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Önsöz

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- Ivanyshyn Yaman, O. 2018. "Reconstruction of surface impedance functions from the acoustic far field pattern", Book of Abstracts: IABEM 2018 Conference. Paris, p.122.
- Ivanyshyn Yaman, O. 2019. "Reconstruction of generalized impedance functions for 3d acoustic scattering", Journal of Computational Physics, 392, 444 – 455.
- Ivanyshyn Yaman, O., Özdemir, G. 2019. "Integral equation method for the modified Helmholtz equation with generalized impedance boundary condition", Book of Abstracts: ICMME 2019, Konya.
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- Kantekin, B. 2018. Boundary value problems with Laplace equation, IYTE BSc Thesis report.

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Özet

Bu projede geçirgen ince bir malzeme ile kaplanmış bir cismin ve/veya fiziksel özelliklerinin birkaç düzlem dalga ile sabit bir frekansta aydınlatılarak bulunması problemi ele alınmıştır. İteratif regülarize Newton metodu ve doğrusal olmayan integral denklem tabanlı çeşitli tersini alma algoritmaları önerilmiştir. Sınır değer problemlerinin klasik Newton iterasyonları ile ortaya çıkan çözümleri matris vektör çarpımları ile yerdeğiştirildiği için hesaplama açısından metodlar etkindir. Sentetik veri üretmek için spektral olarak hassas bir düz problem çözümü detaylı olarak sunulmuştur. Düz problem çözüm metodunun ve kaplama bulma metodunun uygulanabilirliği sayısal örneklerle gösterilmektedir.

Anahtar kelimeler: integral denklemler, ters problemler, spektral metodlar, GESK, küresel harmonikler, Tikhonov regülarizasyonu, Fréchet türevi, Newton-tipi metodlar

Abstract

In this project the problems of reconstruction an obstacle coated with a thin layer of an penetrable material or/and its physical properties from measurements for a few incident plane waves at a fixed frequency are considered. Several inversion methods based on an iteratively regularized Newton-type method and nonlinear integral equations are proposed. The methods are efficient from the computational point of view since the solutions of boundary value problems appearing in the classical Newton iteration are replaced by matrix-vector products. To generate synthetic data a detailed description of a spectrally accurate method for the direct problem is presented. The feasibility of the direct solution method and the coating reconstruction method is illustrated by numerical examples.

Keywords: integral equations, inverse problems, spectral methods, GIBC, spherical harmonics, Tikhonov regularization, Fréchet derivative, Newton-type methods

Chapter 1

Introduction

The generalized impedance boundary conditions (GIBC's) are used to model obstacles coated with a thin layer of a penetrable material, obstacles with corrugated surfaces or to model more accurately imperfectly conducting obstacles. These boundary condition were introduced in 1940s in the area of electromagnetic for modeling of an electromagnetic wave propagation over irregular terrains, Senior and Volakis (1995). Since then GIBC's are used for simplifying the analytical solutions or reducing the cost of numerical solutions for problems involving complex structures not only in electromagnetics Duruflé et al. (2006) but also in many other disciplines, in particular, three-dimensional acoustic problems, Antoine et al. (2001); Antoine and Barucq (2005); Haddar et al. (2005); Bourgeois and Haddar (2010); Kateb and Le Louër (2016).

The inverse problems we are interested in are to determine the surface impedance functions or/and boundary shape of an obstacle from the knowledge of the far field pattern for a few incident plane waves. This problem appears in practical application such as NDT for detecting porosity, reconstruction of surface roughness or coating thickness, Aslanyürek and Sahintürk (2014), for modeling related to stealth technology or antennas, Bourgeois et al. (2011). Additionally, it is motivated by the need to minimize the wave reflected by the obstacle in some directions what can be achieved by introducing a coating on the surface of the obstacle. The GIBC we are considering is written as follows

$$\frac{\partial u}{\partial \nu} + ik(\lambda - \operatorname{Div} \mu \operatorname{Grad})u = 0, \quad \text{on } \Gamma$$

where Grad and Div are surface gradient and surface divergence operators on the surface Γ and ν is the outward unit normal vector to Γ.

The literature overview for the inverse GIBC related problems in two dimensions reveals the following research results. In the case of Leontovich boundary condition, i.e. $\mu \equiv 0$, the problem for reconstruction the impedance and the shape of the obstacle is well-studied, see e.g. Liu et al. (2007). The numerical solution for the general case $\mu \neq 0$ was investigated by Bourgeois et al. (2011); Bourgeois and Haddar (2010) with the aid of variational formulation for the solution of the direct problem and by Kress (2018) with the solution method completely based on the boundary integral equations. Assuming that the boundary Γ is also unknown the variational approach was extended to the simultaneous reconstruction of the shape and the impedance functions, Bourgeois et al. (2012). A closely related inverse scattering problems with generalized oblique derivative boundary condition was considered in Wang and Liu (2015) and the linear sampling method was developed for the shape reconstruction, Wang and Liu (2016).

Regarding the inverse GIBC related problems in three dimensions the currently available literature is scarce. For the inverse impedance problem a nonlinear boundary integral equation approach was proposed in Ivanyshyn and Kress (2011) for the case of Leontovich boundary condition. Furthermore, the theoretical study on the boundary integral equation methods for the direct scattering problem with generalized impedance boundary condition was recently undertaken by Kress (2016).

During this project we developed reconstruction algorithms based on an iteratively regularized Newton-type method and nonlinear boundary integral equations. Extending the study of Ivanyshyn and Kress (2011) to the second order boundary condition we employ an analogue of the Huygens' principle for GIBC for deriving a system of nonlinear integral equations equivalent to the inverse problem and propose several ways of its stable solution.

The report is organized as follows. In Chapter 2 we formulate a direct boundary value problem for the Helmholtz equation with GIBC in $3D$, present its analytical solution for a special case and a spectrally accurate method for its numerical solution. Additionally, we develop a numerical method for the modified Helmholtz equation with GIBC in $2D$. Chapter 3 is dedicated to the inverse problems under consideration. In particular, the question of uniqueness is addressed and iterative inversion schemes are proposed. Chapter 4 is allocated for the detail descriptions for the implementation of the proposed methods and for the validation of the methods by numerical examples. Discussions and outlook are summarized in Chapter 5.

Chapter 2

Direct boundary value problems with GIBC

In this chapter we review known results for the direct problems, find analytical solutions for special cases and develop numerical solution methods. Whereas the direct and inverse problems for the Helmholtz equation with GIBC in $2D$ is well-studied, there are fewer results for the Helmholtz BVP in $3D$. In addition to the Helmholtz equation in $3D$ we mention some results to the closely related problems such as Laplace equation in $3D$ and the modified Helmholtz equation in $2D$.

2.1 Helmholtz equation in $3D$

To generate synthetic data for the inverse problem we firstly review the known results for the direct problem. Mathematically, the direct scattering problem for an obstacle with generalized impedance boundary condition can be stated as follows. Let $D \subset \mathbb{R}^3$ be a simply connected bounded domain with boundary Γ. Given the incident plane wave $u^i(x) = e^{ik\,x\cdot d}$ with wave number $k > 0$ and the direction of propagation d the scattering problem consists in finding the total field $u = u^s + u^i$ such that u satisfies the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D}, \tag{2.1}$$

the generalized impedance boundary condition (GIBC)

$$\frac{\partial u}{\partial \nu} + ik(\lambda - \operatorname{Div} \mu \operatorname{Grad})u = 0, \quad \text{on } \Gamma \tag{2.2}$$

where Grad and Div are surface gradient and surface divergence operators on Γ and ν is the outward unit normal vector to Γ. For brevity of notations, we introduce the differential operator

$$\mathcal{G}(\lambda, \mu; u) = ik(\lambda - \operatorname{Div} \mu \operatorname{Grad})u.$$

The scattered field has also to satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \quad r = |x|. \tag{2.3}$$

uniformly with respect to all directions. The Sommerfeld radiation condition guarantees the following asymptotic behavior of the scattered field

$$u^s(x) = \frac{e^{ik|x|}}{|x|}\left\{u_\infty\left(\frac{x}{|x|}\right) + O\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty.$$

uniformly in all directions with the far field pattern u_∞ defined on the unit sphere $\mathbb{S}^2$, Colton and Kress (2013b).

2.1.1 Well-posedness

Due to Kress (2016) we have the following well-posedness for the boundary value problem (2.1)-(2.3).

Theorem 2.1 *Let $D \subset \mathbb{R}^3$ be a bounded domain with a connected boundary Γ of Hölder class $C^{4,\alpha}$. Assume $\lambda \in C^1(\Gamma)$, $\mu \in C^2(\Gamma)$ with $\mathrm{Re}\lambda,\ \mathrm{Re}\mu \geq 0$ and $|\mu| > 0$. Then there exists a unique solution $u \in H^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ to (2.1)–(2.3).*

Proof. For the sake of completeness, we recall the ideas of proof presented in Kress (2016, 2018). Since $u|_\Gamma \in H^{\frac{3}{2}}(\Gamma)$ GIBC has to be understood in the weak sense

$$\int_\Gamma \left(\eta\frac{\partial u}{\partial \nu} + ik\lambda\eta u + ik\eta \operatorname{Grad}\eta \cdot \operatorname{Grad} u\right) ds = 0, \quad \forall \eta \in H^{\frac{3}{2}}(\Gamma). \tag{2.4}$$

By Rellich's lemma, (Colton and Kress, 2013b, Theorem 2.13), the direct problem has at most one solution provided $\mathrm{Re}\lambda,\ \mathrm{Re}\mu \geq 0$.

The solution is represented as the combined layer potential

$$u^s(x) = \int_\Gamma \left\{\Phi(x,y) + i\frac{\partial \Phi(x,y)}{\partial \nu(y)}\right\}\varphi(y)\, ds(y),\ x \in \mathbb{R}^3 \setminus \bar{D},\ \varphi \in H^{\frac{3}{2}}(\Gamma),$$

where $\Phi(x,y) = \dfrac{1}{4\pi}\dfrac{e^{ik|x-y|}}{|x-y|}$, $x \neq y$ is the fundamental solution to the Helmholtz equation in $\mathbb{R}^3$.

Substituting it into GIBC (2.2) and using the jump relations of layer potentials yields equivalence of (2.1)–(2.3) to the integro-differential equation

$$\varphi - K'\varphi - iT\varphi - \mathcal{G}(\lambda,\mu; S\varphi + i\varphi + iK\varphi) = 2\left.\frac{\partial u^i}{\partial \nu}\right|_\Gamma + 2\mathcal{G}(\lambda,\mu; u^i|_\Gamma),$$

where S and K are the single- and double-layer potential operators, correspondingly are defined by

$$(S\varphi)(x) := 2\int_\Gamma \Phi(x,y)\varphi(y)\,ds(y), \tag{2.5}$$

$$(K\varphi)(x) := 2\int_\Gamma \frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi(y)\,ds(y), \tag{2.6}$$

and $K'\varphi := \frac{\partial S\varphi}{\partial\nu}$, $\quad T\varphi := \frac{\partial K\varphi}{\partial\nu}$ are their normal derivatives. The operator

$$(K'\varphi)(x) := 2\int_\Gamma \frac{\partial\Phi(x,y)}{\partial\nu(x)}\varphi(y)\,ds(y) \tag{2.7}$$

is adjoint to the operator K with respect to L^2 bilinear form. The statement is completed by showing that the modified Laplace-Beltrami operator $\varphi \mapsto -\operatorname{Div}\operatorname{Grad}\varphi + \varphi$ is an isomorphism from $H^{\frac{3}{2}}(\Gamma)$ onto $H^{-\frac{1}{2}}(\Gamma)$, employing boundedness of $S, K : H^{\frac{3}{2}}(\Gamma) \to H^{\frac{5}{2}}(\Gamma)$ in the case Γ being of Hölder class $C^{4,\alpha}$, Kirsch (1989), using compact embedding $I_{H^{\frac{1}{2}}(\Gamma)\hookrightarrow H^{-\frac{1}{2}}(\Gamma)}$ and finally applying the Riesz theory. □

Newly the following existence result was proved under the weaker conditions on the boundary and the surface impedance functions, Colton and Kress (to appear).

Theorem 2.2 *Let $D \subset \mathbb{R}^3$ be a bounded domain with a connected boundary Γ of Hölder class $C^{3,\alpha}$. Assume $\lambda \in C(\Gamma)$, $\mu \in C^1(\Gamma)$ with* $\mathrm{Re}\lambda$, $\mathrm{Re}\mu \geq 0$ *and $|\mu| > 0$. Then there exists a unique solution $u \in H^{1,1}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D}) = \{u \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D}) : u|_\Gamma \in H^1(\Gamma)\}$ to* (2.1)–(2.3).

2.1.2 Boundary Integral Equation Method

We seek the scattered field $u^s \in H^2_{\text{loc}}(\mathbb{R}^3 \setminus \bar{D})$ in the form of a single-layer potential

$$u^s(x) = \int_\Gamma \Phi(x,y)\varphi(y)\,ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{2.8}$$

where $\Phi(x,y) = \dfrac{1}{4\pi}\dfrac{e^{ik|x-y|}}{|x-y|}$, $x \neq y$ is the fundamental solution to the Helmholtz equation in $\mathbb{R}^3$ and $\varphi \in H^{\frac{1}{2}}(\Gamma)$. Substituting the total field to the boundary condition (2.2) and using the jump relations for the single-layer potential, Colton and Kress (2013b), we obtain the following integro-differential equation

$$\varphi - K'\varphi - \mathcal{G}(\lambda,\mu;S\varphi) = 2\left.\frac{\partial u^i}{\partial \nu}\right|_\Gamma + 2\mathcal{G}(\lambda,\mu;u^i|_\Gamma) \tag{2.9}$$

It is shown, Kress (2016), that the operator $A(\lambda,\mu;\cdot) : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ defined by

$$A(\lambda,\mu;\varphi) := \varphi - K'\varphi - \mathcal{G}(\lambda,\mu;S\varphi) \tag{2.10}$$

has a bounded inverse provided k^2 is not a Dirichlet eigenvalue for the negative Laplacian in D. Hence, our synthetic data $u_\infty \in L^2(\mathbb{S}^2)$ can be found as following

$$u_\infty(\widehat{x}) = \frac{1}{4\pi}\int_\Gamma e^{-ik\widehat{x}\cdot y}\varphi(y)\,ds(y), \quad \widehat{x} := \frac{x}{|x|} \in \mathbb{S}^2.$$

2.2 Analytical solutions

In this section we present analytic solutions to the exterior boundary value problems for the Helmholtz and Laplace equation with GIBC in the case of a ball and constant surface impedance functions.

2.2.1 Helmholtz BVP with constant impedances

Let D be a ball of radius R centered at the origin and let λ, μ be constants such that $\operatorname{Re}\lambda \geq 0$, $\operatorname{Re}\mu \geq 0$, $|\mu| > 0$.

To start, we use the facts that spherical harmonics form a complete orthonormal system in $L^2(\mathbb{S}^2)$ and that any radiating solution can be represented in terms of spherical Hankel functions of the first kind $h_n^{(1)}$ and spherical harmonics Y_n^m, Colton and Kress (2013b).

Theorem 2.1 *The functions* $\{Y_n^m : |m| \le n, n \in \mathbb{N} \cup \{0\}\}$ *are complete in* $L^2(\mathbb{S}^2)$*; i.e. every function* $f \in L^2(\mathbb{S}^2)$ *can be expanded into a generalized Fourier series in the form*

$$f = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (f, Y_n^m)_{L^2(\mathbb{S}^2)} Y_n^m.$$

The series can also be written as

$$f(x) = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1) \int_{\mathbb{S}^2} f(y) P_n(y \cdot x) ds(y), \quad x \in \mathbb{S}^2.$$

The convergence has to be understood in the L^2*-sense.*

Theorem 2.2 *Let* $k > 0$ *and* $u \in C^2\left(\mathbb{R}^3 \setminus \bar{B}[0,R]\right)$ *satisfy* $\Delta u + k^2 u = 0$ *in the exterior of the ball of radius* R *centered at the origin,* $B[0,R]$*. Furthermore, assume that* u *satisfies the Sommerfeld radiation condition* (2.3)*. There exist unique* $\alpha_n^m \in \mathbb{C}, |m| \le n, n = 0, 1, 2, 3, \ldots$ *with*

$$u(r\hat{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_n^m h_n^{(1)}(kr) Y_n^m(\hat{x}), \quad r > R, \quad \hat{x} \in \mathbb{S}^2.$$

The series converges uniformly with all of its derivatives on compact subsets of $\mathbb{R}^3 \setminus B[0,R]$.

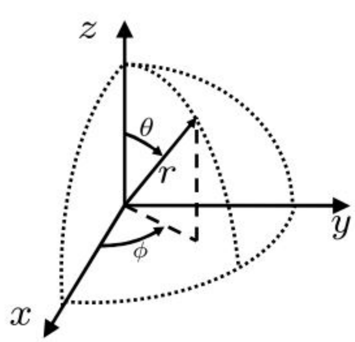


Figure 2.1: Spherical coordinates

Since μ is a constant we can rewrite

$$\operatorname{Div}_{\mathbb{S}^2_R} \mu \operatorname{Grad}_{\mathbb{S}^2_R} Y_n^m(\hat{x}) = \mu/R^2 \Delta_{\mathbb{S}^2} Y_n^m(R\hat{x}),$$

where $\Delta_{\mathbb{S}^2}$ is the Laplace-Beltrami operator

$$\Delta_{\mathbb{S}^2} u(\theta,\phi) = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial u(\theta,\phi)}{\partial\theta}\right] + \frac{1}{\sin^2\theta}\frac{\partial^2 u(\theta,\phi)}{\partial\phi^2}.$$

Substituting the Jacobi-Anger expansion for an incident plane wave

$$e^{ikR\hat{x}\cdot d} = 4\pi\sum_{n=0}^{\infty}\sum_{m=-n}^{n} i^n j_n(kR) Y_n^m(\hat{x}) Y_n^{-m}(d)$$

with incident direction d to the generalized boundary condition (2.2) and taking into account that the spherical harmonics are eigenvalues of the Laplace-Beltrami operator, i.e.

$$\mathrm{Div}_{\mathbb{S}^2_R}\,\mu\,\mathrm{Grad}_{\mathbb{S}^2_R}\,Y_n^m(\hat{x}) = -n(n+1)\,\mu/R^2\,Y_n^m(R\hat{x})$$

we obtain an equation for the unknown coefficients.

$$a_n^m = -4\pi i^n Y_n^{-m}(d)\frac{(kj_n'(kR) + ik\left(\lambda j_n(kR) + \mu n(n+1) j_n(kR)/R^2\right))}{kh_n^{(1)\prime}(kR) + ik(\lambda h_n^{(1)}(kR) + \mu n(n+1) h_n^{(1)}(kR)/R^2)}.$$

By the following we find the expression for the unknown far field patter.

Theorem 2.3 *The far field pattern of the radiating solution to the Helmholtz equation with the expansion*

$$u(x) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_n^m h_n^{(1)}(k|x|) Y_m^n(\hat{x}), \hat{x}\in\mathbb{S}^2$$

is given by the uniformly convergent series

$$u_\infty = \frac{1}{k}\sum_{n=0}^{\infty}\frac{1}{i^{n+1}}\sum_{m=-n}^{n} a_n^m Y_n^m.$$

2.2.2 Laplace BVP with constant impedances

In order to cover the case of a very small wave number, the Laplace equation is added to the project study. Consider the following boundary value problem with GIBC

$$\Delta u = 0 \quad \text{in} \quad \mathbb{R}^3\setminus\bar{D}$$

$$\frac{\partial u}{\partial \nu} + (\lambda - \mu \Delta_\Gamma) u = f, \quad \text{on} \quad \Gamma$$
$$u(x) = o(1), \quad |x| \to \infty$$

Using the the weak form of the boundary condition we obtain and Green's theorem

$$\int_D |\nabla u|^2 \, dx + \int_\Gamma \left(\lambda |u|^2 + \mu |\operatorname{Grad}_{\mathbb{S}^2} u|^2 \right) \, ds = 0$$

and hence the stated BVP has at most one solution for positive impedances.

Using separation of variables and Theorem 2.1 we find hat the solution is represented by

$$u(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_l^m r^{-(l+1)} Y_l^m(\theta, \phi)$$

with the coefficients given below

$$B_l^m = \frac{\displaystyle\int_0^\pi \int_0^{2\pi} f(\theta, \phi) \overline{Y_{l'}^{m'}(\theta, \phi)} \sin\theta d\phi d\theta}{-(l+1) R^{-(l+1)-1} + \lambda R^{-(l+1)} + l(l+1) \mu R^{-(l+1)}}.$$

2.3 Modified Helmholtz equation in $2D$

Let D be simply connected and bounded domain in $\mathbb{R}^2$ with boundary ∂D. We denote by ν the unit normal vector directed into exterior of D. Given $g \in H^{-\frac{1}{2}}(\partial D), \mu > 0$ and $\lambda > 0$ find a solution $u \in H^2(D)$ to

$$\Delta u - k^2 u = 0 \quad \text{in} \quad D \tag{2.11}$$

with wave number $k > 0$, that satisfies

$$\frac{\partial u}{\partial \nu} + k \left(\lambda u - \frac{d}{ds} \mu \frac{du}{ds} \right) = g \quad \text{on}\, \partial D \tag{2.12}$$

In the weak sense (2.12) can be considered as

$$\int_{\partial D} \left(\zeta \frac{\partial u}{\partial \nu} + k \lambda \zeta u + k \mu \frac{d\zeta}{ds} \frac{du}{ds} \right) ds = \int_{\partial D} \zeta g ds, \quad \forall \zeta \in H^{\frac{3}{2}}(\partial D) \tag{2.13}$$

$\mu, \lambda \in C^1(D)$ and $\frac{d}{ds}$ is tangential derivative and s is arc length. The derivative $\frac{d}{ds}$ with respect to arc length in (2.12) has to be understood in weak sense.

This problem arises in implicit marching schemes for the heat equation, in Debye–Huckel theory, and in the linearization of the Poisson–Boltzmann equation Juffer et al. (1991); Liang and Subramaniam (1997); Russell et al. (1989).

2.3.1 Boundary Integral Equation Method

We seek solution in the form of a single layer potential

$$u(x) = \int_{\partial D} \Phi(x, y)\varphi(y)ds(y), \quad x \in D,$$

where $\varphi \in H^{\frac{1}{2}}(\partial D)$ and

$$\Phi(x, y) = \frac{1}{2\pi} K_0(k|x - y|)$$

is fundamental solution of modified Helmholtz equation in $\mathbb{R}^2$

$$K_0(x) = -\left(\ln \frac{x}{2} + \alpha\right) I_0(x) + 2\sum_{k=1}^{\infty} \frac{I_{2k}(x)}{k}$$

with Euler constant $\alpha = .5772156\ldots$ and K_0, I_0 are modified Bessel functions. The boundary condition (2.12) is satisfied provided φ solves the boundary integral equation

$$K'\varphi + \frac{1}{2}\varphi + k\left(\lambda - \frac{d}{ds}\mu\frac{d}{ds}\right) S\varphi = g \tag{2.14}$$

where $S : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ and $K' : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ are bounded integral operators defined by

$$(S\varphi)(x) = 2\int_{\partial D} \Phi(x, y)\varphi(y)ds(y), \quad x \in \partial D$$

and

$$(K'\varphi)(x) = \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(x)}\varphi(y)ds(y), \quad x \in \partial D.$$

Theorem 2.1 *For each $g \in H^{-\frac{1}{2}}(\partial D)$, the boundary integral equation* (2.14) *has a unique solution $\varphi \in H^{\frac{1}{2}}(\partial D)$*

Proof. The boundary value problem (2.11)-(2.12) has at most one solution. Assume that $\varphi \in H^{\frac{1}{2}}(\partial D)$ and define

$$
\begin{aligned}
A_1\varphi &= \frac{d^2}{ds^2} S\varphi + \int_{\partial D} S\varphi ds \\
A_2\varphi &= \frac{1}{\mu}\frac{d\mu}{ds} S\varphi - \frac{\lambda}{\mu} S\varphi - \frac{1}{\mu}(K'\varphi + \varphi) - \int_{\partial D} S\varphi
\end{aligned}
$$

The operator $A_1 : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is invertible with a bounded inverse and the operator $A_2 : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is compact. Uniqueness of BVP yields $u = 0$ in D. Taking the boundary trace of u we obtain $\varphi = 0$. The proof is completed by Riesz theorem. $\square$

Assume that the boundary ∂D, of the bounded domain $D \in \mathbb{R}^2$ is analytic and has a $2\pi-$periodic parametric representation of the form

$$\partial D = \{z(t) = (z_1(t), z_2(t)) : 0 \leq t \leq 2\pi\} \tag{2.15}$$

where $z : \mathbb{R} \to \mathbb{R}^2$ is analytic and 2π-periodic with $|z'(t)| > 0$ for all t. We denote $\varphi := \varphi \circ z$. The boundary integral equation (2.14) contains integrals with smooth kernel, a weakly singular and a strongly singular kernel. We split off each of this singularity with the aim to compute the corresponding integral analytically. Fo instance, we represent the tangential derivative of the single-layer potential in the form

$$\frac{d(S\varphi)(t)}{dt} = \frac{1}{4\pi}\int_0^{2\pi} \cot\frac{\tau - t}{2}\varphi(\tau)|z'(\tau)|d\tau + \int_0^{2\pi} L(t,\tau)\varphi(t,\tau)|z'(\tau)|d\tau$$

where

$$L(t,\tau) = L_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + L_2(t,\tau)$$

and L_1, L_2 are smooth. The second tangential derivative of the single-layer operator with aid of partial integration reduces to

$$\frac{d^2(S\varphi)(t)}{dt^2} = \frac{1}{4\pi}\int_0^{2\pi} \cot\frac{\tau - t}{2}(\varphi(\tau)|z'(\tau)|)'d\tau + \int_0^{2\pi} N(t,\tau)\varphi(\tau)|z'(\tau)|d\tau$$

where N is a weakly singular kernel with the logarithmic singularity. The parametrized form of the integral equation (2.14) can be summarized in the following compact form

$$T\varphi + B\varphi = g, \tag{2.16}$$

where T is bounded from $H^{p+1}[0, 2\pi] \to H^p[0, 2\pi], p \geq 0$ and has a bounded inverse $T^{-1} : H^p[0, 2\pi] \to H^{p+1}[0, 2\pi]$. The operator $B : H^{p+1}[0, 2\pi] \to H^p[0, 2\pi]$ can be seen to be compact since all kernels are either continuous or weakly singular. Hence from the Riesz theory (Kress, 2014, Corollary 3.6), $T + B$ has bounded inverse and it is injective, therefore the solution exists. For details we refer to Ivanyshyn Yaman and Özdemir (to appear).

Chapter 3

Inverse Problems

The inverse problems we are concerned with are formulated as following.

(IP1) Reconstruction of surface impedance functions

Given the shape Γ and the location of the obstacle D and the far field pattern u_∞ for several incident plane waves determine the surface impedance functions λ and μ.

(IP2) Reconstruction of the shape of the obstacle

Given the location of the obstacle D, the surface impedance functions λ and μ and the far field pattern u_∞ for several incident plane waves determine the shape Γ of the obstacle D.

(IP3) Reconstruction of the shape and the properties of the obstacle

Given the location of the obstacle D and the far field pattern u_∞ for several incident plane waves determine the surface impedance functions λ and μ and the shape Γ.

3.1 Uniqueness

Before starting to solve the inverse problem we need to figure out what is the minimal number of far field patterns to guarantee the uniqueness of the solution. Due to the result of Bourgeois, Chaulet, Haddar Bourgeois et al. (2012) it is known that both the shape and the impedance functions are uniquely determined by the far field patterns for an infinite number of incident waves with distinct incident directions and one fixed wave number.

The more suitable result for the numerical methods was recently found by Cakoni and Kress, Cakoni and Kress (2013); Kress (2018), in the case of two dimensions. They have shown that three far field patterns corresponding to the scattering of plane waves with different incident directions uniquely determine the impedance function for a given shape Γ. Unfortunately, there is no straightforward conclusion for the uniqueness in three dimensions. Moreover, extending the counter example given in Kress (2018), we can show non-uniqueness for the inverse impedance problem with finite number of far field patterns corresponding to incident spherical wave.

Let D be a ball of radius R centered at the origin and let λ, μ be constants such that $\operatorname{Re}\lambda \geq 0$, $\operatorname{Re}\mu \geq 0$, $|\mu| > 0$. We consider incident spherical waves $u^i(x) = j_n(k|x|)Y_n^\ell(\widehat{x})$, $|\ell| \leq n$, $n \in \mathbb{N}$ where j_n is the spherical Bessel function and Y_n^ℓ is the spherical harmonic of n degree and ℓth order. The corresponding total fields are given by

$$u_\ell(x) = \left(j_n(k|x|) - a_n\, h_n^{(1)}(k|x|)\right) Y_n^\ell(\widehat{x}).$$

Substituting the total fields in the generalized impedance boundary condition (2.2) and recalling that the spherical harmonics are eigenfunctions of the Laplace-Beltrami operator we find the coefficients

$$a_n = \frac{k\, j_n'(kR) + ik\left(\lambda + n(n+1)\,\mu/R^2\right) j_n(kR)}{k\, h_n^{(1)\prime}(kR) + ik\left(\lambda + n(n+1)\,\mu/R^2\right) h_n^{(1)}(kR)}. \tag{3.1}$$

The denominator in (3.1) does not vanish due to Rellich's lemma and the assumptions on the impedance coefficients. Hence, we can choose different combinations of impedances λ and μ giving the same value of a_n and $2n+1$ linear independent total fields. **Uniqueness for *(IP1)–(IP3)* with finite number of incident plane waves is an open question in $3D$.**

Note, this model example demonstrates also another difficulties of the inverse impedance problem such as nonlinearity and severe ill-posedness.

In the case when the far fields associated to plane waves with all incident directions are known there is a uniqueness result, Bourgeois et al. (2011), concerning identification of both the obstacle D and the impedances λ, μ – *(IP3)*.

3.2 Nonlinear boundary integral equations

The main idea of the methods we propose is to replace the solution of the inverse problems (*IP1*)–(*IP3*) by the solution of systems of nonlinear integral equation. To derive the system we employ Green's formula, (Colton and Kress, 2013b, Theorem 2.5), to the scattered field

$$u^s(x) = \int_\Gamma \left\{ u^s(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u^s(y)}{\partial \nu(y)} \Phi(x,y) \right\} ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}$$

and Green's theorem to the entire solution u^i and $\Phi(x, \cdot)$

$$0 = \int_\Gamma \left\{ u^i(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u^i(y)}{\partial \nu(y)} \Phi(x,y) \right\} ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}.$$

Substituting the total field to the boundary condition (2.2) we find its representation in terms of the boundary traces, i.e., for $x \in \mathbb{R}^3 \setminus \bar{D}$

$$u(x) = u^i(x) + \int_\Gamma \left\{ \frac{\partial \Phi(x,y)}{\partial \nu(y)} u(y) + \Phi(x,y)\, \mathcal{G}(\lambda, \mu; u)(y) \right\} ds(y). \tag{3.2}$$

Considering that the influence of a given object on an incident field is described by a distribution of the so-called "secondary sources" along the surface we may interpret (3.2) as Huygens' principle, Colton and Kress (2013b), for generalized impedance scattering.

As the next step we recall the single- double-layer operators $S, K : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ defined by (2.5) and introduce the far fields for single- and double-layer operators $S_\infty, K_\infty : H^{-\frac{1}{2}}(\Gamma) \to L^2(\mathbb{S}^2)$ defined by

$$(S_\infty \varphi)(\widehat{x}) := \frac{1}{4\pi} \int_\Gamma e^{-ik\widehat{x}\cdot y} \varphi(y)\, ds(y),$$

$$(K_\infty \varphi)(\widehat{x}) := \frac{-ik}{4\pi} \int_\Gamma e^{-ik\widehat{x}\cdot y}\, \widehat{x} \cdot \nu(y) \varphi(y)\, ds(y).$$

From the jump relations for single- and double-layer potentials, the asymptotic behavior of the Hankel functions we derive the following theorem which lays a foundation for the inversion method.

Theorem 3.1 *For a given boundary Γ, an incident field u^i and the corresponding far field pattern u_∞, assume that the surface impedance functions*

λ, μ and the density φ satisfy the system of nonlinear boundary integral equations

$$\varphi - K\varphi - S\,\mathcal{G}(\lambda, \mu; \varphi) = 2u^i|_\Gamma, \tag{3.3}$$

$$K_\infty\varphi + S_\infty\,\mathcal{G}(\lambda, \mu; \varphi) = u_\infty. \tag{3.4}$$

Then λ, μ are the solutions to the inverse impedance problem (IP1).

Proof
Consider the total field expressed via (3.2). Then its far field u_∞ is represented by $K_\infty\varphi + S_\infty\,\mathcal{G}(\lambda, \mu; \varphi)$ and hence the data equation (3.4) guarantees that the scattered field has a correct far field pattern. Recalling that $\varphi = u|_\Gamma$ and taking the Dirichlet trace of (3.2) by the jump relation for layer potentials, Colton and Kress (2013b), we find

$$2\varphi = 2u^i|_\Gamma + \varphi + K\varphi + S\,\mathcal{G}(\lambda, \mu; \varphi).$$

Hence the field equation (3.3) ensures the boundary condition (2.2). □
Note, the obtained system is ill-posed due to the data equation which contains the compact operators with exponentially decreasing singular values.

The analogous theorems hold for the inverse problems (*IP2*) and (*IP3*).

3.3 Reconstruction of surface impedance functions

We list several methods to solve the system (3.3)–(3.4), i.e. (*IP1*).

1. Introducing the new unknown $\chi = \mathcal{G}(\lambda, \mu; \varphi)$ we can interpret the system as linear which can be solved by the Tikhonov regularization. The unknown impedance functions are then solved from the above differential equation.

2. Solving the density φ from the field equation and linearizing the data equation.

3. Reversing the order of the equations in the 2nd method.

4. Simultaneous linearization of both equations with respect to all unknowns

The first method resembles in some way the direct approach for the inverse problem with the Leontovich boundary condition, Ivanyshyn and Kress (2011). The overview of the methods 2-4 for the boundary shape reconstruction can be found for example in (Colton and Kress, to appear, Section 5), Ivanyshyn et al. (2010). Since the first method was found to be less stable as compare to the method based on the simultaneous linearization we continue the study on the latter one, method 4.

In order to investigate the properties of the boundary integral operators appearing in (3.3)–(3.4) we introduce a bounded linear operator $A'(\lambda, \mu; \cdot) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ defined by

$$A'(\lambda, \mu; \varphi) := \varphi - K\varphi - S\,\mathcal{G}(\lambda, \mu; \varphi)$$

and a bounded linear operator $A'_\infty(\lambda, \mu; \cdot) : H^{-1/2}(\Gamma) \to L^2(\mathbb{S}^2)$ defined by

$$A'_\infty(\lambda, \mu; \varphi) := K_\infty\varphi + S_\infty\,\mathcal{G}(\lambda, \mu; \varphi).$$

Note that the operator $A'(\lambda, \mu; \cdot)$ is adjoint of the operator $A(\lambda, \mu; \cdot)$ defined by (2.10) in the dual system $< H^{1/2}(\Gamma), H^{-1/2}(\Gamma) >$ with respect to L^2 bilinear form. Applying the Fredholm alternative we find that

Theorem 3.1 *The operator $A'(\lambda, \mu; \cdot) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ has a bounded inverse.*

Having found the Fréchet derivatives of the operators in (3.3)–(3.4) we are ready to formulate the fully linearized system. The inverse problem is not uniquely solvable for one incident plane wave and uniqueness for the finite number of incident direction is an open problem in three dimensions. Motivated by the uniqueness result in two dimensions we consider the nonlinear system (3.3)–(3.4) for $p \geq 3$ incident plane waves

$$u^i_\ell(x) = e^{ik\,x\cdot d_\ell},\ d_\ell \in \mathbb{S}^2.$$

Given the current approximation $\varphi_\ell = u_\ell|_\Gamma$, $\ell = \overline{1,p}$, λ, μ the fully linearized system reads

$$\begin{aligned} A'(\lambda, \mu; \psi_\ell) - S\,\mathcal{G}(\eta, \zeta; \varphi_\ell) &= 2u^i_\ell|_\Gamma - A'(\lambda, \mu; \varphi_\ell),\ 1 \leq \ell \leq p, \quad (3.5) \\ A'_\infty(\lambda, \mu; \psi_\ell) + S_\infty\,\mathcal{G}(\eta, \zeta; \varphi_\ell) &= u_{\infty,\ell} - A'_\infty(\lambda, \mu; \varphi_\ell),\ 1 \leq \ell \leq p, \quad (3.6) \end{aligned}$$

for the unknown updates ψ_ℓ, η, ζ of the functions φ_ℓ, λ, μ, correspondingly.

In order to apply the Tikhonov regularization we need to investigate injectivity of the system (3.5)–(3.6)at the exact solution. Unfortunately, this issue is not resolved yet since it is directly related to the question of unique reconstruction of the surface impedances. To demonstrate the connection between the uniqueness issue and the injectivity of the operator in the system system (3.5)–(3.6) we define a function V_ℓ for $x \in \mathbb{R}^3 \setminus \bar{D}$ by

$$V_\ell(x) = 2\int_\Gamma \left\{ \frac{\partial \Phi(x,y)}{\partial \nu(y)} \psi_\ell(y) + \Phi(x,y)\mathcal{G}(\lambda,\mu;\psi_\ell)(y) + \Phi(x,y)\mathcal{G}(\eta,\zeta;\varphi_\ell)(y) \right\} ds(y),$$

where $\varphi_\ell = u_\ell|_\Gamma$ is the restriction of the total field corresponding to the incident plane wave u^i_ℓ to the boundary Γ. The equation (3.6) guarantees $V_{\infty,\ell} = 0$, and by Rellich's lemma we obtain $V_\ell = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. From the jump relations for single- and double-layer potentials, Colton and Kress (2013b), it follows that $\psi_\ell \equiv 0$. From (3.6) and the assumption k^2 is not a Dirichlet eigenvalue for the negative Laplacian in D we can conclude that $\mathcal{G}(\eta,\zeta;\varphi_\ell) = 0$, i.e.

$$(\eta - \operatorname{Div} \zeta \operatorname{Grad})\varphi_\ell = 0$$

This leads us to the unsolved uniqueness problem. Indeed, choosing D to be a ball of radius R centered at the origin, η and ζ to be constants, recalling that the spherical harmonics are dense in $L^2(\mathbb{S}^2)$ we arrive to the equation

$$\eta + n(n+1)\zeta = 0.$$

Since η and ζ are the unknown updates, we cannot put positivity restriction on their real or imaginary parts and hence there is a nontrivial solution.

The remedy to this might lie in a special choice on incident directions. However, there is no solution available at the moment and this issue will be closed automatically once the uniqueness question is resolved.

3.4 Investigation of inverse solution methods for a ball

For a special case of a radially symmetric incident wave we derive a system of nonlinear equation for the unknown radius R and the unknown scattered field on the boundary.

We consider a superposition of plane waves and by Funk-Hecke formula we obtain

$$\int_{\mathbb{S}^2} e^{ikx\cdot d}\, ds(d) = \frac{4\pi \sin k|x|}{|x|}, \quad x \in \mathbb{R}^3.$$

Scattering of this incident field

$$u^i(x) = \frac{\sin k|x|}{|x|}$$

by a ball of radius R generates a radially symmetric scattered field for constant impedance functions. The scattered field is equal constant C on the sphere, consequently $\Delta_{\mathbb{S}^2} u = 0$ and the system of integral equations simplifies to

$$(I - K - S\, ik\lambda)C = 2u^i\big|_{|x|=R} \tag{3.7}$$

$$(K_\infty + S_\infty\, ik\lambda)C = u_\infty. \tag{3.8}$$

Using this fact and Green's theorem for spherical Bessel functions $u_0 = j_0(k|z|)$ we find, (Colton and Kress, 2013b, Section 2.4),

$$\int_{|z|=R} \left\{ u_0(z)\frac{\partial \Phi(x,z)}{\partial \nu(z)} - \frac{\partial u_0(z)}{\partial \nu(z)}\Phi(x,z) \right\} ds(z) = 0, x > R, \tag{3.9}$$

and by Green's formula applied to spherical Hankel functions of the first kind $v_0(z) = h_0^{(1)}(k|z|)$ we obtain

$$\int_{|z|=R} \left\{ v_0(z)\frac{\partial \Phi(x,z)}{\partial \nu(z)} - \frac{\partial v_0(z)}{\partial \nu(z)}\Phi(x,z) \right\} ds(z) = v_0(x), |x| > R. \tag{3.10}$$

Taking the difference between (3.9) multiplied by v_0 and (3.10) multiplied by u_0 and noting that

$$j_0(t)y_0'(t) - j_0'(t)y_0(t) = \frac{1}{t^2} \tag{3.11}$$

we find that

$$\frac{1}{ikR^2}\int_{|z|=R} \Phi(x,z)ds(z) = j_0(kR)h_0^{(1)}(k|x|), \quad |x| > R. \tag{3.12}$$

Due to $j_0(t) = \dfrac{\sin t}{t}$ and $h_0^{(1)}(t) = \dfrac{e^{it}}{it}$ we have

$$\int_{|z|=R} \Phi(x,z)ds(z) = R\frac{\sin kR}{k}\frac{e^{ik|x|}}{|x|}, \quad |x| > R.$$

Since the single-layer potential can be continuously extended up to the boundary we have

$$S1 = 2R\frac{\sin kR}{k}\,\frac{e^{ikR}}{R}.$$

Taking the difference between (3.10) multiplied by u_0' and (3.9) multiplied by v_0' we have

$$\int_{|z|=R} i\frac{\partial\Phi(x,z)}{\partial\nu(z)}(j_0'(k|z|)y_0(k|z|) - y_0'(k|z|)j_0(k|z|))ds(z) = j_0'(kR)v_0(x)$$

and due to (3.11) we find that

$$-\frac{i}{k^2R^2}\int_{|z|=R} \frac{\partial\Phi(x,z)}{\partial\nu(z)}\,ds(z) = j_0'(kR)h_0^{(1)}(k|x|), \quad |x| > R$$

and

$$K1 = 2\left(kR\cos kR - \sin kR\right)\frac{e^{ikR}}{R}.$$

Furthermore, from the asymptotic behaviors of the fundamental solution and spherical wave functions we find the values of the far field operators applied to a constant

$$S_\infty 1 = R\frac{\sin kR}{k}, \quad K_\infty 1 = kR\cos kR - \sin kR$$

In this way, a system of nonlinear boundary integral equations (3.7)-(3.8) for the unknown radius and the unknown constant density is reduced to

$$\left\{\begin{array}{rcl} \left(1 - 2f(R)e^{ikR}\right)C & = & 2\dfrac{\sin kR}{R} \\ f(R)C & = & u_\infty, \end{array}\right.$$

where $f(R) = kR(\cos kR + i\lambda\frac{1}{k}\sin kR) - \sin kR.$

The two-by-two system of algebraic equations is linear with respect to C and nonlinear with respect to R. We rewrite the system in the operator form

$$F(C,R) = [0, u_\infty]^T, \quad F(C,R) = \left[\begin{array}{r} \left(1 - 2f(R)e^{ikR}\right)C - \dfrac{2\sin kR}{R} \\ f(R)C \end{array}\right]$$

The operator F is twice continuously differentiable for $R \neq 0$,

$$F'_{[C,R]} = \left[\begin{array}{cr} \left(1 - 2f(R)e^{ikR}\right) & -2Cf'(R)e^{ikR} - 2Cikf(R)e^{ikR} - 2\frac{kR\cos kR - \sin kR}{R^2} \\ f(R) & Cf'(R) \end{array}\right],$$

$$\det F'(C,R) = Cf'(R) + 2f(R)\left(ikCf(R)e^{ikR} + \frac{kR\cos kR - \sin kR}{R^2}\right)$$

Considering the case of the exact solution, i.e. C found from (3.7) using the symbolic computations we find that $\det F'_{[C,R]} \neq 0$. By the Newton-Kantorovich theorem, Kantorovich (1948); Ciarlet and Mardare (2012), the Newton method for this model example converges quadratically.

3.5 Reconstruction of the boundary shape and impedances

In this section we propose several iterative scheme for the shape and impedance functions reconstruction.

We will assume that the surface Γ is $C^{4,\alpha}$-smooth, homeomorphic to the unit sphere $\mathbb{S}^2$ and has a star-shaped representation

$$\Gamma := \Gamma_r = \{z(\hat{x}) = r(\hat{x})\hat{x} : \hat{x} \in \mathbb{S}^2\}$$

with $r(\hat{x}) > 0$ for $\hat{x} \in \mathbb{S}^2$ and the Jacobian of the transformation r is given by

$$J_r = r\sqrt{r^2 + |\operatorname{Grad}_{\mathbb{S}^2} r|^2}.$$

The subscript r indicates the nonlinear dependence of the operators on the boundary shape. Similar to results presented in Section 3.2. we derive a system of nonlinear integral equation equivalent to the inverse problem *(IP2)*. We replace the integral operators and the right-hand sides by their parametrized form, e.g.

$$\mathsf{S}(r,\varphi)(\hat{x}) = \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{e^{ik|r(\hat{x})\hat{x} - r(\hat{y})\hat{y})|}}{|r(\hat{x})\hat{x} - r(\hat{y})\hat{y})|}\,\varphi(\hat{y})\,J_r(\hat{y})\,ds(\hat{y}), \quad \hat{x} \in \mathbb{S}^2.$$

We use the following transformation

$$(\operatorname{Grad}_\Gamma u) \circ z = [\mathrm{D}z^*]^{-1} \operatorname{Grad}_{\mathbb{S}^2}(u \circ z),$$

$$(\operatorname{Div}_\Gamma v) \circ z = \frac{1}{J_r} \operatorname{Div}_{\mathbb{S}^2}\left(J_r[\mathrm{D}z]^{-1}(v \circ z)\right),$$

with $\mathrm{D}z$ the total derivative which maps the tangent plane to $\mathbb{S}^2$ to the tangent plane to Γ and the operator $[\mathrm{D}z^*]^{-1}$ maps the cotangent plane to the unit sphere $\mathbb{S}^2$ to the cotangent plane to the given surface Γ. The parametrized version of the operator $\mathcal{G}$ takes the form

$$\mathsf{G}(r,\varphi)(\hat{x}) = \left(\lambda(\hat{x}) - \frac{1}{J_r(\hat{x})}\operatorname{Div}_{\mathbb{S}^2}\left(J_r(\hat{x})\mu(\hat{x})\operatorname{Grad}_{\mathbb{S}^2}\varphi(\hat{x})\right)\right).$$

The first group of methods is based on the direct integral equations.

3.5.1 Method based on the direct integral equation approach

The system of nonlinear boundary integral equations then reads

$$\varphi - \mathsf{K}(r,\varphi) - \mathsf{S}(r,\mathsf{G}(r,\varphi)) = 2u^i(r), \tag{3.13}$$

$$\mathsf{K}_\infty(r,\varphi) + \mathsf{S}_\infty(r,\mathsf{G}(r,\varphi)) = u_\infty. \tag{3.14}$$

Firstly, we would like to note the difference in the Fréchet differentiability of the boundary integral operators w.r.t. impedance functions and shape. Although, the inverse problem for impedance reconstruction *(IP1)* is nonlinear, the integral operators occurring in the proposed method are linear with respect to each impedance. The situation is different for the differentiability w.r.t. the shape, since all integral operators are defined over the unknown boundary.

The Fréchet differentiability of the boundary integral operators in (3.13)-(3.14) is obtained by proving the Fréchet differentiability of their kernels, Potthast (1994). By this result, the exact representations of the derivatives of nonlinear integral operators with respect to shape are found.

Theorem 3.1 *The operator* $\mathsf{S} : C^3(\mathbb{S}^2,\mathbb{R}) \times H^{-1/2}(\mathbb{S}^2,\mathbb{C}) \to H^{1/2}(\mathbb{S}^2,\mathbb{C})$ *is Fréchet differentiable and the first derivatives at* r *in the direction* $q \in C^3(\mathbb{S}^2)$ *is a bounded linear integral operator from* $H^{-1/2}(\mathbb{S}^2,\mathbb{C})$ *to* $H^{1/2}(\mathbb{S}^2,\mathbb{C})$ *defined by*

$$\begin{aligned}\mathsf{S}'(r,\varphi;q)(\hat{x}) = \frac{1}{2\pi}\int_{\mathbb{S}^2} &\varphi(\hat{y})\left(ik - \frac{1}{|r(\hat{x})\hat{x}-r(\hat{y})\hat{y}|}\right)e^{ik|r(\hat{x})\hat{x}-r(\hat{y})\hat{y}|}\\ &\times\frac{< r(\hat{x})\hat{x}-r(\hat{y})\hat{y}, q(\hat{x})\hat{x}-q(\hat{y})\hat{y} >}{|r(\hat{x})\hat{x}-r(\hat{y})\hat{y}|^2}J_r(\hat{y})\,ds(\hat{y})\\ &+\frac{1}{2\pi}\int_{\mathbb{S}^2}\frac{e^{ik|r(\hat{x})\hat{x}-r(\hat{y})\hat{y})|}}{|r(\hat{x})\hat{x}-r(\hat{y})\hat{y})|}\varphi(\hat{y})\,J_r'q(\hat{y})\cdot\hat{y}\,ds(\hat{y}),\end{aligned}$$

where

$$J_r'q(\hat{x}) = \frac{J_r(\hat{x})}{r(\hat{x})}q(\hat{x}) + \frac{1}{J_r(\hat{x})}\left(r(\hat{x})q(\hat{x}) + \operatorname{Grad}_{\mathbb{S}^2} r(\hat{x}) \cdot \operatorname{Grad}_{\mathbb{S}^2} q(\hat{x})\right).$$

The operator $\mathsf{S}'(r,\cdot;q)$ is linear w.r.t. q and it represents a linearization of $\mathsf{S}(r,\cdot)$ in the sense that

$$\|\mathsf{S}(r+q,\varphi) - \mathsf{S}(r,\varphi) - \mathsf{S}'(r,\varphi;q)\|_{H^{1/2}(\mathbb{S}^2)} = o(\|q\|^2_{C^2(\mathbb{S}^2)}).$$

Recalling the operator

$$\mathsf{G}(r,\varphi)(\hat{x}) = \left(\lambda(\hat{x}) - \frac{1}{J_r(\hat{x})}\operatorname{Div}_{\mathbb{S}^2}\left(J_r(\hat{x})\mu(\hat{x})\operatorname{Grad}_{\mathbb{S}^2}\varphi(\hat{x})\right)\right).$$

we find its Fréchet derivative G represented by

$$\begin{aligned}\mathsf{G}'(r,\varphi;q)(\hat{x}) = \quad & \frac{1}{J_r^2(\hat{x})}J_r'q(\hat{x})\operatorname{Div}_{\mathbb{S}^2}\left(J_r(\hat{x})\mu(\hat{x})\operatorname{Grad}_{\mathbb{S}^2}\varphi(\hat{x})\right) \\ & -\frac{1}{J_r(\hat{x})}\operatorname{Div}_{\mathbb{S}^2}\left(J_r'q(\hat{x})\mu(\hat{x})\operatorname{Grad}_{\mathbb{S}^2}\varphi(\hat{x})\right).\end{aligned}$$

Similarly we find the parametrization and the derivatives of the operators S_∞, K, K_∞.

The Fréchet derivative operators are bounded and linear in the corresponding Soboles spaces inherited from the original nonlinear operators.

By the same token as reconstructing the surface impedance functions we can generate the following four iterative schemes for (*IP2*).

1. Solving the density φ from the field equation and linearizing the data equation with respect to r
2. Reversing the order of the equations in the 2nd method.
3. Simultaneous linearization of both equations with respect to φ and r. We note that by introducing the new unknown $\chi = \mathcal{G}(\lambda,\mu;\varphi)$ the linearized system simplifies substantially.

Combining the results for shape and impedance problems several iterative scheme for (*IP3*) can be constructed, e.g.

1. Introducing the new unknown $\chi = \mathcal{G}(\lambda, \mu; \varphi)$ we can can find the unknowns shape via one of the stated above methods. The unknown impedance functions are then solved from the above differential equation.

2. Simultaneous linearization of both equations with respect to all unknowns.

3. Using the data for 1-2 incident directions solving the density φ from the field equation and linearizing the data equation with respect to r. Then updating the impedance functions employing the remaining data and one of the method presented for (*IP1*).

3.5.2 Method based on the indirect integral equation approach

Recalling the ideas presented in Section 2.2.1, we seek the scattered field $u^s \in H^2_{\text{loc}}(\mathbb{R}^3 \setminus \bar{D})$ in the form of a single-layer potential

$$u^s(x) = \int_\Gamma \Phi(x, y)\varphi(y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}.$$

Substituting the total field to the boundary condition (2.2) and using the jump relations for the single-layer potential, Colton and Kress (2013b), we can prove that a system of integro-differential equations is equivalent to (*IP2*).

Theorem 3.2 *For given surface impedance functions λ, μ, an incident field u^i and the corresponding far field pattern u_∞, assume that the surface Γ and the density φ satisfy the system of nonlinear boundary integral equations*

$$\varphi - K'\varphi - \mathcal{G}(\lambda, \mu; S\varphi) = 2\left.\frac{\partial u^i}{\partial \nu}\right|_\Gamma + 2\mathcal{G}(\lambda, \mu; u^i|_\Gamma)$$

$$S_\infty\varphi = u_\infty$$

Then Γ is the solution to the inverse impedance problem (IP2).

Proof
Consider the total field expressed via the single-layer potential. Then its far field u_∞ is represented by $S_\infty\varphi$ and hence the data equation guarantees that

the scattered field has a correct far field pattern. Taking the Dirichlet and Neumann traces of the total fields by the jump relation for layer potentials, Colton and Kress (2013b), we find that the field equation ensures the boundary condition (2.2). $\square$

To simplify the computations we introduce a new variable $\psi = \varphi J_r$ and the corresponding operators $\tilde{\mathsf{S}}$, $\tilde{\mathsf{S}}_\infty$, $\tilde{\mathsf{K}}'$, e.g.

$$\tilde{\mathsf{S}}_\infty(r,\psi)(\hat{x}) = \frac{1}{4\pi}\int_{\mathbb{S}^2} e^{-ik\hat{x}\cdot r(\hat{y})\hat{y}}\,\psi(\hat{y})\,ds(\hat{y}), \quad \hat{x}\in\mathbb{S}^2.$$

The parametrized system of nonlinear integro-differential equations reads

$$\frac{\psi}{J_r} - \tilde{\mathsf{K}}'(r,\psi) - \mathsf{G}(r,\tilde{\mathsf{S}}(r,\psi)) = 2\left.\frac{\partial u^i}{\partial\nu}\right|_{\Gamma_r} + 2\mathsf{G}(r;u^i|_{\Gamma_r}), \tag{3.15}$$

$$\tilde{\mathsf{S}}_\infty(r,\psi) = u_\infty$$

Theorem 3.3 *The operator $\tilde{\mathsf{S}}_\infty : C^3(\mathbb{S}^2,\mathbb{R})\times H^{-1/2}(\mathbb{S}^2,\mathbb{C}) \to L^2(\mathbb{S}^2,\mathbb{C})$ is Fréchet differentiable and the first derivatives at r in the direction $q\in C^3(\mathbb{S}^2)$ is a bounded linear integral operator from $H^{-1/2}(\mathbb{S}^2,\mathbb{C})$ to $L^2(\mathbb{S}^2,\mathbb{C})$ defined by*

$$\tilde{\mathsf{S}}'_\infty(r,\psi;q)(\hat{x}) = \frac{-ik}{4\pi}\int_{\mathbb{S}^2} e^{-ik\hat{x}\cdot r(\hat{y})\hat{y}}\,\hat{x}\cdot q(\hat{y})\hat{y}\,\psi(\hat{y})\,ds(\hat{y})$$

Recalling that the boundary integral equation (2.9) is uniquely solvable, Kress (2016), we obtain that (3.15) is uniquely solvable and the following theorem can be stated, which justifies application of the Tikhonov regularization.

Theorem 3.4 *Assume that k^2 is neither Dirichlet nor Neumann eigenvalue of the negative Laplacian in D, the boundary ∂D can be parametrized as follows*

$$\partial D = \{r(\hat{x})\hat{x},\ \hat{x}\in\mathbb{S}^2\}$$

and ψ be a solution to (3.15). Then the equation

$$\tilde{\mathsf{S}}'_\infty(r,\psi;q) = 0$$

has only the zero solution among three times differentiable and periodic functions.

Proof. Using the idea of parallel surfaces, Colton and Kress (2013a) and extending results of Ivanyshyn and Kress (2006); Ivanyshyn and Johansson (2008) to three dimensions it is possible to search the update in the form $\tilde{q}(\hat{x})\nu(\hat{x})$ instead of $q(\hat{x})\hat{x}$. The proof then follows the ideas presented in Ivanyshyn and Johansson (2008). Consider the double-layer potential

$$v(x) = \int_{\mathbb{S}^2} \nu(r(\hat{y})) \cdot \operatorname{grad}_x \Phi(x, r(\hat{y}))\psi(\hat{y})\tilde{q}(\hat{y})\, ds(\hat{y}).$$

Due to the assumption of the theorem it has a vanishing far field pattern. Hence, by Rellich's lemma we can conclude that v vanishes outside of D.

The potential v solves the homogeneous Neumann problem in D and therefore vanishes identically in D by our assumption on k. By the jump relations we obtain $\psi\tilde{q} = 0$ on $\mathbb{S}^2$. Consequently, by smoothness of $\tilde{q}$ we have $\tilde{q} = 0$ on $\mathbb{S}^2$. $\square$

To summarize, we presented several iterative schemes for each of the problem (*IP1*)-(*IP3*) and investigated the question of injectivity for the operator of the linearized problem. In the case of unknown surface impedance function the injectivity cannot be established due to the lack of uniqueness results for the problem under consideration. In the case of shape problem the injectivity is settled for the iterative scheme based on the linearization of the data equation.

Chapter 4

Numerical implementations and examples

In this chapter we described detailed implementation of the considered problems in $2D$ and $3D$.

4.1 Numerical implementation for the solution of the direct GIBC problem in $2D$

The boundary integral equation is solved by a collocation method using P_n a trigonometric interpolation operators a projection, trapezoidal and the following quadrature rules (see Kress (2014))

$$\frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\tau-t}{2}\varphi'(\tau)d\tau \approx \sum_{i=0}^{2n-1} T_{1i}(t)\varphi(t_i^{(n)})$$

$$\frac{1}{2\pi}\int_0^{2\pi} \ln 4\sin^2\frac{\tau-t}{2}\varphi(\tau) \approx \sum_{i=0}^{2n-1} R_j(t)\varphi(t_i^{(n)})$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\tau-t}{2}\varphi(\tau) \approx \sum_{i=0}^{2n-1} T_{2i}(t)\varphi(t_i^{(n)})$$

with the quadrature weights given by

$$T_{1i}^{(n)}(t) = -\frac{1}{n}\sum_{m=1}^{n-1} m\cos m(t-t_i^{(n)}) - \frac{1}{2}\cos n(t-t_i^{(n)})$$

$$R_i^{(n)}(t) = -\frac{1}{n}\sum_{m=1}^{n-1} \frac{1}{m}\cos m(t-t_i^{(n)}) - \frac{1}{2n^2}\cos n(t-t_i^{(n)})$$

$$T_{2i}^{(n)}(t) = \frac{1}{2n}\left\{1-\cos n(t-t_i^{(n)})\right\}\cot\frac{t-t_i^{(n)}}{2}.$$

Recalling the error estimate for the trigonometric interpolation

$$||P_n h - h||_q \le \frac{C}{n^{p-q}}||h||_p, \quad 0 \le q \le p,\ \frac{1}{2} > p \tag{4.1}$$

for all $g \in H^p[0,2\pi]$ and some constant C depending p and q (see (Kress, 2014, Theorem 11.8)), we establish the following theorem

Theorem 4.1 *Let φ_n be an approximate solution found by the collocation method and φ be the exact solution to* (2.14)*. Then*

$$||\varphi_n - \varphi||_{p+1} \le C(||P_n g - g||_p + ||P_n B_n \varphi - B\varphi||_p)$$

for some constant C depending on $p > \frac{1}{2}$

In case of analytic functions data and the boundary, the interpolation error decays exponentially.

4.2 Direct GIBC problem in 2D

Let ∂D be parametrized by

$$z(t) = (2\cos(t) - 2\cos^2(t) + 1, 5\sin(t) - \cos(t)\sin(t)),\ 0 \le t \le 2\pi$$

and the impedance function given by

$$\lambda(x) = -\sin(|x|) + 4.5,$$

$$\mu(x) = -2\cos(|x|) + 4.5.$$

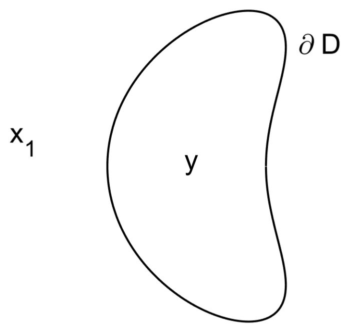


Figure 4.1: Planar domain D

Example 1. (Test by the exact solution)
Consider a point source located at $x_1 = (2, 0.4)$. Choose the parameter $k = \frac{\pi}{3}$. The exact solution is represented by $u^\dagger(y) = \Phi(y, x_1), y \in D, x_1 \in \mathbb{R}^2 \setminus \overline{D}$ and u is an approximate solution. As can be seen from the Table 4.1 the absolute error at $y = (0, 0.5)$ converges exponentially.

Table 4.1: Error analysis, $2D$

n	$\lvert u - u^\dagger \rvert$
8	0.002246206691377
16	0.000143708893458
32	0.000000009790216
64	0.000000000000001

Example 2. (Exact solution is unknown)
Let $k = \frac{1}{2}$ and $x_1 = (3, 2)$. The right-hand side is chosen as

$$g(x) = \Phi(x_1, x), \quad x \in \partial D.$$

In Table 4.2 we present values of the approximate solution at $y = (0, 0.5)$ for different degrees n of trigonometric interpolation.

Table 4.2: Approximate solution, $2D$

n	u_n
8	0.012063279277905
16	0.012284634729342
32	0.012285858740215
64	0.012285858683054
128	0.012285858683054

4.3 Numerical implementation in $3D$

To obtain a fully discrete system we apply the fully discrete Galerkin method by Ganesh and Graham, Ganesh and Graham (2004), to the parametrized linear boundary integral equations. The method is based on approximations by spherical harmonics and converges super algebraically in the case of smooth boundaries. We start with the numerical integration formula over the unit sphere for a continuous function, the so called Gauss trapezoidal product rule,

$$\int_{\mathbb{S}^2} u(\widehat{x})ds(\widehat{x}) \approx \sum_{\rho=0}^{2n+1}\sum_{\tau=1}^{n+1} \mu_\rho \nu_\tau u(\widehat{x}_{\rho\tau}), \quad \widehat{x}_{\rho\tau} = \widehat{x}(\theta_\tau, \phi_\rho), \tag{4.2}$$
$$\mu_\rho = \frac{\pi}{n+1},\ \phi_\rho = \frac{\rho\pi}{n+1},\ \theta_\tau = \arccos\zeta_\tau,$$

where ζ_τ are the zeros of the Legendre polynomial P^0_{n+1} of degree $n+1$ and ν_τ are the corresponding Gauss-Legendre weights. The formula (4.2) is exact for the scalar spherical polynomials of order less than or equal to $2n+1$. This induces the discrete inner product $(\cdot,\cdot)_n$

$$(\varphi_1, \varphi_2)_n = \sum_{\rho=0}^{2n+1}\sum_{\tau=1}^{n+1} \mu_\rho \nu_\tau \varphi_1(\widehat{x}_{\rho\tau})\overline{\varphi_2(\widehat{x}_{\rho\tau})},$$

on the space of all scalar spherical harmonics Y_{lj}, for $j = -l, \dots, l$ and $l = 1, 2, \dots$, of degree less than or equal to n. We introduce a projection operator ℓ_n defined by

$$\ell_n u = \sum_{l=0}^{n}\sum_{j=-l}^{l} (u, Y_{lj})_n Y_{lj}.$$

For the numerical approximation of the integral operators with smooth kernels K_∞, S_∞, we apply the Gauss trapezoidal product rule. The singularity in the weakly singular kernels of the operators K, K′, S, is moved by an orthogonal transformation $T_{\widehat{x}}$ to the north pole $\widehat{n} = (0,0,1)^T$, such that $T_{\widehat{x}}\widehat{x} = \widehat{n}$. For $\widehat{z} = T_{\widehat{x}}\,\widehat{y}$ we have the identity

$$|\widehat{x} - \widehat{y}| = |T_{\widehat{x}}^{-1}(\widehat{n} - \widehat{z})| = |\widehat{n} - \widehat{z}|.$$

To approximate the resulting integrals we use the modified Gauss trapezoidal rule

$$\int_{\mathbb{S}^2} \frac{u(\widehat{x})}{|\widehat{n} - \widehat{x}|} ds(\widehat{x}) \approx \int_{\mathbb{S}^2} \frac{(\ell_n u)(\widehat{x})}{|\widehat{n} - \widehat{x}|} ds(\widehat{x}) = \sum_{\rho=0}^{2n+1} \sum_{\tau=1}^{n+1} \alpha_\tau \mu_\rho \nu_\tau u(\widehat{x}_{\rho\tau}),\ \ \alpha_\tau = \sum_{l=0}^{n} P_l^0(\zeta_\tau),$$

which is based on the fact that the scalar spherical harmonics are eigenfunctions of the single layer potential on the sphere, Colton and Kress (2013b).

To approximate the surface differential operators we introduce the vectorial spherical harmonics

$$\mathcal{Y}_{lj}^{(1)} = \frac{1}{\sqrt{l(l+1)}} \operatorname{Grad}_{\mathbb{S}^2} Y_{lj}, \quad \mathcal{Y}_{lj}^{(2)} = \frac{1}{\sqrt{l(l+1)}} \operatorname{Curl}_{\mathbb{S}^2} Y_{lj},$$

for $j = -l, \dots, l$ and $l = 1, 2, \dots$ which form a complete orthonormal system in the spaces of tangent vectors fields $L_t^2(\mathbb{S}^2)$. The corresponding projection operator $\mathcal{L}_n$ on the space generated by the orthonormal basis of tangential vector spherical harmonics is defined by

$$\mathcal{L}_n \upsilon = \sum_{i=1}^{2} \sum_{l=1}^{n} \sum_{j=-l}^{l} (\upsilon | \mathcal{Y}_{lj}^{(i)})_n \, \mathcal{Y}_{lj}^{(i)},$$

where $(\upsilon_1|\upsilon_2)_n = \displaystyle\sum_{\rho=0}^{2n+1} \sum_{\tau=1}^{n+1} \mu_\rho \nu_\tau \upsilon_1(\widehat{x}_{\rho\tau}) \cdot \overline{\upsilon_2(\widehat{x}_{\rho\tau})}$.

Both impedance functions are approximated by the scalar spherical harmonics of degree K, i.e.

$$\mu \approx \sum_{lj,l=1}^{K} \mu_{lj} Y_{lj}.$$

The main difficulty in calculations arose due to numerical calculations of surface differential operators for an arbitrary surface. To overcome this problem we use the transformation formulas for the surface differential operators which replace the derivatives over an arbitrary surface by the surface derivatives over a unit sphere. Recalling the parametrization $z : \mathbb{S}^2 \to \Gamma$,

$$\Gamma := \{z(\widehat{x}) : \widehat{x} \in \mathbb{S}^2\},$$

of the surface Γ we introduce the total derivative $\mathrm{D}z$ which maps the tangent plane to $\mathbb{S}^2$ to the tangent plane to Γ. The operator $[\mathrm{D}z^*]^{-1}$ maps the cotangent plane to the unit sphere $\mathbb{S}^2$ to the cotangent plane to the given surface Γ. Taking advantage of the following transformation formulas

$$(\operatorname{Grad}_\Gamma u) \circ z = [\mathrm{D}z^*]^{-1} \operatorname{Grad}_{\mathbb{S}^2}(u \circ z),$$

$$(\operatorname{Div}_\Gamma v) \circ z = \frac{1}{J_z} \operatorname{Div}_{\mathbb{S}^2} \big(J_z [\mathrm{D}z]^{-1}(v \circ z)\big),$$

we can reduce substantially the computational (technical) challenge.

Next, the surface differential operator $\mathrm{G}(\lambda, \mu; \cdot)$ can be approximated as following

$$\operatorname{Div} \mu \operatorname{Grad}_{\mathbb{S}^2} Y_{lj} \approx \frac{1}{J_z} \operatorname{Div}_{\mathbb{S}^2} \mathcal{L}_n \, \mu \operatorname{Grad}_{\mathbb{S}^2} Y_{lj}.$$

Since

$$\operatorname{Div}_{\mathbb{S}^2} \operatorname{Grad}_{\mathbb{S}^2} = \Delta_{\mathbb{S}^2}, \ \operatorname{Div}_{\mathbb{S}^2} \operatorname{Curl}_{\mathbb{S}^2} = 0, \text{ and } \Delta_{\mathbb{S}^2} Y_{lj} = -l(l+1) Y_{lj}$$

we obtain

$$(\operatorname{Div} \mu \operatorname{Grad}_{\mathbb{S}^2} Y_{lj}, Y_{l'j'}) \approx \sum_{pq,p=1}^{n} \left(\frac{1}{J_z} Y_{pq}, Y_{l'j'}\right)_n \big(\mu \operatorname{Grad}_{\mathbb{S}^2} Y_{lj} | \mathcal{Y}^{(1)}_{pq}\big)_n \, (-\sqrt{p(p+1)}).$$

For the representation of the tangential gradient of the spherical harmonics we refer to Louer (2018).

Considering the inverse shape problem we assume that the unknown boundary is star-shaped, i.e.

$$\Gamma := \{r(\widehat{x})\widehat{x} : \widehat{x} \in \mathbb{S}^2\},$$

and we approximate the unknown radial function by real valued spherical harmonics

$$r \approx \sum_{lj,l=1}^{N} r_{lj} Y^{\mathbb{R}}_{lj}.$$

The fully discrete systems are solved by the conjugate gradient (CG) method. In the CG algorithm we compute L^2 adjoint and evaluate norms in corresponding $H^s(\mathbb{S}^2)$ spaces for $s \in \mathbb{R}$ which can be characterized by

$$H^s(\mathbb{S}^2) = \left\{ v = \sum_{l=0}^{\infty} \sum_{j=-l}^{l} v_{lj} Y_{lj},\ v_{lj} \in \mathbb{C},\ \sum_{l=0}^{\infty} \sum_{j=-l}^{l} (1+l^2)^s |v_{lj}|^2 < \infty \right\}.$$

4.4 Direct GIBC problem in $3D$

We present two examples to demonstrate convergence for the solution of integral equation based on the single-layer approach (2.8).
Example 1.
We consider one very simple domain, i.e. D a unit sphere, and a complicated domain which has a concave part, e.g. a bean shown on the figure
The impedance functions are $\lambda(x) = x_1^2 + 0.2 + i \sin x_3$, $\mu(x) = x_2^3 + 2$.
To test the method we assume that the incident field is given by a point

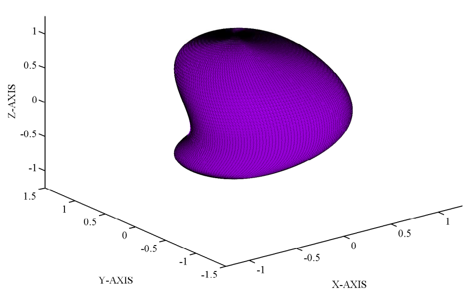


Figure 4.2: $3D$ bean-shaped domain

source $u^p(x) = \Phi(x, x^*)$, where the source $x^* \in D$. In this case the boundary condition is given by

$$\left.\frac{\partial u^s}{\partial \nu}\right|_{\Gamma} + \mathcal{G}(\lambda, \mu; u^s|_{\Gamma}) = \left.\frac{\partial u^p}{\partial \nu}\right|_{\Gamma} + \mathcal{G}(\lambda, \mu; u^p|_{\Gamma})$$

The exact solution is obviously $u^{s,\ddagger}(x) = \Phi(x, x^*)$. In this way we test both the correctness of the method and the convergence.

Table 4.3: $\|u_\infty^\ddagger - \Phi_\infty(x, x^*)\|_{L^2(\mathbb{S}^2)}$

	D – ball		D – bean-shape domain	
n	$k = \pi/2$	$k = 3\pi/2$	$k = \pi/2$	$k = 3\pi/2$
5	5.1656e-14	5.3626e-07	0.0035	0.0161
10	7.6864e-16	1.6489e-15	2.0547e-04	3.2044e-04
20	–	–	1.5697e-06	2.2790e-06
40	–	–	6.9326e-11	2.2877e-10

The parameter n appears in the dicsretized linear integral equation, i.e. we have $(n+1)^2$ algebraic equations and the unknown density function is approximated by spherical harmonics of order less or equal n. The far field is measured at $2(5+1)^2$ points.

Example 2.
We set $k = \pi/2$ and present the far field pattern at two points for an incident plane wave $u^i = e^{ikx\cdot d}$, where $d = (0, 0, 1)$

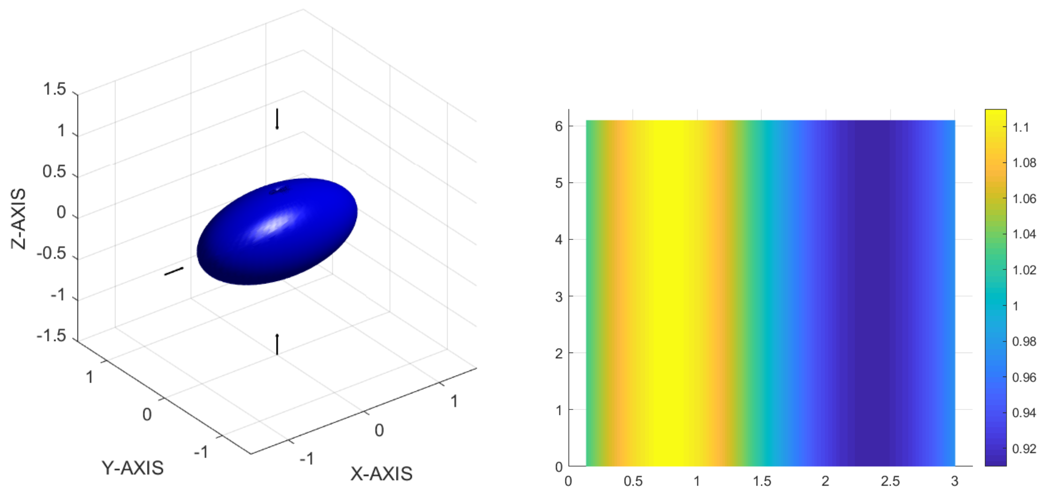
Table 4.4: Convergence for modulus of the far field, $|u_\infty|$

	D – ball		D – bean-shape domain	
n	$\hat{x} = (1, 0, 0)$	$\hat{x} = (-1, 0, 0)$	$\hat{x} = (1, 0, 0)$	$\hat{x} = (-1, 0, 0)$
5	1.51567288	0.32517080	1.10434899	0.32497011
10	1.51566786	0.32516661	1.10509488	0.32390658
20	1.51566785	0.32516661	1.10506971	0.32386016
40	–	–	1.10506954	0.32386035

4.5 Inverse Problem

We illustrate the feasibility of the method by numerical experiments. The synthetic data are obtained by solving the boundary integral equation (2.9) with $n_{syn} = 20$ for an ellipsoid and the incident directions $(1, 0, 0)$, $(0, 0, 1)$, $(0, 0, -1)$ depicted on Fig. 4.3. The performance of the method is investigated on the examples where the impedance functions are linear combinations of a constant and the function g given by

$$g(\theta, \phi) = \frac{1}{1 - 0.1 \sin 2\theta}, \quad \theta \in [0, \pi].$$



surface $\qquad\qquad\qquad\qquad$ g

Figure 4.3: The scatterer D and the impedance

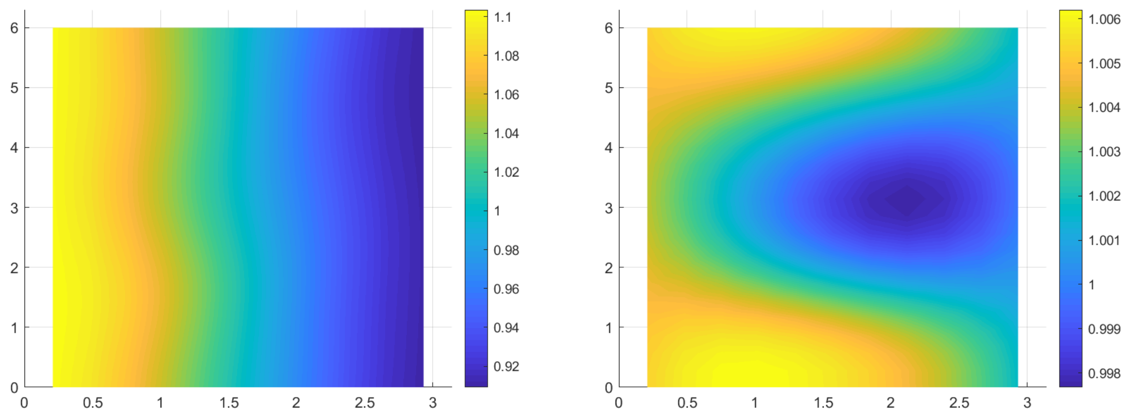
The perturbed far field pattern was generated as detailed below

$$u^{\delta}_{\infty,\ell} = u_{\infty,\ell} + \Theta_\ell, \quad \sum_{\ell=1}^{p} \|\Theta_\ell\|^2_{L^2} < \delta^2, \ \delta = 0.02, \ p = 3,$$

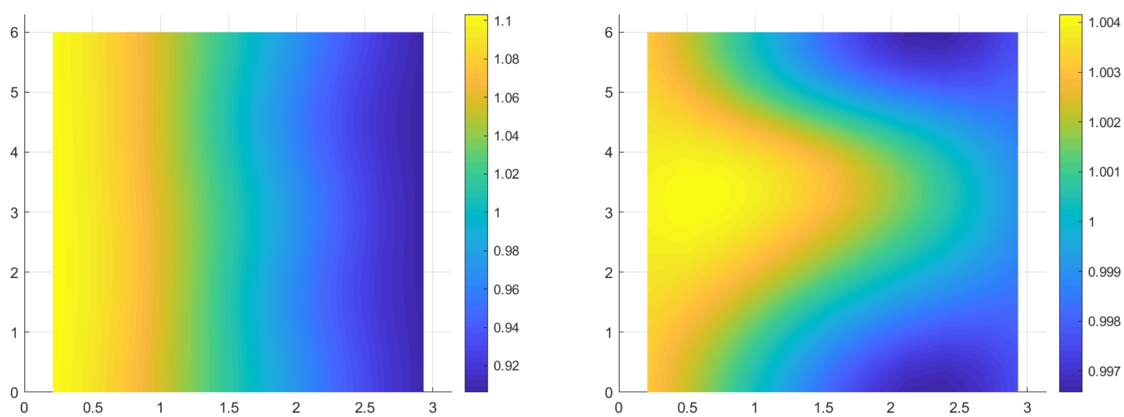
where Θ_ℓ is a random complex variable with normally distributed real and imaginary parts. The wave number was chosen $k = \pi/2$ such that the wavelength $2\pi/k$ is of a comparable size to the diameter of the obstacle D since we consider scattering for frequencies in the resonance region. The discretization parameters were chosen as follows: $n = 15$ and $K = 5$. The regularization parameters in are selected as $\alpha_N = \beta_N = \gamma_N = \alpha_0\chi^N, \ \alpha_0 = 0.001, \ \chi = 10/11$. The indexes for the Sobolev space used in the parametrization of impedance functions λ and μ are chosen as $s_\lambda = 1.1$ and $s_\mu = 2.1$, correspondingly. The iterations are terminated according to the Morozov's discrepancy principle with $\tau = 1.001$. The reconstructions are obtained from the far field pattern for 3 incident directions if not stated otherwise.

In the example we consider the case when the imaginary parts $\operatorname{Im}\lambda$, $\operatorname{Im}\mu$ of the surface impedance functions are known. The sought surface impedance functions are chosen as following

$$\lambda = g, \quad \mu = \lambda.$$



$a)$ Reλ, $\text{err}_\lambda^7 = 0.030661$ Reμ, $\text{err}_\mu^7 = 0.072075$

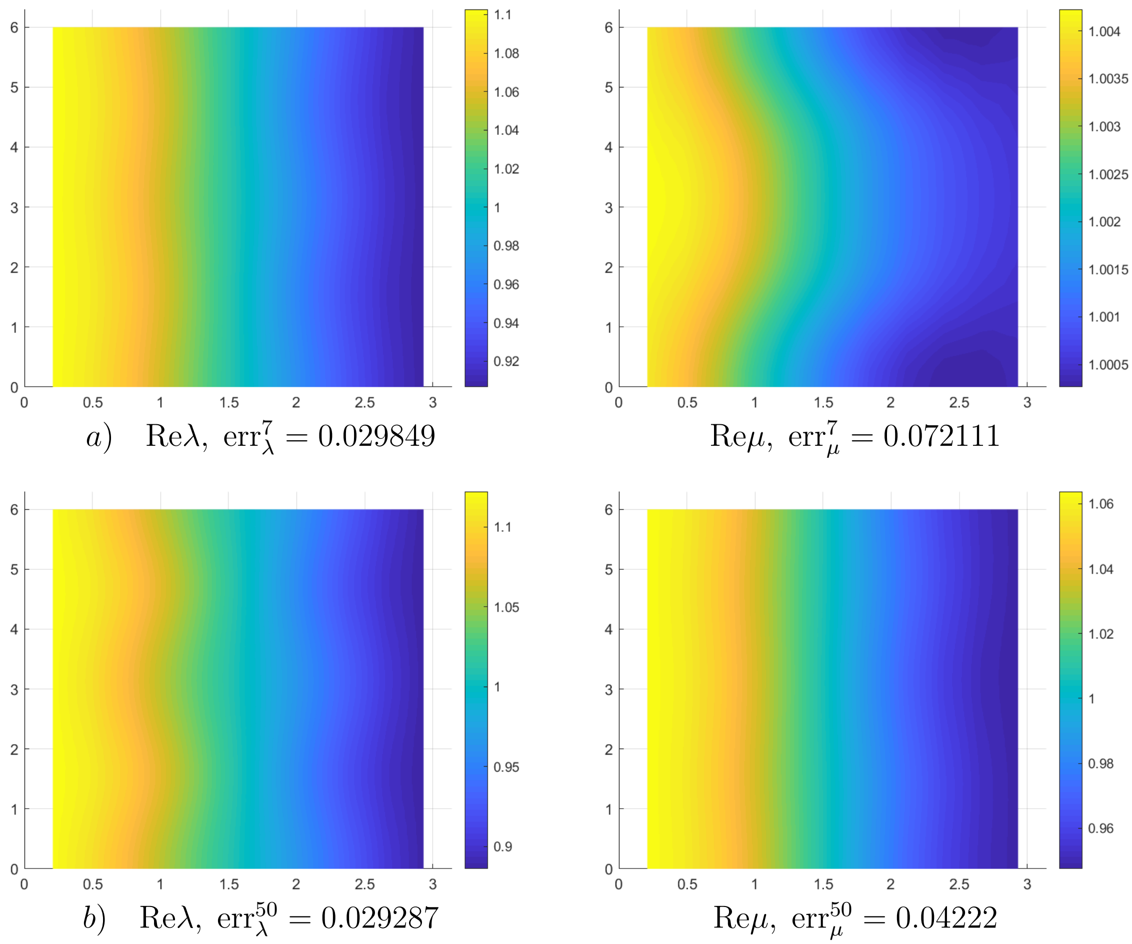


$b)$ Reλ, $\text{err}_\lambda^5 = 0.030138$ Reμ, $\text{err}_\mu^5 = 0.071784$

Figure 4.4: Reconstruction for $\lambda = g$, $\mu = \lambda$ with 3 a) and 6 b) incident waves

We test the inversion algorithm for the initial guess $\lambda^0 = 1, \mu^0 = 1$. As can be seen from the Fig. 4.4a) the impedance function λ is accurately reconstructed whereas μ is affected by the noise more substantially. Increasing the number of incident directions to 6 improves slightly the quality of the reconstruction and reduces the number of iterations, see Fig. 4.4b). Under the plots we included the relative errors for surface impedance functions, defined by $\text{err}_\lambda^N := \|\lambda^N - \lambda\|_{L^2}/\|\lambda\|_{L^2}$, $\text{err}_\mu^N := \|\mu^N - \mu\|_{L^2}/\|\mu\|_{L^2}$, where N is the iteration number.

Fig. 4.4 illustrates results for noise free data after 7 and 50 iteration steps. The reconstruction improves when the number of iteration increases. As can be seen from the figure even in the case of noise free data the second impedance function needs more iterations to find its accurate reconstruction.



$a)$ $\mathrm{Re}\lambda,\ \mathrm{err}_{\lambda}^{7} = 0.029849$ $\mathrm{Re}\mu,\ \mathrm{err}_{\mu}^{7} = 0.072111$

$b)$ $\mathrm{Re}\lambda,\ \mathrm{err}_{\lambda}^{50} = 0.029287$ $\mathrm{Re}\mu,\ \mathrm{err}_{\mu}^{50} = 0.04222$

Figure 4.5: Reconstruction for $\lambda = g$, $\mu = \lambda$ with 3 incident waves from the exact data

All the parameters are kept exactly the same although for the accurate and fast reconstruction of impedance functions we should have decreases the regularization parameters. Employing the algorithm twice, i.e. finding the proper initial guess as a constant ($K = 0$) at the first step, improves the quality of reconstructions and allows to use a less accurate initial guess.

The algorithm provides accurate reconstructions of the first impedance function λ and satisfactory identifications of μ under some restrictions on the surface impedance functions. In general, in the agreement with results Bourgeois et al. (2011); Kress (2018) for two dimensional case, we note that the simultaneous reconstruction of both impedance function is sensitive to noise, especially the identification of the second impedance function. For more details we refer to Ivanyshyn Yaman (2019).

Chapter 5

Discussion and Outlook

In this study we investigated solution of the boundary value problems with second order boundary condition, the so-called generalized impedance boundary condition. In particular, the numerical solution methods for the direct boundary value problems based on boundary integral equation were designed for the Helmholtz equation in $3D$ and the modified Helmholtz equation in $2D$ are developed and their feasibility is confirmed by numerical examples. Analytical solutions are presented for the case of specially chosen boundary shape and impedances. Furthermore, the inverse problems for the shape and surface impedances are investigated and several iterative inversion algorithms are proposed. The proposed inversion methods are efficient from the computational point of view since the solutions of boundary value problems appearing in the classical Newton iteration are replaced by matrix-vector products.

The reconstruction algorithm for surface impedance is presented with deep technical details and its feasibility/limitations are demonstrated by numerical examples, (published Ivanyshyn Yaman (2019)). The methods for reconstruction of coated obstacles, i.e. recovery of impedances and shape of the obstacle are presented. Moreover, the obtained preliminary results for the modified Helmholtz equation, (to be submitted ivanyshyn Yaman and Özdemir) lays down the foundation for numerical solution of the closely related inverse problems of recovering the surface properties and/or the shape of an object from the exterior measurements where the surrounding medium is both conductive and homogeneous. This problem arises in engineering science such as heat conduction and electronics. The investigation of the transmission eigenvalue problem for Maxwell equation (published Cakoni etal

(2018)) opens a new research direction such as direct and inverse problem with GIBC for the Maxwell system.

To conclude, the study shows that the surface impedances or boundary shape can be recovered from finite number of far field patterns and a fixed wave number. The result is encouraging for tackling an important open problem of uniqueness, i.e. whether it is possible to uniquely recover the surface impedance functions with finite number of incident plane waves in three dimensions.

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- Ivanyshyn Yaman O. 2019. "Reconstruction of generalized impedance functions for 3D acoustic scattering", Journal of Computational Physics, 392, 444 – 455.

- Cakoni F., Ivanyshyn Yaman O., Kress R., Le Louër F. 2018. "A boundary integral equation for the transmission eigenvalue problem for Maxwell equation", Math. Methods Appl. Sci., 41(4), 1316–1330.

- Ivanyshyn Yaman O., Özdemir G. "Integral equation methods for modified Helmholtz equations in two dimensions", *(preprint)*

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TÜBİTAK
PROJE ÖZET BİLGİ FORMU

Proje Yürütücüsü:	Dr. Öğr. Üyesi OLHA IVANYSHYN YAMAN
Proje No:	116F299
Proje Başlığı:	İnhomojen Geçirgen Kaplamalı Akustik Cismin Uzak Alan Örüntüsünden Belirlenmesi
Proje Türü:	3001 - Başlangıç AR-GE
Proje Süresi:	24
Araştırmacılar:	
Danışmanlar:	
Projenin Yürütüldüğü Kuruluş ve Adresi:	İZMİR YÜKSEK TEKNOLOJİ ENS. FEN F. MATEMATİK B.
Projenin Başlangıç ve Bitiş Tarihleri:	01/04/2017 - 01/04/2019
Onaylanan Bütçe:	99963.0
Harcanan Bütçe:	44810.0
Öz:	Bu projede geçirgen ince bir malzeme ile kaplanmış bir cismin ve/veya fiziksel özelliklerinin birkaç düzlem dalga ile sabit bir frekansta aydınlatılarak bulunması problemi ele alınmıştır. İteratif regülarize Newton metodu ve doğrusal olmayan integral denklem tabanlı çeşitli tersini alma algoritmaları önerilmiştir. Sınır değer problemlerinin klasik Newton iterasyonları ile ortaya çıkan çözümleri matris vektör çarpımları ile yerdeğiştirildiği için hesaplama açısından metodlar etkindir. Sentetik veri üretmek için spektral olarak hassas bir düz problem çözümü detaylı olarak sunulmuştur. Düz problem çözüm metodunun ve kaplama bulma metodunun uygulanabilirliği sayısal örneklerle gösterilmektedir.
Anahtar Kelimeler:	integral denklemler, ters problemler, spektral metodlar, GESK, Frechet türevi, Newton-tipi metodlar
Fikri Ürün Bildirim Formu Sunuldu Mu?:	Hayır
Projeden Yapılan Yayınlar:	1- A boundary integral equation for the transmission eigenvalue problem for Maxwell equation (Makale - Diğer Hakemli Makale), 2- Reconstruction of generalized impedance functions for 3D acoustic scattering (Makale - Diğer Hakemli Makale), 3- Reconstruction of surface impedance functions from the acoustic far field pattern (Bildiri - Uluslararası Bildiri - Sözlü Sunum),