KRULL-SCHMIDT PROPERTIES OVER NON-NOETHERIAN RINGS

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ABSTRACT

KRULL-SCHMIDT PROPERTIES OVER NON-NOETHERIAN RINGS

Let R be a commutative ring and C a class of indecomposable R-modules. The Krull-Schmidt property holds for C if, whenever $G_1 \oplus \cdots \oplus G_n \cong H_1 \oplus \cdots \oplus H_m$ for $G_i, H_j \in C$, then $n = m$ and, after reindexing, $G_i \cong H_i$ for all $i \leq n$. The main purpose of this thesis is to investigate Krull-Schmidt properties of certain classes of modules over Non-Noetherian rings. Particularly weakly Matlis domains, strong Mori domains and Marot rings, all of which are among the class of Non-Noetherian rings, are studied. w-weak isomorphism types are defined and the conditions when they coincide for torsionless modules over weakly Matlis domains are discussed. With the help of this comparison, the Krull-Schmidt property of w-ideals of a strong Mori domain is characterized. Also, the same property for overrings of a strong Mori domain is examined. Some useful results for a Marot ring with ascending condition on its regular ideals are obtained. Krull-Schmidt property on regular ideals of such a ring is studied and a characterization is given. Furthermore, the same property is discussed for overrings of a Marot ring.

ÖZET

NOETHER OLMAYAN HALKALAR ÜZERİNDE KRULL-SCHMIDT ÖZELLİKLERİ

R bir değişmeli halka ve C parçalanamaz R-modüllerin bir sınıfı olsun. C için, Krull-Schmidt özelliği sağlanmasının koşulu, $G_i, H_j \in C$ için, $G_1 \oplus \cdots \oplus G_n \cong H_1 \oplus \cdots \oplus H_m$ ise $n = m$ ve yeniden indekslemeden sonra, tüm $i \leq n$ için $G_i \cong H_i$'dir. Bu tezin temel amacı Noether olmayan halkalar üzerindeki belirli modüllerin Krull-Schmidt özelliklerini incelemektir. Özellikle, Noether olmayan halkalar sınıfındaki zayıf Matlis bölgeleri, güçlü Mori bölgeleri ve Marot halkaları çalışılmıştır. w-zayıf izomorfizmaları tanımlanmıştır ve zayıf Matlis bölgeleri üzerindeki burulmasız modüller için çakıştıkları koşullar tartışılmıştır. Bu kıyaslama ile bir Mori bölgesinin w-idealleri için Krull-Schmidt özelliği karakterize edilmiştir. Ayrıca aynı özellik güçlü Mori halkalarının üst halkaları için incelenmiştir. Regüler idealleri üzerinde artan zincir koşulu olan bir Marot halkası için bazı kullanışlı sonuçlar elde edilmiştir. Bu halkaların regüler idealleri için Krull-Schmidt özelliği çalışılmıştır ve bir karakterizasyon verilmiştir. Ayrıca, aynı özellik Marot halkalarının üst halkaları için tartışılmıştır.

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LIST OF SYMBOLS

R	commutative ring with identity
$\tilde{R}$	the integral closure of R
$Q(R)$	the total quotient ring of R
$S^{-1}R$	the localization of R at S
R_P	the localization of R at a prime ideal P
$R_{(P)}$	the regular localization of R at a prime ideal P
$S^{-1}M$	the localization of R-module M at S
M_P	thelocalization of R-module M at a prime ideal P
$M_{(P)}$	the regular localization of an R-module M at a prime ideal P
$J(R)$	the Jacobson radical of R
$\subseteq$	submodule
$\subset$	proper submodule
$\cong$	isomorphic
$\otimes$	tensor product
$\mathbb{Z}$	the ring of integers
$\bigoplus_{i\in I} M_i$	the direct sum of R- modules M_i
$\prod_{i\in I} M_i$	the direct product of R- modules M_i
$\dim(V)$	the dimension of the vector space V
$\mathrm{Ann}_R(X)$	the annihilator of the set X
$\mathrm{Ker}\, f$	the kernel of the map f
$\mathrm{Coker}\, f$	the cokernel of the map f
$\mathrm{Im}\, f$	the image of the map f
$\mathrm{Hom}(M, N)$	all R-module homomorphisms from M to N
$\mathrm{End}(M)$	the endomorphism ring of a module M
$\mathrm{E}(M)$	the injective envelope (hull) of a module M
$t(M)$	the torsion submodule of M
$\mathrm{Ext}^1_R(C, A)$	the set of all equivalence classes of short exact sequences starting with the R-module A and ending with the R-module C

CHAPTER 1

INTRODUCTION

Let R be a commutative ring and C a class of indecomposable R-modules. The Krull-Schmidt property holds for C if, whenever $G_1 \oplus \cdots \oplus G_n \cong H_1 \oplus \cdots \oplus H_m$ for $G_i, H_j \in C$, then $n = m$ and, after reindexing, $G_i \cong H_i$ for all $i \leq n$. This property fails generally for modules over commutative rings, and even the weaker property of cancellation, $A \oplus B \cong A \oplus C \Rightarrow B \cong C$ for R-modules A, B, C, holds only in special situations. Due to that reason, concern about this property is common in both commutative and non-commutative algebra. According to the class C, in each section we give a different name to this property for convenience. We are going to deal with weakly Matlis domains, strong Mori domains and Marot rings which are among the class of non-Noetherian rings.

If the class of indecomposable ideals of R has Krull-Schmidt property, we say that R has the UDI property. In other words, let R be a ring and C the class of indecomposable ideals of R. We say that R has the unique decomposition into ideals (UDI) property if, whenever

$$I_1 \oplus I_2 \oplus \cdots \oplus I_n \cong J_1 \oplus J_2 \oplus \cdots \oplus J_m$$

for $I_i, J_j \in C$, then $n = m$ and, after a possible reindexing, $I_i \cong J_i$ for all $i \leq n$.

In (Goeters & Olberding, 2001), the authors prove that, for Noetherian integral domains, the UDI property is almost local, in the sense that a Noetherian integral domain R has the UDI property if and only if R has at most one non-principal maximal ideal and R has the UDI property locally at every maximal ideal [(Goeters & Olberding, 2001), Theorem 2.8]. They also characterize the UDI property for local Noetherian integral domains in terms of the integral closure, showing that the local domain R has the UDI property if and only if its integral closure $\tilde{R}$ has at most three maximal ideals, and, if $\tilde{R}$ has more than one maximal ideal, the maximal ideals of $\tilde{R}$ stand in certain restrictive relations to the maximal ideal of R.

In (Ay & Klingler, 2011), Ay and Klingler prove the same almost local nature of the UDI property for indecomposable reduced Noetherian rings, and they also characterize the UDI property for local reduced Noetherian rings in terms of the integral closure and its maximal ideals.

In a recent paper of Klingler and Omairi, the same results are proven for any arbitrary indecomposable Noetherian ring (Klingler & Omairi, 2020). They examined the UDI property for arbitrary commutative Noetherian rings, establishing the same almost local nature of the property and giving an example which shows that the local results do not extend to commutative Noetherian rings, in general. Moreover, it has been proven that the UDI property extends to overrings which are finitely generated as modules and arbitrary Noetherian integral overrings (Theorem 3.3 and Theorem 3.4, (Klingler & Omairi, 2020)).

Two torsion-free R-modules G and H are said to be locally isomorphic if $G_M \cong H_M$ for all maximal ideals M of R, they are nearly isomorphic if, for every nonzero ideal I of R, there exists a monomorphism $f : G \to H$ such that $\mathrm{Ann}_R(\mathrm{Coker}(f))$ and I are comaximal, that is, $\mathrm{Ann}_R(\mathrm{Coker}(f)) + I = R$, they are stable isomorphic if $G \oplus R^{(n)} \cong H \oplus R^{(n)}$ for some integer $n > 0$. These isomorphism types are called weak isomorphism types. In (Goeters & Olberding, 2002), the authors compare weak isomorphism types for torsionless modules over an h-local domain (every nonzero nonunit element of R is contained in only finitely many maximal ideals of R, and every prime ideal of R is contained in a unique maximal ideal of R). By using these results, they were able to discuss Krull-Schmidt properties of torsionless modules and ideals over h-local domains [(Goeters & Olberding, 2002) Theorem 3.4]. This study is followed by (Ay Saylam & Klingler, 2019), and the authors compare weak isomorphism types for torsionless modules over a finite character domain R (every nonzero nonunit element of R is contained in only finitely many maximal ideals of R).

Inspired by all these studies mentioned above, we define and discuss w-weak isomorphism types, the UDwI and UDRI properties. We will introduce all the definitions in sequel.

The outline of this thesis is as follows:

In Chapter 2, some basic definitions, results and preliminary notions are given.

In Chapter 3, w-weak isomorphism types are compared for torsionless modules over weakly Matlis domains.

Let R be a domain. R is said to be of finite t-character if every nonzero nonunit of R is contained in only finitely many maximal t-ideals of R. An integral domain R is a weakly Matlis domain if R is of finite t-character and each prime t-ideal of R is contained in a unique maximal t-ideal. Many useful properties of weakly Matlis domains are given in (Anderson & Zafrullah, 1999). In Chapter 3, we compare w-weak isomorphism types, w-versions of weak isomorphism types which are w-locally isomorphism, w-nearly isomorphism, and stable isomorphism for torsionless w-modules over weakly Matlis domains . This comparison between w-weak isomorphism types helps us to discuss the Krull-Schmidt properties over some certain modules in the next chapter. We investigate under what conditions these w-weak isomorphism types for torsionless w-modules over weakly Matlis domains coincide, which helps us to discuss the UDwI property for strong Mori domains which will be introduced in the next chapter.

In Chapter 4, UDwI property is defined and characterized for strong Mori domains.

An integral domain R (with quotient field K) has the unique decomposition into ideals property with respect to the w-operation (abbreviated, the UDwI property) if for any ideals $I_1, \ldots, I_n$, $J_1, \ldots, J_m$ of R with

$$(I_1)_w \oplus \ldots \oplus (I_n)_w \cong (J_1)_w \oplus \ldots \oplus (J_m)_w,$$

then $n = m$ and after reindexing, $(I_i)_w \cong (J_i)_w$ for each index i. For an ideal I of R, $I_w := \bigcup (I :_K J)$, where the union is taken over all finitely generated ideals J of R with $J^{-1} = R$. If $I_w = I$, then I is called a w-ideal. An integral domain R is said to be a strong Mori domain if R satisfies the ascending chain condition on w-ideals (Wang & McCasland, 1997), (Wang & McCasland, 1999). The main purpose of this chapter is to study and to characterize the UDwI property for strong Mori domains. We show that R has the UDwI property if and only if R is a w-PID, or R has a unique non-principal maximal w-ideal M such that R_M has the UDI property (Ay Saylam & Gürbüz & Hamdi, 2022). After examining the properties of a strong Mori domain with a unique non-principal maximal w-ideal, we provide an example. Also it is shown that the UDwI property on R implies the equivalence of w-weak isomorphisms and isomorphism in a class of R-modules. We end the chapter by investigating overrings (rings between R and K) of strong Mori domains with UDwI property.

In Chapter 5, the UDRI property is defined and characterized for Marot rings.

Let R be a ring and C the class of regular ideals of R. R has the unique decomposition into regular ideals (UDRI) property if, whenever

$$I_1 \oplus I_2 \oplus \cdots \oplus I_n \cong J_1 \oplus J_2 \oplus \cdots \oplus J_m$$

for $I_i, J_j \in C$, then $n = m$ and, after a possible reindexing, $I_i \cong J_i$ for all $i \leq n$. We note that regular ideals cannot be written as a sum of two regular ideals, without loss of generality, we assume regular ideals are indecomposable in the class C. Elements of R that are not zero divisors are called regular. An ideal of R is called regular if it contains a regular element. If R is a ring and P is a prime ideal of R, then the regular localization of R at P, is the ring $R_{(P)} = \{a/b \ : \ a, b \in R$ with $b \notin P$, b is regular$\}$. A ring R is said to be a Marot ring if every regular ideal of R is generated by its regular elements. In this chapter, we investigate the UDRI property over a Marot ring whose regular ideals are finitely generated. First, some preliminary results concerning such a ring are gathered. Then we extend a previous result by Goeters and Olberding Theorem 2.8 (Goeters & Olberding, 2001), and prove that R has the UDRI property if and only if R has at most one non-principal maximal ideal M and $R_{(M)}$ has the UDRI property (Ay Saylam & Gürbüz, 2022). Two examples of Marot rings, one satisfying and one not satisfying the property, are provided. In this chapter we define local isomorphism and near isomorphism in a different way. We show that the UDRI property on R implies the equivalence of isomorphism, local and near isomorphism in a class of R-modules. The last section of the chapter is dedicated to the study of the UDRI property on overrings (rings between R and quoetient ring of R) of R.

CHAPTER 2

PRELIMINARIES

This chapter consists of some basic tools about commutative algebra which are used in this thesis. All rings mentioned below are commutative with identity. All the stated propositions and theorems can be found in (Wang & Kim, 2016), (Atiyah & Macdonald, 1969) and (Dummit & Foote 2004).

2.1. Localization

Let R be a ring. A multiplicatively closed subset of R is a subset S of R such that $1 \in S$ and S is closed under multiplication: in other words S is a subsemigroup of the multiplicative semigroup of R. Define a relation $\sim$ on $R \times S$ as follows:

$$(r, s) \sim (r', s') \Leftrightarrow (s'r - sr')u = 0 \text{ for some } u \in S.$$

We can verify that this is an equivalence relation on $S \times R$. Let r/s denote the equivalence classes of (r, s), and let $S^{-1}R$ denote the set of equivalence classes. We put a ring structure on $S^{-1}R$ by defining addition and multiplication of these fractions r/s as:

$$\begin{aligned}(r_1/s_1) + (r_2, s_2) &= (r_1 s_2 + r_2 s_1)/s_1 s_2, \\ (r_1/s_1)(r_2/s_2) &= r_1 r_2/s_1 s_2.\end{aligned}$$

The ring $S^{-1}R$ is called the **ring of fractions** of R. If S is the set of all non-zero-divisors of R, $S^{-1}R$ is called the **total quotient ring** of R, denoted by $T(R)$. If R is a domain, then $T(R)$ is a field, which is called **the quotient field** of R.

Let P be a prime ideal of R. Then $S = R \setminus P$ is a multiplicatively closed subset of R. We write R_P in this case. One can show that R_P is a local ring with unique maximal ideal PR_P.

The construction of $S^{-1}R$ can be carried through with an R-module M in place of the ring R. Define a relation on $\sim$ on $M \times S$ as follows:

$$(m, s) \sim (m', s') \Leftrightarrow t(sm' - s'm) = 0 \text{ for some } t \in S.$$

This is also an equivalence relation. Let m/s denote the equivalence class of the pair (m, s), let $S^{-1}M$ denote the set of such fractions, and make $S^{-1}M$ into an $S^{-1}R$-module with the addition and multiplication defined as:

$$\begin{aligned} (m_1/s_1) + (m_2/s_2) &= (s_2m_1 + s_1m_2)/s_1s_2, \\ (m_1/s_1)(m_2/s_2) &= m_1m_2/s_1s_2. \end{aligned}$$

Proposition 2.1 *Let M be an R-module, N and P submodules of M. The following holds:*

1. $S^{-1}M \cong S^{-1}R \otimes_R M$.
2. $S^{-1}(N + P) = S^{-1}N + S^{-1}P$.
3. $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$.
4. $S^{-1}(M/N) \cong (S^{-1}M)/(S^{-1}N)$.
5. $\mathrm{Ann}(S^{-1}M) \subseteq S^{-1}(\mathrm{Ann}\, M)$, *and if M is finitely generated* $\mathrm{Ann}(S^{-1}M) = S^{-1}(\mathrm{Ann}\, M)$.

2.2. Projective, Injective and Flat Modules

An R-module P is called **projective** if for every R-module epimorpisim $f : N \to M$ and R-module homomorphism $g : P \to M$ there exists an R-homomorphism $h : P \to N$ such that $f \circ h = g$. In other words; the following diagram commutes.

$$\begin{tikzcd} & P \arrow[dl, dotted, "h"'] \arrow[d, "g"] & \\ N \arrow[r, two heads, "f"] & M \arrow[r] & 0 \end{tikzcd}$$

Theorem 2.1 *The following statements are equivalent:*

1. *P is projective.*

2. *Every exact sequence such as $0 \to A \to B \to P \to 0$ is split.*

3. *P is a direct summand of a free module, that is, there exist a module P' and a free module F such that $F \cong P \oplus P'$.*

4. $\mathrm{Hom}_R(P, -)$ *is an exact functor, that is, if* $0 \to A \to B \to C \to 0$ *is an exact sequence, then the sequence* $0 \to \mathrm{Hom}_R(P, A) \to \mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C) \to 0$ *is also exact*

An R-module E is called **injective** if for every R-module monomorphism $f : K \to N$ and R-module homomorphism $g : K \to E$ there exists an R-homomorphism $h : N \to E$ such that $h \circ f = g$. In other words; the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "g"] & N \arrow[dl, dashed, "h"] \\ & E & \end{tikzcd}$$

Theorem 2.2 *The following statements are equivalent:*

1. *E is injective.*

2. *Every exact sequence such as $0 \to E \to B \to C \to 0$ is split.*

3. $\mathrm{Hom}_R(-, E)$ *is an exact functor, that is, if* $0 \to A \to B \to C \to 0$ *is an exact sequence, then the sequence* $0 \to \mathrm{Hom}_R(C, E) \to \mathrm{Hom}_R(B, E) \to \mathrm{Hom}_R(A, E) \to 0$ *is also exact.*

Proposition 2.2 *Let R be an integral domain with quotient field K. Then*

1. *K is an injective R-module.*

2. *Every vector space E over K is an injective R-module.*

Definition 2.1 *An essential extension of a module M is a module E containing M such that if N is a nonzero submodule of E, then $M \cap N \neq 0$. In addition, if $M \neq E$, then E is called a proper essential extension of M.*

Theorem 2.3 *The following statements are equivalent for an extension $M \subseteq E$ of modules:*

1. *E is a maximal essential extension of M.*

2. *E is essential over M and E is injective.*

3. *E is injective, and there is no injective modules E' with $M \subseteq E' \subseteq E$.*

A module E satisfying one of the equivalent conditions in Theorem 2.3 is called the **injective hull** or **injective envelope** of M.

Theorem 2.4 *The following statements hold for an R-module M.*

1. *Any module M has an injective hull.*

2. *Let E be an injective hull of M, and let E' be an injective module containing M. Then there exists a monomorphism $g \colon E \to E'$ such that $g|_M = f$.*

3. *The injective hull of M is uniquely determined up to isomorphism.*

Let us denote the injective hull of a module M by $E(M)$.

Example 2.1 Let R be an integral domain with quotient field K. Then K is injective by Proposition 2.2. Then for any nonzero ideal I of R, K is an essential extension of I. In fact, for any element $x = a/b$ of K, taking a nonzero element $c \in I$, we have $(bc)x = ac \in I$. Therefore, $E(I) = K$.

An R-module M is said to be **flat** if for every monomorphism $f \colon A \to B$, the induced homomorphism $f \otimes 1 \colon A \otimes M \to B \otimes M$ is also a monomorphism.

Theorem 2.5 *The following statements are equivalent:*

1. *M is flat.*
2. *For any ideal I of R, $0 \to I \otimes_R M \to R \otimes_R M$ is exact.*
3. *For any finitely generated ideal I of R, $0 \to I \otimes_R M \to R \otimes_R M$ is exact.*
4. *For any finitely generated ideal I of R, the natural homomorphism $\sigma\colon I \otimes_R M \to IM$ is an isomorphism.*
5. *For any ideal I of R, the natural homomorphism $\sigma\colon I \otimes_R M \to IM$ is an isomorphism.*

2.3. Fractional Ideals

Let R be an integral domain with quotient field Q and A be a nonzero R-submodule of Q. Then A is called a **fractional ideal** of R if there exists a nonzero element $b \in R$ such that $bA \subseteq R$. It is easy to see that every nonzero finitely generated submodule of Q is a fractional ideal.

Let A, B be R-submodules of Q. Define

$$AB = \{\sum_{i=1}^{n} a_i b_i \mid n \text{ is a positive integer, } a_i \in A, b_i \in B\}.$$

which is called the product of submodules A and B.

For an R-submodule of Q, define

$$A^{-1} := \{x \in Q \mid xA \subseteq R\}.$$

If A is a fractional ideal, then A^{-1} is also a fractional ideal.

Definition 2.2 *Let A be an R-submodule of Q. Then A is called an* ***invertible fractional ideal*** *if there exists a submodule B of Q such that $AB = R$. Note that in this case, A is necessarily a fractional ideal. Take $b \in B$ with $b \neq 0$. Then $bA \subseteq R$.*

Theorem 2.6 *Let A be a non-zero R-submodule of Q. Then the following are equivalent:*

1. *A is invertible.*
2. *A is projective.*
3. *A is finitely generated and flat.*
4. *A is finitely generated, and A_P is principal over R_P for every prime ideal P of R.*
5. *A is finitely generated, and A_M is principal over R_M for every maximal ideal M of R.*

2.4. Chain Conditions

An R-module M is called a **Noetherian module** if every ascending chain of submodules of M is stationary, that is, if $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \cdots$ is an ascending chain of submodules of M, there exists a positive integer m such that for all $n \geq m$, then $M_n = M_m$.

A ring R is called Noetherian if R itself is a Noetherian R-module.

Theorem 2.7 *The following statements are equivalent for a ring R:*

1. *R is Noetherian.*
2. *R satisfies the ascending chain condition on ideals, that is, any ascending chain of ideals of R is stationary.*
3. *R has the maximal condition on ideals, that is, every nonempty set of ideals of R possesses a maximal element.*
4. *Every prime ideal of R is finitely generated.*
5. *Every finitely generated R-module is Noetherian.*

Lemma 2.1 *Let M be a Noetherian R-module and $f\colon M \to M$ a homomorphism. Then f is an isomorphism if and only if f is surjective.*

An R-module M is called an **Artinian module** if every descending chain of submodules of M is stationary, that is, if $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \cdots$ is a descending chain of submodules of M, there exists a positive integer m such that for all $n \geq m$, then $M_n = M_m$.

A ring R is called Artinian if R itself is a Artinian R-module.

Theorem 2.8 *(Nakayama's Lemma):) Let I be an ideal of R such that $I \subseteq J(R)$, and M a finitely generated module. If $M = IM$, then $M = 0$.*

Theorem 2.9 *Let I be an ideal of R such that $I \subseteq J(R)$, and N be a submodule of an R-module M.*

1. *If M/N is finitely generated and $M/N = I(M/N)$, then $N = M$.*
2. *If M is finitely generated and $N + IM = M$, then $N = M$.*

2.5. Discrete Valuation Rings

A discrete valuation on a field K is a function $v\colon K \setminus \{0\} \to \mathbb{Z}$ satisfying

(i) v is surjective,

(ii) $v(x + y) = v(x) + v(y)$ for all $x, y \in K \setminus \{0\}$,

(iii) $v(x + y) \geq min\{v(x), v(y)\}$ for all $x, y \in K \setminus \{0\}$ with $x + y \neq 0$.

The subring $\{x \in K \mid v(x) \geq 0\}$ is called the valuation ring of v.

An integral domain R is called a **Discrete Valuation Ring (D.V.R)** if R is the valuation ring of a discrete valuation v on $T(R)$.

The valuation v is often extended to all of K by defining $v(0) = +\infty$, in which case (ii) and (iii) hold for all $a, b \in K$.

Example 2.2 The localization $\mathbb{Z}_{<p>}$ of $\mathbb{Z}$ at any nonzero prime ideal $< p >$ is a D.V.R with respect to the discrete valuation v_p on $\mathbb{Q}$ defined as follows. Every element $a/b \in \mathbb{Q} \setminus \{0\}$ can be written uniquely in the form $p^n(a_1)/b_1$ where $n \in \mathbb{Z}$, $a_1/b_1 \in \mathbb{Q} \setminus \{0\}$ and both a_1 and b_1 are relatively prime to p. Define

$$v_p\left(\frac{a}{b}\right) = v_p\left(p^n\frac{a_1}{b_1}\right) = n.$$

One can easily check that the axioms for a D.V.R. are satisfied.

Theorem 2.10 *The following properties of a ring R are equivalent:*

1. *R is a Discrete Valuation Ring.*

2. *R is a P.I.D. (Principal Ideal Domain) with a unique maximal ideal.*

3. *R is a U.F.D. (Unique Factorization Domain) with a unique (up to associates) irreducible element.*

4. *R is a Noetherian integral domain that is also a local ring whose unique maximal ideal is nonzero and principal.*

5. *R is a Noetherian, integrally closed, integral domain that is also a local ring of Krull dimension 1 i.e. R has a unique nonzero prime ideal.*

2.6. Star Operations

In this section, we will introduce the concept of star operations and mention fundamental definitions and properties. Mainly, we are interested in the most used star operations v, t and w.

2.6.1. Basic Properties of Star Operations

Let R be an integral domain with quotient field K and let $F(R)$ denote the set of all fractional ideals of R. A **star operation** is a mapping $*: F(R) \to F(R)$ satisfying: for any $A, B \in F(R)$ and $0 \neq c \in K$, we have

1. $< c >_* =< c >$ and $(cA)_* = cA_*$;

2. If $A \subseteq B$, then $A_* \subseteq B_*$;

3. $A \subseteq A_*$ and $(A_*)_* = A_*$.

For any fractional ideal A of R, A is called a **fractional $*$-ideal** if $A_* = A$; A is called a **$*$-ideal** of R if A is an ideal of R and $A_* = A$.

Let $*$ be a star operation over R. Then $*$ is said to be of **finite character** if for any fractional ideal A of R,

$$A_* = \bigcup\{B_* \mid B \text{ is taken over all finitely generated fractional subideal of } A\}.$$

Let R be an integral domain with quotient field K. For nonzero fractional ideals I and J of R, let $(I :_K J) := \{x \in K \mid xJ \subseteq I\}$, Define the operations v, t and w as follows :

- $I_v := (I^{-1})^{-1}$;
- $I_t := \bigcup J_v$, where J ranges over the set of finitely generated subideals of I;
- $I_w := \bigcup(I :_K J)$, where the union is taken over all finitely generated ideals J of R with $J^{-1} = R$.

One can check that v, t and w-operations are $*$-operations. We note that t and w are star operations of finite character over R.

Proposition 2.3 *Let $*$ be a star operation over an integral domain R, and let A and B be fractional ideals of R.*

1. $(AB)_* = (A_*B)_* = (AB_*)_* = (A_*B_*)_*$.
2. $(A^{-1})_* = A^{-1}$.
3. $(A_*)^{-1} = A^{-1}$. *Thus, if $A_* = B_*$, then $A^{-1} = B^{-1}$.*

Definition 2.3 *Let $*$ be a star operation over an integral domain R, and let A be a fractional ideal of R. Then A is said to be of* ***$*$-finite type*** *if there is a finitely generated fractional subideal B of A such that $A_* = B_*$.*

Definition 2.4 *A fractional ideal I of R is a* ***maximal $*$-ideal*** *if it is maximal among all proper integral $*$-ideals of R.*

Theorem 2.11 *Let $*$ be a star operation of finite character over an integral domain R.*

1. *If A is a proper $*$-ideal of R, then there exists a maximal $*$-ideal of R containing A.*
2. *Every maximal $*$-ideal of R is prime.*

3. *Let $A \subseteq R$. If $A_* = R$, then $A_P = R_P$ for any maximal $*$-ideal P of R.*

4. *Let A be a fractional ideal of R. Then $A_* = \bigcap (A_*)_P$, where P is taken over all maximal $*$-ideals. In particular, $A = \bigcap A_M$, where M is taken over all maximal ideals.*

2.6.2. $*$-Invertible Fractional Ideals

Let $*$ be a star operation over an integral domain R, and let A be a fractional ideal of R. Then A is said to be **$*$-invertible** if there is a fractional ideal B of R such that $(AB)_* = R$.

Theorem 2.12 *Let A and B be fractional ideals of R.*

1. *A is $*$-invertible if and only if $(AA^{-1})_* = R$.*

2. *If A is a $*$-invertible fractional ideal, then $A_* = A_v$. Thus, every $*$-invertible fractional $*$-ideal is a v-ideal.*

3. *If A is a $*$-invertible fractional ideal and $(AB)_* = R$, then $B_v = A^{-1}$. Thus, every fractional v-ideal B satisfying $(AB)_* = R$ is uniquely determined.*

4. *AB is $*$ invertible if and only if A and B are $*$-invertible.*

Theorem 2.13 *Let $*$ be a star operation of finite character over an integral domain R and let A be a fractional ideal of R. Then the following are equivalent:*

1. *A is $*$-invertible.*

2. *A_* is $*$-invertible.*

3. *A is of $*$-finite type, and A_P is a principal ideal of R_P for any maximal $*$-ideal P of R.*

4. *There exists a finitely generated fractional ideal B of R such that $(AB)_* = R$.*

Corollary 2.1 *Let $*$ be a star operation of finite character over an integral domain R, and let I be a fractional ideal of R.*

1. *If I is $*$-invertible, $(I_*)_M = I_M$ for any maximal $*$-ideal M of R.*

2. *If I is a maximal $*$-ideal of R, then I is $*$-invertible if and only if $II^{-1} \nsubseteq I$.*

Example 2.3 *Let R be an integral domain with quotient field K.*

- *If A is an R-submodule of K, then A^{-1} is a w-module.*

- *If A is a fractional t-ideal of R, then A_t is a w-module. Thus, every t-ideal is a w-ideal.*

Theorem 2.14 *Let R be an integral domain.*

1. *If I is an ideal of R, then $I_w = R$ if and only if $I_t = R$.*

2. *If M is a prime ideal of R, then M is a maximal w-ideal if and only if M is a maximal t-ideal.*

3. *If A is a fractional ideal of R, then A is w-invertible if and only if A is t-invertible.*

Let $Inv_t(R)$ be the set of all fractional t-invertible t-ideals of R. Then $Inv_t(R)$ becomes an Abelian group with identity R, under the t-product $I * J = (IJ)_t$, for each $I, J \in Inv_t(R)$. The factor group $Inv_t(R)/Prin(R)$, where $Prin(R)$ is the group of all principal ideals of R, is called the **t-class group** of R and denoted by $Cl_t(R)$.

Remark 2.1 t-class group of R is trivial if and only if every w-invertible w-ideal is principal. Assume that t-class group of R is trivial. Let I be a w-invertible w-ideal of R. Then I is t-invertible by Theorem 2.14 (3) . Since t is a star operation of finite character, Corollary 2.1 implies that $(I_t)_M = I_M$ for any maximal t-ideal M of R. By Theorem 2.11, we have that $I_t = \bigcap_{M \in t-\mathrm{Max}(R)} (I_t)_M$. So, we have $I_t = \bigcap_{M \in t-\mathrm{Max}(R)} (I)_M$. On the other hand, since I is a w-ideal, and w is a star operation of finite character, we obtain that $I = I_w = \cap_{M \in w-\mathrm{Max}(R)} (I)_M$. Again by Theorem 2.14, $t-\mathrm{Max}(R) = w-\mathrm{Max}(R)$ implies that $I = I_w = I_t$, that is, I is a t-ideal. Thus, I is principal by assumption. For the converse, let I be a t-invertible t-ideal. Then I is w-invertible by Theorem 2.14. Also $I \subseteq I_w \subseteq I_t$ implies that I is a w-ideal. Hence, I is principal, and so t-class group of R is trivial.

2.7. w-Modules

An ideal J of R is called a **Glaz-Vasconcelos ideal**, denoted by $J \in GV(R)$, if J is finitely generated and $J^{-1} = R$. An R-module M is called a w**-module** if $\mathrm{Ext}_R^1(R/J, M) = 0$ for any $J \in GV(R)$. An R-module M is called a GV**-torsion-free module** if whenever $Jx = 0$ for some $J \in GV(R)$ and $x \in M$, we have that $x = 0$.

Proposition 2.4 *The following statements hold for an R-module M:*

1. *If M is a GV-torsion-free module, then $E(M)$ is a w-module.*
2. *Let $\{M_i \mid i \in \Gamma\}$ be a family of modules. Then $\prod_i M_i$ is a w-module if and only $\bigoplus_i M_i$ is a w-module if and only if each M_i is a w-module.*
3. *Every projective module is a w-module.*

Theorem 2.15 *Let M be a GV-torsion-free module. Then M is a w-module if and only whenever $Jx \subseteq M$, where $J \in GV(R)$ and $x \in E(M)$, then $x \in M$.*

Theorem 2.16 *Let M be a GV-torsion-free module. Then M is a w-module if and only whenever $Jx \subseteq M$, where $J \in GV(R)$ and $x \in E(M)$, then $x \in M$.*

Theorem 2.17 *The following statements are equivalent for a GV-torsion-free module M:*

1. *M is a w-module.*
2. *If $0 \to M \to N \to C \to 0$ is an exact sequence in which N is a w-module, then C is GV-torsion-free.*
3. *There exists an exact sequence $0 \to M \to N \to C \to 0$ such that N is a w-module and C is GV-torsion-free.*

Proposition 2.5 *Let M be a GV-torsion-free module. Then for any $x \in M$, $\mathrm{Ann}(x)$ is a w-ideal of R. Thus, for any nonempty subset X of M, $\mathrm{Ann}(X)$ is also a w-ideal of R.*

Proposition 2.6 *Let M be a w-module, N a GV-torsion-free module, and $f\colon M \to N$ a homomorphism. Then $\mathrm{Ker}(f)$ is a w-submodule of M.*

Let M be a GV-torsion-free module. Define

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in GV(R)\}.$$

Then it is easy to see that M_w is a submodule of $E(M)$, which is called the w-envelope (or w-closure) of M.

Theorem 2.18 *Let M be a GV-torsion-free module.*

1. *If N is a w-module with $M \subseteq N$, then $M_w \subseteq N$.*
2. *M is a w-module if and only if $M_w = M$.*
3. *If A is a submodule of M, then $A_w \subseteq M_w$.*
4. *$(M_w)_w = M_w$. Thus, M_w is the smallest w-submodule of $E(M)$.*
5. *If I is an ideal of R, then $(IM)_w = (I_w M_w)_w$.*

Theorem 2.19 *The following statements are equivalent for a module M:*

1. *M is GV-torsion.*
2. *If $0 \to A \to B \to M \to 0$ is an exact sequence in which B is a w-module, then $A_w = B$.*
3. *There exists an exact sequence $0 \to A \to F \to M \to 0$ such that F is a w-module and $A_w = F$.*

Theorem 2.20 *If I is a proper w-ideal of R, then there is a maximal w-ideal M of R such that $I \subseteq M$. Therefore, R has at least one maximal w-ideal, and every maximal w-ideal is prime.*

Theorem 2.21 *An R-module M is a GV-torsion module if and only $M_P = 0$ for any maximal w-ideal P of R.*

Theorem 2.22 *Let P be a prime w-ideal of R, and let M be a GV-torsion-free module. Then $M_P = (M_w)_P$.*

Theorem 2.23 *Let M be a GV-torsion-free module and let A and B be submodules of M. Then $A_w = B_w$ if and only if $A_M = B_M$ for any maximal w-ideal M of R.*

Proposition 2.7 *Let P be prime w-ideal of R, and let N be an R_P-module. Then N as an R-module is a w-module.*

Theorem 2.24 *Let R be an integral domain, and let M be a torsion-free R-module. Then $M_w = \bigcap_{P \in w-\max(R)} M_P$.*

An R-module M is called w**-projective** if M is of w-finite type, that is, $M_w = N_w$ for some finitely generated submodule N of M, and M_P is a free R_P-module for every maximal w-ideal P of R.

Theorem 2.25 *Let R be an integral domain and let I be a finite type fractional ideal of R. Then I is a w-projective module if and only if I is w-invertible.*

2.7.1. w-Exact Sequences

Let M and N be R-modules. A homomorphism $f: M \to N$ is called a $w-$**monomorphism** (respectively, a w**-epimorphism**, a w**-isomorphism**) if $f_M : A_M \to B_M$ is a monomorphism (respectively, an epimorphism, an isomorphism) for any maximal w-ideal M of R.

Theorem 2.26 *Let A and B be GV-torsion-free modules, and let $f: A \to B$ be a homomorphism.*

1. *f can be uniquely extended to a homomorphism from A_w to B_w.*
2. *If f is a w-isomorphism and $g: A_w \to B_w$ is an extension of f, then g is an isomorphism.*

A sequence $A \to B \to C$ of modules and homomorphisms is called a w**-exact sequence** if the sequence $A_M \to B_M \to C_M$ is exact for any maximal w-ideal M of R.
Let M be an R-module. Then M is said to be w**-flat** if for any w-monomorphism $f: A \to B$, the induced sequence $1 \otimes f: M \otimes_R A \to M \otimes_R B$ is a w-monomorphism.

Theorem 2.27 *The following statements are equivalent for an R-module M:*

1. *M is w-flat.*

2. *For any w-exact sequence $0 \to A \to B \to C \to 0$, the induced sequence $0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ is w-exact.*

3. *M is w-locally flat, that is, M_P is a flat R_P-module for any maximal w-ideal P of R.*

4. *The natural homomorphism $M \otimes_R I \to IM$ is a w-isomorphism for any finite type ideal I of R.*

5. *The natural homomorphism $M \otimes_R I \to M$ is a w-isomorphism for any finite type ideal I of R.*

CHAPTER 3

W-LOCALLY ISOMORPHIC TORSIONLESS MODULES OVER WEAKLY MATLIS DOMAINS

In this chapter, we will compare w-weak isomorphism types that are w-locally isomorphism, w-nearly isomorphism, and stable isomorphism for torsionless w-modules over weakly Matlis domains (definitions are given below). This comparison will help us to discuss Krull-Schmidt properties over some certain modules in the next chapter.

We first introduce some definitions and notations. An R-module G is called torsionless if it is isomorphic to a submodule of a finitely generated free module. Any torsionless module over an integral domain is torsion-free. If G is a torsion-free R-module, then the divisible hull KG of G is $K \otimes_R G$. We identify G with its image in KG. The rank of G is the dimension of the K-vector space KG. We write $G^{(n)}$ for a direct sum of n copies of G. We define w-weak isomorphism types which are w-versions of weak isomorphism types which are local, stable and near isomorphism. Two torsion-free R-modules G and H are said to be nearly isomorphic if for every nonzero ideal I of R, there exists a monomorphism $f : G \to H$ such that $\mathrm{Ann}_R(\mathrm{Coker}(f))$ and I are comaximal, that is, $\mathrm{Ann}_R(\mathrm{Coker}(f)) + I = R$, and two torsion-free R-modules G and H are said to be w-nearly isomorphic if for every nonzero w-ideal I of R, there exists a monomorphism $f : G \to H$ such that $\mathrm{Ann}_R(\mathrm{Coker}(f)$ and I are w-comaximal, that is, $(\mathrm{Ann}_R(\mathrm{Coker}(f)) + I)_w = R$. The R-modules G and H are called locally isomorphic if $G_M \cong H_M$ for all maximal ideals M of R, and they are called w-locally isomorphic if $G_M \cong H_M$ for all maximal w-ideals M of R. The R-modules G and H are called stable isomorphic if $G \oplus R^{(n)} \cong H \oplus R^{(n)}$ for some integer $n > 0$. The R-modules G and H are called power isomorphic if $G^{(n)} \cong H^{(n)}$ for some integer $n > 0$.

Let R be an integral domain. R is said to be of finite t-character if every nonzero nonunit of R is contained in only finitely many maximal t-ideals of R. Noetherian domains and Krull domains (i.e., integral domains in which each nonzero ideal is t-invertible) are domains of finite t-character. An integral domain R is a **weakly Matlis domain** if R is of finite t-character and each prime t-ideal of R is contained in a unique maximal t-ideal.

Krull domains are weakly Matlis, and an integral domain of t-dimension one is a weakly Matlis domain if and only if it is of finite t-character. We know that the set of maximal t-ideals and maximal w-ideals coincide. We denote this set by $w - \text{Max}(R)$.

Recall that a GV-torsion-free R-module M (if whenever $Jx = 0$ for some $J \in GV(R)$ and $x \in M$, we have that $x = 0$) is called a w-module if $M_w = M$, where the w-envelope of M is the set given by $M_w = \{x \in M \otimes K \mid Jx \subseteq M \text{ for some } J \in GV(R)\}$.

There are various properties of weakly Matlis domains but we frequently use the following ones in this study:

1. $R_P \otimes R_{P'} = K$, where K is the field of fractions of R, for any distinct maximal w-ideals P and P' [(Anderson & Zafrullah, 1999), Lemma 4.1].

2. If M is a GV-torsion-free torsion R-module, then $M_w \cong \oplus_{P \in w-\text{Max}(R)} M_P$ [(El Baghdadi & Kim & Wang, 2014), Corollary 2.4].

In (Goeters & Olberding, 2002), the authors compare weak isomorphism types which are local, near, and stable isomorphisms for torsionless modules over an h-local domain R (every nonzero nonunit element of R is contained in only finitely many maximal ideals of R, and every prime ideal of R is contained in a unique maximal ideal of R). By using these results, they were able to discuss the Krull-Schmidt properties of torsionless modules and ideals over h-local domains [(Goeters & Olberding, 2002), Theorem 3.4]. This study is followed by (Ay Saylam & Klingler, 2019), and the authors compare these isomorphisms for torsionless modules over a finite character domain R (every nonzero nonunit element of R is contained in only finitely many maximal ideals of R). We will investigate under what conditions these w-weak isomorphism types for torsionless w-modules over weakly Matlis domains coincide. This enables us to characterize the Krull-Schmidt properties of w-ideals over strong Mori domains which will be introduced in the next chapter.

Throughout this chapter, R will denote an integral domain with quotient field K, if otherwise stated.

3.1. Torsionless w-Modules over Weakly Matlis Domains

In this section, we will deal with torsionless w-modules over weakly Matlis domains. The results we achieve will be a step towards the results we aim for.

Definition 3.1 Let $R \subseteq T$ be an extension of integral domains. Then T is called a **w-linked extension** of R if T is a w-module over R. In the case that $R \subseteq T \subseteq K$, we say that T is a **w-overring** of R. Let T be a w-linked extension of R. For any fractional ideal of T, define $\dot{w} : A \mapsto A_w$. By the properties of w-modules, $\dot{w}$ is a w-operation over R, which is the induced finite character star operation over T. An ideal A of T with $A_w = A$ is called a $\dot{w}$ ideal. For any ideal A of T, $A_{\dot{w}} = A_w$.

Lemma 3.1 *Let R be an integral domain with quotient field K, $\mathcal{M}$ a nonempty collection of maximal w-ideals of R and $S := \bigcap_{M\in\mathcal{M}} R_M$. If R is a weakly Matlis domain, then S is a weakly Matlis domain, and*

$$\dot{w}\text{-}Spec(S) = \{PS \mid P \text{ is a prime } w\text{-ideal of } R \text{ such that } P \subseteq M \text{ for some } M \in \mathcal{M}\}.$$

Proof We first show that S is a w-linked overring, that is, $S_w = S$. Let Q be a maximal w-ideal of R. If $Q \in \mathcal{M}$, then $SR_Q = (\bigcap_{M\in\mathcal{M}} R_M)R_Q = R_Q$. If $Q \notin \mathcal{M}$, then $SR_Q = (\bigcap_{M\in\mathcal{M}} R_M)R_Q = \bigcap_{M\in\mathcal{M}} R_M R_Q = K$ since R is a weakly Matlis domain. Hence,

$$\begin{aligned} S_w = \bigcap_{Q\in w\text{-Max}(R)} SR_Q &= \bigcap_{Q\in\mathcal{M}} SR_Q \cap \bigcap_{Q\in w\text{-Max}(R)\setminus\mathcal{M}} SR_Q \\ &= \bigcap_{Q\in\mathcal{M}} R_Q \cap K = \bigcap_{Q\in\mathcal{M}} R_Q = S. \end{aligned}$$

Next, let P be a prime w-ideal of R such that $P \subseteq M$ for some $M \in \mathcal{M}$. Then $PSR_M = PR_M$. Let Q be a maximal w-ideal of R distict from M. If $Q \in \mathcal{M}$, then $PSR_Q = PR_Q = R_Q$ since R is a weakly Matlis domain and M is the unique maximal

w-ideal of R containing P. If $Q \notin \mathcal{M}$, then $PSR_Q = K$. Therefore,

$$\begin{aligned}
(PS)_w = & \bigcap_{Q \in w\text{-Max}(R)} PSR_Q \\
= & PSR_M \cap \bigcap_{Q \in \mathcal{M} \setminus \{M\}} PSR_Q \cap \bigcap_{Q \in w\text{-Max}(R) \setminus \mathcal{M}} PSR_Q \\
= & PR_M \cap \bigcap_{Q \in \mathcal{M}} R_Q \cap K \\
= & PR_M \cap S.
\end{aligned}$$

We note that $(PS)_w \neq S$ because if $(PS)_w = S$, then $PR_M \cap S = S$ which implies that $PR_M \cap SR_M = SR_M$. Thus, $PR_M = R_M$; a contradiction. Hence, $(PS)_w \neq S$. Since S is a w-linked extension and PS is a prime ideal of S, [(Wang & Kim, 2016), Theorem 7.7.7(2)] implies that $(PS)_{\dot{w}} = (PS)_w = PS$, so PS is a prime $\dot{w}$-ideal of S. Now, for the reverse containment, let Q be a prime $\dot{w}$-ideal of S and put $P := Q \cap R$. It suffices to show $QR_M = PSR_M$ for each maximal w-ideal M of R. If $M \in \mathcal{M}$, then $SR_M = R_M$ and $PSR_M = (Q \cap R)R_M = QR_M$. If $M \notin \mathcal{M}$, then $SR_M = K$ and $PSR_M = PK = QR_M$. Thus, $Q = (PS)_w \neq S$, and hence $Q = (PS)_w = PS$ [(Wang & Kim, 2016), Theorem 7.7.7(2)].

Since for every $M \in \mathcal{M}$, $(MS)_w \neq S$, every maximal $\dot{w}$-ideal Q of S has the form MS for some $M \in \mathcal{M}$ Also, $M \subseteq MS \cap R \subseteq MR_M \cap R = M$ which implies that $M = MS \cap R$. Thus, S is a $(w, \dot{w})$-flat overring of R by [(El Baghdadi & Fontana, 2004), Proposition 4.4 (iv)]. Also, we note that $\dot{w}$-Max(S) = w'-Max(S), where w' is the w-operation on S; to see this, let Q be a maximal $\dot{w}$-ideal of S. Since $\dot{w}$ and w' are two star operations on S such that $\dot{w} \leq w'$, $Q = Q_{\dot{w}} \subseteq Q_{w'} = (Q_{\dot{w}})_{w'} = (Q_{w'})_{\dot{w}}$. Thus, $Q_{w'}$ is a $\dot{w}$-ideal, and hence $Q = Q_{w'}$. Let Q' be a maximal w'-ideal of S such that $Q \subseteq Q'$. Then $Q \subseteq Q' = Q'_{w'} = (Q'_{\dot{w}})_{w'} = (Q'_{w'})_{\dot{w}} = Q'_{\dot{w}}$ implies that $Q = Q'$. Hence, Q becomes a maximal w'-ideal.

Moreover, S is a weakly Matlis domain because it is clear that each element of S is contained in only finitely many maximal $\dot{w}$-ideals, and hence maximal w'-ideals of S by the previous paragraph. To see the second property, let Q be a prime $\dot{w}$-ideal of S. Then $Q = (Q \cap R)S$, where $Q \cap R \subseteq M$ for some $M \in \mathcal{M}$. If $M'S$ is a maximal $\dot{w}$-ideal of S distinct from MS containing Q, then $Q \cap R \subseteq M'S \cap R = M'$; a contradiction. $\square$

Lemma 3.2 *Let R be a weakly Matlis domain with quotient field K, and let G and H torsionless R-modules such that H is a w-module. Then the canonical homomorphism*

$$\mathrm{Hom}_R(G,H)_M \to \mathrm{Hom}_{R_M}(G_M,H_M)$$

is an isomorphism for all maximal w-ideals M of R.

Proof We first note that for torsionless R-modules G and H, the canonical homomorphism

$$\mathrm{Hom}_R(G,H)\otimes K \to \mathrm{Hom}_R(G,HK)$$

is an isomorphism. Let M be a maximal w-ideal of R, and $T := H_M/H$ which is a GV-torsion-free torsion R-module. Consider the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(G,H)_M \xrightarrow{\alpha} \mathrm{Hom}_R(G,H_M) \xrightarrow{\beta} \mathrm{Hom}_R(G,T)_M.$$

Since $\mathrm{Hom}_R(G,H_M)$ is an R_M-module, $\mathrm{Hom}_R(G,H_M)_M \cong \mathrm{Hom}_R(G,H_M)$. Since

$$\mathrm{Hom}_R(G,H_M) \subseteq \mathrm{Hom}_R(G,HK) \cong \mathrm{Hom}_R(G,H)\otimes K,$$

$\mathrm{Hom}_R(G,H_M)/\,\mathrm{Im}(\alpha)$ is a torsion module because for each $f \in \mathrm{Hom}_R(G,H_M)$, there exists a nonzero $r \in R$ such that $rf \in \mathrm{Hom}_R(G,H) \subseteq \mathrm{Hom}_R(G,H)_M = \mathrm{Im}(\alpha)$, which implies that $r(f+\mathrm{Im}(\alpha)) = \mathrm{Im}(\alpha)$. Hence, $\mathrm{Im}(\beta)$ is torsion because

$$\begin{aligned}\mathrm{Im}(\beta) &\cong \frac{\mathrm{Hom}_R(G,H_M)}{\mathrm{Ker}(\beta)}\\ &\cong \frac{\mathrm{Hom}_R(G,H_M)}{\mathrm{Im}(\alpha)}.\end{aligned}$$

Since T is a GV-torsion-free torsion module, (El Baghdadi & Kim & Wang, 2014)[Theorem 2.3 (2)] and (Wang & Kim, 2016), Theorem 6.2.6 imply that $T_w = \oplus_{N\in w\text{-}\mathrm{Max}(R)} T_N$. By [(Anderson & Chun, 2014), Theorem 2.3] and the assumption that R is a weakly Matlis domain,

$$\begin{aligned} t(\mathrm{Hom}_R(G, T_w)) &= t(\mathrm{Hom}_R(G, \oplus_{N\in w\text{-}\mathrm{Max}(R)} T_N)) \\ &\cong t(\oplus_{N\in w\text{-}\mathrm{Max}(R)} \mathrm{Hom}_R(G, T_N)) \\ &\cong \oplus_{N\in w\text{-}\mathrm{Max}(R)}\, t(\mathrm{Hom}_R(G, T_N)). \end{aligned}$$

Hence,

$$(t(\mathrm{Hom}_R(G, T_w)))_M \cong \oplus_{N\in w\text{-}\mathrm{Max}(R)} (t(\mathrm{Hom}_R(G, T_N)))_M.$$

Let N be a maximal w-ideal of R distinct from M. Then

$$\begin{aligned} (t(\mathrm{Hom}_R(G, T_N)))_M &= t(\mathrm{Hom}_R(G, T_N)) \otimes_R R_M \\ &\cong t(\mathrm{Hom}_R(G, T_N)) \otimes_R R_N \otimes_R R_M \\ &\cong t(\mathrm{Hom}_R(G, T_N)) \otimes K \\ &= 0, \end{aligned}$$

where the first isomorphism follows from the fact that $t(\mathrm{Hom}_R(G, T_N))$ is an R_N-module. Since $T_M = 0$, then $(t(\mathrm{Hom}_R(G, T_M)))_M = 0$. Therefore,

$$(t(\mathrm{Hom}_R(G, T_w)))_M = 0.$$

Since $\mathrm{Hom}_R(G, T) \subseteq \mathrm{Hom}_R(G, T_w)$, we have

$$t(\mathrm{Hom}_R(G, T)) \subseteq t(\mathrm{Hom}_R(G, T_w))$$

which implies that

$$(t(\mathrm{Hom}_R(G, T)))_M \subseteq (t(\mathrm{Hom}_R(G, T_w)))_M = 0,$$

and hence by [(Anderson & Chun, 2014), Theorem 2.3 (3)],

$$t(\mathrm{Hom}_R(G,T)_M) = (t(\mathrm{Hom}_R(G,T)))_M = 0.$$

Since $\mathrm{Im}(\beta)$ is torsion and a subset of $\mathrm{Hom}_R(G,T)_M$,

$$\begin{aligned}\mathrm{Im}(\beta) =& t(\mathrm{Im}(\beta))\\ \subseteq& t(\mathrm{Hom}_R(G,T)_M)\\ =& 0,\end{aligned}$$

which implies that α is an isomorphism since $\mathrm{Im}(\beta) \cong \dfrac{\mathrm{Hom}_R(G,H_M)}{\mathrm{Hom}_R(G,H)_M}$. Therefore,

$$\begin{aligned}\mathrm{Hom}_{R_M}(G_M,H_M) &\cong \mathrm{Hom}_{R_M}(G\otimes_R R_M, H_M)\\ &\cong \mathrm{Hom}_R(G, \mathrm{Hom}_{R_M}(R_M,H_M))\\ &\cong \mathrm{Hom}_R(G,H_M)\\ &\cong \mathrm{Hom}_R(G,H)_M.\end{aligned}$$

□

Lemma 3.3 *Let R be a weakly Matlis domain and $S := \bigcap_{M\in\mathcal{M}} R_M$, where $\mathcal{M}$ is a nonempty collection of maximal w-ideals of R. Then for torsionless R-modules G and H with H a w-module, we have the isomorphism*

$$[\mathrm{Hom}_R(G,H)\otimes_R S]_Q \cong [\mathrm{Hom}_S(GS,HS)]_Q$$

for every maximal $\dot{w}$ ideal Q of S.

Proof It suffices to show that the canonical map

$$\phi_M : (\mathrm{Hom}_R(G,H)\otimes_R S)_Q \to (\mathrm{Hom}_S(GS,HS))_Q$$

is an isomorphism for each maximal $\dot{w}$-ideal Q of S. Let Q be a maximal $\dot{w}$-ideal of S. Then by Lemma 3.1, $Q = MS$ for some $M \in \mathcal{M}$. Also, note that $S_{MS} = S_Q = R_M$. Therefore,

$$\begin{aligned}
(\mathrm{Hom}_R(G, H) \otimes_R S)_Q &\cong \mathrm{Hom}_R(G, H) \otimes_R S \otimes_S S_Q \\
&\cong \mathrm{Hom}_R(G, H) \otimes_R S \otimes_S S_{MS} \\
&\cong \mathrm{Hom}_R(G, H) \otimes_R R_M \\
&\cong \mathrm{Hom}_{R_M}(G_M, H_M) \\
&\cong \mathrm{Hom}_{S_{MS}}(GS_M, HS_M) \\
&\cong \mathrm{Hom}_S(GS, HS) \otimes_S S_{MS} \\
&\cong (\mathrm{Hom}_S(GS, HS))_Q,
\end{aligned}$$

□

Lemma 3.4 *[(Anderson & Zafrullah, 1999), Theorem 3.3] Let R be an integral domain of finite t-character and I an ideal of R. Then I_w is of finite type if and only if I_M is a finitely generated ideal of R_M for every maximal w-ideal M of R.*

Proof Take any nonzero element $x \in I$. Since R is of finite t-character, x is contained in only finitely maximal w-ideals, say $P_1, \ldots, P_n$. Let M be any maximal w-ideal of R different from P_i for $i = 1, 2, \ldots, n$. Then $xR_M = R_M$ implies $IR_M = R_M$. By assumption, there exists a finite subset A contained in I which generates I_{P_i}. Let X be the finite set $A \cup \{x\}$ and J the ideal of R which is generated by X. By construction of X, we have $I_{P_i} = J_{P_i}$. Hence, by [(Wang & Kim, 2016), Theorem 7.2.11], $I_w = J_w$ that is I_w is of finite type. The converse part is true without the condition that R is of finite t-character. To see that, if I is of w-finite type, then there exists a finitely generated subideal J of I such that $I_w = J_w$. Since $(I_w)_M = I_M$ for every w-maximal ideal M, the result is clear. □

Let us recall the definition of w-projective module. An R-module M is called ***w*-projective** if M is of w-finite type, that is, $M_w = N_w$ for some finitely generated submodule N of M, and M_P is a free R_P-module for every maximal w-ideal P of R.

Lemma 3.5 *Let R be a finite t-character domain and G a torsionless w-module. If G_P is free R_P-module for all maximal w-ideal P of R, then G is a w-projective module.*

Proof If G is a torsionless R-module, there is an embedding $\phi : G \to R^{(n)}$ for some positive integer n. If G_P is free for all maximal w-ideals P of R, then G_P is finitely generated for all $P \in w\text{-Max}(R)$. Since G is a w-module, each coordinate of $\phi(G)$ is a w-ideal by [(Wang & Kim, 2016), Proposition 6.1.13]. Since R is of finite t-character, localization of coordinates of $\phi(G)$ are proper in only finitely many maximal w-ideals of R and are finitely generated. Hence, by Lemma 3.4, each coordinate is a w-finite type module and this yields G is a w-projective R-module [(Wang & Kim, 2016), Theorem 6.7.21]. $\square$

Proposition 3.1 *Let R be a weakly Matlis domain and G a torsionless w-module with rank$(G) = 2$. If G_P is a free R_P-module for all maximal w-ideal P of R, then $G \cong R \oplus N$ for some R-module N.*

Proof Since G is a torsionless module of rank two, there is a map $\varphi = (f, g) : G \to R \oplus R$ such that the cokernel $C := \text{Coker}(\varphi)$ is a torsion R-module, and hence $I := \text{Ann}(C) \neq 0$. For all maximal w-ideals P of R such that $I \nsubseteq P$, the map $\varphi_P = (f_P, g_P)$ is surjective because $C_P = 0$. Since R is of finite t-character, I is contained in only finitely many maximal w-ideals $P_1, \dots, P_n$ of R, for which f_{P_i} is not surjective. Since R is a weakly Matlis domain, the map $\tau : \text{Hom}_R(G, R) \otimes_R G \to R$ by $\tau(f, x) = f(x)$ for $f \in \text{Hom}_R(G, R)$ and $x \in G$, is w-surjective because for all maximal w-ideals P of R,

$$
\begin{aligned}
(\text{Hom}(G, R) \otimes_R G)_P &\cong \text{Hom}(G, R) \otimes_R G_P \\
&\cong \text{Hom}_R(G, R) \otimes_R R_P^{(n)} \\
&\cong \oplus_n \text{Hom}_R(G, R) \otimes_R R_P \\
&\cong \oplus_n \text{Hom}_{R_P}(G_P, R_P),
\end{aligned}
$$

and hence $\tau_P : (\text{Hom}_R(G, R) \otimes_R G)_P \to R_P$ is surjective which implies that $(\text{Im}(\tau))_w = R$. Thus, $\text{Im}(\tau) \nsubseteq P$ for all maximal w-ideal P of R. Hence, there exists $y \in \text{Im}(\tau) \setminus P$ such that $y = \tau(h \otimes x) = h(x)$ for some $h \in \text{Hom}_R(G, R)$ and $x \in P$. Since $h(x) \notin P$, it implies that h_P is surjective. In particular, for each $i = 1, \dots, n$, there exists a homomorphism $h_i \in \text{Hom}_R(G, R)$ such that $(h_i)_{P_i}$ is surjective. For each $i = 1, \dots, n$, let $r_i \in \prod_{j \neq i} P_j$, and set $h := r_1 h_1 + \dots + r_n h_n$. Then h_{P_i} is surjective for each $i = 1, \dots, n$. Define $\psi = (f, h) : G \to R \oplus R$. It follows that $\text{Im}(\psi)$ has rank two. As above, the cokernel

$D := \mathrm{Coker}(\psi)$ is a torsion R-module, and hence $J := \mathrm{Ann}(D) \neq 0$. For each maximal w-ideal Q of R not containing J, ψ_Q is surjective since $D_Q = 0$. Let $Q_1, \ldots, Q_t$ be the maximal w-ideals of R containing J. Note that, by choice of f and h, for every maximal w-ideal P of R, at least one of f_P and h_P is surjective. Number the maximal w-ideals $Q_1, \ldots, Q_t$ so that h_{Q_i} is surjective for $i = 1, \ldots, s$ but not for $i = s+1, \ldots, t$. Let $u = (1, 0)$ and $v = (0, 1)$ in $R \oplus R$, and denote by $\bar{u}_i$ and $\bar{v}_i$ the images of u and v in D_{Q_i}. Then for $i = 1, \ldots, s$, there is an element $x_i \in G_{Q_i}$ such that $h_{Q_i}(x_i)$ is unit in R_{Q_i} so that $\psi_{Q_i}(x_i)$ and u generate $R_{Q_i} \oplus R_{Q_i}$ and hence D_{Q_i} is cyclic generated by $\bar{u}_i$. Similarly, for $i = s+1, \ldots, t$, there is an element $x_i \in G_{Q_i}$ such that $f_{Q_i}(x_i)$ is unit in R_{Q_i} so that $\psi_{Q_i}(x_i)$ and v generate $R_{Q_i} \oplus R_{Q_i}$ and hence D_{Q_i} is cyclic generated by $\bar{v}_i$. Choose $r \in Q_1 \cdots Q_s \setminus (Q_{s+1} \cup \ldots \cup Q_t$. We claim that $\bar{u}_i + r\bar{v}_i$ is a generator for D_{Q_i} for each $i = 1, \ldots, t$. For $i = 1, \ldots, s$, the claim follows from Nakayama's Lemma because $\bar{u}_i$ generates D_{Q_i} and $r \in Q_i R_{Q_i}$. For $i = s+1, \ldots, t$, there is an element $x_i \in G_{Q_i}$ such that $f_{Q_i}(x_i)$ is a unit while $h_{Q_i}(x_i)$ is a non-unit in R_{Q_i}; since r is a unit in R_{Q_i}, again it follows from Nakayama's Lemma that $\psi_{Q_i}(x_i) = (f_{Q_i}(x_i), h_{Q_i}(x_i))$ and $u + rv$ generate $R_{Q_i} \oplus R_{Q_i}$ and hence D_{Q_i} is generated by $\bar{u}_i + r\bar{v}_i$. Since $D_P = 0$ for maximal w-ideals P of R not containing J, it follows from the claim that D_w is cyclic, generated by the coset $u + rv + (\psi(G))_w$ in D_w. Considering the exact sequences $0 \to (\mathrm{Im}(\psi))_w \to R \oplus R \to R \oplus R/(\mathrm{Im}(\psi))_w \to 0$, and $0 \to (\mathrm{Im}(\psi))_w \to R \oplus R \to D_w$, it follows that $D_w = R \oplus R/(\mathrm{Im}(\psi))_w$.

Thus, for some scalar $a \in R$, $\beta := v - a(u + rv) \in (\psi(G))_w$. Hence, there is a $J \in GV(R)$ such that $\beta J \subseteq \psi(G)$. Let $j \in J$ and $\alpha := j\beta = \psi(g)$ for some $g \in G$. Then $R(u + rv) + R\alpha = Ru + Rv = R \oplus R$ so that $R(u + rv) + R\alpha = R(u + rv) \oplus R\alpha$ with $R\alpha \cong R$ as R-modules. Finally, composing ψ with the projection map to $R\alpha$ in the direct sum $R(u + rv) \oplus R\alpha$ yields a surjection from G onto $R\alpha$ (because $\psi(g) = \alpha$), and since $R\alpha \cong R$ is projective, so the surjection $G \to R$ splits which implies that $G \cong N \oplus R$ for some R-module N. □

Remark 3.1 Let R be an integral domain with quotient field K. If M is a w-projective GV-torsion-free module of rank 1, then M is isomorphic to a w-invertible ideal of R. First we note that M is torsion-free [(Wang & Kim, 2016), Theorem 6.7.11]. Since M is torsion free, $f\colon M \to M \otimes K$ is injective. This gives M is isomorphic to an R-submodule of K because $dim(M \otimes K) = 1$ implies $M \otimes K \cong K$. Suppose $M \cong Y$ where Y is an R-submodule of K. Then $M_w \cong Y_w$, and since M is w-finite type, $Y_w = X_w$ for some finitely generated submodule $X \subset Y$. Since X is finitely generated, there exists an element $s \in R$ such that $sX \subseteq R$. From the containment $(sX)_w = sX_w \subseteq R$, we get $sY \subseteq sY_w \subseteq R$. Thus, we see that Y is a fractional ideal of R. Over any domain, a fractional ideal is isomorphic to an integral ideal of R, so $M \cong I$, where I is an ideal of R. Since I is w-projective by assumption, I is w-invertible by [(Wang & Kim, 2016), Theorem 7.2.18].

Corollary 3.1 *Let R be a weakly Matlis domain and G a torsionless w-module with rank$(G) \geq 2$. If G_P is free R_P-module for all maximal w-ideal P of R, then G is isomorphic to a direct sum of a free R-module and a w-invertible ideal of R.*

Proof By Proposition 3.1, $G \cong R \oplus G'$ for some torsionless module such that G'_P is free for each maximal w-ideal P of R and $\text{rank}(G') = \text{rank}(G) - 1$. By applying the same argument to G', we obtain that G is isomorphic to a direct sum of a free R-module and a w-invertible ideal of R. □

3.2. Comparision of w-Weak Isomorphism Types

In this section, we will compare w-weak isomorphism types for torsionless w-modules over weakly Matlis domains. Under some assumptions, these isomorphisms coincide, and under some certain conditions they imply even isomorphism.

First let us recall two important theorems.

Theorem 3.1 ((Wang & Kim, 2016),Theorem 6.2.6) *The following statements are equivalent for a module M:*

1. *M is GV-torsion.*
2. *If $0 \to A \to B \to M \to 0$ is an exact sequence in which B is a w-module, then $A_w = B$.*
3. *There exists an exact sequence $0 \to A \to F \to M \to 0$ such that F is a w-module and $A_w = F$.*

In the following theorem, for an R-module M, let Φ_M be the canonical map

$$\Phi_M \colon M \to \bigoplus_{P \in w\text{-}\mathrm{Max}\, R} M_P \text{ defined as } \Phi(m) = (\cdots, m \otimes 1, \cdots), x \in M.$$

Theorem 3.2 ((El Baghdadi & Kim & Wang, 2014), Theorem 2.3) *Let R be an integral domain. The following conditions are equivalent:*

1. *R is a weakly Matlis domain;*
2. *For every GV-torsion-free torsion module A, there exists an exact sequence*

$$0 \longrightarrow A \xrightarrow{\Phi_A} \bigoplus_{P \in w-\mathrm{Max}(R)} A_P \longrightarrow B \to 0.$$

   for some GV-torsion R-module B;

3. *For every nonzero w-submodule L of K, there exists an exact sequence*

$$0 \longrightarrow (K/L) \xrightarrow{\Phi_{K/L}} \bigoplus_{P \in w-\mathrm{Max}(R)} (K/L)_P \longrightarrow B \to 0$$

   for some GV-torsion R-module B;

4. *There exists an exact sequence*

$$0 \longrightarrow (K/R) \xrightarrow{\Phi_{K/R}} \bigoplus_{P \in w-\mathrm{Max}(R)} K/R_P \longrightarrow B \to 0$$

   for some GV-torsion R-module B.

The following two lemmas are the main tools which are used broadly in a series of results to compare w-local, w-near, and stable isomorphism of torsionless w-modules over weakly Matlis domains.

Lemma 3.6 *If R is a weakly Matlis domain, and I is w-ideal of R, then $(R/I)_w$ is a finite direct sum of local rings.*

Proof Let I be a w-ideal of R. We first claim that (R/I) is GV-torsion-free. Suppose that $J(R/I) = 0$ for some $J \in GV(R)$. Then for any $j \in J$ and $x + I \in R/I$, we have that $j(x + I) = I$. This implies $Jx \subseteq I$. Since I is a w-ideal, we must have $x \in I$. Thus if I is a w-ideal of a weakly Matlis domain R, then we have the following exact sequence

$$0 \longrightarrow (R/I) \xrightarrow{\Phi_{R/I}} \bigoplus_{P\in w-\mathrm{Max}(R)} (R/I)_P \longrightarrow B \to 0,$$

where B is a GV-torsion-free module by Theorem 3.2. Thus, by Theorem 3.1, we conclude that $(R/I)_w = \oplus_{P\in w-\mathrm{Max}(R)}(R/I)_P$ Since R is a weakly Matlis domain, I is contained in only finitely many maximal w-ideals of R. Hence $(R/I)_w$ is a finite direct sum local rings.

□

Lemma 3.7 *Let R be a weakly Matlis domain, F a finitely generated free R-module and I a w-ideal of R. If ϕ is an $(R/I)_w$- automorphism of $(F/IF)_w$ such that* $\det(\phi) = 1$*, then ϕ lifts to an automorphism of F.*

Proof Since R is a weakly Matlis domain, $(R/I)_w$ is a finite direct sum of quasilocal rings by Lemma 3.6. Therefore, the result follows from the same technique used in [Chase (1962), Lemma 3.1]. □

Lemma 3.8 *Let $A_1, \cdots, A_n$ be GV-torsion-free R-modules. Then we have*

$$(A_1 \oplus \ldots \oplus A_n)_w = (A_1)_w \oplus \ldots (A_n)_w.$$

Proof Take any element $a \in (A_1 \oplus \ldots \oplus A_n)_w$. Then $x \in (A_1 \oplus \ldots \oplus A_n) \otimes K$ such that $xJ \subseteq A_1 \oplus \ldots \oplus A_n$ for some $J \in GV(R)$. Let $x = (a_1, \ldots, a_n)$ for some $a_i \in A_i \otimes K$. Now $aJ = (a_1, \ldots, a_n)J = (a_1J, \ldots, a_nJ) \subseteq A_1 \oplus \ldots \oplus A_n$ implies that $a_iJ \subseteq A_i$ for each $i = 1, \ldots, n$. Thus, $a_i \in (A_i)_w$ for $i = 1, \ldots, n$, and hence $a \in (A_1)_w \oplus \ldots \oplus (A_n)_w$. For the reverse containment, assume that $(a_1, \ldots, a_n) \in (A_1)_w \oplus \ldots \oplus (A_n)_w$. Then for each $i = 1, \ldots, n$, $a_i \in A_i \otimes K$ and $a_iJ_i \subseteq A_i$ for some $J_i \in GV(R)$. Set $J := J_1 \cdots J_n$. Then $J \in GV(R)$ by [(Wang & McCasland, 1997), Lemma 1.1], and $(a_1, \ldots, a_n)J \subseteq A_1 \oplus \ldots \oplus A_n$. That is, $(a_1, \ldots, a_n) \in (A_1 \oplus \ldots \oplus A_n)_w$. Thus $(A_1)_w \oplus \ldots \oplus (A_n)_w \subseteq (A_1 \oplus \ldots \oplus A_n)_w$. □

Before the next lemma, we note that if M is an R-module and N is a w-submodule of M, one can easily show that M/N is a GV-torsion-free R-module.

Lemma 3.9 *Let R be a weakly Matlis domain, and let F_1 and F_2 be finitely generated free R-modules of the same rank n. Suppose that G is a w-submodule of F_1 with rank n and H is w-submodule of F_2 with rank n. If $(F_1/G)_w \cong (F_2/H)_w$, then there is an isomorphism $\alpha\colon (F_1 \oplus R) \to (F_2 \oplus R)$ such that $\alpha(G \oplus R) = H \oplus R$.*

Proof Since F_1 and F_2 are finitely generated free modules of the same rank, there is an isomorphism $\phi\colon F_1 \to F_2$. So, $F_1/G \cong F_2/\phi(G)$ implies $(F_1/G)_w \cong (F_2/\phi(G))_w \cong (F_2/H)_w$. To prove the lemma, it is enough to assume that there is a rank n free R-module F such that $(F/G)_w \cong (F/H)_w$, where $G, H \subseteq F$. Let $I = \mathrm{Ann}_R(F/G)_w$. Then I is a w-ideal. Since $\mathrm{Ann}(M) = \mathrm{Ann}(M_w)$ for an R-module M, we have that $IF \subseteq G$. Since R is a weakly Matlis domain, I is contained in only finitely many maximal w-ideals, say $P_1, \ldots, P_n$. By [(El Baghdadi & Kim & Wang, 2014), Corollary 2.4], we have the following primary decomposition of the modules , $(F/IF)_w = \oplus_{i=1}^{n}(F/IF)_{P_i}$, $(G/IF)_w = \oplus_{i=1}^{n}(G/IF)_{P_i}$ and $(H/IF)_w = \oplus_{i=1}^{n}(H/IF)_{P_i}$. $(F/G)_w \cong (F/H)_w$ implies $(F/G)_p \cong (F/H)_P$ for every maximal w-ideal P of R. Thus, we have

$$\frac{(F/IF)_w}{(G/IF)_w} \cong \bigoplus_{i=1}^{n}\left(\frac{F}{G}\right)_{P_i} \cong \bigoplus_{i=1}^{n}\left(\frac{F}{H}\right)_{P_i} \cong \frac{(F/IF)_w}{(H/IF)_w}.$$

It is clear that $(F/IF)_w$ is a finitely generated free $(R/I)_w$-module, and we know that $(R/I)_w$ is a direct sum of quasilocal rings. By [(Goeters & Olberding, 2001), Lemma 2.1], there exists an automorphism $\beta\colon (F/IF)_w \to (F/IF)_w$ such that $\beta((G/IF)_w) = (H/IF)_w$. Since $u = \det\beta$ is a unit in $(R/I)_w$, β can be extended to an automorphism β' of $(F/IF)_w \oplus (R/I)_w$ such that $\beta'((G/IF_w)) \oplus (R/I)_w = (H/IF)_w \oplus (R/I)_w$, where $\beta' = \beta \oplus u^{-1}$, u^{-1} is the multiplication map by u^{-1}. We note that $det\beta' = 1$. Let $F' = F \oplus R$, then $F'/IF' = (F \oplus R)/(IF \oplus I)$ implies $(F'/IF')_w = ((F \oplus R)/(IF \oplus I))_w$ by [(Wang & Kim, 2016), Theorem 6.3.2]. Thus, by Lemma 3.7, β' lifts to an automorphism of α of $F \oplus R$ such that $\beta'(((F \oplus R)/(IF \oplus I)))_w = (\alpha(F \oplus R)/(IF \oplus I))_w$. Since $(\alpha(G \oplus R)/(IF \oplus I))_w = \beta'(((G \oplus R)/(IF \oplus I)))_w = ((H \oplus R)/(IF \oplus I))_w$, for any maximal w-ideal, we have $(\alpha(G \oplus R)/(IF \oplus I)_P = (H \oplus R)/(IF \oplus I)_P$ which gives $(\alpha(G \oplus R))_P = (H \oplus R)_P$. Since $G \oplus R \cong \alpha(G \oplus R)$ and G and H are w-modules, so are $\alpha(G \oplus R)$ and $H \oplus R$. Hence $\alpha(G \oplus R) = (H \oplus R)$. □

Lemma 3.10 *Let R be a weakly Matlis domain, and let F be a free R-module of rank n. If G is a w-submodule of F with rank n and H is a torsionless w-module, that is w-*

locally isomorphic to G, then there exists a w-projective w-submodule P of KH such that $(F/G)_w \cong (P/H)_w$.

Proof Since G has rank n, we have $dim(K \otimes G) = dim(K \otimes F) = n$. So KG is an n-dimensional subspace of KF, and hence $KF = KG$. Since F is finitely generated, there exist $x_1, \dots, x_n \in F$ such that $F = Rx_1 + \dots + Rx_n$. For each $x_i/1 \in KF = KG$, so $x_i/1 = g_i/s_i$ for some $g_i \in G, s_i \in R$. Then we can write $s_i x_i = g_i$ since F is torsionless. Let $s = s_1 \cdots s_n$. It is clear that $sF \subseteq G$. Since G and H are w-locally isomorphic, H is a torsionless module of rank n. Thus, there exists an injection of H into F, and composing this with multiplication by s yields an injection of H into G, that is, H is isomorphic to an R-submodule of G, so replacing H by its image under this isomorphism, we can assume $H \subseteq G$. Since H has rank n, we have $KF = KH$, and by a similar argument, there is an element $b \in R$, $bF \subseteq H$. Therefore, $I = \mathrm{Ann}(F/H)_w \neq 0$.

Since R is a weakly Matlis domain, I is contained in only finitely many maximal w-ideals, say $M_1, \dots, M_t$. If $M \neq M_i$, then $F_M = H_M = G_M$ since F/G is finitely generated. On the other hand, for each index i, $1 \leq i \leq t$, there is an isomorphism of R_{M_i}-modules $\phi_i \colon G_{M_i} \to H_{M_i}$. Since $\mathrm{Hom}(G, H) \otimes R_{M_i} \cong \mathrm{Hom}_{R_{M_i}}(G_{M_i}, H_{M_i})$ by Lemma 3.2, ϕ_i corresponds to a sum $f_1 \otimes \frac{r_1}{s_1} + \dots + f_k \otimes \frac{r_k}{s_k}$ where $f_1, \dots, f_k \in \mathrm{Hom}(G, H)$, $r_1, \dots, r_k \in R$, $s_1, \dots, s_k \in R \setminus M_i$. Let $b_i = s_1 \cdot \ldots \cdot s_k \in R \setminus M_i$ so that $b_i \phi_i$ corresponds to $(f_1 \frac{b_i r_1}{s_1} + \dots + f_k \frac{b_i r_k}{s_k}) \otimes 1_{R_{M_i}} = \sigma_i \otimes 1_{R_{M_i}} = b_i \phi_i = \varphi_i$.

Since ϕ_i is an isomorphism and b_i is a unit in R_{M_i}, φ_i is an isomorphism between G_{M_i} and H_{M_i}. Also $\varphi_i = \sigma_i \otimes 1_{R_{M_i}}$, $\sigma_i \in \mathrm{Hom}(G, H)$ implies that $\varphi_i(G) \subseteq H$. Here, we identify G and H with their images in $G \otimes R_{M_i} = G_{M_i}$ and $H \otimes R_{M_i} = H_{M_i}$. Identifying $G_{M_i} \otimes K$ and $H_{M_i} \otimes K$ with $KH = KG = KF$, let $\tilde{\varphi}_i = \varphi_i \otimes 1_K$. Then $\tilde{\varphi}_i$ is an automorphism of KF because φ_i is an isomorphism between two R_{M_i}-modules of rank n. Since F_{M_i} is a free R_{M_i}-module of rank n, so is $\tilde{\varphi}_i(F_{M_i}) = F_i$. Since $\tilde{\varphi}_i(F_{M_i}) = F_i$ and $\tilde{\varphi}_i(G_{M_i}) = H_{M_i}$, $\tilde{\varphi}_i$ induces an isomorphism $F_{M_i}/G_{M_i} \cong F_i/H_{M_i}$. Let $P = F_1 \cap \ldots \cap F_t \cap F$. Since each $F_i \cong R_{M_i}^{(n)}$, P is w-module by [(Wang & McCasland, 1997), Proposition 1.4] and [(Wang & Kim, 2016), Proposition 6.2.18]. For any $M_i \neq M \in w-\mathrm{Max}(R)$, $P_M = F_M$ since $(F_i)_M = \tilde{\varphi}_i((F_{M_i})_M) = \tilde{\varphi}_i(KF) = KF$. For each index i, $1 \leq i \leq t$, $P_{M_i} = F_i$ since $(F_i)_{M_j} = \tilde{\varphi}_i((F_{M_i})_{M_j} = \tilde{\varphi}_i(KF) = KF$ when $i \neq j$ and $(F_i)_{M_i} = \tilde{\varphi}_i(F_{M_i}) = F_i$. Therefore, P_M is a free R_M-module for every maximal w-ideal M of R.

Now, we need to show that P is torsionless. If $M \neq M_i$ is a maximal w-ideal of

R, then we have $H_M = P_M = F_M$ by construction of P. For each index i, $1 \leq i \leq t$, $H_{M_i} = \varphi_i(G_{M_i}) \subseteq \tilde{\varphi}_i(F_{M_i}) = F_i = P_{M_i}$. Therefore, $H \subseteq P$ because P and H are w-modules. We also have $bP \subseteq H$ since $bP_M = bH_M$ and $bP_{M_i} = bF_i = \tilde{\varphi}_i(bF_{M_i}) \subseteq \varphi(G_{M_i}) = H_{M_i}$. Thus, bP is a torsionless module, and $bP \cong P$ implies that P is also torsionless. Hence, by Lemma 3.5, P is a w-projective module. Finally, since R is a weakly Matlis domain, we have the decompositions of the modules $(F/G)_w = \bigoplus_{M \in w-\mathrm{Max}(R)}(F/G)_M$ and $(P/H)_w = \bigoplus_{M \in w-\mathrm{Max}(R)}(P/H)_M$, and this gives the desired isomorphism.

□

Recall the t-class group of R. Let $Inv_t(R)$ be the set of all fractional t-invertible t-ideals of R and $Prin(R)$ be the set all principal fractional ideals of R. Then $Inv_t(R)$ becomes an Abelian group with identity R, under the t-product $I * J = (IJ)_t$, for each $I, J \in Inv_t(R)$. The factor group $Inv_t(R)/Prin(R)$ is called the **t-class group** of R and denoted by $Cl_t(R)$. Note that t-class group of R is trivial if and only if every w-invertible w-ideal is principal by Remark 2.1.

The next proposition will be needed in the next chapter when determining Krull-Schmidt properties of w-ideals of a strong Mori domain. It shows that under some assumptions, w-locally isomorphism implies isomorphism. The assumptions on R with trivial t-class group and one of the modules has a direct summand isomorphic to an ideal of R make a remarkable contribution to this result.

Proposition 3.2 *Let R be a weakly Matlis domain with trivial t-class group. If G and H are w-locally isomorphic torsionless w-modules, and G has a direct summand isomorphic to an ideal of R, then $G \cong H$.*

Proof Since G has a direct summand isomorphic to a nonzero ideal of R, we can assume $G = X \oplus Ig$, where I is an ideal of R and $g \in G$. Since G is torsionless, X is also torsionless. So, there exists a finitely generated free module F' such that $A \subseteq F'$. By the proof of Lemma 3.10, we can write $A \subseteq F' \subseteq KA \subseteq KG$. Define the free module $F = F' \oplus Rg$. Clearly, $G \subseteq F$. Then, there exists a w-projective w-module P such that $(F/G)_w \cong (P/H)_w$ again by Lemma 3.10. Then $P = \tilde{F} \oplus J$, where J is a w-invertible ideal of R, and $\tilde{F}$ is a free R-module by Corollary 3.1. Since the t-class group of R is trivial, J is principal. So, there exists a free module F_2 such that $(F/G)_w \cong (F_2/G)_w$ such that $(F/G)_w \cong (F_2/H)_w$. Since F and F_2 has rank n, $F \cong F_2$, and without loss of generality, we may assume $H \subseteq F$ and $(F/G)_w \cong (F/H)_w$. Now, we have $(F/G)_w \cong (F'/X)_w \oplus (R/J)_w$ by [(Wang & Kim, 2016), Theorem 6.3.2]. Let $I = \mathrm{Ann}(F/G)_w$. Since R is a weakly Matlis domain, following a similar argument as in Lemma 3.9, we get $(F/IF)_w/(G/IF)_w \cong (F/IF)_w/(H/IF)_w$. Thus, by (Goeters & Olberding, 2001), Lemma 2.1, there is an automorphism $\phi\colon (F/IF)_w \to (F/IF)_w$ such that $\phi((G/IF)_w) = (H/IF)_w$. If $u = \det\phi$, then u is a unit in $(R/I)_w$, and we can define a new automorphism $\psi\colon (F/IF)_w \to (F/IF)_w$ such that $\psi = (a, b) = (a, u^{-1}b)$ for all $a \in (F/IF)_w$ and $b \in (R/I)_w$, where u^{-1} is the multiplication map by u^{-1}. Then $\psi((G/IF)_w) = (H/IF)_w$. Since $\det\psi = u \cdot u^{-1} = 1$, by [(Goeters & Olberding, 2001), Lemma 2.2], ψ lifts to an automorphism α of F such that $\psi((F/IF)_w) = (\alpha(F)/IF)_w$. Therefore, $\alpha(G) = H$ by a similar argument as in the proof of Lemma 3.9. □

Lemma 3.11 *Let R be a weakly Matlis domain, I a nonzero ideal of R and n a positive integer. If $(I^n)_w \cong R$, then $I_w^{(n)} \cong R^{(n)}$.*

Proof Assume that $(I^n)_w \cong R$. Then $(I^n)_w = xR$ for some $x \in R$. Hence, I^n is of finite type. Since I^n is w-locally free, I^n is w-projective, and hence I^n is w-invertible by [(Wang & Kim, 2016), Theorem 7.2.18]. Thus, I is w-invertible, and hence $I_w^{(n)}$ is w-projective. By Corollay 3.1, there exist a free module F and a w-invertible ideal J such that $I_w^{(n)} \cong F \oplus J$. Taking the n-th exterior powers of $I_w^{(n)}$ and $F \oplus J$, $(I_w)^n \cong J \cong R$, and hence $I_w^{(n)} \cong R^{(n)}$ by [(Kaplansky, 1952), Lemma 1]. □

Now, we are going to show that given two torsionless w-modules over a weakly Matlis domain with torsion t-class group, w-locally isomorphism implies power isomorphism.

Proposition 3.3 *Let R be a weakly Matlis domain with torsion t-class group. If G and H are w-locally isomorphic torsionless w-modules , then there exists $n > 0$ such that $G^{(n)} \cong H^{(n)}$.*

Proof By Lemma 3.10, there exist w-projective w-modules P_1 and P_2 such that $(P_1/G)_w \cong (P_2/H)_w$. Since P_1 and P_2 are w-projective modules, there exist free R-modules F_1 and F_2 and w-invertible ideals J_1 and J_2 such that $P_1 \cong F_1 \oplus J_1$ and $P_2 \cong F_2 \oplus J_2$ by Corollary 3.1. We note that J_1 and J_2 are w-ideals by [(Wang & Kim, 2016), Proposition 6.1.13]. Let J be a w-invertible w-ideal. Since t-class group of R is torsion, there exists a positive integer k such that $(J^k)_w = R$. Thus, $J^{(k)} \cong R^{(k)}$ by Lemma 3.11. Following this argument, there exists a positive integer k such that $P_1^{(k)}$ and $P_2^{(k)}$ are free R-modules. Set $F := P_1^{(k)}, A := G^{(k)}$ and $B := H^{(k)}$. Since $(P_1/G)_w \cong (P_2/H)_w$, we may assume that $(F/A)_w \cong (F/B)_w$ as in the proof of Lemma 3.9. Put $I := \mathrm{Ann}(F/A)_w$. Then $(F/IF)_w/(A/IF)_w \cong (F/IF)_w/(B/IF)_w$ by the proof of Lemma 3.9. Hence, there exists an automorphism $\phi\colon (F/IF)_w \to (F/IF)_w$ such that $\phi((A/IF)_w) = (B/IF)_w$ by [(Goeters & Olberding, 2001), Lemma 2.1]. Let m be the rank of $(F/IF)_w$ as a finitely generated free $(R/I)_w$-module and $u := \det\phi$. Since u is a unit in $(R/I)_w$, we can define an automorphism $\psi\colon (F^{(m)}/IF^{(m)})_w \to (F^{(m)}/IF^{(m)})_w$ such that $\psi((x_1, x_2 \dots, x_m)) = \left(u^{-1}\phi(x_1), \phi(x_2), \dots, \phi(x_m)\right)$. Then $\det\psi = u^{-m}(\det\phi)^m = 1$. Thus, ψ lifts to an automorphism α of $F^{(m)}$ such that $\psi(F^{(m)}/IF^{(m)}) = (\alpha(F^{(m)})/IF^{(m)})_w$ by Lemma 3.7. We note that $\psi(A^{(m)}/IF^{(m)})_w = (B^{(m)}/IF^{(m)})_w$. Hence, $\alpha(G^{(m)}) = H^{(m)}$ again by a similar argument as in the prooof of Lemma 3.9. □

Now, we are ready to prove one of the main theorems of this chapter. This theorem shows that w-locally isomorphism for two torsionless w-modules coincides with stable isomorphism for weakly Matlis domains with trivial t-class group, and they imply power isomorphism. In the next section, we will provide an example to mention the signifance of these assumptions.

Theorem 3.3 *(Stable Isomorphism) Let R be a weakly Matlis domain with trivial t-class group, and let G and H be rank n torsionless w-modules. The following statements are equivalent :*

(a) *G and H are w-locally isomorphic.*

(b) *$(F_1/G)_w \cong (F_2/H)_w$ for some free R-modules F_1 and F_2 with $G \subseteq F_1 \subseteq KG$ and $H \subseteq F_2 \subseteq KH$.*

(c) *G and H are stably isomorphic.*

(d) *$G \oplus A \cong H \oplus A$ for some finitely generated R-module A.*

Also, the statements $(a) - (d)$ imply

(e) *$G^{(m)} \cong H^{(m)}$ for some $m > 0$.*

Proof $(a) \Rightarrow (b)$ Since the t-class group of R is trivial, every w-projective w-module is free by the proof of Lemma 3.2. So the proof follows from Lemma 3.10 and Lemma 3.1. $(b) \Rightarrow (c)$ is a consequence of Lemma 3.9. $(c) \Rightarrow (a)$ follows from the fact that if M and N are modules over a quasilocal ring S, then $M \oplus S \cong N \oplus S$ implies $M \cong N$ [

(Vasconcelos, 1965), Proposition 1.7]. This proves the equivalence of $(a)-(c)$. $(c) \Rightarrow (d)$ is clear. $(d) \Rightarrow (a)$ is a consequence of that fact finitely generated modules cancel over quasilocal domains [(Estes & Guralnick, 1982), Theorem 2.5]. $(a) \Rightarrow (e)$ follows from Proposition 3.3.

□

Recall that two torsion-free R-modules G and H are said to be w-nearly isomorphic if for every nonzero w-ideal I of R, there exists a monomorphism $f : G \to H$ such that $\mathrm{Ann}_R(\mathrm{Coker}(f))$ and I are w-comaximal, that is, $(\mathrm{Ann}_R(\mathrm{Coker}(f)) + I)_w = R$.

Remark 3.2 Let D be an integral domain with finitely many maximal w-ideals. Then, D is a semilocal domain such that these maximal w-ideals are maximal ideals of D by [(Zafrullah, 2006), Proposition 7]. Let J be a w-invertible w-ideal of D. Then $(JJ^{-1})_w = D$. We have $JJ^{-1} \subseteq (JJ^{-1})_w = D$. If JJ^{-1} is contained in a maximal ideal M of D, then $(JJ^{-1})_w \subseteq M_w = M$ contradicts with the assumption. So, $JJ^{-1} = D$, that is, J is invertible. Since D is semilocal, J is principal. Hence the t-class group of D is trivial.

One of our aims is to determine when w-locally isomorphism implies w-nearly isomorphism for torsionless w-modules over weakly Matlis domains, and we have accomplished our purpose with the next theorem. We note that w-nearly isomorphism implies w-locally isomorphism without the assumption R is a weakly Matlis domain.

Theorem 3.4 *(w-Near Isomorphism) Let R be a weakly Matlis domain, and let G and H be torsionless w-modules of rank n. The following are equivalent :*

(a) *G and H are w-locally isomorphic.*

(b) *$(P_1/G)_w \cong (P_2/H)_w$ for some w-projective w-modules P_1 and P_2 with $G \subseteq P_1 \subseteq KG$ and $H \subseteq P_2 \subseteq KH$.*

(c) *G is w-nearly isomorphic to H.*

Proof $(a) \Rightarrow (b)$ is a consequence of Lemma 3.10. $(b) \Rightarrow (a)$ Suppose $(P_1/G)_w \cong (P_2/H)_w$. Let M be a maximal w-ideal of R. Then, we have $(P_1/G)_M \cong (P_2/H)_M$. Since G and H have rank n, $G \subseteq P_1 \subseteq KG$ and $H \subseteq P_2 \subseteq KH$, $(P_1)_M$ and $(P_2)_M$ have rank n. So, without loss of generality, we can assume that $F/G_M \cong F/H_M$ where F is a finitely generated R_M-module of rank n. Thus, by [(Goeters & Olberding, 2001), Lemma 2.1], we get an isomorphism between G_M and H_M.

$(a) \Rightarrow (c)$ Now, assume that G and H are w-locally isomorphic. Let I be a w-ideal of R. Since R is a weakly Matlis domain, I is contained in finitely many maximal w-ideals, say $M_1, \ldots, M_n$. By Lemma 3.1 maximal $\dot{w}$-ideals of $\tilde{S}$ are of the form $\tilde{S}M_i$, so $\tilde{S}$ is a semilocal domain by [(Zafrullah, 2006), Proposition 7]. By the Remark 3.2, $\tilde{S}$ has trivial t-class group. Consider the $\tilde{S}$-modules $G\tilde{S}$ and $H\tilde{S}$. We claim that $G\tilde{S}$ and $H\tilde{S}$ are w'-locally isomorphic modules. We can write $\tilde{S} = S^{-1}R$, where $S = R \setminus \cup_{i=1}^{n} M_i$, so we have $G\tilde{S} = S^{-1}G$ and $H\tilde{S} = S^{-1}H$. Since $S^{-1}G \otimes (S^{-1}R)_{S^{-1}M_i} \cong S^{-1}G \otimes R_{M_i} \cong S^{-1}R \otimes G \otimes R_{M_i} \cong G_{M_i}$ and $S^{-1}H \otimes (S^{-1}R)_{S^{-1}M_i} \cong S^{-1}H \otimes R_{M_i} \cong S^{-1}R \otimes H \otimes R_{M_i} \cong H_{M_i}$ we conclude that $G\tilde{S}$ and $H\tilde{S}$ are $\dot{w}$-locally isomorphic. Since $\tilde{S}$ has trivial t-class group, by Theorem 3.3, $G\tilde{S} \oplus \tilde{S} \cong H\tilde{S} \oplus \tilde{S}$. Now, we can cancel $\tilde{S}$ from both sides since $\tilde{S}$ is semilocal [(Estes & Guralnick, 1982), Theorem 2.5]. By Lemma 3.3, there exists an embedding $f : G \to H$ such that $f_{M_i}(G_{M_i}) = H_{M_i}$ for all $i = 1, 2, \ldots, n$. Since $(H/\operatorname{Im} f)_{M_i} = 0$ for all i and $\operatorname{Ann}((H/\operatorname{Im} f)_{M_i}) \subseteq (\operatorname{Ann}(H/\operatorname{Im} f))_{M_i}$, we get $(\operatorname{Ann}(H/\operatorname{Im} f))_{M_i} = R_{M_i}$. Consequently, $\operatorname{Ann}\operatorname{Coker} f \nsubseteq M_i$ for each i. Hence, I is w-comaximal with $\operatorname{Ann}\operatorname{Coker} f$.

$(c) \Rightarrow (a)$ Suppose G is w-nearly isomorphic to H, and let M be a maximal w-ideal. Then, there exists a monomorphism $f : G \to H$ such that $(M + \operatorname{Ann}\operatorname{Coker} f)_w = R$. Localizing at M, we get $(M + \operatorname{Ann}\operatorname{Coker} f)_M = R_M$. Since R_M is a local ring with a unique maximal ideal M_M, we must have $(\operatorname{Ann}\operatorname{Coker} f)_M = R_M$. So, $\operatorname{Ann}\operatorname{Coker} f$ is not contained in M. Thus, there exists an element $x \in \operatorname{Ann}\operatorname{Coker} f$ such that $x \notin M$. Since $x \in \operatorname{Ann}\operatorname{Coker} f$, $xH \subseteq \operatorname{Im} f$. Again localizing at M, we have $H_M = xH_M \subseteq (\operatorname{Im} f)_M = \operatorname{Im} f_M \subseteq H_M$, and this yields $\operatorname{Im} f_M = H_M$. Hence, G and H are w-locally isomorphic.

□

Finally, we can give a necessary and sufficient condition on a weakly Matlis domain R to prove when w-weak isomorphism types are equivalent.

Corollary 3.2 *Let R be a weakly Matlis domain with trivial t-class group. Suppose that G and H are torsionless w-modules of the same rank. Then the following are equivalent:*

(a) *G and H are w-locally isomorphic.*

(b) *G and H are w-nearly isomorphic.*

(c) *G and H are stable isomorphic.*

Proof The proofs follow immediately from Theorems 3.3 and 3.4. □

3.3. Examples

In this section, we will provide two examples to point out the importance of the assumptions in Theorems 3.3 and 3.4. Furthermore, for domains that are not weakly Matlis domains, we mention another approach to w-locally isomorphism that asserts something that is generally weaker than w-nearly isomorphism.

Example 3.1 Let D be a w-Dedekind domain, that is, D is strong Mori and D_M is a valuation domain for every $M \in w - \mathrm{Max}(R)$. D is of finite t-character by [(Wang & McCasland, 1999), Theorem 1.9]. Let P be a prime t-ideal of R, and suppose $P \subset M$ for some $M \in t - \mathrm{Max}(R)$. Then P_M is a prime ideal of R_M. Since R_M is a valuation domain, we must have $P_M = M_M$, and this implies $P = M$. Thus, a w-Dedekind domain is a weakly Matlis domain. Let I be non-pincipal w-invertible w-ideal of D. Since I is w-invertible, I and R are w-locally isomorphic. But $I \oplus R \ncong R \oplus R$, that is, I and R are not stable isomorphic by [(Kaplansky, 1952), Lemma 1]. Since I is w-invertible w-ideal but not principal, t-class group of D is not trivial. Thus, the assumption of trivial t-class group in Theorem 3.3 cannot be dropped.

Example 3.2 There exist integral domains which are not weakly Matlis domains for which Theorem 3.4 fails. An example of such a domain is a domain R such that there is a w-invertible w-ideal I such that $I_w = J_w$ for some finitely generated subideal J which J is not two generated. To see that, assume w-locally isomorphic torsionless R-modules are w-nearly isomorphic. Let I be a w-invertible w-ideal. Then I is w-isomorphic to R. Take any non-zero element $a \in I$. Then there exists a monomorphism $f : R \to I$ such that $(Ra + \mathrm{Ann}\,\mathrm{Coker}\, f)_w = R$. Let $b = f(1)$. We claim that $I = (aR + bR)_w$. We show that these two ideals are w-locally equal. Take any $M \in w - \mathrm{Max}(R)$. If $\mathrm{Ann}\,\mathrm{Coker}\, f \nsubseteq M$, we have $I_M = bR_M$, and if $\mathrm{Ann}\,\mathrm{Coker}\, f \subseteq M$, we have $Ra \nsubseteq M$, which implies $I_M = R_M = aR_M$. Thus, $I_w = I = (aR + bR)_w$ by Theorem 2.23, as we desired. Also, we note that R is not a weakly Matlis domain by [(Anderson & Zafrullah, 1993), Theorem 2.2]. Thus, the comparisons in Theorem 3.4 cannot be true if R is not a weakly Matlis domain.

Before our next example, let us recall the following corollary :

Corollary 3.3 ((Wang, 1997),Corollary 2.8) *Let M be a torsion-free module such that*

M is of w-finite type, that is, $M_w = N_w$ for some finitely generated submodule N of M. Let C be a w-module. Then

$$\mathrm{Hom}_R(M, C)_P = \mathrm{Hom}_{R_P}(M_P, C_P)$$

for each $P \in w - \mathrm{Max}(R)$.

Example 3.3 For domains which are not weakly Matlis domains, we mention another approach to w-locally isomorphism which asserts something that is generally weaker than w-nearly isomorphism: For any given torsionless R-modules G and H are w-locally isomorphic if and only if for any given finite set of maximal w-ideals $\{M_1, \ldots, M_n\}$, there is an embedding $f\colon G \to H$ such that $\mathrm{Ann}(\mathrm{Coker}\, f) \nsubseteq M_j$ for all $j \leq n$. Suppose G and H are torsionless R-modules, and H is of w-finite type. Suppose that G is w-locally isomorphic to H, and the set of maximal w-ideals $\{M_1, \ldots, M_n\}$ is given. Since $\mathrm{Hom}(G_{M_j}, H_{M_j}) = [\mathrm{Hom}(G, H)]_{M_j}$, for every $j \in \{1, 2, \ldots, n\}$, there exists $f_j\colon G \to H$ such that $(f_j)_{M_j}$ is an isomorphism. Let $r_i \in \prod_{j \neq i} M_j \setminus M_i$. Set $f := r_1 f_1 + \ldots + r_n f_n$. Note that $(r_i(f_i))_{M_j} \subseteq M_j H_{M_j}$, and $(r_j(f_j))_{M_j}(G_{M_j}) = r_j H_{M_j} = H_{M_j}$ since r_j is a unit in R_{M_j}. So, we can define a surjective homomorphism

$$\varphi_j\colon \frac{G_{M_j}}{M_j G_{M_j}} \to \frac{H_{M_j}}{M_j H_{M_j}}$$

where $\varphi_j(x + M_j G_{M_j}) = f_{M_j}(x) + M_j H_{M_j}$. Since $f_{M_j}(G_{M_j}) + M_j H_{M_j} = H_{M_j}$, and H_{M_j} is finitely generated, by Nakayama's Lemma we obtain that $f_{M_j}(G_{M_j}) = H_{M_j}$. So f_{M_j} is surjective. Since G and H are w-locally isomorphic, they have the same rank. Thus f_{M_j} is an isomorphism, and so f is a monomorphism. Consequently, $\mathrm{Ann}(\mathrm{Coker}\, f) \nsubseteq M_j$. The converse part is clear.

CHAPTER 4

UNIQUE DECOMPOSITIONS INTO *W*-IDEALS FOR STRONG MORI DOMAINS

A commutative ring R has the unique decompositions into ideals (UDI) property if for any indecomposable ideals $I_1, \ldots, I_n, J_1, \ldots, J_m$ of R, such that

$$I_1 \oplus \ldots \oplus I_n \cong J_1 \oplus \ldots \oplus J_m,$$

then $n = m$ and after reindexing, $I_i = J_i$ for each index i. In (Goeters & Olberding, 2001), the UDI property has been classified for Noetherian integral domains. Goeters and Olberding showed that a Noetherian integral domain R has the UDI property if and only if R is a PID or R has exactly one non-principal maximal ideal M and R_M has the UDI property. Also, some characterizations of Noetherian local domains with the UDI property have been investigated in [(Goeters & Olberding, 2001), Theorem 3.2]. In (Ay & Klingler, 2012,A), the UDI property has generalized for reduced Noetherian rings, and recently in (Klingler & Omairi, 2020), Klingler and Omairi examined the UDI property for arbitrary commutative Noetherian rings, establishing the same almost local nature of the property and giving an example which shows that the local results do not extend to commutative Noetherian rings in general. Moreover, in [(Klingler & Omairi, 2020), Theorems 3.3 and 3.4], it was proven that the UDI property extends to overrings which are finitely generated as modules and which are arbitrary Noetherian integral overrings.

We first introduce the definitions, basic concepts and main theorems that will be convenient. Throughout this chapter, R will denote an integral domain with quotient field K. Let us recall the following definitions. A ring S is said to be an overring of R if $R \subseteq S \subseteq K$. An ideal J of R is called a **Glaz-Vasconcelos ideal**, denoted by $J \in GV(R)$, if J is finitely generated and $J^{-1} = R$. A GV-torsion-free R-module is called a w-module if whenever $Jx \subseteq M$ $(J \in GV(R), x \in M \otimes K)$, then $x \in M$. I is called a w-ideal if I is a w-module over R. A module M is said to be of w-**finite type** if M is a w-module and

$M = N_w$, where N is a finitely generated submodule of M.

In (Wang & McCasland, 1997) and (Wang & McCasland, 1999), Wang and McCasland introduced the concept of strong Mori domains; an integral domain R is said to be a **strong Mori domain** if R satisfies the ascending chain condition on w-ideals. A module M is said to be a **strong Mori module** if M satisfies ascending chain condition on w-submodules. The following theorems and propositions are mainly used in our proofs.

Theorem 4.1 *((Wang & McCasland, 1997), Theorem 4.3) The following are equivalent for a domain R:*

1. *R is a strong Mori domain.*

2. *Each w-ideal is of w-finite type.*

3. *Each prime w-ideal is of w-finite type.*

Proposition 4.1 *((Wang & McCasland, 1997), Proposition 4.6) If R is a strong Mori domain, then R_P is Noetherian for every maximal w-ideal P of R.*

Theorem 4.2 *((Wang & McCasland, 1997), Theorem 4.4) For a w-module M the following are equivalent:*

1. *M is a strong Mori module.*

2. *Each w-submodule of M is of finite type.*

3. *M and each prime w-submodule of M are of finite type.*

Proposition 4.2 *((Wang & McCasland, 1997), Theorem 4.5) A domain R is a strong Mori domain if and only if every finite type w-module over R is a strong Mori module.*

It is worth noting that a strong Mori domain need not be Noetherian; the polynomial ring $R = F[X_1, X_2, \ldots]$ in countably many indeterminates over any field F provides us an example of a strong Mori domain which is not Noetherian. In fact, a strong Mori domain with Krull dimension one is Noetherian.

The main purpose of this chapter is to study and to characterize the UDwI property (Krull-Schmidt property of w-ideals) for strong Mori domains. We show that if a domain R has the UDwI property, then R has at most one non-principal maximal w-ideal. For that reason, we examine the features of such domains. Then we continue with investigating overrings of strong Mori domains with UDwI property.

4.1. UDwI Property for Strong Mori Domains

We say that an integral domain R has the unique decomposition into ideals property with respect to the w-operation (abbreviated, UDwI property) if for any ideals $I_1, \ldots, I_n$, $J_1, \ldots, J_m$ of R with

$$(I_1)_w \oplus \ldots \oplus (I_n)_w \cong (J_1)_w \oplus \ldots \oplus (J_m)_w,$$

then $n = m$ and after reindexing, $(I_i)_w \cong (J_i)_w$ for each index i.

The trivial examples of integral domains with the UDwI property are PIDs and Noetherian domains with the UDI property. Also, strong Mori domains with dimension one which have UDwI property satisfy UDI property because the d (identity operation) and w operations are the same on one dimensional Noetherian domians (Mimouni, 2005).

We start by an important tool. We note that this proposition holds for any integral domain R.

Proposition 4.3 *Let R be an integral domain with the UDwI property. Then at most one maximal w-ideal of R is non-principal.*

Proof Suppose that P_1 and P_2 are two distinct maximal w-ideals of R. Then $(P_1+P_2)_w = R$ implies that $P_1 \cap P_2 = P_1P_2$, and $P_1 \cap P_2$ is also a w-ideal. Let us recall that for any GV-torsion-free R-module M, we have that $M_w = \bigcap_{P \in w-\mathrm{Max}(R)} M_P$. By using this equality for $P_1 \oplus P_2$ and $R \oplus (P_1P_2)$, we obtain $P_1 \oplus P_2 \cong R \oplus P_1P_2$. Since R has the UDwI-property, P_1 or P_2 must be isomorphic to R. Hence R has at most one non-principal maximal w-ideal. $\square$

We recall from (El-Baghdadi, 2010) that an integral domain R is called a w-principal ideal domain (for short, w-PID) if every w-ideal is principal. The polynomial ring $F[X, Y]$ in two indeterminates over any field F is an example of a w-PID which is not a PID [(El-Baghdadi, 2010), Theorem 2.5].

Lemma 4.1 *((Wang & McCasland, 1999), Lemma 1.5) Let M be a torsion-free R-module, and let A, B be submodules of M. Then $(A + B)_w = (A_w + B_w)_w$.*

Lemma 4.2 *Let R be a strong Mori domain. Then the following statements hold.*

1. *R is a w-PID if and only if every maximal w-ideal of R is principal.*

2. *For every principal maximal w-ideal N of R, R_N is a DVR.*

Proof (1) Assume that each maximal w-ideal of R is principal. To show that R is a w-PID, it is enough to show that for each nonzero element a, b of R, $(aR + bR)_w$ is principal by Lemma 4.1 If $(aR + bR)_w = R$, then there is nothing to prove. If $(aR + bR)_w = R$ is not cyclic, then it is contained in a maximal w-ideal, say P. Then by assumption $P = cR$ for some $c \in R$. Since $aR + bR \subseteq (aR + bR)_w$, $a = ca_1$, $b = cb_1$ for some $a_1, b_1 \in R$. Note that aR and bR are contained in a_1R and b_1R, respectively. Since R is a domain, these containments are proper. Also note that if $(a_1R + b_1R)_w$ is principal, then $(aR + bR)_w$ is principal because of the equalities $a = ca_1, b = cb_1$, so that $(a_1R + b_1R)_w$ is not principal either. Iterating this process, we obtain two sequences a_nR and b_nR which ascend properly. Thus, we get a contradiction. Hence, by induction this is equivalent to every w-ideal of finite type being principal.

(2) Let N be a principal maximal w-ideal of R. Then R_N is a Noetherian local domain whose unique maximal ideal is principal. Hence, R_N is DVR by Theorem 2.10. □

Recall from the previous chapter that an integral domain R is a weakly Matlis domain if R is of finite t-character (every nonzero nonunit of R is contained in only finitely many maximal t-ideals of R) and each prime t-ideal of R is contained in a unique maximal t-ideal.

Lemma 4.3 *Let R be a strong Mori domain with a unique non-principal maximal w-ideal M. Then the following statements hold.*

1. *R is a weakly Matlis domain.*

2. *Every w-ideal of R not contained in M is principal.*

3. *Every w-invertible w-ideal of R is principal, that is, the t-class group of R is trivial.*

Proof (1) First note that R is of finite t-character by [(Wang & McCasland, 1999), Theorem 1.9]. Let P be a nonzero prime t-ideal of R contained in a principal maximal

t-ideal $N = aR$. Since R_N is a DVR by Lemma 4.2, prime ideals and maximal ideals of R_N coincide. Since PR_N is a prime ideal of R_N, and R_N is local, we must have $PR_N = NR_N$ which implies that $P = N$. Otherwise, P is contained in M exclusively, and it follows that R is a weakly Matlis domain.

(2) Let I be a nonzero w-ideal of R such that $I \nsubseteq M$. Since R is a weakly Matlis domain by part (1), I is contained in only finitely many maximal w-ideals which are all principal. Say $I \subseteq N_1, \ldots, N_t$, where $N_i = a_iR$ for all $i = 1, \ldots, t$. Since R_{N_i} is a DVR, $IR_{N_i} = a_i^{k_i}R_{N_i}$ for some $k_i > 0$. Set the ideal $J := (a_1^{k_1} \cdots a_t^{k_t})R$. Since J is principal, $J_w = J$. Also note that $J_P = I_P$ for every maximal w-ideal P of R. Recall that $M_w = \cap_{P \in w-\mathrm{Max}(R)} M_P$ holds for any GV-torsion-free R-module M. Hence, $I = I_w = J_w = J$ implying that I is principal.

(3) Let I be a w-invertible w-ideal of R. Then $(II^{-1})_w = R$ implies that there exists an element $q \in I^{-1}$ such that $qI \nsubseteq M$. Since $(qI)_w = qI_w = qI$, we can say that qI is also a w-ideal. Thus from part (2), qI is a principal ideal. Since $qI \cong I$, we conclude that I is also principal.

□

Lemma 4.4 *Let R be a strong Mori domain with a unique non-principal maximal w-ideal M. If G and H are torsion-free R-modules of finite type such that $G_M \cong H_M$, then G and H are w-locally isomorphic.*

Proof Since G and H are of w-finite type, there exist finitely generated submodules G', H' of G and H, respectively, such that $G = G'_w$, and $H = H'_w$. Then for any maximal w-ideal P, $G_P = (G')_P$ implies that G_P and H_P are finitely generated R_P-modules. Let N be a principal maximal w-ideal of R. Then R_N is a PID and G_N, and H_N are finitely generated torsion-free R_N-modules. We note that G and H have the same rank. Since every finitely generated torsion-free module over a PID is free, G_N and H_N are free modules with the same rank. Hence, $G_N \cong H_N$. Therefore, by assumption, G and H are w-locally isomorphic. □

From the previous chapter, without any restriction we know that if G and H are w-nearly isomorphic w-modules, then G and H are w-locally isomorphic by Theorem 3.4. Also, if R is weakly Matlis domain with trivial t-class group, then w-locally isomorphism implies nearly w-isomorphism again by Theorem 3.4. Since a strong Mori domain with a unique non-principal maximal w-ideal is a weakly Matlis domain with trivial t-class group by Lemma 4.3, the following lemma is immediate from Lemma 4.4 and Theorem 3.4. However, we will prove it with a new approach. Before proving the result, we state a proposition and a lemma which will be useful in our proof.

Proposition 4.4 *((Wang, 1997), Proposition 2.1) Let M be a torsion-free module and C a w-module over R. Then* $\mathrm{Hom}_R(M, C)$ *is a w-module, and* $\mathrm{Hom}_R(M, C) = \mathrm{Hom}_R(M_w, C)$.

Corollary 4.1 *((Wang, 1997),Corollary 2.8) Let M be a torsion-free module such that M is of w-finite type, that is, $M_w = N_w$ for some finitely generated submodule N of M. Let C be a w-module. Then*

$$\mathrm{Hom}_R(M, C)_P = \mathrm{Hom}_{R_P}(M_P, C_P)$$

for each $P \in w - \mathrm{Max}(R)$.

Lemma 4.5 *Let R be a strong Mori domain with a unique non-principal maximal w-ideal M. If G and H are torsion-free modules of finite type such that $G_M \cong H_M$, then G and H are w-nearly isomorphic.*

Proof Let I be a nonzero w-ideal of R and $\Omega = \{N_0, N_1, \ldots, N_n\}$ the set of maximal w-ideals of R containing I. By Lemma 4.3, $M = N_i$ for some $i = 0, \ldots, n$; say N_0, and $G_{N_i} \cong H_{N_i}$ for each i by Lemma 4.4. Proposition 4.4 and Corollary 4.1 imply that for each i, there exists a map $f_i : G \to H$ such that $(f_i)_{N_i} : G_{N_i} \to H_{N_i}$ is an isomorphism. Since $(N_i + \prod_{j \neq i} N_j)_w = R$ for each i, there exists a $J \in GV(R)$ such that $J \subseteq N_i + \prod_{j \neq i} N_j$. Note that $J \nsubseteq N_i$ for each i because $J_w = J_t = J_v = R$. Pick $a \in J$ such that $a \notin N_i$ and hence $aH_{N_i} = H_{N_i}$ for each i. Then $a = a_i + b_i$ for some $a_i \in N_i$ and $b_i \in \prod_{j \neq i} N_j$, and hence $1 = \lambda a_i + \lambda b_i$, where $\lambda = \frac{1}{a}$. Hence, $(f_i)_{N_i} = \lambda a_i (f_i)_{N_i} + \lambda b_i (f_i)_{N_i}$. Since $\mathrm{Im}(a_i \lambda (f_i)_{N_i}) \subseteq \lambda N_i H_{N_i}$, we have

$$H_{N_i} = \mathrm{Im}((f_i)_{N_i}) \subseteq \lambda N_i H_{N_i} + \mathrm{Im}(\lambda b_i (f_i)_{N_i}) \subseteq H_{N_i}.$$

Hence,

$$H_{N_i} = aH_{N_i} = N_i H_{N_i} + \mathrm{Im}(b_i (f_i)_{N_i}).$$

Thus, by Nakayama's Lemma, $\mathrm{Im}(b_i (f_i)_{N_i}) = H_{N_i}$. Let $g := b_0 f_0 + \ldots + b_n f_n$. Note that for $j \neq i$, $\mathrm{Im}(b_j (f_j)_{N_i}) \subseteq N_i H_{N_i}$, and $\mathrm{Im}(b_i (f_i)_{N_i}) = H_{N_i}$ since b_i is a unit in R_{N_i}. Thus, we can define a surjection

$$\phi_i\colon \frac{G_{M_i}}{M_iG_{M_i}} \to \frac{H_{M_i}}{M_iH_{M_i}},$$

where $\phi_i(x + M_iG_{M_i}) = g_{M_i}(x) + M_iH_{M_i}$. Hence, $\mathrm{Im}(g_{N_i}) = H_{N_i}$ again by Nakayama's Lemma. Since G_{N_i} is a Noetherian R_{N_i}-module and $G_{N_i} \cong H_{N_i}$, g_{N_i} is an isomorphism. Now we show that $g : G \to H$ is injective. Since g_{N_i} is an isomorphism from G_{N_i} onto H_{N_i}, $(\mathrm{Ker}(g))_{N_i} = \mathrm{Ker}(g_{N_i}) = 0$. Since R is a strong Mori domain, and G is w-finite type w-module, G is a strong Mori module by [(Wang & McCasland, 1997), Theorem 4.5]. That is, for every submodule X of G, there exists a finitely generated submodule of X such that $X_w = X'_w$. So $0 = (\mathrm{Ker}(g))_{N_i} = Y_{N_i}$ for some finitely generated submodule Y of $\mathrm{Ker}\, g$, where $Y_w = (\mathrm{Ker}\, g)_w$. Thus, there exists an element $t \in R \setminus N_i$ such that $tY = 0$. Since G is torsion-free, we have that $Y = 0$, and so $\mathrm{Ker}\, f = 0$. Therefore, there is an embedding $g : G \to H$ such that g_{N_i} is an isomorphism for each $i = 0, \ldots, n$. Hence, $(\mathrm{Coker}(g))_{N_i} = \mathrm{Coker}(g_{N_i}) = 0$, and hence $\mathrm{Ann}(\mathrm{Coker})g \not\subseteq N_i$ for each i. Therefore, $(\mathrm{Ann}\,\mathrm{Coker}\, f + I)_w = R$. □

Now, we are ready to prove the equivalence w-weak isomorphism types for torsion-free w-modules of finite type.

Theorem 4.3 *Assume that R is a strong Mori domain with the UDwI property with a unique non-principal maximal w-ideal M. Let G and H be torsion-free modules of finite type such that H is isomorphic to a finite direct sum of ideals of R. Then the following statements are equivalent.*

1. *$G_M \cong H_M$.*
2. *G and H are w-locally isomorphic.*
3. *G and H are w-nearly isomorphic.*
4. *G and H are isomorphic.*

Proof (1) ⇒ (2) follows from Lemma 4.4, (1) ⇒ (3) follows from Lemma 4.5, and (3) ⇒ (2) follows from Lemma 4.3 and Theorem 3.4. (2) ⇒ (1) and (4) ⇒ (1) are trivial. Since every w-invertible w-ideal of R is principal by Lemma 4.3, (1) ⇒ (4) follows from Lemma 4.3 and Proposition 3.2. □

Theorem 4.4 *Assume that R is a strong Mori domain.*

1. *If R has the UDwI property, then R_Q has the UDI property for every maximal w-ideal Q of R.*
2. *R has the UDwI property if and only if R is a w-PID, or R has a unique non-principal maximal w-ideal M such that R_M has the UDI property.*

Proof (1) First note that by Proposition 4.3, R has at most one non-principal maximal w-ideal. For each maximal w-ideal N of R which is principal, R_N is a DVR by Lemma 4.2, and hence has the UDI property. Assume that M is the unique non-principal maximal w-ideal of R. Let $I'_1, \ldots, I'_n, J'_1, \ldots, J'_m$ be ideals of R_M such that

$$I'_1 \oplus \ldots \oplus I'_n \cong J'_1 \oplus \ldots \oplus J'_m.$$

Put $I_i := I'_i \cap R$, $J_j := J'_j \cap R$ for each $i = 1, \ldots, n$, $j = 1, \ldots, m$. Then $I'_i = I_i R_M$ and $J'_j = J_j R_M$ for each i, j, and

$$(\oplus_{i=1}^n I_i)_w R_M = (\oplus_{i=1}^n I_i) R_M \cong (\oplus_{j=1}^m J_j) R_M = (\oplus_{j=1}^m J_j)_w R_M.$$

Since every w-ideal of R is of finite type, Lemma 3.8 implies that $(\oplus_{i=1}^n I_i)_w$ and $(\oplus_{j=1}^m J_j)_w$ are modules of finite type. Thus,

$$(\oplus_{i=1}^n I_i)_w \cong (\oplus_{j=1}^m J_j)_w$$

by Theorem 4.3. Hence, by assumption, $n = m$ and $(I_i)_w \cong (J_i)_w$ for each i. Therefore,

$$(I_i)_w R_M \cong (J_i)_w R_M$$

which implies $I'_i = I_i R_M \cong J_i R_M = J'_i$.

(2) If R has the UDwI property, then R has at most one non-principal maximal w-ideal by Proposition 4.3. If all the maximal w-ideals of R are principal, then R is a

w-PID by Lemma 4.2. Otherwise, let M be the unique non-principal maximal w-ideal of R. Then R_M has the UDI property by (1).

If R is a w-PID, then clearly R has UDwI. Assume that R is not a w-PID and M is the unique non-principal maximal w-ideal of R such that R_M has the UDI property. Let $I_1, \ldots, I_n, J_1, \ldots, J_m$ be ideals of R with

$$(I_1)_w \oplus \ldots \oplus (I_n)_w \cong (J_1)_w \oplus \ldots \oplus (J_m)_w.$$

Then

$$\oplus_{i=1}^{n} I_i R_M \cong \oplus_{j=1}^{m} J_j R_M.$$

Hence, by assumption, $n = m$ and $(I_i)_w R_M = I_i R_M \cong J_i R_M = (J_i)_w R_M$ for each i. Therefore, $(I_i)_w \cong (J_i)_w$ by Theorem 4.3. $\square$

4.2. Examples

By Theorems 2.4 and 2.5 of (Badawi, 2003) and Theorem 2.12 of (Kim & Wang, 2012), it is possible to construct some examples of non-Noetherian strong Mori rings by means of trivial extensions. Let D be a commutative ring with 1 and M a unitary D-module. Then $D \propto M$ with coordinate-wise addition and multiplication

$$(d_1, m_1)(d_2, m_2) = (d_1 d_2, d_1 m_2 + d_2 m_1)$$

is a commutative ring with 1 called the *idealization of M* or the *trivial extension of D* by M. If R is a Noetherian domain with quotient field K such that

1. $\dim R = 1$ and R has infinitely many maximal ideals (for example, $\mathbb{Z}$), or

2. $\dim R \geq 2$,

Then $D = R \propto K$ is a non-Noetherian strong Mori ring.

Lemma 4.6 *Let R be an integral domain with quotient field K and $D = R \propto K$ the trivial extension of R by K. Then w-*$\mathrm{Max}(D) = \{P \propto K \mid P \in w\text{-}\mathrm{Max}(R)\}$.

Proof Let P be a maximal w-ideal of R. Then $(P \propto K)_w = P \propto K$ since for each $(a, b) \in (P \propto K)_w$, $(a, b)J \subseteq P \propto K$ for some $J \in GV(D)$. Note that $J = I \propto K$ for some $I \in GV(R)$ by [(Huckaba, 1988), Theorem 25.10]. For each $(j_1, j_2) \in J$, $(a, b)(j_1, j_2) = (p, k)$ for some $p \in P$ and $k \in K$. Thus, $a \in P$ and $b = \frac{k-aj_2}{j_1} \in K$ which implies that $(a, b) \in P \propto K$. Thus, there exists a maximal w-ideal Q of D such that $P \propto K \subseteq Q$. Note that $Q = P' \propto K$ for some prime ideal P' of R by (Anderson & Winders, 2009), Theorem 3.2. Also, $Q = I_w \propto K$ where $I = \{r \in R \mid (r, k) \in R \propto K \text{ for some } k \in K\}$ by (Chang & Kim 2017), Proposition 2.2. Hence, $Q = P' \propto K$ where P' is a prime w-ideal of R. It implies that $P \subseteq P'$ and hence $P = P'$, and $P \propto K = Q$ is a maximal w-ideal of D. For the converse, let Q be a maximal w-ideal of D. Then $Q = P \propto K$ for some prime w-ideal P of R. Let P' be a maximal w-ideal of R containing P. Since $(P' \propto K)_w = P' \propto K$, $P \propto K = P' \propto K$ implies that $P = P'$ which is a maximal w-ideal of R. □

Proposition 4.5 *Assume that R is a Noetherian domain with quotient field K such that* $\dim R = 1$ *and R has infinitely many maximal ideals. If R has the UDI property, then $D = R \propto K$ is a non-Noetherian strong Mori ring such that D has at most one non-principal maximal w-ideal.*

Proof If R is a Noetherian domain of dimension one with infinitely many maximal ideals, then $D = R \propto K$ is a non-Noetherian strong Mori ring by [(Badawi, 2003), Theorem 2.4] and [(Kim & Wang, 2012), Theorem 2.12]. Let Q be a maximal w-ideal of D. Since $\dim R = 1$, the w-operation is the identity operation on R and hence $Q = P \propto K$ for some maximal ideal P of R by Lemma 4.6. Since R has the UDI property, D has at most one non-principal maximal w-ideal. □

Example 4.1 Let $R = \mathbb{Z}[2i]$, where $i = \sqrt{-1}$. Then the ring of integers in $\mathbb{Q}(i)$ is $\mathbb{Z}[i]$, the Gaussian integers. Example 4.6(b) of (Goeters & Olberding, 2001), for the cases $d = -1$ and $p = 2$, implies that R has the UDI property and hence R has at most one non-principal maximal ideal. Since $i \notin \mathbb{Z}[2i]$, R is not integrally closed. Hence, R has a unique non-principal maximal ideal which is $(2, 2i) = 2\mathbb{Z}[i]$. Since the w-operation is the identity operation on R by [(Anderson & Zafrullah, 1991), Theorem 4.17], [(Mimouni, 2005), Corollary 2.11] and Proposition 4.5 implies that the trivial extension of R by its quotient field $\mathbb{Q}(i)$, $D = R \propto \mathbb{Q}(i)$, is a non-Noetherian strong Mori ring such that D has at most one non-principal maximal w-ideal.

4.3. UDwI for Overrings

Let $R \subseteq T$ be an extension of integral domains. Then T is called a w-linked extension of R if T is a w-module over R. In the case that $R \subseteq T \subseteq K$, we say that T is a w-overring of R.

It is well-known that the restriction of w to the set of the T-submodules of K is a star operation on T, denoted by $\dot{w}$ such that for a T-submodule I of K, $I_{\dot{w}} = I_w$.

Recall that w' is the w-operation over T, that is, for any fractional ideal A of T, we have that

$$A_{w'} = \{x \in K \mid xJ \subseteq A \text{ for some } J \in GV(T)\}$$

In general, $\dot{w} \leq w'$ on T which means that for any fractional ideal A of T, we have that $A_{\dot{w}} \subseteq A_{w'}$ or $A_w \subseteq A_{w'}$. Since $GV(R)$ and $GV(T)$ need not be equal, w' and $\dot{w}$ are not the same.

Lemma 4.7 *Let R be a strong Mori domain and T a w-overring of R. If N is a maximal w'-ideal of T that contracts to a principal maximal w-ideal of R, then N is principal.*

Proof Assume that P is a principal maximal w-ideal of R such that $P = N \cap R$; say $P = xR$ for some $x \in P$. Since R_P is a DVR by Lemma 4.2, and T_P is an overring of R_P, $R_P = T_P$. Thus, $PR_P = xR_P \subseteq N_P \subsetneq T_P = R_P$ which implies that $(xT)_P = N_P$. Now, let Q be a maximal w-ideal of R different from P. Then $x \notin Q$, and hence $T_Q = (xT)_Q \subseteq N_Q \subseteq T_Q$ which implies $N_Q = (xT)_Q$. Therefore, $N_Q = (xT)_Q$ for each maximal w-ideal Q of R, and hence $N = xT$ by [(Wang & McCasland, 1997), Proposition 3.4]. □

Lemma 4.8 *Assume that R is a strong Mori domain with a unique non-principal maximal w-ideal M and T is a w-overring of R. Let N be a maximal w'-ideal of T, lying over M such that N_M is principal. Then N is principal.*

Proof Let $N_M = aT_M$ for some $a \in T$. We may assume that $a \in N$ since $\frac{a}{1} = \frac{n}{t}$ for some $n \in N$ and $t \in R \setminus M$, hence there exists $s \in R \setminus M$ such that $sat = sn \in N$ which implies $a \in N$. We claim that $N = aT$. To see this, it suffices to show that $N_Q = (aT)_Q$ for every maximal w-ideal Q of R [(Wang & McCasland, 1997), Proposition 3.4]. Let $a = \frac{b}{c}$ and $X = \{P_\alpha\}$ be the set of maximal w-ideals of R which contain b. Since R is of finite t-character, X is finite. Also, we note that $M \in X$ since $b = cn$ for some $n \in N$ which implies $b \in N \cap R = M$. Now, let $P \in X$ be a maximal w-ideal of R which is different from M, hence $P = \pi R$ for some $\pi \in P$. Since R is a strong Mori domain, we may assume that n is the largest positive integer such that $\pi^n | b$. Thus $b = \pi^n r$ for some $r \in R \setminus P$. Since $b \in P$ and $\pi \notin M$, $r \in M$ and $aT_M = \frac{b}{c}T_M = \frac{\pi^n r}{c}T_M = \frac{r}{c}T_M$. So, by replacing a by $\frac{r}{c}$, we conclude that $N_Q = (aT)_Q$ for every maximal w-ideal Q of R. □

Proposition 4.6 *Assume that T is a w-overring of an integral domain R which is of finite type. If R has the UDwI property, then T has the UDw'I property.*

Proof Since T is a w-module of finite type, $T = X_w$ for some finitely generated submodule X of T. Suppose $X = R(a_1/b_1) + \ldots + R(a_k/b_k)$ for some $a_i/b_i \in X$. Setting $b = b_1 \cdots b_n$, we get $bX \subseteq R$ which implies $bT \subseteq R$. Suppose that $(I_1)_{w'} \oplus \ldots \oplus (I_n)_{w'} \cong (J_1)_{w'} \oplus \ldots \oplus (J_m)_{w'}$, where $I_1, \ldots, I_n, J_1, \ldots, J_m$ are ideals of T. Let $(I_i)_{w'} = X_i$ and $(J_k)_{w'} = Y_k$ for $i = 1, \cdots, n$, $k = 1, \cdots, m$. Note that X_i, Y_k are w-modules over R. Then we have $bX_1 \oplus \ldots \oplus bX_n \cong bY_1 \oplus \ldots \oplus bY_m$. From the above argument, bX_i, bY_k are w-ideals of R. Since R has the UDwI property, we have $n = m$, and $bX_i \cong bY_i$ for each index i as R-modules. Therefore, $X_i \cong Y_i$ as R-modules which are also T-isomorphisms by [(Ay & Klingler, 2012), Lemma 1.1]. Thus, T has UDw'I property. □

Theorem 4.5 *Let R be a strong Mori domain with w-$\dim R = 1$ and T a w-overring of R. If R has the UDwI property, then T has the UDw'I property.*

Proof First, T is a strong Mori domain with w'-$\dim T \leq 1$ by [(Wang & McCasland, 1999), Theorem 3.4]. If every maximal w-ideal of R is principal, then R is a w-PID by Lemma 4.2. This implies that T is w'-Bezout domain by [(El-Baghdadi, 2010), Theorem 3.7]. Since T is strong Mori w'-Bezout domain, T is a w'-PID. So, we may assume that R has a unique non-principal maximal w-ideal; say M. It suffices to show that T has at most one non-principal maximal w'-ideal and T_Q has the UDI property for every maximal w'-ideal Q of T so that Theorem 4.4 implies that T has the UDw'I property. Let Q be a maximal w'-ideal of T, and put $P := Q \cap R$. Then P is a prime w-ideal of R by Theorem 7.7.4. (Wang & Kim, 2016). Since w-dim$(R) = 1$, then P is a maximal w-ideal of R. If Q is a principal ideal of T, then T_Q is a DVR by Lemma 4.2 and hence it has the UDI property. So we may assume that Q is a non-principal maximal w'-ideal of T. Then $P = M$ and Q_M is non-principal by Lemmas 4.7 and 4.8. We note that both T_Q and T_M, as overrings of one dimensional Noetherian domain R_M, have the UDI property by [(Goeters & Olberding, 2001), Proposition 4.2]. Hence, T_M has at most one non-principal maximal ideal by (Goeters & Olberding, 2001), Theorem 2.8. Let Q_1 and Q_2 be two distinct non-principal maximal w'-ideals of T. Then, for $i = 1, 2$, Q_iT_M is non-principal by Lemmas 4.7 and 4.8. Let P_iT_M be a maximal ideal of T_M such that $Q_iT_M \subseteq P_iT_M$, where P_i is a prime ideal of T maximal with respect to $P_i \cap (R \setminus M) = \emptyset$. Since $\dim T_{Q_i} = 1$ and $T_{Q_i} \cong (T_M)_{Q_iT_M}$, $Q_iT_M = P_iT_M$; a contradiction. Therefore, T has at most one non-principal maximal w'-ideal. □

Let T be a w-overring of R. Following (Wang & Kim, 2016), we say that $R \subseteq T$ is a **w-extension** if every element $x \in T$ is w-integral over R, that is, there is a nonzero finitely generated ideal I of R such that $xI_w \subseteq I_w$. The set of elements of T which are w-integral over R is called the **w-integral closure** of R in T, denoted by R_T^w. As in (Wang & Kim, 2016), the **w-global transform** of R, R^{wg}, is defined as follows:

$$R^{wg} = \{x \in K \mid xP_1 \cdots P_k \subseteq R \text{ for some } P_1, \cdots, P_n \in w\text{-Max}(R)\}.$$

Theorem 4.6 *Let T be a w-extension of a strong Mori domain R such that $T \subseteq R^{wg}$. If R has the UDwI property, then T has the UD$\dot{w}$I property.*

Proof Since $T \subseteq R^{wg}$, T is also a strong Mori domain by (Wang & Kim, 2016), Theorem 7.10.12. By the same method as in the proof of Theorem 4.5, we may assume that R has a unique non-principal maximal w-ideal M. Let Q be a maximal $\dot{w}$-ideal of T. Since $\dot{w}$ and w' are two star operations on T such that $\dot{w} \leq w'$, $Q = Q_{\dot{w}} \subseteq Q_{w'} = (Q_{\dot{w}})_{w'} = (Q_{w'})_{\dot{w}}$. Thus $Q_{w'}$ is a $\dot{w}$-ideal, and hence $Q = Q_{w'}$. Let Q' be a maximal w'-ideal of T such that $Q \subseteq Q'$. Then $Q \subseteq Q' = Q'_{w'} = (Q'_{\dot{w}})_{w'} = (Q'_{w'})_{\dot{w}} = Q'_{\dot{w}}$ implies that $Q = Q'$. Hence, Q becomes a maximal w'-ideal. So, by Lemmas 4.7 and 4.8, we may take Q and QT_M to be non-principal and assume that $Q \cap R = M$. So, T_Q is a Noetherian domain.

Since $Q \cap R = M$, $M \cap (T \setminus Q) = \emptyset$, and hence $T_M = (T_Q)_{MT_Q}$ is a Noetherian domain. Then $R_M \subseteq T_M$ is an integral extension by (Wang & Kim, 2016), Theorem 7.7.13. Hence, T_M has the UDI property by [(Klingler & Omairi, 2020), Theorem 3.4]. Therefore, T_M has at most one non-principal maximal ideal by [(Goeters & Olberding, 2001), Theorem 2.8]. To show that T has at most one non-principal maximal $\dot{w}$-ideal, let Q_1 and Q_2 be two distinct non-principal maximal $\dot{w}$-ideals of T. Then, for $i = 1, 2$, Q_iT_M is non-principal by Lemmas 4.7 and 4.8. Let P_iT_M be a maximal ideal of T_M such that $Q_iT_M \subseteq P_iT_M$, where P_i is a prime ideal of T maximal with respect to $P_i \cap (R \setminus M) = \emptyset$. Then $P_i \cap R = M = Q_i \cap R$ by [(Wang & Kim, 2016), Theorems 7.7.18 and 7.7.9 (3)]. Hence, for each i, $(P_i)_{\dot{w}} = P_i$ by [(Wang & Kim, 2016), Theorem 7.7.7 (3)] which implies that $P_i = Q_i$ since P_i and Q_i are incomparable $\dot{w}$-ideals (Wang & Kim, 2016), Theorem 7.7.18. Therefore, Q_1T_M and Q_2T_M are two distinct non-principal maximal ideals of T_M, which is a contradiction.

$\square$

CHAPTER 5

UNIQUE DECOMPOSITIONS INTO REGULAR IDEALS FOR MAROT RINGS

In the previous chapter, we recall that the UDI property and define the UDwI property which is a Krull-Schmidt property of w-ideals. In a similar way, we define UDRI property in this chapter. Let R be a ring and C the class of regular ideals of R. R has the unique decomposition into regular ideals (UDRI) property if, whenever

$$I_1 \oplus I_2 \oplus \cdots \oplus I_n \cong J_1 \oplus J_2 \oplus \cdots \oplus J_m$$

for $I_i, J_j \in C$, then $n = m$ and, after a possible reindexing, $I_i \cong J_i$ for all $i \leq n$. We note that regular ideals cannot be written as a sum of two regular ideals, wlog, we assume regular ideals are indecomposable in class C.

In this chapter, we show that a Marot ring R has the UDRI property if and only if R has a unique non-principal regular maximal ideal M and $R_{(M)}$ has UDRI property (Ay Saylam & Gürbüz, 2022). We emphasize that if every ideal is regular then these rings can be considered as Noetherian domains, and our result coincide with the characterization given in (Goeters & Olberding, 2001). We also provide an example satisfying the UDRI property and an example which does not satisfy this property. Next, we compare local and near isomorphisms (definitions are given below) for some classes of modules and prove they imply isomorphism if R has the UDRI property. We also show that if R has the UDRI property and R' is an overring of R which is a finitely generated R-module, then R' has the UDRI property. Lastly, we prove that if R has the UDRI property with $reg - dim(R) \leq 2$, then $\tilde{R}$, the integral closure of R, has the UDRI property.

5.1. Definitions and Fundamental Tools

Let R be a commutative ring with unity. Elements of R that are not zero divisors are called **regular**. An ideal of R is called regular if it contains a regular element. The integral clouse of R, denoted by $\tilde{R}$, is a ring which is the set of all integral elements of R. The total quotient ring $Q(R)$ of R is defined as $Q := Q(R) = \{a/b \,:\, a, b \in R \text{ with } b \text{ regular}\}$. A ring S is called an overring of R if $R \subseteq S \subseteq Q$. If I is a nonzero ideal of R, $(R : I) = \{q \in Q \,|\, qI \subseteq R\}$ is an R-submodule of Q. If R is a ring and P is a prime ideal of R, then the **regular localization** of R at P, is the ring $R_{(P)} = \{a/b \,:\, a, b \in R \text{ with } b \notin P, \, b \text{ is regular}\}$. If an ideal is contained in a unique maximal ideal, then it is called **colocal**. A ring R is said to be of **finite character** if every nonzero regular ideal is contained in only finitely many maximal ideals of R. We call R ***h*-local** if R is of finite character and each nonzero regular prime ideal of R is contained in a unique maximal ideal of R. The regular height of a regular prime ideal P of R, abbreviated $reg - ht(P)$, is defined to be the supremum of the length of chains consisting of regular prime ideals contained in P plus 1. The regular dimension of R, abbreviated $reg - dimR$, is $\sup\{reg - ht(P) | P \text{is a regular prime ideal of } R\}$.

Two torsion-free R-modules G and H are called nearly isomorphic if for every regular ideal I of R, there is an embedding $f : G \to H$ such that $I + \mathrm{Ann}(\mathrm{Coker}\, f) = R$. Two R-modules G and H are called locally isomorphic if for every regular maximal ideal M of R, $G_{(M)} \cong H_{(M)}$

A ring R is called a **Marot ring** if every regular ideal can be generated by a set of regular elements. This propery was defined by Marot in (Marot, 1977).

Theorem 5.1 *((Huckaba, 1988),Theorem 7.1) The following conditions on a ring R are equivalent:*

1. *R is a Marot ring.*
2. *Every pair of elements $\{a, b\}$ in R with b regular has the property that the ideal $< a, b >$ admits a finite system of regular elements as generators.*
3. *Every regular R-module contained in Q admits a system of regular elements as generators.*

We note that each overring of a Marot ring is also a Marot ring by [(Huckaba, 1988), Corollary 7.3].

Valuation rings with zero divisors were defined by Manis (Manis, 1967). A valuation is a map v from a ring K onto a totally ordered group G and a symbol ∞ such that for all $x, y \in K$:

- $v(xy) = v(x) + v(y)$
- $v(x + y) \geq min\{v(x), v(y)\}$
- $v(1) = 0$ and $v(0) = \infty$.

The ring $R = R_v = \{x \in K : v(x) \geq 0\}$ together with the ideal $P = P_v = \{x \in K : v(x) > 0\}$ denoted (R, P) is called a valuation pair of K. R is called a **valuation ring** of K and G is called the value group of G. If G is isomorphic to the group of integers, R is called a **discrete rank one valuation ring**. In the presence of the Marot property, valuation rings have some similar properties with valuation domains :

Proposition 5.1 *(Glaz, 2002)[Proposition 4.1] Let R be a Marot ring. Assume that $R \neq Q$. Then the following conditions are equivalent:*

1. *R is a valuation ring.*
2. *For each regular element $x \in Q$, either $x \in R$ or $x^{-1} \in R$.*
3. *R has only one regular maximal ideal and each of its finitely generated regular ideal is principal.*

5.2. Properties of Marot Rings Whose Regular Ideals are Finitely Generated

In this section, we will give some properties of Marot rings whose regular ideals are finitely generated. We use some of these properties for the proofs of our results in the next section. We start with an important tool which we use in the proofs of our results.

Theorem 5.2 *For a commutative ring R, the following are equivalent :*

(i) Every regular prime ideal is finitely generated.

(ii) Every regular ideal is finitely generated.

(iii) Every ascending chain of regular ideals is stationary.

(iv) Every nonempty set of regular ideals in R has a maximal element.

Proof $(i) \Rightarrow (ii)$ Suppose that every regular prime ideal of R is finitely generated and there exists a regular ideal which is not finitely generated. Let Γ be the set regular ideals which are not finitely generated.

By assumption $\Gamma \neq \emptyset$. We consider ordering Γ by inclusion. Let Φ be a totally ordered subset of Γ. Set $J = \bigcup_{I \in \Phi} I$ which is a regular ideal of R. We claim that J is not finitely generated. To prove it, suppose that J is finitely generated, say J is generated by $j_1, \ldots, j_n$. Since Φ is a chain, there exists an index N such that $Rj_1 + \ldots + Rj_n = J \subseteq I_N \subseteq J$ impyling I_N is finitely generated which is a contradiction. So, J is not finitely generated. Hence, $J \in \Phi$ and is an upper bound for Φ in Γ. By Zorn's Lemma, Γ has a maximal element, say P.

We claim that P is a prime ideal. Let a, b be elements of R such that $a, b \in R \setminus P$ with $ab \in P$. Since $P \subset P + Ra$ and P is maximal, we have that $P + Ra$ is finitely generated. Suppose that $P + Ra$ is generated by $p_1 + r_1 a, \ldots, p_n + r_n a$ where $p_i \in P$ and $r_i \in R$. Set the ideal $K = (P : a) = \{r \in R \,|\, ra \in P\}$. We have that $P \subset P + Rb \subseteq K$, and, by the maximality of P, K is finitely generated implying that aK is finitely generated.

Now we claim that $P = Rp_1 + \ldots + Rp_n + aK$. Clearly , $Rp_1 + \ldots + Rp_n + aK \subseteq P$. Take any $p \in P \subset P + Ra$. Then $p = c_1(p_1 + r_1 a) + \ldots + c_n(p_n + r_n a)$ implies $(c_1 r_1 + \ldots + c_n r_n)a \in P$ i.e $c_1 r_1 + \ldots + c_n r_n \in (P : a) = K$, so $p \in Rp_1 + \ldots + Rp_n + aK$. Thus we have that $P = Rp_1 + \ldots + Rp_n + aK$ that is P is finitely generated which is a contradiction. So $ab \notin P$ and P is prime ideal. Thus we get a regular prime ideal which is not finitely generated. This contradicts to the assumption. So, all regular ideals of R must be finitely generated.

$(ii) \Rightarrow (i)$: Clear.

$(iii) \Rightarrow (ii)$ Suppose that I is a regular ideal and I is not finitely generated. Then there exists a regular element $x_1 \in I$. Set $I_1 = Rx_1$. Then $I_1 \subset I$ and $I_1 \neq I$ since I is not finitely generated. So, there exists an element $x_2 \in I \setminus I_1$. Set $I_2 = Rx_1 + Rx_2$. By continuing this way, we obtain a chain of regular ideals, and since this chain is not stationary, the ascending chain condition does not hold for this chain.

$(ii) \Rightarrow (iii)$ Suppose that every regular ideal of R is finitely generated. Let

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots$$

be an arbitrary infinite chain of regular ideals. Let $I = \bigcup_{i=1}^{\infty}$. Then I is also a regular ideal. By assumption I is finitely generated, say $I = Rx_1 + \ldots + Rx_n$ such that $x_i \in I_{n_i}$ for some n_i. So, there exists $N \in \mathbb{N}$ such that $x_1, \ldots, x_n \in I_N$. But then $I_N = I$ and this shows that $I_m = I_N$ for all $m \geq N$. Hence, the ascending chain condition holds for regular ideals.

$(ii) \Leftrightarrow (iv)$: Immediate from Zorn's lemma. □

From now on, R denotes a Marot ring which has the ascending chain condition on its regular ideals, and therefore it also satisfies all the equivalent conditions stated in Theorem 5.2.

Proposition 5.2 *Every nonzero regular ideal I of R contains a finite product of regular prime ideals.*

Proof Let Λ be the set of regular ideals I_i of R such that I_i does not contain a finite product of regular prime ideals, and order Λ by inclusion. Since R has the ascending chain condition on its regular ideals, Λ has a maximal element, say I. Then I is regular but not a prime ideal. So, there exist elements $x, y \in R$ such that $xy \in I$ but $x \notin I$, $y \notin I$. Since $I \subsetneq I + Rx$ and $I \subsetneq I + Ry$ and I is maximal, we have that $I + Rx \notin \Lambda$, $I + Ry \notin \Lambda$, and they contain a finite product of regular prime ideals. But then $(I + Rx)(I + Ry) = I^2 + Iy + Ix + Rxy \subset I$ implies that I contains a finite product of regular prime ideals, too. This contradicts to the assumption. □

Corollary 5.1 *R is h-local if and only if every regular prime ideal is colocal.*

Proof Suppose that every regular prime ideal of R is colocal, and let I be a regular ideal of R. Then I contains a finite product of regular prime ideals by Proposition 5.2, that is, there exist prime ideals $P_1, \ldots, P_n$ such that $P_1 \cdots P_n \subseteq I$. Suppose $I \subseteq M_1, \ldots, M_n, \ldots$, then, without loss of generality, $P_1 \subseteq M_1, P_2 \subseteq M_2, \ldots, P_n \subseteq M_n$ and $P_i \nsubseteq M_j$ when $i \neq j$ since P_i's are colocal prime ideals. Thus, I can be contained in at most n maximal ideals. □

Lemma 5.1 *Every regular ideal of R is a finite intersection of irreducible regular ideals.*

Proof Let Γ be the set regular ideals I_i of R such that I_i is not a finite intersection of irreducible regular ideals and order Γ by inclusion. Suppose that $\Gamma \neq \emptyset$, so Γ has a maximal element, say M. Then M is reducible, that is, $M = M_1 \cap M_2$. Since $M \subset M_1$ and $M \subset M_2$, M_1 and M_2 are also regular ideals but they are not in Γ by the maximality of M. Thus, M_1 and M_2 are finite intersection of irreducible regular ideals, and so is M. This gives a contradiction. □

For our next result, we give a definition. An ideal P of any commutative ring R is primary for its regular elements if whenever x and y are regular elements of R such that $xy \in P$, then $x \in P$ or $y \in \mathrm{Rad}(P)$.

Theorem 5.3 *((Huckaba, 1988), Theorem 7.10) Let R be Marot ring. Then a regular ideal Q of R is primary if and only if Q is primary for its regular elements.*

Proposition 5.3 *Every irreducible regular ideal of R is primary.*

Proof Let I be a regular ideal of R. We claim that I is primary for its regular elements. Suppose that a, b are regular elements of R such that $ab \in I$ and $a \notin I$. We will show that $b^n \in I$ for some $n \in \mathbb{Z}$. Define the ideals $I_i = \{x \mid xb^i \in I\}$. Then $I_0 = I \subseteq I_1 \subseteq I_2 \subseteq \ldots$ is an ascending chain of regular ideals of R. So, this chain must be stationary, that is, there exists $n \in \mathbb{Z}$ such that $I_n = I_m$ for all $m \geq n$. Let $Q = I + Ra$ and $J = I + Rb^n$. We claim that $I = Q \cap J$. One direction is clear. For the other direction, take $y \in Q \cap J$. Then $y = i_1 + ua = i_2 + vb^n$ where $i_1, i_2 \in I, u, v \in R$. This implies that $ua - vb^n \in I$ and $uab - vb^{n+1} \in I$. Since $ab \in I$ and $I_n = I_m$ for all $m \geq n$, we get $vb^n \in I$. Thus, $y \in I$ and $I = Q \cap J$. Since I is irreducible, $I = Q$ or $I = J$. $I = Q$ is not possible since $a \in Q$ but $a \notin I$. Hence $I = J$ and $b^n \in I$. By Theorem 5.3, I is primary. □

Corollary 5.2 *Every regular ideal of R has a primary decomposition.*

Proof Immediately follows from Lemma 5.1 and Proposition 5.3. □

Proposition 5.4 *If every regular maximal ideal of R is principal, then every regular ideal is principal.*

Proof Let I be a regular ideal. Then I is finitely generated, and since R is Marot ring, it must be generated by finitely many regular elements. So, we only need to show that $aR + bR$ is principal, where $a, b \in R$ are regular elements. If $aR + bR = R$, then we are done. If not, $aR + bR \subseteq M$, where M is a maximal ideal of R. Since a and b are regular elements, M is also a regular ideal. So, by assumption, $M = cR$ for some regular element $c \in R$. Then, we can write $a = a_1c, b = b_1c$ for some $a_1, b_1 \in R$. We note that a_1, b_1 are also regular elements, and aR is properly contained in a_1R. Now, if $a_1R + b_1R = R$, then $aR + bR = cR$. If not, we repeat the process. After a finite number of steps we will find that $aR + bR$ is principal. □

Lemma 5.2 *For every regular ideal I of R, the number of minimal prime ideals over I is finite.*

Proof Let Γ be the set of regular ideals I_i such that the number of minimal prime ideals over I_i is infinite. Suppose that $\Gamma \neq \emptyset$. Since R has the ascending chain condition on its regular ideals, Γ has a maximal element, say Q. Clearly, Q is not a prime ideal. So, there exist $a, b \in R$ such that $ab \in Q$ with $a \notin Q, b \notin Q$. Define the ideals $I = Q + aR$ and $J = Q + bR$. Since $Q \subsetneq I, Q \subsetneq J$, we get $I, J \notin \Gamma$ but $IJ \subseteq Q$. This implies that any prime ideal minimal over Q is minimal over either I or J. But the number of minimal prime ideals over I and J are finite. So, we get a contradiction. □

5.3. The UDRI Property

In this section, R denotes a Marot ring whose regular ideals are finitely generated unless otherwise stated. In these rings, we will characterize the UDRI property. We start with an important tool.

Lemma 5.3 *If R has the UDRI property, then R has at most one non-principal regular maximal ideal.*

Proof Let M_1 and M_2 be two distinct regular maximal ideals of R. Then we have the exact sequence

$$0 \to M_1 \cap M_2 \xrightarrow{f} M_1 \oplus M_2 \xrightarrow{g} M_1 + M_2 = R \to 0$$

where the homomorphisms f and g are defined as $f(x) = (x, x)$ for every $x \in M_1 \cap M_2$, and $g((a, b)) = a - b$ for every $(a, b) \in M_1 \oplus M_2$. Since g is onto, there exists an element $x \in M_1 \oplus M_2$ such that $g(x) = 1$. Now define a homomorphism $\phi\colon R \to M_1 \oplus M_2$ such that

$\phi(a) = ax$ for every $a \in R$. One can easily show that $g \circ \phi = \mathrm{id}_R$ which implies that this exact sequence splits. Thus, we have $M_1 \oplus M_2 \cong R \oplus (M_1 \cap M_2)$. Since R has UDRI, one of M_1 or M_2 is isomorphic to R, that is, one of them is principal. □

Now, we give an example of a Marot ring whose regular ideals are finitely generated with more than two non-principal regular maximal ideals. First of all, let us recall the $A + B$ construction. Let D be an integral domain and $\mathcal{P}$ a nonempty set of prime ideals with index set $\mathcal{A}$. Let $\mathcal{I} = \mathcal{A} \times \mathbb{N}$, and for each $i = (\alpha, n) \in \mathcal{I}$, let $K_i = K_\alpha$ be the quotient field of D/P_α. For $B = \sum K_i$, form a ring $R = D + B$ from $D \times B$ by defining addition and multiplication as $(r, b) + (s, c) = (r + b, s + c)$, and $(r, b)(s, c) = (rs, rb + sc + bc)$. (See (Lucas, 2016) for further details.)

Example 5.1 *((Lucas, 2016), Example 2.4)* Let $D = \mathbb{Z}[\sqrt{10}]$. This is a Dedekind domain which is not a PID. Both $M = 2D + \sqrt{10}D$ and $N = 5D + \sqrt{10}D$ are maximal ideals, and neither is principal. Let $\mathcal{P} = Max(D) \setminus \{M, N\}$, and let $R = D + B$ be the ring of the form $A + B$ corresponding to D and $\mathcal{P}$. Then R is a Marot ring whose regular ideals are finitely generated and the only regular maximal ideals are $MR = M + B$ and $NR = N + B$, and neither is principal. So, by Lemma 5.3, R does not satisfy UDRI.

Proposition 5.5 *Let M be the unique regular maximal ideal of R. If M is principal, then every regular ideal of R is of the form M^n, where $n \in \mathbb{N}$.*

Proof Suppose that $M =< m >$, where m is a regular element of R, and let J be a nonzero regular ideal. Assume that $J \subseteq M^n$ for every $n \in \mathbb{Z}^+$. Then $J \subseteq \cap_{i=1}^{\infty} M^i$. First, we note that $M^i \neq M^j$ whenever $i \neq j$. Let $I = \cap_{i=1}^{\infty} M^i$. Then $IM = I$. If $I \neq 0$, then $I =< t >$ for some regular element t. Since $t \in IM$, there exist $n \in \mathbb{Z}^+$ and $x_i \in I, y_i \in M$ such that $t = \sum_{i=1}^{n} x_i y_i$, and this implies that M contains a unit. So, $I = 0$. Hence, there exists $n \in \mathbb{Z}$ such that $J \subseteq M^n$ but $J \nsubseteq M^{n+1}$. Since $J \nsubseteq M^{n+1}$ and R is a Marot ring, there exists a regular element $j \in J \setminus M^{n+1}$. Then, we have that $j = m^n t$ for some $t \in R$, and $t \neq 0$. If t is a unit, we are done. If t is not unit and $t \in M$, then $j \in M^{n+1}$. So, $t \notin M$, and this implies t is not regular. But then this would contradict with being j regular. Hence, t must be a unit. Therefore, $< j >=< m^n >= M^n$. Since $< j >\subseteq J \subseteq M^n =< m^n >$, we get $J = M^n$. □

Remark 5.1 *Let R be a Marot ring and M a regular maximal ideal of R. Then we have the following:*

(1) Every regular ideal of $R_{(M)}$ is of the form $IR_{(M)}$ where I is a regular ideal of R with $I \subseteq M$.

(2) $MR_{(M)}$ is the unique regular maximal ideal of $R_{(M)}$.

The prooof of the remark is as follows. Let I be a regular ideal of R. Then there exists a regular element $x \in I$. If $x \notin M$, then $IR_{(M)} = R_{(M)}$. If $x \in M$, then $IR_{(M)}$ is an ideal of $R_{(M)}$, and $x/1$ is a regular element contained in $IR_{(M)}$. For the converse part, let X be a regular ideal of $R_{(M)}$. Then $X = JR_{(M)}$ for some ideal J of R. Since $JR_{(M)}$ is regular, it contains a regular element x/y, where $x \in J$, and x must be regular in R. This implies J is a regular ideal of R. If J is not contained in M, then there exists a regular element $j \in J$ such that $j \notin M$ which yields $JR_{(M)} = R_{(M)}$. From part (1), we can conclude that $MR_{(M)}$ is a regular ideal of $R_{(M)}$. Assume $MR_{(M)} \subseteq YR_{(M)}$. If $Y \neq M$, then there exists a regular element $y \in Y \setminus M$ since R is a Marot ring. Thus, $YR_{(M)} = R_{(M)}$. Suppose $A_{(M)}$ is a proper regular ideal of $R_{(M)}$. Then A is a regular ideal of R. Take any regular element $a \in A$. Since $A_{(M)}$ is proper, $a \in M$. Thus, every regular element of A is contained in M. Since R is a Marot ring, we must have $A \subseteq M$. Hence, every regular ideal of $R_{(M)}$ is contained in $M_{(M)}$.

Lemma 5.4 *For any commutative ring R, if P is a regular principal prime ideal of R such that $P \subseteq N$, where N is a principal maximal ideal, then $P = N$.*

Proof Suppose that $P =< p >$ and $N =< n >$, where p, n are regular elements. Then $p = nt$ for some $t \in R$. Since P is a prime ideal, we have $n \in P$ or $t \in P$. If $n \in P$, then we are done. If $t \in P$, then $t = kp$, for some $k \in R$. This gives $p = nkp$. Since p is regular, this implies $N = R$. Hence, $P = N$. □

Lemma 5.5 *If N is a principal regular maximal ideal of R, then N does not contain a regular prime ideal properly.*

Proof If P is a regular prime ideal such that $P \subset N$, then $PR_{(N)}$ is a regular prime ideal of $R_{(N)}$. Since $R_{(N)}$ has a unique regular maximal ideal $NR_{(N)}$ which is principal , $PR_{(N)}$ must be principal by Proposition 5.4. So, by Lemma 5.4, we must have $PR_{(N)} = NR_{(N)}$, and this implies $P = N$. □

Proposition 5.6 *If R has finitely many non-principal regular maximal ideals, then R is of finite character.*

Proof Let I be a regular ideal of R, and suppose that I is contained in a principal maximal ideal of R, say N. In R/I, N/I is both a maximal ideal and a minimal prime ideal of R/I by Lemma 5.5. Since there are only finitely many minimal prime ideals containing I by Lemma 5.2, I can be contained in only finitely many principal regular maximal ideals. Hence, I can be contained in only finitely many regular maximal ideals.

□

Lemma 5.6 *If P is a principal regular maximal ideal of R, then $R_{(P)}$ is a discrete rank one valuation ring.*

Proof Let P be a principal regular maximal ideal of R. Since $R_{(P)}$ has a unique regular maximal ideal $PR_{(P)}$ which is principal, every regular ideal of $R_{(P)}$ is of the form $P^nR_{(P)}$, where n is a positive integer by Proposition 5.5. For any regular element $x \in R_{(P)}$, define a valuation map as follows:

$$v(x) = \begin{cases} 0 & \text{if} \quad x \in R_{(P)} \setminus PR_{(P)}, \\ n & \text{if} \quad x \in P^nR_{(P)} \setminus P^{n+1}R_{(P)}, \\ \infty & \text{if} \quad x \in \cap P^nR_{(P)}. \end{cases}$$

It can be easily shown that $v(0) = \infty, v(1) = 0$ and $v(xy) = v(x) + v(y)$, and $v(x + y) \geq min\{v(x), v(y)\}$ for any regular elements $x, y \in R$. So, each regular element in $R_{(P)}$ has a finite v-value, and v may be extended to a valuation on $Q(R_{(P)})$. We can see that $R_{(P)} = \{x \in Q(R_{(P)}) \mid v(x) \geq 0\}$ and $PR_{(P)} = \{x \in Q(R_{(P)}) \mid v(x) > 0\}$. Thus, $(R_{(P)}, PR_{(P)})$ is a discrete rank one valuation ring. □

Lemma 5.7 *If M is a regular maximal ideal of R such that every other regular maximal ideal other than M is principal, then R is h-local.*

Proof It is enough to show that every regular prime ideal is colocal by Corollary 5.1. Let P be a regular prime ideal. If $P \subseteq N$, where $N \neq M$ is maximal ideal, then $PR_{(N)}$ is a regular prime ideal of $R_{(N)}$. Since $NR_{(N)}$ is the unique regular maximal ideal of $R_{(N)}$ and principal, we have $PR_{(N)}$ is also principal by Proposition 5.4. So by 5.4 we get $PR_{(N)} = NR_{(N)}$ which implies $P = N$. Otherwise, P is contained in M. Thus, R is h-local. □

Theorem 5.4 *(, Huckaba, 1988, Theorem 6.1) Let R be a ring and S an R-submodule of Q. If $\{M_\sigma\}$ is the set of regular maximal ideals of R and S contains a regular element of R, then $S = \bigcap S R_{(M_\sigma)}$.*

Lemma 5.8 *If M is a regular maximal ideal of R such that every other regular maximal ideal other than M is principal, then every regular ideal of R which is not contained in M is principal.*

Proof Let I be a regular ideal which is not contained in M. Since R is h-local by Lemma 5.7, I is contained in only finitely many maximal ideals , say $N_1, \ldots, N_t$, all of which are principal. Then $IR_{(N_i)} = a_i^{k_i} R_{(N_i)}$ for some $k_i \in \mathbb{N}$ where $N_i =< a_i >$ by Proposition 5.5. So, $IR_{(N)} = (a_1^{k_1} \cdots a_t^{k_t} R)_{(N)}$ for all regular maximal ideals of R. Therefore, by Theorem 5.4, $I = a_1^{k_1} \cdots a_t^{k_t} R$, that is, I is principal. $\square$

Now, we prove that, for direct sum of regular ideals, nearly isomorphism implies locally isomorphism. We note that the following lemma holds for all Marot rings without the assumption that their regular ideals are finitely generated.

Lemma 5.9 *Let R be a Marot ring, and let G and H direct sum of regular ideals. If G and H are nearly isomorphic, then they are locally isomorphic.*

Proof Suppose that G and H are nearly isomorphic, where $G = I_1 \oplus \cdots \oplus I_n$ and $H = J_1 \oplus \cdots \oplus J_k$ for some regular ideals I_i, J_k. Let M be a regular maximal ideal of R and $f : G \to H$ an embedding such that $M + \mathrm{Ann}(\mathrm{Coker}\, f) = R$. Since $I_1 \oplus \cdots \oplus I_n = G \cong Imf = \tilde{J}_1 \oplus \cdots \oplus \tilde{J}_k$, where $\tilde{J}_i$ are regular ideals, $\mathrm{Ann}(H/Imf)$ is regular. By regular localization, we obtain $M_{(M)} + \mathrm{Ann}(\mathrm{Coker}\, f)_{(M)} = R_{(M)}$. We conclude that $R_{(M)} = \mathrm{Ann}(\mathrm{Coker}\, f)_{(M)}$ since $R_{(M)}$ has a unique regular maximal ideal $M_{(M)}$. So $\mathrm{Ann}(\mathrm{Coker}\, f)$ is not contained in M. Since R is a Marot ring, there exists a regular element $x \in \mathrm{Ann}(\mathrm{Coker}\, f) \setminus M$. Since $xH \subseteq \mathrm{Im}\, f$, we get that $H_{(M)} = xH_{(M)} \subseteq \mathrm{Im}\, f_{(M)} \subseteq H_{(M)}$. Hence, $H_{(M)} = Imf_{(M)}$, and $f_{(M)}$ is surjective. $\square$

Before the next proposition, we recall an important result that is related to h-local Marot rings. An R-module T is called torsion R-module if it is annihilated by a regular element of R.

Theorem 5.5 *(Klingler & Omairi, 2021) The following conditions are equivalent for a Marot ring R.*

1. *R is an h-local ring.*

2. *Every torsion R-module T is canonically isomorphic to $\oplus T_{(M)}$, where M ranges over the maximal ideals M of R.*

The following proposition plays an important role in classifying the UDRI property. It shows that for the modules that are direct sum of regular ideals of R, locally isomorphism implies isomorphism.

Proposition 5.7 *Let G and H be R-modules that are direct sum of regular ideals, and suppose R has a unique non-principal regular maximal ideal M. If $G_{(M)} \cong H_{(M)}$, then $G \cong H$.*

Proof First, suppose that G and H are regular ideals. Since $G_{(M)} \cong H_{(M)}$, there exists an isomorphism $\phi : G_{(M)} \to H_{(M)}$. Since G is torsion-free, $G \subseteq G_{(M)}$, and so we can consider its restriction $\phi : G \to H_{(M)}$. Since G is finitely generated, there exists a regular element $s \notin M$ such that $f = s\phi$ maps G into H. The map $f_{(M)}$ is also an isomorphism because s is a unit in $R_{(M)}$. So, we have $(\operatorname{Ker} f)_{(M)} = \operatorname{Ker} f_{(M)} = 0$, and this implies $\operatorname{Ker} f = 0$.

Now let $I = \operatorname{Ann}(H/\operatorname{Im} f)$. Since $f_{(M)}$ is onto, we have $IR_{(M)} = R_{(M)}$ and so I is a regular ideal which is not contained in M. So, I is principal by Theorem 5.8. Let $I = tR$. By Lemma 5.7, t is contained in only finitely many maximal ideals which all are principal, say N_i, $i = \{1, 2, \ldots, n\}$. Also, again, since R is h-local, R/tR and H/tH have a decomposition such that $R/tR \cong \oplus_{i=1}^{n}(R/tR)_{(N_i)}$ and $H/tH \cong \oplus_{i=1}^{n}(H/tH)_{(N_i)}$ (Klingler & Omairi, 2021). We know that $H_{(N_i)} \cong R_{(N_i)}$, and this implies $(tH)_{(N_i)} \cong (tR)_{(N_i)}$. Thus, we get $R/tR \cong H/tH$. Since $t \in I$, $tH \subseteq \operatorname{Im} f$. So, there is a surjection $\alpha : R \to H/\operatorname{Im} f$ defined by $\alpha(r) = rx$, where x is a generator of $H/\operatorname{Im} f$. We note that $\operatorname{Ker} \alpha = tR = I$, and we have $H/tH \cong H/\operatorname{Im} f$. Since R/tR is Artinian, H/tH and $H/\operatorname{Im} f$ have the same finite length but since $tH \subseteq \operatorname{Im} f$, we conclude that $H \cong tH = \operatorname{Im} f \cong G$.

We claim that G and H cannot have different number of summands. Suppose that $G = J_1 \oplus \cdots \oplus J_n$ and $H = I_1 \oplus \cdots \oplus I_k$ with $G_{(M)} \cong H_{(M)}$. By the first part, there exists an injection $\varphi : G \to H$, and so $G \cong \operatorname{Im} \varphi = X_1 \oplus \cdots \oplus X_k$ where X_i are regular ideals. Take any principal regular maximal ideal N. Then we have $R_{(N)}^{(n)} \cong R_{(N)}^{(k)}$ which implies $k = n$.

Assume that $G = J_1 \oplus J_2$ and $H = I_1 \oplus I_2$ with $G_{(M)} \cong H_{(M)}$. Let f be the injective map from G into H such that $f_{(M)}$ is an isomorphism. Set $\operatorname{Im} f = X_1 \oplus X_2$ and $\Pi : I_1 \oplus I_2 \to I_1$. Define $g = \Pi \circ f$. Since $\Pi_{(M)}, f_{(M)}$ are onto, we get $(X_1)_{(M)} \cong (I_1)_{(M)}$. By the first part of the proof, we get $X_1 \cong I_1$. Similary, $I_2 \cong X_2$ and so $G \cong H$. By induction, we are done. □

Proposition 5.8 *Assume R has a unique non-principal regular maximal ideal M and G and H be direct sum of regular ideals of R. Then the following are equivalent:*

(a) G is nearly isomorphic to H.

(b) $G_{(M)} \cong H_{(M)}$.

(c) G and H are locally isomorphic.

(d) G is isomorphic to H.

Proof Follows from Lemma 5.9 and Proposition 5.7. □

Finally, we are ready to give a necessary and sufficient condition for R to satisfy the UDRI property.

Theorem 5.6 *R has the UDRI property if and only if R has at most one non-principal regular maximal ideal M and $R_{(M)}$ has the UDRI property.*

Proof R must have at least one regular maximal ideal since if every regular maximal ideal of R is principal, then every regular ideal is principal by Proposition 5.4, and in this case R has UDRI. By Lemma 5.3, R has at most one regular maximal ideal, say M. Suppose that $I'_1 \oplus \cdots \oplus I'_n \cong J'_1 \oplus \cdots \oplus J'_m$, where I'_i and J'_k are regular ideals of $R_{(M)}$. Since every regular ideal of $R_{(M)}$ is of the form $I_{(M)}$, where I is a regular ideal of R, we can write R-modules $G = I_1 \oplus \cdots \oplus I_n$ and $H = J_1 \oplus \cdots \oplus J_m$, where $I'_i = I_iR_{(M)}, J'_i = J_iR_{(M)}$ so that $G_{(M)} \cong H_{(M)}$. By Proposition 5.8, $G \cong H$, and by assumption, $m = n$, and after a possible reindexing, $I_i \cong J_i$ implying that $I'_i \cong J'_i$.

For the converse, suppose that the R-modules $G = I_1 \oplus \cdots \oplus I_n$ and $H = J_1 \oplus \cdots \oplus J_m$ are isomorphic, where I_i, J_k are regular ideals of R. Then, we can write $G_{(M)} \cong H_{(M)}$ so that $G_{(M)}$ and $H_{(M)}$ are direct sum of regular ideals of $R_{(M)}$. By assumption, $m = n$, and after a possible reindexing, $I_iR_{(M)} \cong J_iR_{(M)}$. Again, by Proposition 5.8, $I_i \cong J_i$. □

Example 5.2 Let $D = \mathbb{Z}[\sqrt{10}]$. Define the set

$$X = \{I \in Max(D) \mid I \text{ is a principal maximal ideal}\}.$$

Since $17D \in X$, $X \neq \emptyset$. Choosing $\mathcal{P} = Max(D) \setminus (X \cup \{M\})$, where $M = 2D + \sqrt{10}D$, we construct the ring $R = D + B$ of the form $A + B$ corresponding to D and $\mathcal{P}$. Then R is a Marot ring whose regular ideals are finitely generated and R has a unique regular non-principal maximal ideal which is $MR = M + B$. Since $R_{(MR)}$ is a discrete valuation ring [(Lucas, 2016),Example 2.7], $R_{(MR)}$ has UDRI property. Thus, R has UDRI by Theorem 5.6.

5.4. UDRI for Overrings

Througout this section, R is assumed to be a Marot ring whose regular ideals are finitely generated. Our purpose is to show that if R has the UDRI property and $reg-dim(R) \leq 2$ then $\tilde{R}$, the integral closure of R, has the UDRI property. We start by the following useful lemma for the next result.

Lemma 5.10 *Let R' be an overring of R and N a regular maximal ideal of R which contracts to a regular principal maximal ideal of R. Then N is principal.*

Proof Let $P = N \cap R = xR$ be the maximal ideal where $x \in P$ is a regular element of R. Since $R_{(P)}$ has a unique regular maximal ideal which is principal, $R_{(P)}$ is a discrete rank one valuation ring. Since $(Q(R))_{(P)} \subset Q(R_{(P)})$ and R' is an overring of R, we have the inclusions $R_{(P)} \subset R'_{(P)} \subset (Q(R))_{(P)} \subset Q(R_{(P)})$ which implies $R'_{(P)}$ is an overring of $R_{(P)}$. So, we conclude that $R'_{(P)} = R_{(P)}$ or $R'_{(P)} = Q(R_{(P)})$ because $R_{(P)}$ is a Marot discrete rank one valuation ring by [(Huckaba, 1988), Lemma 8.1]. Suppose that $R'_{(P)} = Q(R_{(P)})$. Since N is a regular maximal ideal in R', $N_{(P)}$ is regular maximal ideal in $R'_{(P)} = Q(R_{(P)})$. But every regular element in $Q(R_{(P)})$ is a unit, and this implies $N_{(P)} = R'_{(P)}$, which is a contradiction. Therefore, we must have $R_{(P)} = R'_{(P)}$. We claim that $N = xR'$. Since N is a regular R-submodule of $Q(R)$, we need to show that N and xR' are locally equal by Theorem 5.4. If $M \neq P$ is a regular maximal ideal, $x \notin M$ implies $(xR')_{(M)} = R'_{(M)}$, and since $R'_{(M)} = xR'_{(M)} \subseteq N_{(M)} \subseteq R'_{(M)}$, we get $N_{(M)} = R'_{(M)} = (xR')_{(M)}$. Also, we have

$$P_{(P)} = (xR)_{(P)} = xR_{(P)} = xR'_{(P)} \subseteq N_{(P)} \subseteq R'_{(P)} = R_{(P)}.$$

Since $P_{(P)}$ is maximal in $R_{(P)}$, we have $P_{(P)} = N_{(P)}$, and it follows that $N_{(P)} = xR_{(P)} = xR'_{(P)} = (xR')_{(P)}$. Thus, $N = \cap N_{(M)} = \cap (xR')_{(M)} = xR$ implies that $N = xR'$.

$\square$

We are ready to show that if R has the UDRI property and $reg - dim(R) \leq 2$ then $\tilde{R}$, the integral closure of R, has the UDRI property. For the proof of this result, we state some helpful lemmas which are valid for a commutative ring R.

Lemma 5.11 *((Klingler & Omairi, 2020), Lemma 3.1) If $\phi : I \to J$ is an isomorphism of R-ideals, and R' is an overring of R finitely generated as an R-module, then ϕ extends uniquely to an isomorphism $\phi' : R'I \to R'J$ of R'-ideals.*

We show that the UDRI property passes to finitely generated overrings.

Lemma 5.12 *If R has the UDRI property and R' is an overring of R which is a finitely generated R-module, then R' has the UDRI property.*

Proof Suppose $I_1 \oplus \ldots I_n \cong J_1 \oplus \ldots J_m$, where I_i, J_k are regular ideals of R'. Since R' is finitely generated as an R-module, there exists a regular element $d \in R$ such that $dR' \subseteq R$. So, we get the isomorphism $dI_1 \oplus \ldots \oplus dI_n \cong dJ_1 \oplus \ldots \oplus dJ_m$ where dI_i, dJ_k are regular ideals of R. Since R has UDRI, we get $n = m$ and after reindexing $dI_i \cong dJ_i$. Since d is regular, we have $I_i \cong J_i$ as R-modules. By Lemma 5.11, $I_i \cong J_i$ as R'-ideals which completes the proof. $\square$

Proposition 5.9 *If R has the UDRI property and $reg - dim(R) \leq 2$, then $\tilde{R}$ has the UDRI property.*

Proof Suppose that $\phi\colon H_1 \oplus H_2 \oplus \cdots \oplus H_m \to K_1 \oplus K_2 \oplus \cdots \oplus K_n$ is an isomorphism, where H_i, $1 \leqslant i \leqslant m$, and K_t, $1 \leqslant t \leqslant n$, are regular ideals of $\tilde{R}$. Let $\{h_\alpha\}$ be the union of generating sets of the ideals H_i and $\{k_\beta\}$ the union of generating sets of the ideals K_t. These sets are finite and subsets of regular elements of $\tilde{R}$ by [(Chang, 1999), Theorem 7]. Then, there are elements $r_{\alpha,\beta} \in \tilde{R}$ such that $\phi(h_\alpha) = \sum_\beta r_{\alpha,\beta} k_\beta$ for each index α and elements $s_{\beta,\alpha} \in \tilde{R}$ such that $k_\beta = \sum_\alpha s_{\beta,\alpha}\phi(h_\alpha)$. Let R' be the overring of R defined as $R' = \sum R[h_\alpha] + \sum R[k_\beta] + \sum R[r_{\alpha,\beta}] + \sum R[s_{\beta,\alpha}]$. Since these elements are integral over R, R' is finitely generated as an R-module by [(Atiyah & Macdonald, 1969), Proposition 5.1] and so R' has UDRI by Lemma 5.12. Hence, we can replace R' by R and assume that all of the elements $h_\alpha, k_\beta, r_{\alpha,\beta}, s_{\beta,\alpha}$ are regular elements in R.

For each index α, let I_α be the R-ideal generated by the elements of $\{h_\alpha\}$ which generate H_i as an $\tilde{R}$ ideal and for each index β, let J_β be the R-ideal generated by the elements of $\{k_\beta\}$ which generate K_t as an $\tilde{R}$ ideal. Then we see that $\tilde{R}I_\alpha = H_\alpha$ and $\tilde{R}J_\beta = K_\beta$. Let φ be the restriction of ϕ to $I_1 \oplus \cdots \oplus I_m$. Then we get $\varphi(I_1 \oplus \cdots \oplus I_m) = J_1 \oplus \cdots \oplus J_n$, and φ is an isomorphism of direct sums of regular ideals of R.

By assumption, we have $m = n$ and renumbering if necessary, there is an R-isomorphism $\varphi_i : I_i \to J_i$. Take any $x \in H_\alpha$ and $y \in K_\beta$. Then we can write these elements as $x = \sum_\alpha r_\alpha h_\alpha$ and $y = \sum_\beta s_\beta k_\beta$ for some $r_\alpha, s_\beta \in \tilde{R}$ using the generating sets given above. Let R' be the overring defined as $R' = \sum R[r_\alpha] + \sum R[s_\beta]$. Since these elements are integral over R, R' is finitely generated as an R-module, and therefore φ_i extends to an R'-isomorphism $\varphi_i' : R'I_i \to R'J_i$. Since $\varphi_i'(x)$ is independent of the choice of finite extension of R containing x, we define $\tilde{\varphi}_i(x) = \varphi_i'(x)$. Then $\tilde{\varphi}_i$ is injective since each extension of φ to a finitely generated overring is injective, and since $y \in \mathrm{Im}(\varphi_i')$, it follows that $\tilde{\varphi}_i$ is surjective. Also for any $x, y \in H_\alpha$ and $r \in \tilde{R}$, we have $\tilde{\varphi}_i(x + y) = \tilde{\varphi}_i(x) + \tilde{\varphi}_i(y)$ and $\tilde{\varphi}_i(rx) = r\tilde{\varphi}_i(x)$ implying that $\tilde{\varphi}_i$ is an $\tilde{R}$-isomorphism. □

CHAPTER 6

CONCLUSION

In this thesis Krull-Schmidt properties over Non-Noetherian rings are investigated. Mainly, weakly Matlis domains, strong Mori domains, and Marot rings are studied, all of which are among the group of Non-Noetherian rings. Firstly w-weak isomorphism types are defined, and the connections between these isomorphism types are characterized for torsionless modules over weakly Matlis domains. By using the comparison of w-weak isomorphism types, Krull-Schmidt property on w-ideals of a strong Mori domain is examined. Applying the obtained results, these properties are also discussed for overrings of strong Mori domains. Some preliminary results about Marot rings satisfying ascending chain condition on regular ideals are obtained. Furthermore, Krull-Schmidt property on regular ideals of a Marot ring and overrings of a Marot ring are studied.

REFERENCES

Anderson, D. D., Winders, M. 2009: Idealization of a module. *Journal of Commutative Algebra*, **1(1)**, 3-56.

Anderson, D. D., Zafrullah, M. 1999: Independent locally-finite intersections of localizations. *Houston J. Math*, **25(3)**, 433-452.

Anderson, D. D., Zafrullah, M. 1991: Almost Bezout domains. *Journal of Algebra*, **142(2)**, 285-309.

Anderson, D. D., Chun, S. 2014: The set of torsion elements of a module. *Communications in Algebra* , **42(4)**, 1835-1843.

Anderson, D. D., Zafrullah, M. 1993: On t-invertibility III. *Communications in Algebra*, **21(4)**, 1189-1201.

Atiyah, M.F., Macdonald, I.G. 1969: Introduction to Commutative Algebra. *Boston, MA, USA: Addition-Wesley Publishing Company.*

Ay, B., Klingler, L. (2011): Unique decomposition into ideals for reduced commutative Noetherian rings. *Trans. Am. Math. Soc.* **363 (7)**: 3703-3716.

Ay, B. and Klingler, L. 2012: Unique decomposition into ideals for reduced commutative Noetherian rings II. *J. Pure Appl. Algebra*, **216(4)**, 743-751.

Ay Saylam, B., Klingler, L. 2019: Locally isomorphic torsionless modules over domains of finite character. *J. Algebra and Its Applications*, **18(07)**, 1950138.

Ay Saylam, B. and Gürbüz, E. 2022: Unique decomposition into regular ideals for Marot rings. *Communications in Algebra*

Ay Saylam, B., Gürbüz, E. and Hamdi,H. 2022: Unique decomposition into w ideals for strong Mori domains. *J. Algebra and Its Applications*. (to appear)

Ay Saylam, B., Gürbüz, E. and Hamdi,H. 2022: w-locally isomorphic torsionless modules over weakly Matlis domain.(in preparation)

Badawi, A. 2003: On nonnil-Noetherian rings. *Communications in Algebra*, **31**, 1669-1677.

Chang, G.W. 1999: Integral closure of a Marot ring whose regular ideals are finitely generated. *Communications in Algebra*, **27(4)**: 1783-1795.

Chang, G. W., Kim, H. 2017: The w-FF property in trivial extensions. *Bull. Iranian Math. Soc.* **43(7)**, 2259-2267.

Chase, S. U. 1962: Torsion-free modules over $K[X,Y]$. *Pacific Journal of Math.*, **12**, 437-447.

Dummit, D. S., Foote, R.M. 2004: Abstract Algebra. *John Wiley and Sons Inc.*

El Baghdadi, S., Kim, H., and Wang, F. 2014. Injective modules over Prüfer v-multiplication domains. *Communications in Algebra*, **42(1)**, 286-298.

El Baghdadi, S., Fontana, M. 2004: Semistar linkedness and flatness, Prüfer semistar multiplication domains. *Communications in Algebra*, **32(3)**, 1101-1126.

El-Baghdadi, S. 2010: Semistar GCD domains. *Communications in Algebra*, **38(8)**, 3029-3044.

Estes, D., Guralnick, R. 1982: Module Equivalences : local to global when primitive polynomials represents units. *Journal of Algebra*, **77**, 138-157.

Evans, E. 1973: Krull-Schmidt and cancellation over local rings. *Pac. J. Math.*, **46(1)**, 115-121.

Glaz, S. 2002: Controlling the zero divisors of a commutative ring. *Marcel Dekker Lecture Notes in Pure and Applied Mathematics*, **231**: 191-212.

Goeters, P., Olberding, B. 2001: Unique decomposition into ideals for Noetherian domains. *Journal of Pure and Applied Algebra*, **165**, 169-182.

Goeters, P., Olberding, B. 2002: On Locally Isomorphic Torsion-Free Modules. *Commutative Rings (Nova Sci. Publ., Hauppauge, NY)*, 83-94.

Goeters, P., Olberding, B. 2002: The Krull-Schmidt Property for Ideals and Modules over Integral Domains. *Rocky Mountain J. Math.*, **32**, 1409-1429.

Glaz, S., Vasconcelos, W. V. 1977: Flat ideals II. *Manuscripta Math.* **22(4)**, 325-341.

Huckaba, J.A. 1988: Commutative rings with zero divisors. *Marcel Dekker, New York*

Kaplansky, I. 1949: Elementary divisors and modules. *Trans. Am. Math. Soc.*, **66 (2)**, 464-491.

Kaplansky, I. 1952: Modules over Dedekind rings and valuation rings. *Trans. Amer. Math. Soc.*, **72(2)**, 327-340.

Kim, H., Wang, F. G. 2012: On φ-strong Mori rings. *Houston J. Math*, **38**, 359-371.

Klingler, L., Omairi, A. 2020: Unique decomposition into ideals for commutative Noetherian rings. *Journal of Pure and Applied Algebra*, **224(9)**:106364.

Klingler, L., Omairi, A. 2021: *h*-local Rings. *Commutative Algebra: 150 years with Roger and Sylvia*, **773**.

Lucas, T.G. 2016: Addivitely regular rings and Marot rings. *Palestine Journal of Mathematics*, **5 (1)**, 90-99.

Manis, M.E. 1967: Extension Of valuation theory. *Bull. Amer. Math. Soc.*, **73** 735-736.

Marot, J. 1977: Ph.D Thesis: Sur Les Anneaux Universeillement Japonais, Univ. Paris-Sud.

Mimouni, A. 2005: Integral domains in which each ideal is a *w*-ideal. *Communications in Algebra*, **33(5)**, 1345-1355.

Serre, J. P. 1979: Local Fields. *Berlin, Springer-Verlag*

Wang, F. 1997: On *w*-projective modules and *w*-flat modules. *Algebra Colloq.*, *4(1)*, 111-120.

Yin, H., Wang, F., Zhu, X., Chen, Y. 2011: *w*-modules over commutative rings. *J. Korean Math. Soc*. **48**, 207-222.

Vasconcelos, W.V. 1965: On local and stable cancellation, *An. Acad. Bras. Ciencios*, **37**, 389-393.

Wang, F., McCasland, R. 1997: On *w*-modules over strong Mori domains. *Commun. Algebra*, **25 (4)**, 1285-1306.

Wang, F., McCasland, R. 1999: On strong Mori domains. *J. Pure Appl. Algebra*, **135 (2)**, 155-165.

Wang, F., Kim, H. 2016: Foundations of commutative rings and their modules. *Springer Singapore*

Zafrullah, M. 2006: What *v*-coprimality can do for you. *Multiplicative Ideal Theory in Commutative Algebra, Springer US*, **387-404**.

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