pe - 2a_1q_1e - 2a_2q_2e - J(q_{1x}e + q_1a_1Je) - J(q_{2y}e + q_2a_2Je)\\
&\quad = pe - 2a_1q_1e - 2a_2q_2e - \mathrm{i}q_{1x}e + a_1q_1e - \mathrm{i}q_{2y}e + a_2q_2e\\
&\quad = (p - \mathrm{i}q_{1x} - \mathrm{i}q_{2y} - a_1q_1 - a_2q_2)e = 0 \qquad (\text{by } 1.9), \qquad (3.53)
\end{aligned}$$

which implies that $C = 0$. Therefore we conclude that the solution constructed in proposition 3.2 indeed solves (1.2). □

Remark 3.2. All our computations are carried out for smooth solutions, however, for $u \in C(\mathbb{R}, H^2(\mathbb{R}^2))$, we have $e \in C(\mathbb{R}, H^2(\mathbb{R}^2))$, $q_1, q_2, a_1, a_2 \in C(\mathbb{R}, H^1(\mathbb{R}^2))$ and $p, a_0 \in C(\mathbb{R}, L^2(\mathbb{R}^2))$. Therefore, all our equations hold in the sense of distributions.

Combining theorems 3.1 and 3.2 we have proved theorem 1.1. Now we can establish the main result of this paper.

Proof of theorem 1.2. Given $u_0(x) \in H^2(\mathbb{R}^2)$, by theorem 1.1 we obtain the system (1.6)–(1.13) for q_1 and q_2 with the initial data q_{10} and $q_{20} \in H^1(\mathbb{R}^2)$. Then theorem 2.1 implies that (1.6)–(1.13) admits a unique global solution. Finally, theorem 3.2 and remark 3.2 conclude that the spin-liquid model (1.2) has a unique global solution in the space

$$\begin{aligned}
&u \in C(\mathbb{R}, H^2(\mathbb{R}^2)) \cap L^4_{\mathrm{loc}}(\mathbb{R}, W^{2,4}(\mathbb{R}^2)),\\
&\nabla v_1, \nabla v_2 \in L^4(\mathbb{R}, L^{4/3}(\mathbb{R}^2)) \cap L^2(\mathbb{R}, L^2(\mathbb{R}^2)).
\end{aligned}$$

□

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