CLASS NUMBER AND THE SPECIAL VALUES OF L-FUNCTIONS

HAYDAR GÖRAL

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We give infinitely many explicit new representations of the class number of imaginary quadratic fields in terms of certain trigonometric series. Our result relies on a hybrid between power series and trigonometric series. Furthermore, in some cases we prove that the special values of Dirichlet L-functions can be evaluated as certain finite sums.

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1. INTRODUCTION

Let K be a number field and $\mathcal{O}_K$ its ring of integers. It is well-known that $\mathcal{O}_K$ is a Dedekind domain. In other words, the fundamental theorem of arithmetic works when we consider the ideals of $\mathcal{O}_K$. However, in general $\mathcal{O}_K$ is not a unique factorization domain (equivalently, a principal ideal domain). Let I_K be the group of all fractional ideals of K and P_K its subgroup of principal ideals. The order of the group I_K/P_K, denoted by h_K, is finite and called the class number of K. Note that $h_K = 1$ if and only if $\mathcal{O}_K$ is a principal ideal domain. Computation of h_K has been studied extensively as it measures how far $\mathcal{O}_K$ is from being a principal ideal domain, hence a unique factorization domain. The number h_K is related to the pole of the Dedekind zeta function $\zeta_K(s)$ of K at 1, and it is called the analytic class number formula, see [7, Chapter 8, Section 2, Theorem 5] or [6, Theorem 10.9].

Now we investigate a particular case, namely when K is an imaginary quadratic field. To illustrate, it is known that the class numbers $h(d)$ of the fields $K = \mathbb{Q}(\sqrt{-d})$ are all 1, where $d = 1, 3, 7$. Gauss conjectured that

$$\lim_{d\to\infty} h(d) = \infty. \tag{1.1}$$

In particular, he anticipated that there are only finitely many d such that $h(d) = 1$, and this is called the class number 1 problem. By Siegel's theorem [5, Chapter 21], which is a lower estimation of the value of Dirichlet L-function at 1, one sees that (1.1) holds. Therefore, Siegel's theorem implies the class number 1 problem. In fact, it known that there are exactly nine values of d such that $h(d) = 1$, see [5, Chapter 21].

Trigonometric series are directly connected with the class number of an imaginary quadratic field and in particular with the special values of L-functions, namely at 1. For this purpose, let

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \Re(s) > 1$$

be the L-function associated to the Dirichlet character χ. Note that actually the sum above converges in $\Re(s) > 0$ if χ is not the trivial character. Let $q > 3$ be a prime with $q \equiv 3 \pmod 4$ and let $h(q) = h_K$ denote the class number of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-q})$. By [5, Section 6] we know the following striking relation between $L(1,\chi)$ and $h(q)$:

$$\frac{\pi h(q)}{\sqrt{q}} = L(1,\chi), \tag{1.2}$$

where $\chi(n) = \left(\frac{n}{q}\right)$ is the Legendre symbol modulo q which is a non-trivial real (quadratic) character. Let

$$G = \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) e^{\frac{2\pi i m}{q}}$$

be the Gauss sum associated to the Legendre symbol. In this case, one has $G = i\sqrt{q}$ [5, Section 2] and by [2, Chapter 8] (which is reminiscent of the inverse Fourier transform) we get that

$$\left(\frac{n}{q}\right) = \frac{1}{\sqrt{q}} \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) \sin\left(\frac{2\pi m n}{q}\right).$$

Hence, by (1.2) the trigonometric series representation

$$h(q) = \frac{1}{\pi} \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{2\pi m n}{q}\right)$$

holds. By the previous formula, the trigonometric series

$$\sum_{n=1}^{\infty} \frac{\sin(2\pi n t)}{n},$$

which is the imaginary part of

$$-\log(1 - e^{2\pi i t}) = -\log(2\sin \pi t) - i\pi\left(t - \frac{1}{2}\right)$$

yields a finite expression for the class number. To illustrate, the argument above yields

$$L(1,\chi) = -\frac{\pi}{q^{\frac{3}{2}}}\sum_{m=1}^{q-1} m\left(\frac{m}{q}\right).$$

By (1.2), we see that $L(1,\chi) > 0$ and the previous formula gives that

$$\sum_{m=1}^{q-1} m\left(\frac{m}{q}\right) < 0. \tag{1.3}$$

It is very interesting that there is no known elementary proof of the inequality (1.3). Dirichlet also obtained another finite representation of $L(1,\chi)$ as

$$L(1,\chi) = \frac{\pi}{(2-(\frac{2}{q}))q^{\frac{1}{2}}}\sum_{m<\frac{q}{2}}\left(\frac{m}{q}\right).$$

This yields that the number of quadratic residues is bigger than the number of non-residues in the first half period. There is no known elementary proof again. Further interactions between class numbers and trigonometric sums can be found in [4].

By exploiting the transition from power series to trigonometric series having the Legendre symbol as coefficients, we obtain infinitely many explicit new representations of the class number of imaginary quadratic fields in terms of trigonometric series. It is somewhat surprising that the series

$$\sum_{n=1}^{\infty}\left(\frac{n}{q}\right)\frac{\cos nt}{n}$$

is a non-zero constant for all but finitely many t in $[0,2\pi]$.

Theorem 1. *Let $q > 3$ be a prime with $q \equiv 3 \pmod 4$ and $\left(\frac{\cdot}{q}\right)$ denote the Legendre symbol modulo q. Let $h(q)$ represent the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-q})$. Then for all t with $0 \le t < 2\pi$ and $t \neq \frac{2\pi m}{q}$ for any $1 \le m \le q-1$, the class number $h(q)$ can be computed as*

$$h(q) = \frac{\sqrt{q}}{\pi}\sum_{n=1}^{\infty}\left(\frac{n}{q}\right)\frac{\cos nt}{n}.$$

Our second theorem is about evaluations of Dirichlet L-functions at the positive integers. For related results on the topic, we refer the reader to [1, 3].

Theorem 2. *Let q be an odd prime number, $\left(\frac{\cdot}{q}\right)$ denote the Legendre symbol modulo q and $k \ge 2$ be a positive integer. Suppose that either k is even* *and* $q \equiv 1 \pmod 4$, *or* k *is odd and* $q \equiv 3 \pmod 4$. *Then in either case,* $L\left(k, \left(\frac{\cdot}{q}\right)\right)$ *can be computed as a finite sum of the form*

$$\pi^k \sum_{j=1}^{k} \frac{a_{k,j}}{q^{j+\frac{1}{2}}} M_j$$

where

$$M_j = \sum_{m=1}^{q-1} m^j \left(\frac{m}{q}\right) \quad \textit{and} \quad a_{k,j} \in \mathbb{Q}.$$

As a corollary to the previous theorem, if $q \equiv 1 \pmod 4$ then

$$L\left(2, \left(\frac{\cdot}{q}\right)\right) = \frac{\pi^2}{q^{5/2}} \sum_{m=1}^{q-1} m^2 \left(\frac{m}{q}\right). \tag{1.4}$$

In particular, if $q \equiv 1 \pmod 4$ then the sum

$$\sum_{m=1}^{q-1} m^2 \left(\frac{m}{q}\right)$$

is always positive. To see equation (1.4), note that

$$S = \sum_{m=1}^{q-1} m\left(\frac{m}{q}\right) = \sum_{m=1}^{q-1} (q-m)\left(\frac{q-m}{q}\right) = q\sum_{m=1}^{q-1} \left(\frac{m}{q}\right) - \left(\frac{-1}{q}\right)S.$$

If $q \equiv 1 \pmod 4$, then we know that $\left(\frac{-1}{q}\right) = 1$. By orthogonality,

$$\sum_{m=1}^{q-1} \left(\frac{m}{q}\right) = 0.$$

This yields that $S = -S$ and hence

$$\sum_{m=1}^{q-1} m\left(\frac{m}{q}\right) = 0$$

as well. Moreover, from equation (3.2) in the proof of Theorem 2, we have $a_{2,2} = 1$. Therefore, we deduce (1.4) as desired.

Remark 1. Under the conditions of Theorem 2, one sees that

$$\frac{L\left(k, \left(\frac{\cdot}{q}\right)\right)}{\pi^k} \in \mathbb{Q}(\sqrt{q}),$$

so it is an algebraic number. Moreover, we see that

$$\sum_{j=1}^{k} \frac{a_{k,j}}{q^{j+\frac{1}{2}}} M_j \sim \pi^{-k}$$

as k tends to infinity.

2. PROOF OF THEOREM 1

Let q be a prime satisfying $q \equiv 3 \pmod 4$. Then, the exact value of the Gauss sum G corresponding to the Legendre symbol modulo q is $i\sqrt{q}$ [5, Page 13]. As the Legendre symbol is a primitive character modulo q, we also know that [2, Theorem 8.19]

$$\left(\frac{n}{q}\right) = \frac{1}{i\sqrt{q}} \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) e^{\frac{2\pi i m n}{q}}$$

for all n. It follows that

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{n}{q}\right) \frac{z^n}{n} &= \frac{1}{i\sqrt{q}} \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) \sum_{n=1}^{\infty} \frac{(z e^{\frac{2\pi i m}{q}})^n}{n} \\ &= -\frac{1}{i\sqrt{q}} \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) \log(1 - z e^{\frac{2\pi i m}{q}}) \end{aligned} \tag{2.1}$$

for $|z| = 1$ and $z \neq e^{\frac{2\pi i m}{q}}$, $1 \leq m \leq q-1$. Inserting $z = e^{it}$ in (2.1) with $0 \leq t < 2\pi$ and $t \neq \frac{2\pi m}{q}$ for any $1 \leq m \leq q-1$, we see that

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{n}{q}\right) \frac{e^{int}}{n} = \frac{1}{i\sqrt{q}} \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) \Bigg(&- \log\left(2\left|\sin\left(\frac{t}{2} + \frac{\pi m}{q}\right)\right|\right) \\ &- \frac{i}{2}\left(t + \pi\left(\frac{2m}{q} - 1\right)\right)\Bigg), \end{aligned} \tag{2.2}$$

as

$$\log(1 - e^{it} e^{\frac{2\pi i m}{q}}) = -\log\left(2\left|\sin\left(\frac{t}{2} + \frac{\pi m}{q}\right)\right|\right) - \frac{i}{2}\left(t + \pi\left(\frac{2m}{q} - 1\right)\right)$$

holds. Now by comparing real parts of both sides of (2.2), one infers that

$$\sum_{n=1}^{\infty} \left(\frac{n}{q}\right) \frac{\cos nt}{n} = -\frac{1}{2\sqrt{q}} \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) \left(t + \pi\left(\frac{2m}{q} - 1\right)\right) \tag{2.3}$$

is satisfied for all t with $0 \leq t < 2\pi$, $t \neq \frac{2\pi m}{q}$ and $1 \leq m \leq q-1$. As we have

$$\sum_{m=1}^{q-1} \left(\frac{m}{q}\right) = 0,$$

equation (2.3) yields that

$$\sum_{n=1}^{\infty} \left(\frac{n}{q}\right) \frac{\cos nt}{n} = -\frac{\pi}{q^{\frac{3}{2}}} \sum_{m=1}^{q-1} m \left(\frac{m}{q}\right). \tag{2.4}$$

As mentioned in the introduction, by [5, Page 8], we know that the value of the L-function of $\left(\frac{\cdot}{q}\right)$ at 1 is given as

$$L\left(1, \left(\frac{\cdot}{q}\right)\right) = -\frac{\pi}{q^{\frac{3}{2}}} \sum_{m=1}^{q-1} m \left(\frac{m}{q}\right).$$

Combining (2.4) with the formula above, one has that

$$L\left(1, \left(\frac{\cdot}{q}\right)\right) = \sum_{n=1}^{\infty} \left(\frac{n}{q}\right) \frac{\cos nt}{n} \tag{2.5}$$

holds for all t with $0 \leq t < 2\pi$, $t \neq \frac{2\pi m}{q}$ and $1 \leq m \leq q-1$. Hence, by (1.2) we deduce that

$$h(q) = \frac{\sqrt{q}}{\pi} \sum_{n=1}^{\infty} \left(\frac{n}{q}\right) \frac{\cos nt}{n}$$

is verified for all t with $0 \leq t < 2\pi$, $t \neq \frac{2\pi m}{q}$ and $1 \leq m \leq q-1$. □

3. PROOF OF THEOREM 2

Since the Legendre symbol is a primitive character modulo q, for all $n \geq 1$ we see that [2, Theorem 8.19]

$$\left(\frac{n}{q}\right) = \frac{1}{G} \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) e^{\frac{2\pi i m n}{q}}$$

where G is the Gauss sum as before. Thus, we obtain that

$$L\left(k, \left(\frac{\cdot}{q}\right)\right) = \sum_{n=1}^{\infty} \left(\frac{n}{q}\right) \frac{1}{n^k} = \frac{1}{G} \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) \sum_{n=1}^{\infty} \frac{e^{\frac{2\pi i m n}{q}}}{n^k}. \tag{3.1}$$

Now let $Sl_k(t)$ be the Clausen function defined by

$$Sl_k(t) = \sum_{n=1}^{\infty} \frac{\cos nt}{n^k}, \quad k \text{ is even}$$

or

$$Sl_k(t) = \sum_{n=1}^{\infty} \frac{\sin nt}{n^k}, \quad k \text{ is odd.}$$

It is known that $Sl_k(t)$ is a polynomial of degree k. For example, some of the Clausen polynomials are given by

$$Sl_1(t) = \frac{\pi}{2} - \frac{t}{2},\ Sl_2(t) = \frac{\pi^2}{6} - \frac{\pi t}{2} + \frac{t^2}{4},\ Sl_3(t) = \frac{\pi^2 t}{6} - \frac{\pi t^2}{4} + \frac{t^3}{12},$$

$$Sl_4(t) = \frac{\pi^4}{90} - \frac{\pi^2 t^2}{12} + \frac{\pi t^3}{12} - \frac{t^4}{48}.$$

Next, suppose that either k is even and $q \equiv 1 \pmod 4$, or k is odd and $q \equiv 3 \pmod 4$. Then the Gauss sum $G = \sqrt{q}$ or $i\sqrt{q}$, respectively. In either case, from (3.1) one deduces that

$$L\left(k, \left(\frac{\cdot}{q}\right)\right) = \frac{1}{\sqrt{q}} \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) Sl_k\left(\frac{2\pi m}{q}\right). \tag{3.2}$$

Claim: $Sl_k(t) = \sum_{j=0}^{k} b_{k,j}\pi^{k-j}t^j$ where $b_{k,j} \in \mathbb{Q}$.

We will prove the claim by induction on k. From the examples above if $k = 1, 2, 3, 4$ then the claim holds. Observe that

$$(Sl_{2k+1}(t))' = Sl_{2k}(t) \ \textit{and} \ (Sl_{2k+2}(t))' = -Sl_{2k+1}(t). \tag{3.3}$$

By (3.3) and induction, we see that

$$Sl_{k+1}(t) = p(t) + c_k$$

where c_k is a constant and $p(t) = \sum_{j=1}^{k+1} d_{k+1,j}\pi^{k+1-j}t^j$ with $d_{k+1,j}$ in $\mathbb{Q}$. Taking $t = \pi$ when k is even, we see that $c_k = \pi^{k+1}h$ for some $h \in \mathbb{Q}$. Similarly, taking $t = \pi$ when k is odd, we conclude again that $c_k = \pi^{k+1}g$ for some $g \in \mathbb{Q}$, as the Riemann zeta function is of the form $\pi^r v$ at the even integers r where $v \in \mathbb{Q}$. So, we have the claim.

Now, by (3.2) and the claim above, we obtain that

$$L\left(k, \left(\frac{\cdot}{q}\right)\right) = \frac{1}{\sqrt{q}} \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) \sum_{j=0}^{k} b_{k,j}\pi^{k-j}(2\pi m/q)^j$$

$$= \pi^k \sum_{j=0}^{k} \frac{2^j b_{k,j}}{q^{j+\frac{1}{2}}} \sum_{m=1}^{q-1} m^j \left(\frac{m}{q}\right)$$
$$= \pi^k \sum_{j=0}^{k} \frac{a_{k,j}}{q^{j+\frac{1}{2}}} M_j$$

where

$$M_j = \sum_{m=1}^{q-1} m^j \left(\frac{m}{q}\right) \quad \text{and} \quad a_{k,j} = 2^j b_{k,j} \in \mathbb{Q}.$$

Finally, by orthogonality

$$M_0 = \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) = 0,$$

and this completes the proof of the theorem. □

Remark 2. From the proofs of Theorem 1 and Theorem 2, one sees that the proofs work for more general characters, namely for the Kronecker symbol. This follows from the exact evaluation of the corresponding Gauss sums and the Kronocker symbol is a real primitive character. Details can be found in [8, Chapter 9].

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Izmir Institute of Technology
Department of Mathematics
35430 Urla, Izmir, Turkey
haydargoral@iyte.edu.tr