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ABSTRACT

The time-dependent Schrödinger equation describing a generalized two-dimensional quantum parametric oscillator in the presence of time-variable external fields is solved using the evolution operator method. For this, the evolution operator is found as a product of exponential operators through the Wei–Norman Lie algebraic approach. Then, the propagator and time-evolution of eigenstates and coherent states are derived explicitly in terms of solutions to the corresponding system of coupled classical equations of motion. In addition, using the evolution operator formalism, we construct linear and quadratic quantum dynamical invariants that provide connection of the present results with those obtained via the Malkin–Man'ko–Trifonov and the Lewis–Riesenfeld approaches. Finally, as an exactly solvable model, we introduce a Cauchy–Euler type quantum oscillator with increasing mass and decreasing frequency in time-dependent magnetic and electric fields. Based on the explicit results for the uncertainties and expectations, squeezing properties of the wave packets and their trajectories in the two-dimensional configuration space are discussed according to the influence of the time-variable parameters and external fields.

I. INTRODUCTION

In the present work, we consider the time-evolution problem for a quantum system described by a generalized two-dimensional quadratic Hamiltonian of the form

$$\hat{\mathcal{H}}_{gen}(t) = \sum_{j=1}^{2}\left(\frac{\hat{p}_j^2}{2\mu(t)} + \frac{\mu(t)\omega^2(t)}{2}\hat{q}_j^2 + \frac{B(t)}{2}(\hat{q}_j\hat{p}_j + \hat{p}_j\hat{q}_j) + D_j(t)\hat{p}_j + E_j(t)\hat{q}_j\right) + \lambda(t)(\hat{q}_1\hat{p}_2 - \hat{q}_2\hat{p}_1), \tag{1}$$

where $\mu(t)$, $\omega^2(t)$, $B(t)$, $D_j(t)$, $E_j(t)$, and $\lambda(t)$ are real-valued parameters depending on time t. This Hamiltonian is usually used to describe quantum particles in two-dimensional space and comprises many fundamental physical systems as subcases. A significant physical and mathematical distinction can be done according to the coupling parameter $\lambda(t)$. When $\lambda(t) = 0$, one has a two-dimensional quantum parametric oscillator with time-dependent mass $\mu(t) > 0$, frequency $\omega(t)$, squeezing parameter $B(t)$, and driving forces $D_j(t)$, $E_j(t)$ $(j = 1, 2)$. Since in that case Hamiltonian (1) is separable, formally one can speak about two independent one-dimensional oscillators. As known, for solving one-dimensional non-stationary quantum oscillator problems, there are many powerful approaches, such as Feynman path integral,[1] Husimi,[2] Lewis–Riesenfeld (LR),[3] Malkin–Man'ko–Trifonov (MMT),[4,5] and Wei–Norman (WN) approaches.[6] All these methods were used for a long time and in many research articles.[7–15] Recently, in Ref. 16, by a straightforward application of the Wei–Norman technique and by properly choosing the ordering of the exponential operators, we found the evolution operator for a quantum parametric oscillator described by a Hamiltonian with a $SU(1,1)\oplus h(4)$ group structure. The significance of our results is that for a time-dependent one-dimensional Schrödinger equation (SE) with the most general quadratic in the position and momentum Hamiltonian, we were able to determine the evolution operator explicitly in terms of two linearly independent homogeneous solutions and a particular solution to the corresponding classical equation of motion. This allowed us to give an exact description of the quantum dynamics and its relation with the corresponding classical motion. Later, based on these results, in Ref. 17, we discussed the squeezing and resonance properties of coherent states of a generalized Caldirola–Kanai type dissipative model, and in Ref. 18, time-evolution of squeezed coherent states for the most general one-dimensional quadratic parametric oscillator were obtained.

Clearly, wave function solutions of the N-dimensional harmonic oscillator, and, in particular, of the two-dimensional oscillator described by (1) when $\lambda(t) = 0$, can be easily written as a product of solutions to the one-dimensional problem.[4,19,20] Since the dynamics in higher dimensions is always more interesting and brings new questions, the N-dimensional oscillator always remains under research, and for some recent works, one can see Refs. 21–24.

On the other hand, when $\lambda(t) \neq 0$, that is in the presence of the angular momentum operator $\hat{L} = \hat{q}_1\hat{p}_2 - \hat{q}_2\hat{p}_1$, Hamiltonian (1) can be used to describe the motion of a charged particle in time-dependent magnetic and electric fields. In that context, parameter $\lambda(t)$ is known as the Larmor frequency, $\omega(t)$ is the modulated frequency, and $E_j(t)$ $j = 1, 2$ are parameters of the external electric field. The problem of a charged particle in magnetic and electric fields is addressed in numerous research articles and has applications in electromagnetic theory, quantum optics, plasma physics, etc. For non-stationary systems, including a charged particle in a time-dependent electromagnetic field, long time ago Lewis and Riesenfeld[3] derived explicitly time-dependent quadratic invariants. Soon after, Malkin, Man'ko, and Trifonov suggested the use of linear in position and momentum invariants[4,5,26–29] and constructed two-dimensional coherent states of Gaussian type, which can be seen as a generalization of the Glauber coherent states of the one-dimensional harmonic oscillator.[25,31] Then, time-dependent coherent states in a magnetic field were discussed in many related articles.[26–30] For a recent review of various families of coherent states, squeezed states, and their generalizations for a charged particle in a magnetic field, including Gaussian and non-Gaussian states, one can see the work of Dodonov.[32]

Although significant progress has been done, the problem of a charged particle in an electromagnetic field is still an active area of research. Different approaches were proposed and various generalizations were discussed. For example, in Ref. 33, the time-evolution problem of a charged oscillator under the combined action of arbitrary electric and magnetic fields was solved in Heisenberg picture, and then Green's function and coherent states were found. In Ref. 34, authors employed linear and quadratic invariants to find coherent states from which number states and propagator were derived for the time-dependent isotropic charged oscillator. A similar approach was developed in Ref. 35 to solve the problem for an anisotropic charged oscillator in a constant magnetic field. In Ref. 36, an algebraic approach was used to find the evolution operator and wave functions of the two-dimensional harmonic oscillator with time-dependent mass and frequency in a static magnetic field. In Ref. 37, wave functions of a time-dependent coupled oscillator in a variable magnetic field were found by a unitary transformation approach and the method of quadratic invariants. For more recent results, one can see Refs. 38–40, where a Lie algebraic approach is used to find the evolution operator for generalized quadratic oscillators, including oscillators in the presence of magnetic and electric fields. In those works, the technique is based on step by step application of unitary transformations that reduce the Floquet operator, and symbolic calculations of most expressions are done using the Mathematica program. More recently, in Ref. 41, unitary transformations were used to reduce the Hamiltonians, and a separation of variables method was proposed to solve problems for time-dependent quadratic Hamiltonians.

In the present work, we solve the two-dimensional quantum parametric oscillator described by the generalized quadratic Hamiltonian (1) using the evolution operator approach. We find the exact evolution operator by first applying a simple unitary transformation to decouple the Schrödinger equation and then using the Wei–Norman Lie algebraic technique, as in Ref. 16. This gives the evolution operator of the problem as a finite product of unitary exponential operators being generators of a Lie group associated with the closed Lie algebra describing the Hamiltonian.

A crucial point in the Lie algebraic techniques is that of finding all time-variable coefficients that completely determine the evolution operator as a product of Lie group generators. Usually this requires solution of a large nonlinear system of ordinary differential equations, which is not always an easy task, and in most works, it is usually solved by quadratures. The utility of our results is that all time-variable coefficients in the formulation of the evolution operator for the quantum problem are found explicitly in terms of solutions to the corresponding system of classical equations of motion. Then, the propagator (Green's function), time-evolution of the wave functions, expectations of the position and momentum, and their uncertainties are also found in terms of the classical solutions. Furthermore, using the evolution operator formalism, we also construct linear and quadratic quantum invariants and compare our results with those obtained using the MMT- and the LR-approaches.

The main goal of this work is to provide exact and explicit results that allow us to investigate the influence of the time-dependent parameters and external terms on the dynamics of the quantum particle described by Hamiltonian (1). Special attention is paid to the study of the squeezing properties of the wave packets and their trajectories in the presence of time-dependent magnetic and electric fields. For this, we organized this paper as follows: In Sec. II, we introduce the problem at the classical level. We begin by the classical Hamiltonian corresponding to (1) and find solutions to the associated system of coupled classical equations of motion. In Sec. III, for the time-dependent Schrödinger equation with Hamiltonian (1), the evolution operator and the propagator (Green's function or fundamental solution) are obtained explicitly in terms of the classical solutions found in Sec. II. In Sec. IV, we describe the exact time-evolution of harmonic oscillator eigenstates and Glauber coherent states under the influence of the generalized Hamiltonian (1). In Sec. V, dynamical invariants for the quantum problem are found and used to compare the results in the present work with those obtained by the MMT- and the LR-techniques. In Sec. VI, as a generalization of the one-dimensional Cauchy–Euler type dissipative oscillator in Ref. 42, we introduce an exactly solvable Cauchy–Euler type quantum parametric oscillator in time-dependent magnetic and electric fields and discuss the dynamical properties of the quantum states, and using concrete numerical values, we draw some illustrative plots. Section VII includes a brief discussion and concluding remarks.

II. THE CLASSICAL PROBLEM

First, we consider a classical two-dimensional oscillator described by the Hamiltonian

$$\mathcal{H}_{cl}(t) = \sum_{j=1}^{2}\left(\frac{P_j^2}{2\mu(t)} + \frac{\mu(t)\omega^2(t)}{2}X_j^2 + B(t)X_jP_j + D_j(t)P_j + E_j(t)X_j\right) + \lambda(t)(X_1P_2 - X_2P_1),$$

where $\mu(t) > 0, \omega^2(t), B(t), D_j(t)$, and $E_j(t)$, $j = 1, 2$, are real-valued parameters depending on time and $\lambda(t)$ is a coupling parameter. The corresponding Hamilton's equations of motion are

$$\begin{aligned}
\dot{X}_1 &= \frac{\partial\mathcal{H}_{cl}}{\partial P_1} \equiv \frac{P_1}{\mu(t)} + B(t)X_1 + D_1(t) - \lambda(t)X_2,\\
\dot{X}_2 &= \frac{\partial\mathcal{H}_{cl}}{\partial P_2} \equiv \frac{P_2}{\mu(t)} + B(t)X_2 + D_2(t) + \lambda(t)X_1,\\
\dot{P}_1 &= -\frac{\partial\mathcal{H}_{cl}}{\partial X_1} \equiv -\left(\mu(t)\omega^2(t)X_1 + B(t)P_1 + E_1(t) + \lambda(t)P_2\right),\\
\dot{P}_2 &= -\frac{\partial\mathcal{H}_{cl}}{\partial X_2} \equiv -\left(\mu(t)\omega^2(t)X_2 + B(t)P_2 + E_2(t) - \lambda(t)P_1\right),
\end{aligned}$$

where "dot" denotes the derivative with respect to time. Then, the system of classical equations of motion in position space becomes

$$\begin{pmatrix}\ddot{X}_1\\ \ddot{X}_2\end{pmatrix} + \begin{pmatrix}\frac{\dot{\mu}}{\mu} & 2\lambda\\ -2\lambda & \frac{\dot{\mu}}{\mu}\end{pmatrix}\begin{pmatrix}\dot{X}_1\\ \dot{X}_2\end{pmatrix} + \begin{pmatrix}\Omega_X(t) & \frac{\dot{\mu}}{\mu}\lambda + \dot{\lambda}\\ -\frac{\dot{\mu}}{\mu}\lambda - \dot{\lambda} & \Omega_X(t)\end{pmatrix}\begin{pmatrix}X_1\\ X_2\end{pmatrix} = \begin{pmatrix}F_{1,X}(t)\\ F_{2,X}(t)\end{pmatrix}, \tag{2}$$

and for $\lambda(t) \neq 0$, it is a system of coupled second-order differential equations. In (2), we have

$$\Omega_X(t) = \omega^2(t) - \left(\dot{B}(t) + B^2(t) + \frac{\dot{\mu}}{\mu}B(t) + \lambda^2(t)\right),$$

and the forcing vector term

$$\mathbf{F}(t) \equiv \begin{pmatrix}F_{1,X}(t)\\ F_{2,X}(t)\end{pmatrix} = \begin{pmatrix}\left(\frac{\dot{\mu}}{\mu} + B(t)\right) & \lambda(t)\\ \lambda(t) & \left(\frac{\dot{\mu}}{\mu} + B(t)\right)\end{pmatrix}\mathbf{D}(t) + \frac{d}{dt}\mathbf{D}(t) - \frac{1}{\mu(t)}\mathbf{E}(t),$$

where we shall use the column vector notations interchangeably,

$$\mathbf{D}(t) = \begin{pmatrix}D_1(t)\\ D_2(t)\end{pmatrix} = (D_1(t), D_2(t))^T, \quad \mathbf{E}(t) = \begin{pmatrix}E_1(t)\\ E_2(t)\end{pmatrix} = (E_1(t), E_2(t))^T.$$

We note that if $D_1 = 0$ and $D_2 = 0$, then $\lambda(t)$ does not influence the forcing vector $\mathbf{F}(t)$. In addition, in momentum space, the system of oscillator equations becomes

$$\begin{pmatrix}\ddot{P}_1\\ \ddot{P}_2\end{pmatrix} + \begin{pmatrix}-\frac{\dot{(\mu\omega^2)}}{\mu\omega^2} & 2\lambda\\ -2\lambda & -\frac{\dot{(\mu\omega^2)}}{\mu\omega^2}\end{pmatrix}\begin{pmatrix}\dot{P}_1\\ \dot{P}_2\end{pmatrix} + \begin{pmatrix}\Omega_P(t) & \dot{\lambda} - \frac{\dot{(\mu\omega^2)}}{\mu\omega^2}\lambda\\ -\dot{\lambda} + \frac{\dot{(\mu\omega^2)}}{\mu\omega^2}\lambda & \Omega_P(t)\end{pmatrix}\begin{pmatrix}P_1\\ P_2\end{pmatrix} = \begin{pmatrix}F_{1,P}(t)\\ F_{2,P}(t)\end{pmatrix}, \tag{3}$$

where

$$\Omega_P(t) = \omega^2(t) + \left(\dot{B}(t) - B^2(t) - \frac{\dot{(\mu\omega^2)}}{\mu\omega^2}B(t) - \lambda^2(t)\right)$$

and

$$\mathbf{F_P}(t) \equiv \begin{pmatrix}F_{1,P}(t)\\ F_{2,P}(t)\end{pmatrix} = -\mu(t)\omega^2(t)\mathbf{D}(t) - \frac{d}{dt}\mathbf{E}(t) + \begin{pmatrix}\frac{\dot{(\mu\omega^2)}}{\mu\omega^2} + B(t) & -\lambda(t)\\ -\lambda(t) & \frac{\dot{(\mu\omega^2)}}{\mu\omega^2} + B(t)\end{pmatrix}\mathbf{E}(t).$$

Here, in momentum space, the forcing $\mathbf{F}_\mathbf{P}(t)$ will be affected by $\lambda(t)$ in the presence of electric fields. To solve systems (2) and (3), it is convenient to introduce the transformation of variables,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = R_\theta(t)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = R_\theta(t)\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, \tag{4}$$

where

$$R_\theta(t) = \begin{pmatrix} \cos\,\theta(t) & \sin\,\theta(t) \\ -\sin\,\theta(t) & \cos\,\theta(t) \end{pmatrix} \tag{5}$$

is a rotation matrix, and the rotation angle is defined as

$$\theta(t) = \int_{t_0}^{t} \lambda(s)ds. \tag{6}$$

Under transformation in (4), the coupled system (2) reduces to the decoupled system of two non-interacting damped oscillators,

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} \frac{\dot{\mu}}{\mu} & 0 \\ 0 & \frac{\dot{\mu}}{\mu} \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \begin{pmatrix} \Omega_x(t) & 0 \\ 0 & \Omega_x(t) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \widetilde{F}_{1,x}(t) \\ \widetilde{F}_{2,x}(t) \end{pmatrix}, \tag{7}$$

with the same damping parameter $\Gamma(t) = \dot{\mu}(t)/\mu(t)$ and the same frequency

$$\Omega_x(t) = \omega^2(t) - \left(\dot{B}(t) + B^2(t) + \frac{\dot{\mu}}{\mu}B(t)\right),$$

which is independent on $\lambda(t)$. On the other hand, the new forcing terms become

$$\widetilde{F}_{j,x}(t) = -\frac{\widetilde{E}_j}{\mu} + \dot{\widetilde{D}}_j + \left(\frac{\dot{\mu}}{\mu} + B\right)\widetilde{D}_j, \quad j = 1, 2,$$

and the relations between parameters $D_j(t), E_j(t)$ and $\widetilde{D}_j(t), \widetilde{E}_j(t)$, $j = 1, 2$ are found as

$$\widetilde{\mathbf{D}}(t) = R_\theta(t)\mathbf{D}(t), \quad \widetilde{\mathbf{E}}(t) = R_\theta(t)\mathbf{E}(t). \tag{8}$$

Similarly, in momentum space, we have

$$\begin{pmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{pmatrix} - \begin{pmatrix} \frac{(\dot{\mu\omega^2})}{\mu\omega^2} & 0 \\ 0 & \frac{(\dot{\mu\omega^2})}{\mu\omega^2} \end{pmatrix}\begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} + \begin{pmatrix} \Omega_p(t) & 0 \\ 0 & \Omega_p(t) \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \widetilde{F}_{1,p}(t) \\ \widetilde{F}_{2,p}(t) \end{pmatrix}, \tag{9}$$

where

$$\Omega_p(t) = \omega^2(t) + \left(\dot{B}(t) - B^2(t) - \frac{(\dot{\mu\omega^2})}{\mu\omega^2}B(t)\right)$$

and

$$\widetilde{F}_{j,p}(t) = -\dot{\widetilde{E}}_j + \left(\frac{(\dot{\mu\omega^2})}{\mu\omega^2} + B\right)\widetilde{E}_j - \mu\omega^2\widetilde{D}_j, \quad j = 1, 2.$$

Since the unforced part of each equation in the decoupled system (7) is same and it is of the form

$$\ddot{x}(t) + \frac{\dot{\mu}}{\mu}\dot{x}(t) + \Omega_x(t)x(t) = 0, \tag{10}$$

let $x_1^{(h)}(t)$ and $x_2^{(h)}(t)$ denote two linearly independent solutions of the homogeneous equation (10), satisfying the initial conditions, respectively,

$$x_1^{(h)}(t_0) = x_0 \neq 0, \quad \dot{x}_1^h(t_0) = x_0 B(t_0),$$
$$x_2^{(h)}(t_0) = 0, \quad \dot{x}_2^{(h)}(t_0) = \frac{1}{x_0 \mu(t_0)}. \tag{11}$$

Then, $\mathbf{x}^{(h)}(t) = (x_1^{(h)}(t), x_2^{(h)}(t))^T$ will denote the solution of the homogeneous part of system (7) with initial conditions (ICs) (11). For system (7) in the presence of forcing terms, we let $\mathbf{x}^{(p)}(t) = (x_1^{(p)}(t), x_2^{(p)}(t))^T$ denote the particular solution satisfying the initial conditions

$$\mathbf{x}^{(p)}(t_0) = \begin{pmatrix} x_1^{(p)}(t_0) \\ x_2^{(p)}(t_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \dot{\mathbf{x}}^{(p)}(t_0) = \begin{pmatrix} \dot{x}_1^{(p)}(t_0) \\ \dot{x}_2^{(p)}(t_0) \end{pmatrix} = \begin{pmatrix} \widetilde{D}_1(t_0) \\ \widetilde{D}_2(t_0) \end{pmatrix}.$$

Furthermore, if $p_1^{(h)}(t)$ and $p_2^{(h)}(t)$ denote two homogeneous solutions of the system of oscillator equations in momentum space given by (9), then they can be found in terms of the solutions of the classical equation in position space as

$$\mathbf{p}^{(h)}(t) = \begin{pmatrix} p_1^{(h)}(t) \\ p_2^{(h)}(t) \end{pmatrix} = \mu(t) \begin{pmatrix} \dot{x}_1^{(h)}(t) - B(t) x_1^{(h)}(t) - \widetilde{D}_1(t) \\ \dot{x}_2^{(h)}(t) - B(t) x_2^{(h)}(t) - \widetilde{D}_2(t) \end{pmatrix},$$

and the particular solution will be

$$\mathbf{p}^{(p)}(t) = \begin{pmatrix} p_1^{(p)}(t) \\ p_2^{(p)}(t) \end{pmatrix} = \mu(t) \begin{pmatrix} \dot{x}_1^{(p)}(t) - B(t) x_1^{(p)}(t) - \widetilde{D}_1(t) \\ \dot{x}_2^{(p)}(t) - B(t) x_2^{(p)}(t) - \widetilde{D}_2(t) \end{pmatrix}.$$

As a result, it follows that

$$\mathbf{X}^{(h)}(t) \equiv \begin{pmatrix} X_1^{(h)}(t) \\ X_2^{(h)}(t) \end{pmatrix} = R_\theta^T(t) \mathbf{x}^{(h)}(t),$$

with $R_\theta^T(t)$ being the transpose of $R_\theta(t)$, is a homogeneous solution to the coupled system (2) satisfying IC's,

$$\mathbf{X}^{(h)}(t_0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}, \quad \dot{\mathbf{X}}^{(h)}(t_0) = \begin{pmatrix} x_0 B(t_0) \\ x_0 \lambda(t_0) + \dfrac{1}{x_0 \mu(t_0)} \end{pmatrix} \tag{12}$$

and

$$\mathbf{X}^{(p)}(t) \equiv \begin{pmatrix} X_1^{(p)}(t) \\ X_2^{(p)}(t) \end{pmatrix} = R_\theta^T(t) \mathbf{x}^{(p)}(t)$$

is a particular solution to the forced coupled system (2), satisfying IC's,

$$\mathbf{X}^{(p)}(t_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \dot{\mathbf{X}}^{(p)}(t_0) = \begin{pmatrix} D_1(t_0) \\ D_2(t_0) \end{pmatrix}. \tag{13}$$

This establishes solutions to the classical problem, whose quantization using the usual replacement $X_j \to \hat{q}_j,\ P_j \to \hat{p}_j,\ X_j P_j \to (\hat{q}_j \hat{p}_j + \hat{p}_j \hat{q}_j)/2$, $j = 1, 2$ is discussed in Sec. III.

III. SOLUTION TO THE GENERALIZED QUANTUM PARAMETRIC OSCILLATOR

Now, we consider the evolution problem for a two-dimensional generalized quantum parametric oscillator given by

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{q}, t) = \hat{\mathcal{H}}_{gen}(t) \Psi(\mathbf{q}, t), \quad \mathbf{q} \in \mathbf{R}^2, \ t > t_0, \tag{14}$$

$$\Psi(\mathbf{q}, t_0) = \Psi^0(\mathbf{q}), \quad \mathbf{q} \in \mathbf{R}^2, \tag{15}$$

where $\Psi(\mathbf{q},t) := \Psi(q_1,q_2,t)$ is the wave function at time $t > t_0$, $\Psi^0(\mathbf{q}) := \Psi^0(q_1,q_2)$ is the initial state at time t_0, and the explicitly time-dependent Hamiltonian $\hat{\mathcal{H}}_{gen}(t)$ is given by (1), that is,

$$\hat{\mathcal{H}}_{gen}(t) = \sum_{j=1}^{2} \hat{\mathcal{H}}_j(t) + \lambda(t)\hat{L},$$

where

$$\hat{\mathcal{H}}_j(t) = \frac{\hat{p}_j^2}{2\mu(t)} + \frac{\mu(t)\omega^2(t)}{2}\hat{q}_j^2 + \frac{B(t)}{2}(\hat{q}_j\hat{p}_j + \hat{p}_j\hat{q}_j) + D_j(t)\hat{p}_j + E_j(t)\hat{q}_j,$$

$\hat{q}_j = q_j$ is the position operator, $\hat{p}_j = -i\hbar\partial/\partial q_j$ is the momentum operator, and $[\hat{q}_j, \hat{p}_j] = i\hbar\delta_{ij}$ for $j = 1,2$. In addition, $\hat{L} = \hat{q}_1\hat{p}_2 - \hat{q}_2\hat{p}_1$ is the angular momentum operator, which satisfies the well-known commutation relations

$$\left[\hat{L}, \sum_{j=1}^{2}\hat{p}_j^2\right] = 0, \quad \left[\hat{L}, \sum_{j=1}^{2}\hat{q}_j^2\right] = 0, \quad \left[\hat{L}, \sum_{j=1}^{2}(\hat{q}_j\hat{p}_j + \hat{p}_j\hat{q}_j)\right] = 0$$

and

$$[\hat{L}, \hat{q}_1] = i\hat{q}_2, \quad [\hat{L}, \hat{q}_2] = -i\hat{q}_1, \quad [\hat{L}, \hat{p}_1] = i\hat{p}_2, \quad [\hat{L}, \hat{p}_2] = -i\hat{p}_1,$$

showing that $\hat{L}$ does not commute with the position and momentum operators. Clearly, in the presence of the angular momentum operator, the Hamiltonian $\hat{\mathcal{H}}_{gen}(t)$ is coupled, but one can overcome this difficulty by introducing a unitary transformation,

$$\hat{U}_\theta(t,t_0) = \exp\left(\frac{i}{\hbar}\theta(t)\hat{L}\right),$$

where $\theta(t)$ is given by (6). Indeed, if we introduce the new wave function as

$$\psi(\mathbf{q},t) = \hat{U}_\theta(t,t_0)\Psi(\mathbf{q},t),$$

then initial value problem (IVP) (14) and (15) transform to the following IVP:

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{q},t) = \hat{H}_{dec}(t)\psi(\mathbf{q},t), \qquad \mathbf{q} \in \mathbf{R}^2, \ t > t_0, \tag{16}$$

$$\psi(\mathbf{q},t_0) = \Psi^0(\mathbf{q}), \qquad \mathbf{q} \in \mathbf{R}^2, \tag{17}$$

with the decoupled Hamiltonian

$$\hat{H}_{dec}(t) = \sum_{j=1}^{2}\hat{H}_j(t), \tag{18}$$

where

$$\hat{H}_j(t) = \frac{\hat{p}_j^2}{2\mu(t)} + \frac{\mu(t)\omega^2(t)}{2}\hat{q}_j^2 + \frac{B(t)}{2}(\hat{q}_j\hat{p}_j + \hat{p}_j\hat{q}_j) + \widetilde{D}_j(t)\hat{p}_j + \widetilde{E}_j(t)\hat{q}_j, \qquad j = 1,2,$$

and parameters $\widetilde{D}_j(t), \widetilde{E}_j(t)$ are defined in terms of $D_j(t), E_j(t)$ by the relations in (8). Therefore, the original IVP (14) and (15) is reduced to solving the IVP (16) and (17).

A. The evolution operator

The dynamics of the quantum system described by Schrödinger equation (14) is contained in the evolution operator defined as

$$i\hbar\frac{d}{dt}\hat{\mathcal{U}}_{gen}(t,t_0) = \hat{\mathcal{H}}_{gen}(t)\hat{\mathcal{U}}_{gen}(t,t_0), \quad \hat{\mathcal{U}}_{gen}(t_0,t_0) = \hat{I}.$$

According to the decoupling procedure discussed before, the evolution operator for IVP (14) and (15) will be of the form

$$\hat{\mathcal{U}}_{gen}(t,t_0) = \hat{U}_\theta^\dagger(t,t_0)\hat{U}_{dec}(t,t_0), \tag{19}$$

where $\hat{U}_{dec}(t,t_0)$ is the evolution operator for IVP (16) and (17) defined as

$$i\hbar\frac{d}{dt}\hat{U}_{dec}(t,t_0)=\hat{H}_{dec}(t)\hat{U}_{dec}(t,t_0),\quad \hat{U}_{dec}(t_0,t_0)=\hat{I}. \tag{20}$$

The exact form of $\hat{U}_{dec}(t,t_0)$ can be found by using the Wei–Norman Lie algebraic process. Indeed, the Hamiltonian $\hat{H}_{dec}(t)$ given by (18) for the decoupled oscillator can be written as time-dependent linear combination of Lie algebra generators as

$$\hat{H}_{dec}(t)=-i\sum_{j=1}^{2}\left(\frac{\hbar^2}{\mu(t)}\hat{\mathcal{K}}_j^{(-)}+\mu(t)\omega^2(t)\hat{\mathcal{K}}_j^{+}+2\hbar B(t)\hat{\mathcal{K}}_j^{(0)}+\hbar\widetilde{D}_j(t)\hat{\mathcal{E}}_j^{(2)}+\widetilde{E}_j(t)\hat{\mathcal{E}}_j^{(1)}\right),$$

where operators

$$\hat{\mathcal{E}}_j^{(1)}=iq_j,\quad \hat{\mathcal{E}}_j^{(2)}=\frac{\partial}{\partial q_j},\quad \hat{\mathcal{E}}_j^{(3)}=i\hat{I},\quad j=1,2,$$

are generators of Heisenberg–Weyl algebra and

$$\hat{\mathcal{K}}_j^{(-)}=-\frac{i}{2}\frac{\partial^2}{\partial q_j^2},\quad \hat{\mathcal{K}}_j^{(+)}=\frac{i}{2}q_j^2,\quad \hat{\mathcal{K}}_j^{(0)}=\frac{1}{2}\left(q_j\frac{\partial}{\partial q_j}+\frac{1}{2}\right),\quad j=1,2,$$

are generators of the su(1,1) algebra. Then, the evolution operator is

$$\hat{U}_{dec}(t,t_0)=\prod_{j=1}^{2}\hat{U}_j(t,t_0), \tag{21}$$

where $\hat{U}_j(t,t_0)$ for each $j=1,2$ can be expressed as a product of exponential operators,

$$\hat{U}_j(t,t_0)=\exp(c_j(t)\hat{\mathcal{E}}_j^{(3)})\exp\left(\frac{a_j(t)}{\hbar}\hat{\mathcal{E}}_j^{(1)}\right)\exp(-b_j(t)\hat{\mathcal{E}}_j^{(2)})\exp(f(t)\hat{\mathcal{K}}_j^{(+)})\exp(2h(t)\hat{\mathcal{K}}_j^{(0)})\exp(g(t)\hat{\mathcal{K}}_j^{(-)}),$$

with $f(t),g(t),h(t)$ and $a_j(t),b_j(t),c_j(t)$ being real-valued functions to be determined. Writing (21) and (18) into (20) and performing necessary calculations, we find that $\hat{U}_{dec}(t,t_0)$ is a solution of (20) if the unknown functions $f(t),g(t),h(t)$ satisfy the nonlinear system,

$$\begin{aligned}\dot{f}+\frac{\hbar}{\mu(t)}f^2+2B(t)f+\frac{\mu(t)\omega^2(t)}{\hbar}&=0, & f(t_0)&=0,\\ \dot{g}+\frac{\hbar}{\mu(t)}e^{2h}&=0, & g(t_0)&=0,\\ \dot{h}+\frac{\hbar}{\mu(t)}f+B(t)&=0, & h(t_0)&=0,\end{aligned} \tag{22}$$

and $a_j(t),b_j(t),c_j(t)$ satisfy the nonlinear system,

$$\begin{aligned}\dot{a}_j+B(t)a_j+\mu(t)\omega^2(t)b_j+\widetilde{E}_j(t)&=0, & a_j(t_0)&=0,\\ \dot{b}_j-B(t)b_j-\frac{1}{\mu(t)}a_j-\widetilde{D}_j(t)&=0, & b_j(t_0)&=0,\\ \dot{c}_j+\frac{1}{2\hbar\mu(t)}a_j^2+\frac{\widetilde{D}_j(t)}{\hbar}a_j-\frac{\mu(t)\omega^2(t)}{2\hbar}b_j^2&=0, & c_j(t_0)&=0,\quad j=1,2.\end{aligned} \tag{23}$$

Then, the solution of system (22) is found in terms of two linearly independent solutions $x_1^{(h)}(t)$ and $x_2^{(h)}(t)$ of the decoupled classical system (7) as

$$\begin{aligned}f(t)&=\frac{\mu(t)}{\hbar}\left(\frac{\dot{x}_1^{(h)}(t)}{x_1^{(h)}(t)}-B(t)\right),\\ g(t)&=-\hbar x_0^2\left(\frac{x_2^{(h)}(t)}{x_1^{(h)}(t)}\right),\\ h(t)&=-\ln\left|\frac{x_1^{(h)}(t)}{x_0}\right|.\end{aligned}$$

On the other hand, for each $j = 1, 2$, the solution of system (23) is obtained in terms of particular solutions of systems (7) and (9) as

$$
\begin{aligned}
a_j(t) &= p_j^{(p)}(t),\\
b_j(t) &= x_j^{(p)}(t),\\
c_j(t) &= \int_{t_0}^{t}\left(\frac{-(p_j^{(p)}(s))^2}{2\hbar\mu(s)} - \frac{\widetilde{D}_j(s)}{\hbar}p_j^{(p)}(s) + \frac{\mu(s)\omega^2(s)}{2\hbar}(x_j^{(p)}(s))^2\right)ds.
\end{aligned}
$$

Therefore, we find

$$
\begin{aligned}
\hat{U}_j(t,t_0) = \exp&\left\{\frac{i}{\hbar}\int_{t_0}^{t}\left[\frac{-1}{2\mu(s)}(p_j^{(p)}(s))^2 - \widetilde{D}_j(s)p_j^{(p)}(s) + \frac{\mu(s)\omega^2(s)}{2}(x_j^{(p)}(s))^2\right]ds\right\}\\
&\times\exp\left(ip_j^{(p)}(t)q_j\right)\times\exp\left(-x_j^{(p)}(t)\frac{\partial}{\partial q_j}\right)\times\exp\left(i\frac{\mu(t)}{2\hbar}\left(\frac{\dot{x}_1^{(h)}(t)}{x_1^{(h)}(t)} - B(t)\right)q_j^2\right)\\
&\times\exp\left(\ln\left|\frac{x_0}{x_1^{(h)}(t)}\right|\left(q_j\frac{\partial}{\partial q_j} + \frac{1}{2}\right)\right)\times\exp\left(\frac{i}{2}\hbar x_0^2\left(\frac{x_2^{(h)}(t)}{x_1^{(h)}(t)}\right)\frac{\partial^2}{\partial q_j^2}\right),\quad j = 1, 2,
\end{aligned}
$$

which determines $\hat{U}_{dec}(t,t_0)$ and $\hat{\mathcal{U}}_{gen}(t,t_0)$ explicitly. We note the action of the shifting and dilatation operators on the given initial function, respectively,

$$\exp\left(\xi_j\frac{\partial}{\partial q_j}\right)\phi_0(q_j) = \phi_0(q_j + \xi_j),\quad \exp\left(\xi_j q_j\frac{\partial}{\partial q_j}\right)\phi_0(q_j) = \phi_0(e^{\xi_j}q_j),\quad j = 1, 2. \tag{24}$$

In addition, we also have

$$\exp\left(-\frac{i\xi_j}{2}\frac{\partial^2}{\partial q_j^2}\right)\phi_0(q_j) = \phi(q_j;\xi_j), \tag{25}$$

where for $j = 1, 2$, the function $\phi(q_j;\ z_j)$ satisfies the free Schrödinger equation,

$$
\begin{cases}
\dfrac{1}{2}\dfrac{\partial^2}{\partial q_j^2}\phi(q_j;z_j) = i\dfrac{\partial}{\partial z_j}\phi(q_j;z_j),\\
\phi(q_j;z_j)|_{z_j=0} = \phi_0(q_j).
\end{cases}
$$

Then, the solution of IVP (14) and (15) is determined as $\Psi(\mathbf{q},t) = \hat{\mathcal{U}}_{gen}(t,t_0)\Psi^0(\mathbf{q})$.

B. The propagator

The solution of the IVP (14) and (15) can also be written in the form

$$\Psi(\mathbf{q},t) = \int_{\mathbb{R}^2}\mathcal{K}_{gen}(\mathbf{q},t;\mathbf{q}',t_0)\Psi^0(\mathbf{q}')d\mathbf{q}',$$

where $\mathcal{K}_{gen}(\mathbf{q},t;\mathbf{q}',t_0)$ denotes the propagator of the system. The propagator is the kernel of the integral transform that converts a given initial function to a wave function solution at later times. Using the evolution operator and relation

$$\mathcal{K}_{gen}(\mathbf{q},t;\mathbf{q}',t_0) = \hat{\mathcal{U}}_{gen}(t,t_0)\delta(\mathbf{q}-\mathbf{q}'),\quad \hat{\mathcal{U}}_{gen}(t_0,t_0) = \hat{I},$$

where $\delta(\mathbf{q})$ denotes the Dirac-delta distribution, one can determine the propagator explicitly. For this, first we find the propagator for the two-dimensional decoupled oscillator as

$$
\begin{aligned}
&\mathcal{K}_{dec}(\mathbf{q},t;\mathbf{q}',t_0) = \hat{U}_{dec}(t,t_0)\delta(\mathbf{q}-\mathbf{q}') \\
&= \frac{-i\omega_0}{2\pi\hbar}\frac{1}{|\epsilon(t)|\sin\eta(t)}\exp\left\{\frac{-i}{2\hbar}\left[\int_{t_0}^{t}\left(\frac{\left|\mathbf{p}^{(p)}(s)\right|^2}{\mu(s)} + 2\tilde{\mathbf{E}}(s)\cdot\mathbf{p}^{(p)}(s) - \mu(s)\omega^2(s)\left|\mathbf{x}^{(p)}(s)\right|^2\right)ds\right.\right. \\
&+\left(\mu(t)\left(B(t)-\frac{d}{dt}\ln|\epsilon(t)|\right)-\omega_0\frac{\cot\eta(t)}{|\epsilon(t)|^2}\right)\left|\mathbf{q}-\mathbf{x}^{(p)}(t)\right|^2 \\
&\left.\left.-2\mathbf{p}^{(p)}(t)\cdot\mathbf{q}-\omega_0\cot\eta(t)|\mathbf{q}'|^2+\frac{2}{\sin\eta(t)|\epsilon(t)|}\left(\mathbf{q}-\mathbf{x}^{(p)}(t)\right)\cdot\mathbf{q}'\right]\right\},
\end{aligned}
$$

where

$$
\epsilon(t) = \frac{x_1^{(h)}(t)}{x_0} + i(\omega_0 x_0)x_2^{(h)}(t) = |\epsilon(t)|e^{i\eta(t)}, \tag{26}
$$

with the modulus and phase

$$
|\epsilon(t)| = \sqrt{\frac{(x_1^{(h)}(t))^2}{x_0^2} + (\omega_0 x_0)^2(x_2^{(h)}(t))^2}, \qquad \eta(t) = \int_{t_0}^{t}\frac{\omega_0}{\mu(s)|\epsilon(s)|^2}ds, \tag{27}
$$

and we use the dot product notation $\mathbf{f}\cdot\mathbf{g} = f_1g_1 + f_2g_2$ for any two vectors $\mathbf{f} = (f_1, f_2)^T$, $\mathbf{g} = (g_1, g_2)^T$, and $|\mathbf{f}|^2 = f_1^2 + f_2^2$. Then,

$$
\mathcal{K}_{gen}(\mathbf{q},t;\mathbf{q}',t_0) = \mathcal{K}_{dec}\left(R_\theta(t)\mathbf{q},t;\mathbf{q}',t_0\right),
$$

where $R_\theta(t)$ is the rotation matrix given by (5), and explicitly in terms of the solutions to the coupled systems (2) and (3), we get

$$
\begin{aligned}
&\mathcal{K}_{gen}(\mathbf{q},t;\mathbf{q}',t_0) = \frac{-i\omega_0}{2\pi\hbar}\frac{1}{|\epsilon(t)|\sin\eta(t)} \\
&\times\exp\left\{\frac{-i}{2\hbar}\left[\int_{t_0}^{t}\left(\frac{\left|\mathbf{P}^{(p)}(s)\right|^2}{\mu(s)} + 2\mathbf{E}(s)\cdot\mathbf{P}^{(p)}(s) - \mu(s)\omega^2(s)\left|\mathbf{X}^{(p)}(s)\right|^2\right)ds\right.\right. \\
&+\left(\mu(t)\left(B(t)-\frac{d}{dt}\ln|\epsilon(t)|\right)-\omega_0\frac{\cot\eta(t)}{|\epsilon(t)|^2}\right)\left|\mathbf{q}-\mathbf{X}^{(p)}(t)\right|^2 \\
&\left.\left.-2\mathbf{P}^{(p)}(t)\cdot\mathbf{q}-\omega_0\cot\eta(t)|\mathbf{q}'|^2+\frac{2}{\sin\eta(t)|\epsilon(t)|}\left(R_\theta(t)\left(\mathbf{q}-\mathbf{X}^{(p)}(t)\right)\right)\cdot\mathbf{q}'\right]\right\},
\end{aligned}
$$

where $\epsilon(t)$ given by (26) can be written also in terms of the homogeneous solution to the coupled classical system (2) as

$$
\epsilon(t) = \frac{1}{x_0}\left(\cos\theta(t)X_1^{(h)}(t) + \sin\theta(t)X_2^{(h)}(t)\right) + i(\omega_0 x_0)\left(-\sin\theta(t)X_1^{(h)}(t) + \cos\theta(t)X_2^{(h)}(t)\right). \tag{28}
$$

In general, the evolution of a state from an arbitrary time t' to t is defined as

$$
\Psi(\mathbf{q},t) = \hat{\mathcal{U}}_{gen}(t,t')\Psi(\mathbf{q},t') = \int_{\mathbb{R}^2}\mathcal{K}_{gen}(\mathbf{q},t;\mathbf{q}',t')\Psi(\mathbf{q}',t')d\mathbf{q}', \quad t_0 \le t' < t,
$$

and it implies that

$$
\mathcal{K}_{gen}(\mathbf{q},t;\mathbf{q}',t') = \hat{U}_{gen}(t,t')\delta(\mathbf{q}-\mathbf{q}') = \hat{U}_\theta^\dagger(t,t')\mathcal{K}_{dec}\left(\mathbf{q},t;\mathbf{q}',t'\right) = \hat{U}_\theta(t',t_0)\mathcal{K}_{dec}\left(R_\theta(t)\mathbf{q},t;\mathbf{q}',t'\right).
$$

After some calculations, we obtain the following result:

$$\mathcal{K}_{gen}(\mathbf{q},t;\mathbf{q}',t') = \frac{-i\omega_0}{2\pi\hbar|\epsilon(t)\|\epsilon(t')|\sin(\eta(t)-\eta(t'))}$$
$$\times\exp\Bigg\{\frac{-i}{2\hbar}\Bigg[\int_{t_0}^{t}\Bigg(\frac{\left|\mathbf{P}^{(p)}(s)\right|^2}{\mu(s)} + 2\mathbf{E}(s)\cdot\mathbf{P}^{(p)}(s) - \mu(s)\omega^2(s)\left|\mathbf{X}^{(p)}(s)\right|^2\Bigg)ds$$
$$+\left(\mu(t)\left(B(t)-\frac{d}{dt}\ln|\epsilon(t)|\right) - \frac{\omega_0\cot\left(\eta(t)-\eta(t')\right)}{|\epsilon(t)|^2}\right)\left|R_\theta^T(t')\mathbf{q}-\mathbf{X}^{(p)}(t)\right|^2$$
$$-\left(\mu(t')\left(B(t')-\frac{d}{dt}\ln|\epsilon(t')|\right) - \frac{\omega_0\cot\left(\eta(t)-\eta(t')\right)}{|\epsilon(t')|^2}\right)\left|R_\theta^T(t')\mathbf{q}'-\mathbf{X}^{(p)}(t')\right|^2$$
$$-2\left[\mathbf{p}^{(P)}(t)\cdot(R_\theta^T(t')\mathbf{q}) - \mathbf{P}^{(p)}(t')\cdot(R_\theta^T(t')\mathbf{q}')\right] + \frac{2\left(R_\theta(t)\left(R_\theta^T(t')\mathbf{q}-\mathbf{X}^{(p)}(t)\right)\right)\cdot\left(\mathbf{q}'-R_\theta^T(t')\mathbf{x}^{(p)}(t')\right)}{|\epsilon(t)\|\epsilon(t')|\sin(\eta(t)-\eta(t'))}\Bigg]\Bigg\}.$$

Usually, the propagator is interpreted as the probability amplitude of finding the particle at point q and time t, given that at the past it was at point q' and time t'. By construction, the propagator $\mathcal{K}_{gen}(\mathbf{q},t;\mathbf{q}',t')$ can be seen as a solution of the time-dependent Schrödinger equation in the variables q,t, with q',t' treated as parameters. It is the solution corresponding to Dirac-delta initial condition $\delta(\mathbf{q}-\mathbf{q}')$, which is highly singular, and due to this, the propagator as a "wave function" is not normalizable. In any case, the propagator like the evolution operator contains all necessary knowledge for describing the dynamics of the quantum system.

IV. TIME-EVOLUTION OF QUANTUM STATES

In this section, for the generalized two-dimensional quantum parametric oscillator, we find time-evolution of eigenstates and coherent states explicitly.

A. Time-evolution of harmonic oscillator eigenstates

First, we solve IVP (14) and (15) by taking the initial function to be an eigenstate $\varphi_n(\mathbf{q})$ of the two-dimensional simple harmonic oscillator, whose Hamiltonian is $\hat{H}_0 = \sum_{j=1}^{2}(\hat{p}_j^2 + \omega_0^2\hat{q}_j^2)/2$. As known, these eigenstates correspond to eigenvalues $E_n = E_{n_1} + E_{n_2} = \hbar\omega_0(n_1 + n_2 + 1)$, and for $n = (n_1, n_2)$, we have

$$\varphi_n(\mathbf{q}) = \varphi_{n_1}(q_1)\varphi_{n_2}(q_2), \quad n_1, n_2 = 0, 1, 2, \ldots,$$

with

$$\varphi_{n_j}(q_j) = N_{n_j} e^{-\frac{\omega_0}{2\hbar}q_j^2} H_{n_j}\left(\sqrt{\frac{\omega_0}{\hbar}}q_j\right), \quad j = 1, 2,$$

where $H_{n_j}(\sqrt{\omega_0/\hbar}q_j)$ are Hermite polynomials and $N_{n_j} = (\omega_0/\pi\hbar)^{1/4}(2^{n_j}n_j!)^{-1/2}$ are the normalization constants. According to this, time-evolved eigenstates of the two-dimensional oscillator (16) and (17) with the Hamiltonian $\hat{H}_{dec}(t)$ are of the form

$$\Psi_n^0(\mathbf{q},t) = \hat{U}_{dec}(t,t_0)\varphi_n(\mathbf{q}) = \prod_{j=1}^{2}\hat{U}_j(t,t_0)\varphi_{n_j}(q_j),$$

and using Eqs. (24) and (25), we obtain explicitly the wave functions,

$$\Psi_n^0(\mathbf{q},t) = \mathbf{N_n}\frac{1}{|\epsilon(t)|}\exp\left(-\frac{iE_n}{\hbar\omega_0}\eta(t)\right)$$
$$\times\exp\left\{\frac{-i}{\hbar}\int_{t_0}^{t}\left(\frac{\left|\mathbf{p}^{(p)}(s)\right|^2}{2\mu(s)} + \tilde{\mathbf{D}}(s)\cdot\mathbf{p}^{(p)}(s) - \frac{\mu(s)\omega^2(s)}{2}\left|\mathbf{x}_j^{(p)}(s)\right|^2\right)ds\right\}$$
$$\times\exp\left\{\frac{i}{\hbar}\left[\frac{-\mu(t)}{2}\left(B(t)-\frac{d}{dt}\ln|\epsilon(t)|\right)\left|\mathbf{q}-\mathbf{x}^{(p)}(t)\right|^2 + \mathbf{p}^{(p)}(t)\cdot\mathbf{q}\right]\right\}$$
$$\times\exp\left[-\frac{\omega_0}{2\hbar}\frac{\left|\mathbf{q}-\mathbf{x}^{(p)}(t)\right|^2}{|\epsilon(t)|^2}\right]\mathbf{H}_n\left(\sqrt{\frac{\omega_0}{\hbar}}\frac{\mathbf{q}-\mathbf{x}^{(p)}(t)}{|\epsilon(t)|}, t\right),$$

and the corresponding probability densities,

$$\rho_n^0(\mathbf{q},t) = \mathbf{N}_n^2 \frac{1}{|\epsilon(t)|^2} \exp\left[-\frac{\omega_0}{\hbar}\frac{\left|\mathbf{q}-\mathbf{x}^{(p)}(t)\right|^2}{|\epsilon(t)|^2}\right]\mathbf{H}_n^2\left(\sqrt{\frac{\omega_0}{\hbar}}\frac{\mathbf{q}-\mathbf{x}^{(p)}(t)}{|\epsilon(t)|},t\right),$$

where $|\epsilon(t)|$ is as defined in (27), and we used the compact notations $\mathbf{N}_n = N_{n_1}N_{n_2}$ and

$$\mathbf{H}_n\left(\sqrt{\frac{\omega_0}{\hbar}}\frac{\mathbf{q}}{|\epsilon(t)|},t\right) \equiv \prod_{j=1}^{2} H_{n_j}\left(\sqrt{\frac{\omega_0}{\hbar}}\left(\frac{q_j}{|\epsilon(t)|}\right)\right).$$

Now, formally time-evolved solutions of the IVP (14) and (15) will be as expected,

$$\Psi_n^\theta(\mathbf{q},t) = \hat{U}_\theta^\dagger(t,t_0)\Psi_n^0(\mathbf{q},t) = \Psi_n^0(R_\theta(t)\mathbf{q},t).$$

Then, in terms of solutions to the coupled systems of classical equations (2) and (3), we have

$$\begin{aligned}\Psi_n^\theta(\mathbf{q},t) &= \mathbf{N}_n\frac{1}{|\epsilon(t)|}\exp\left(-\frac{iE_n}{\hbar\omega_0}\eta(t)\right)\\ &\times\exp\left\{\frac{i}{\hbar}\left[\frac{-\mu(t)}{2}\left(B(t)-\frac{d}{dt}\ln|\epsilon(t)|\right)\left|\mathbf{q}-\mathbf{X}^{(p)}(t)\right|^2+\mathbf{P}^{(p)}(t)\cdot\mathbf{q}\right]\right\}\\ &\times\exp\left\{\frac{-i}{\hbar}\int_{t_0}^{t}\left[\frac{\left|\mathbf{P}^{(p)}(s)\right|^2}{2\mu(s)}+\mathbf{D}(s)\cdot\mathbf{P}^{(p)}(s)-\frac{\mu(s)\omega^2(s)}{2}\left|\mathbf{X}^{(p)}(s)\right|^2\right]ds\right\}\\ &\times\exp\left\{-\frac{\omega_0}{2\hbar}\frac{\left|\mathbf{q}-\mathbf{X}^{(p)}(t)\right|^2}{|\epsilon(t)|^2}\right\}\mathbf{H}_n\left(\sqrt{\frac{\omega_0}{\hbar}}\frac{R_\theta(t)\left(\mathbf{q}-\mathbf{X}^{(p)}(t)\right)}{|\epsilon(t)|},t\right),\end{aligned} \tag{29}$$

and probability densities become

$$\rho_n^\theta(\mathbf{q},t) = \mathbf{N}_n^2\frac{1}{|\epsilon(t)|^2}\exp\left\{-\frac{\omega_0}{\hbar}\frac{\left|\mathbf{q}-\mathbf{X}^{(p)}(t)\right|^2}{|\epsilon(t)|^2}\right\}\mathbf{H}_n^2\left(\sqrt{\frac{\omega_0}{\hbar}}\frac{R_\theta(t)\left(\mathbf{q}-\mathbf{X}^{(p)}(t)\right)}{|\epsilon(t)|},t\right). \tag{30}$$

Here, the expectation values of the position and momentum at states $\Psi_n^\theta(\mathbf{q},t)$ are

$$\langle\hat{\mathbf{q}}\rangle_n^\theta(t) = \begin{pmatrix}\langle\hat{q}_1\rangle_n^\theta(t)\\ \langle\hat{q}_2\rangle_n^\theta(t)\end{pmatrix} = \mathbf{X}^{(p)}(t),\quad \langle\hat{\mathbf{p}}\rangle_n^\theta(t) = \begin{pmatrix}\langle\hat{p}_1\rangle_n^\theta(t)\\ \langle\hat{p}_2\rangle_n^\theta(t)\end{pmatrix} = \mathbf{P}^{(p)}(t), \tag{31}$$

showing that they do not depend on the wave number $n = (n_1, n_2)$ and are completely determined by the external forces. Then, the uncertainties in the position and momentum are found as

$$(\Delta\hat{\mathbf{q}})_n^\theta(t) = \begin{pmatrix}(\Delta\hat{q}_1)_n^\theta(t)\\ (\Delta\hat{q}_2)_n^\theta(t)\end{pmatrix} = \sqrt{\frac{\hbar}{\omega_0}}|\epsilon(t)|\,\mathbf{\Lambda}(n_1,n_2,\theta(t)), \tag{32}$$

$$(\Delta\hat{\mathbf{p}})_n^\theta(t) = \begin{pmatrix}(\Delta\hat{p}_1)_n^\theta(t)\\ (\Delta\hat{p}_2)_n^\theta(t)\end{pmatrix} = \sqrt{\hbar\omega_0}\frac{\Sigma(t)}{|\epsilon(t)|}\,\mathbf{\Lambda}(n_1,n_2,\theta(t)), \tag{33}$$

where

$$\mathbf{\Lambda}(n_1,n_2,\theta(t)) = \begin{pmatrix}\left(\cos^2\theta(t)n_1+\sin^2\theta(t)n_2+1/2\right)^{1/2}\\ \left(\sin^2\theta(t)n_1+\cos^2\theta(t)n_2+1/2\right)^{1/2}\end{pmatrix},$$

$$\Sigma(t) = \sqrt{1+\frac{|\epsilon(t)|^4}{\omega_0^2}\left[\frac{d\ln|\epsilon(t)|}{dt}-B(t)\right]^2},$$

and the uncertainty product becomes

$$(\Delta\hat{\mathbf{q}})(\Delta\hat{\mathbf{p}})_n^\theta(t) = \begin{pmatrix}(\Delta\hat{q}_1)(\Delta\hat{p}_1)_n^\theta(t)\\(\Delta\hat{q}_2)(\Delta\hat{p}_2)_n^\theta(t)\end{pmatrix} = \hbar\,\Sigma(t)\begin{pmatrix}\cos^2\theta(t)n_1+\sin^2\theta(t)n_2+1/2\\\sin^2\theta(t)n_1+\cos^2\theta(t)n_2+1/2\end{pmatrix}.$$

Clearly, uncertainties for some subcases can be easily recovered from above results. For example, in the case $\theta(t)=0$, one gets the uncertainties for the two-dimensional decoupled parametric oscillator. In the case $\theta(t)\neq 0$ and $\mu(t)=1$, $\omega^2(t)=\omega_0^2$, $B(t)=0$, one gets the uncertainties for the simple harmonic oscillator in the electromagnetic field as

$$(\Delta\hat{\mathbf{q}})_n^\theta(t)=\sqrt{\frac{\hbar}{\omega_0}}\mathbf{\Lambda}(n_1,n_2,\theta(t)),\quad (\Delta\hat{\mathbf{p}})_n^\theta(t)=\sqrt{\hbar\omega_0}\mathbf{\Lambda}(n_1,n_2,\theta(t)),$$

and we note that when $n_1=n_2$, then $\mathbf{\Lambda}(n_1,n_2,\theta(t))$ becomes independent of $\theta(t)$.

Finally, it is not difficult to show that the expectation value of the angular momentum operator $\hat{L}$ at wave function $\Psi_n^\theta(\mathbf{q},t)$ is

$$\langle\hat{L}\rangle_n(t)=\langle\Psi_n^\theta(\mathbf{q},t)|\hat{L}|\Psi_n^\theta(\mathbf{q},t)\rangle = X_1^{(p)}(t)P_2^{(p)}(t)-X_2^{(p)}(t)P_1^{(p)}(t),$$

and the matrix elements are

$$\langle\Psi_n^\theta(\mathbf{q},t)|\hat{L}|\Psi_m^\theta(\mathbf{q},t)\rangle = \Big(X_1^{(p)}(t)P_2^{(p)}(t)-X_2^{(p)}(t)P_1^{(p)}(t)\Big)\delta_{nm},$$

where δ_{nm} is the Kronecker delta. In particular, when there are no external fields $[D_j(t)=E_j(t)=0,\ j=1,2]$, for the angular momentum operator, one has expectation $\langle\hat{L}\rangle_n(t)=0$ and uncertainty

$$(\Delta\hat{L})_n(t)=\sqrt{\hbar^2\big((n_1+1)^2(n_2+1)^2+n_1^2n_2^2\big)\Big(X_1^{(h)}(t)P_2^{(h)}(t)-X_2^{(h)}(t)P_1^{(h)}(t)\Big)^2},$$

which is determined by the homogeneous solutions of the classical equations and depends on the wave number $n=(n_1,n_2)$.

B. Time-evolution of Glauber coherent states

Now, we solve IVP (14) and (15) by taking the initial function to be a coherent state of the simple two-dimensional harmonic oscillator with the Hamiltonian $\hat{H}_0$, that is,

$$\phi_\alpha(\mathbf{q})=\phi_{\alpha_1}(q_1)\phi_{\alpha_2}(q_2),$$

where $\alpha=(\alpha_1,\alpha_2)$ and $\alpha_j=\alpha_j^{(1)}+i\alpha_j^{(2)}$, with $\alpha_j^{(1)}$, $\alpha_j^{(2)}$ being real constants, and

$$\phi_{\alpha_j}(q_j)=\left(\frac{\omega_0}{\pi\hbar}\right)^{\frac14}\exp\Big[-i\alpha_j^{(1)}\alpha_j^{(2)}\Big]\exp\left[i\alpha_j^{(2)}\sqrt{\frac{2\omega_0}{\hbar}}q_j\right]\exp\left[-\frac{\omega_0}{2\hbar}\left(q_j-\sqrt{\frac{2\hbar}{\omega_0}}\alpha_j^{(1)}\right)^2\right],\quad j=1,2.$$

Then, time-evolved coherent states $\Phi_\alpha^0(\mathbf{q},t)=\hat{U}_{dec}(t,t_0)\phi_\alpha(\mathbf{q})$ of the decoupled oscillator are found as

$$\begin{aligned}\Phi_\alpha^0(\mathbf{q},t)&=\sqrt{\frac{\omega_0}{\pi\hbar}}\frac{1}{\epsilon(t)}\exp\left\{-\frac12\left(\frac{(\epsilon^*(t))^2}{|\epsilon(t)|^2}\alpha^2+|\alpha|^2\right)\right\}\\&\times\exp\left\{-\frac{i}{\hbar}\int_{t_0}^t\left[\frac{\left|\mathbf{p}^{(p)}(s)\right|^2}{\mu(s)}+2\tilde{\mathbf{D}}(s)\cdot\mathbf{p}^{(p)}(s)-\mu(s)\omega^2(s)\left|\mathbf{x}^{(p)}(s)\right|^2\right]ds\right\}\\&\times\exp\left\{\frac{1}{2\hbar}\left[-\left(i\mu(t)\left(B(t)-\frac{d}{dt}\ln|\epsilon(t)|\right)+\frac{\omega_0}{|\epsilon(t)|^2}\right)\left|\mathbf{q}-\mathbf{x}^{(p)}(t)\right|^2+2i\mathbf{p}^{(p)}(t)\cdot\mathbf{q}\right]\right\}\\&\times\exp\left\{\sqrt{\frac{2\omega_0}{\hbar}}\frac{1}{\epsilon(t)}\Big(\mathbf{q}-\mathbf{x}^{(p)}(t)\Big)\cdot\alpha\right\},\end{aligned}$$

where $\alpha^2=\alpha\cdot\alpha$, $|\alpha|^2=\alpha\cdot\alpha^*$, and we have

$$\rho_\alpha^0(\mathbf{q},t)=|\Phi_\alpha^0(\mathbf{q},t)|^2=\left(\frac{\omega_0}{\pi\hbar}\right)\frac{1}{|\epsilon(t)|^2}\exp\left\{-\frac{\omega_0}{\hbar}\frac{\left|\mathbf{q}-\langle\hat{\mathbf{q}}\rangle_\alpha^0(t)\right|^2}{|\epsilon(t)|^2}\right\}.$$

Here, expectation values at $\Phi_\alpha^0(\mathbf{q},t)$ are obtained as

$$\langle\hat{\mathbf{q}}\rangle_\alpha^0(t) \equiv \begin{pmatrix}\langle\hat{q}_1\rangle_{\alpha_1}^0(t)\\ \langle\hat{q}_2\rangle_{\alpha_2}^0(t)\end{pmatrix} = \sqrt{\frac{2\hbar}{\omega_0}}\mathbf{C}_\alpha^0\mathbf{x}^{(h)}(t) + \mathbf{x}^{(p)}(t), \tag{34}$$

$$\langle\hat{\mathbf{p}}\rangle_\alpha^0(t) \equiv \begin{pmatrix}\langle\hat{p}_1\rangle_{\alpha_1}^0(t)\\ \langle\hat{p}_2\rangle_{\alpha_2}^0(t)\end{pmatrix} = \sqrt{\frac{2\hbar}{\omega_0}}\mathbf{C}_\alpha^0\mathbf{p}^{(h)}(t) + \mathbf{p}^{(p)}(t), \tag{35}$$

where the coefficient matrix $\mathbf{C}_\alpha^0$ is defined as

$$\mathbf{C}_\alpha^0 = \begin{pmatrix}\frac{\alpha_1^{(1)}}{x_0} & \omega_0 x_0\alpha_1^{(2)}\\ \frac{\alpha_2^{(1)}}{x_0} & \omega_0 x_0\alpha_2^{(2)}\end{pmatrix}. \tag{36}$$

The uncertainties at coherent states $\Phi_\alpha^0(\mathbf{q},t)$ are

$$(\Delta\hat{\mathbf{q}})_\alpha^0(t) = \begin{pmatrix}(\Delta\hat{q}_1)_\alpha^0(t)\\ (\Delta\hat{q}_2)_\alpha^0(t)\end{pmatrix},\quad (\Delta\hat{\mathbf{p}})_\alpha^\theta(t) = \begin{pmatrix}(\Delta\hat{p}_1)_\alpha^0(t)\\ (\Delta\hat{p}_2)_\alpha^0(t)\end{pmatrix},$$

where

$$(\Delta\hat{q}_j)_\alpha^0(t) = \sqrt{\frac{\hbar}{2\omega_0}}|\epsilon(t)|,\quad (\Delta\hat{p}_j)_\alpha^0(t) = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{|\epsilon(t)|}\Sigma(t),\quad j=1,2,$$

and the uncertainty product becomes

$$(\Delta\hat{\mathbf{q}})(\Delta\hat{\mathbf{p}})_\alpha^0(t) = \begin{pmatrix}(\Delta\hat{q}_1)(\Delta\hat{p}_1)_\alpha^0(t)\\ (\Delta\hat{q}_2)(\Delta\hat{p}_2)_\alpha^0(t)\end{pmatrix},\quad (\Delta\hat{q}_j\Delta\hat{p}_j)_\alpha^0(t) = \frac{\hbar}{2}\Sigma(t),\quad j=1,2.$$

Now, time-evolved coherent states of the generalized two-dimensional oscillator are

$$\Phi_\alpha^\theta(\mathbf{q},t) = \hat{\mathcal{U}}_{gen}(t,t_0)\phi_\alpha(\mathbf{q}) = \hat{U}_\theta^\dagger(t,t_0)\Phi_\alpha^0(\mathbf{q},t) = \Phi_\alpha^0(R_\theta(t)\mathbf{q},t),$$

and in terms of solutions to the classical systems (2) and (3), we get

$$\begin{aligned}\Phi_\alpha^\theta(\mathbf{q},t) = {}& \sqrt{\frac{\omega_0}{\pi\hbar}}\frac{1}{\epsilon(t)}\exp\left\{-\frac{1}{2}\left(\frac{(\epsilon^*(t))^2}{|\epsilon(t)|^2}\alpha^2 + |\alpha|^2\right)\right\}\\ &\times\exp\left\{-\frac{i}{\hbar}\int_{t_0}^t\left[\frac{\left|\mathbf{P}^{(p)}(s)\right|^2}{\mu(s)} + 2\mathbf{D}(s)\cdot\mathbf{P}^{(p)}(s) - \mu(s)\omega^2(s)\left|\mathbf{X}^{(p)}(s)\right|^2\right]ds\right\}\\ &\times\exp\left\{\frac{1}{\hbar}\left[-\left(i\mu(t)\left(B(t) - \frac{d}{dt}\ln|\epsilon(t)|\right) + \frac{\omega_0}{|\epsilon(t)|^2}\right)\left|\mathbf{q} - \mathbf{X}^{(p)}(t)\right|^2 + 2i\mathbf{P}^{(p)}(t)\cdot\mathbf{q}\right]\right\}\\ &\times\exp\left\{\sqrt{\frac{2\omega_0}{\hbar}}\frac{1}{\epsilon(t)}R_\theta(t)\left(\mathbf{q} - \mathbf{X}^{(p)}(t)\right)\cdot\alpha\right\}.\end{aligned} \tag{37}$$

Then, the probability densities become

$$\rho_\alpha^\theta(\mathbf{q},t) = \left(\frac{\omega_0}{\pi\hbar}\right)\frac{1}{|\epsilon(t)|^2}\exp\left\{-\frac{\omega_0}{\hbar}\frac{\left|\mathbf{q} - \langle\hat{\mathbf{q}}\rangle_\alpha^\theta(t)\right|^2}{|\epsilon(t)|^2}\right\}, \tag{38}$$

with the squeezing coefficient $|\epsilon(t)|$ given by (28). We note that since (28) is equal to (26), then $\epsilon(t)$ does not depend on $\theta(t)$, and thus, uncertainties at $\Phi_\alpha^0(\mathbf{q},t)$ and at $\Phi_\alpha^\theta(\mathbf{q},t)$ are same. On the other hand, expectation values at $\Phi_\alpha^\theta(\mathbf{q},t)$ depend on $\theta(t)$ and are determined as

$$\langle\hat{\mathbf{q}}\rangle_\alpha^\theta(t) = R_\theta^T(t)\langle\hat{\mathbf{q}}\rangle_\alpha^0(t),\quad \langle\hat{\mathbf{p}}\rangle_\alpha^\theta(t) = R_\theta^T(t)\langle\hat{\mathbf{p}}\rangle_\alpha^0(t),$$

where $\langle \hat{\mathbf{q}} \rangle_\alpha^0(t)$, $\langle \hat{\mathbf{p}} \rangle_\alpha^0(t)$ are given by (34) and (35), respectively. In terms of the classical solutions to systems (2) and (3), the expectation values are obtained as

$$\langle \hat{\mathbf{q}} \rangle_\alpha^\theta(t) = \sqrt{\frac{2\hbar}{\omega_0}} \mathbf{C}_\alpha^\theta(t) \mathbf{X}^{(h)}(t) + \mathbf{X}^{(p)}(t),$$

$$\langle \hat{\mathbf{p}} \rangle_\alpha^\theta(t) = \sqrt{\frac{2\hbar}{\omega_0}} \mathbf{C}_\alpha^\theta(t) \mathbf{P}^{(h)}(t) + \mathbf{P}^{(p)}(t),$$

where $\mathbf{C}_\alpha^\theta(t)$ is the similarity matrix,

$$\mathbf{C}_\alpha^\theta(t) = R_\theta^T(t) \mathbf{C}_\alpha^0 R_\theta(t), \tag{39}$$

with $\mathbf{C}_\alpha^0 = \mathbf{C}_\alpha^\theta(t_0)$ being the matrix given by (36).

Thus, time-evolved coherent states of the generalized quantum oscillator in the given external fields are two-dimensional squeezed Gaussian wave packets that follow the trajectory of the classical particles. In general, they do not preserve the minimum uncertainty, and their squeezing properties are controlled by the squeezing coefficient $|\epsilon(t)|$, which depends on the choice of the parameters $\mu(t), \omega^2(t)$ and $B(t)$. On the other hand, the displacement properties of coherent states depend also on parameters $D_j(t), E_j(t), j = 1, 2$, and the rotation angle $\theta(t)$.

Finally, we write the expectation values of angular momentum at coherent states (37), when there are no external fields, as

$$\begin{aligned} \langle \hat{L} \rangle_\alpha(t) &= 2\hbar\Big(\alpha_1^{(1)}\alpha_2^{(2)} - \alpha_1^{(2)}\alpha_2^{(1)}\Big)\Big(X_1^{(h)}(t)P_2^{(h)}(t) - X_2^{(h)}(t)P_1^{(h)}(t)\Big) \\ &= \frac{2\hbar}{\omega_0}\big(detC_\alpha^0\big)\Big(X_1^{(h)}(t)P_2^{(h)}(t) - X_2^{(h)}(t)P_1^{(h)}(t)\Big), \end{aligned}$$

where $\mathbf{C}_\alpha^0$ is given by (36). In that case, uncertainties become

$$(\Delta \hat{L})_\alpha(t) = \sqrt{\hbar^2(|\alpha_1|^2 + |\alpha_2|^2)\Big(X_1^{(h)}(t)P_2^{(h)}(t) - X_2^{(h)}(t)P_1^{(h)}(t)\Big)^2}.$$

Similarly, in the presence of external fields, one can compute expectations and uncertainties of angular momentum by straightforward calculations.

V. QUANTUM DYNAMICAL INVARIANTS

In this section, time-dependent linear and quadratic invariants for the quantum system are constructed using the evolution operator formalism. It is based on the fact that if time-development of a given quantum system is described by the unitary evolution operator $\hat{U}(t, t_0)$, then any operator of the form $\hat{A}(t) = \hat{U}(t, t_0)\hat{A}(t_0)\hat{U}^\dagger(t, t_0)$ is an integral of motion or a dynamical invariant. Using these dynamical invariants, we establish the relation between the present results and those obtained by the MMT- and the LR-approaches.

A. Linear invariants

For the generalized two-dimensional oscillator with the Hamiltonian $\hat{\mathcal{H}}_{gen}(t)$ given by (1), using the evolution operator (19), one can find dynamical invariants that are linear in coordinate and momentum,

$$\hat{A}_{\theta,j}(t) = \hat{\mathcal{U}}_{gen}(t, t_0)\, \hat{a}_j \hat{\mathcal{U}}_{gen}^\dagger(t, t_0), \quad \hat{A}_{\theta,j}^\dagger(t) = \hat{\mathcal{U}}_{gen}(t, t_0)\, \hat{a}_j^\dagger\, \hat{\mathcal{U}}_{gen}^\dagger(t, t_0), \quad j = 1, 2,$$

where

$$\hat{a}_j = \sqrt{\frac{\omega_0}{2\hbar}}\hat{q}_j + \frac{i}{\sqrt{2\omega_0\hbar}}\hat{p}_j, \quad \hat{a}_j^\dagger = \sqrt{\frac{\omega_0}{2\hbar}}\hat{q}_j - \frac{i}{\sqrt{2\omega_0\hbar}}\hat{p}_j, \quad j = 1, 2,$$

are, respectively, the non-Hermitian lowering and raising Dirac operators for the standard two-dimensional harmonic oscillator $\hat{H}_0 = \hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 + 1$. Explicit calculations give us

$$\begin{pmatrix} \hat{A}_{\theta,1}(t) \\ \hat{A}_{\theta,2}(t) \end{pmatrix} = \frac{-i}{\sqrt{2\omega_0\hbar}}\left[\mu(t)(\dot{\epsilon}(t) - B(t)\epsilon(t))\begin{pmatrix} \hat{Q}_1 \\ \hat{Q}_2 \end{pmatrix} - \epsilon(t)\begin{pmatrix} \hat{P}_1 \\ \hat{P}_2 \end{pmatrix}\right] \tag{40}$$

and

$$\begin{pmatrix} \hat{A}_{\theta,1}^\dagger(t) \\ \hat{A}_{\theta,2}^\dagger(t) \end{pmatrix} = \frac{i}{\sqrt{2\omega_0\hbar}}\left[\mu(t)\big(\dot{\epsilon}^*(t) - B(t)\epsilon^*(t)\big)\begin{pmatrix} \hat{Q}_1 \\ \hat{Q}_2 \end{pmatrix} - \epsilon^*(t)\begin{pmatrix} \hat{P}_1 \\ \hat{P}_2 \end{pmatrix}\right], \tag{41}$$

where

$$\begin{pmatrix}\hat{Q}_1\\ \hat{Q}_2\end{pmatrix} = R_\theta(t)\begin{pmatrix}\hat{q}_1 - X_1^{(p)}(t)\\ \hat{q}_2 - X_2^{(p)}(t)\end{pmatrix}, \quad \begin{pmatrix}\hat{P}_1\\ \hat{P}_2\end{pmatrix} = R_\theta(t)\begin{pmatrix}\hat{p}_1 - P_1^{(p)}(t)\\ \hat{p}_2 - P_2^{(p)}(t)\end{pmatrix},$$

and $\epsilon(t)$ is defined by (26). Here, $\epsilon(t)$ is a complex solution of Eq. (10), that is,

$$\ddot{\epsilon}(t) + \frac{\dot{\mu}}{\mu}\dot{\epsilon}(t) + \left(\omega^2(t) - \left(\dot{B}(t) + B^2(t) + \frac{\dot{\mu}}{\mu}B(t)\right)\right)\epsilon(t) = 0, \tag{42}$$

and it satisfies the IC's,

$$\epsilon(t_0) = 1, \quad \dot{\epsilon}(t_0) = B(t_0) + \frac{i\omega_0}{\mu(t_0)}. \tag{43}$$

Therefore, using the Wronskian $W(t) = W(\epsilon(t), \epsilon^*(t)) = \epsilon(t)\dot{\epsilon}^*(t) - \epsilon^*(t)\dot{\epsilon}(t) = -2i\omega_0/\mu(t)$, one can show that these linear invariants (40) satisfy commutation relations,

$$[\hat{A}_{\theta,i}(t), \hat{A}_{\theta,j}^\dagger(t)] = \delta_{ij}, \quad i,j = 1,2,$$

and can be seen also as generalized lowering and rising operators.

Moreover, coherent states $\Phi_\alpha^\theta(q_1,q_2,t)$, $\alpha = (\alpha_1,\alpha_2)$ found in (37) by construction are eigenstates of $\hat{A}_{\theta,j}(t)$ corresponding to complex eigenvalues $\alpha_j, j = 1,2$. Indeed, if $\phi_{\alpha_j}(q_j)$ are eigenstates of $\hat{a}_j$ so that $\hat{a}_j\phi_{\alpha_j}(q_j) = \alpha_j\phi_{\alpha_j}(q_j)$, then

$$\hat{\mathcal{U}}_{gen}(t,t_0)\hat{a}_j\hat{\mathcal{U}}_{gen}^\dagger(t,t_0)\hat{\mathcal{U}}_{gen}(t,t_0)\phi_{\alpha_1}(q_1)\phi_{\alpha_2}(q_2) = \alpha_j\hat{\mathcal{U}}_{gen}(t,t_0)\phi_{\alpha_1}(q_1)\phi_{\alpha_2}(q_2), \quad j = 1,2,$$

from which it follows that

$$\hat{A}_{\theta,j}(t)\Phi_\alpha^\theta(q_1,q_2,t) = \alpha_j\Phi_\alpha^\theta(q_1,q_2,t), \quad j = 1,2.$$

Now, we consider Ref. 4, where Malkin, Man'ko, and Trifonov studied the problem of the N-dimensional nonstationary harmonic oscillator and the problem of a charged particle in an axially symmetric and a uniform time-dependent electromagnetic field. The MMT-approach for solving problems described by a Schrödinger operator $\hat{S}(t) = i\hbar\partial_t - \hat{H}(t)$ is based on finding all independent linear in position and momentum invariants. In that context, an invariant is defined as an operator $\hat{A}(t)$ that commutes with $\hat{S}(t)$, that is, $[\hat{A}(t), \hat{S}(t)] = 0$.

We note that the Hamiltonian in Ref. 4 do not contain damping and external forces so that it is a particular case of Hamiltonian $\hat{\mathcal{H}}_{gen}(t)$ given by (1). Then, if in Ref. 4 one takes $\epsilon(t)$ to satisfy (42) for $\mu(t) = 1$ and $B(t) = 0$ with the specific IC's (43), it will coincide with $\epsilon(t)$ defined in the present work. In addition, one can write

$$\begin{pmatrix}\hat{A}(t)\\ \hat{B}(t)\end{pmatrix} = \frac{1}{2\sqrt{e}}\begin{pmatrix}-i & 1\\ 1 & -i\end{pmatrix}\begin{pmatrix}\hat{A}_{\theta,1}(t)\\ \hat{A}_{\theta,2}(t)\end{pmatrix}, \quad \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix}-i & 1\\ 1 & -i\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}, \tag{44}$$

which shows that the invariants $\hat{A}(t)$, $\hat{B}(t)$ found in Ref. 4 can be written as linear combinations of our invariants $\hat{A}_{\theta,1}(t), \hat{A}_{\theta,2}(t)$, $j = 1,2$. Finally, if one takes α and β as defined in (44), then coherent states $|\alpha,\beta;\ t\rangle$ found in Ref. 4 will coincide with coherent states (37) found in this work.

B. Quadratic invariants

For the quantum system described by the Hamiltonian $\hat{\mathcal{H}}_{gen}(t)$ given by (1), using the evolution operator and $\hat{H}_0 = \hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 + 1$, we can define a quadratic Hermitian invariant

$$\hat{I}_\theta(t) = \hat{\mathcal{U}}_{gen}(t,t_0)\hat{H}_0\hat{\mathcal{U}}_{gen}^\dagger(t,t_0). \tag{45}$$

This invariant can be expressed in terms of the linear invariants (40) and (41) as follows:

$$\hat{I}_\theta(t) = \hat{A}_{\theta,1}^\dagger(t)\hat{A}_{\theta,1}(t) + \hat{A}_{\theta,2}^\dagger(t)\hat{A}_{\theta,2}(t) + 1.$$

We note that the invariants (40) and (41) can be written also in the form

$$\begin{pmatrix}\hat{A}_{\theta,1}(t)\\ \hat{A}_{\theta,2}(t)\end{pmatrix} = \frac{e^{i\eta(t)}}{\sqrt{2\omega_0\hbar}}\left[\left(\frac{\omega_0}{|\epsilon(t)|} + i|\epsilon(t)|\mu(t)\left(B(t) - \frac{d}{dt}\ln|\epsilon(t)|\right)\right)\begin{pmatrix}\hat{Q}_1\\ \hat{Q}_2\end{pmatrix} + i|\epsilon(t)|\begin{pmatrix}\hat{P}_1\\ \hat{P}_2\end{pmatrix}\right]$$

and

$$\begin{pmatrix} \hat{A}^{\dagger}_{\theta,1}(t) \\ \hat{A}^{\dagger}_{\theta,2}(t) \end{pmatrix} = \frac{e^{-i\eta(t)}}{\sqrt{2\omega_0\hbar}}\left[\left(\frac{\omega_0}{|\epsilon(t)|} - i|\epsilon(t)|\mu(t)\left(B(t) - \frac{d}{dt}\ln|\epsilon(t)|\right)\right)\begin{pmatrix} \hat{Q}_1 \\ \hat{Q}_2 \end{pmatrix} - i|\epsilon(t)|\begin{pmatrix} \hat{P}_1 \\ \hat{P}_2 \end{pmatrix}\right],$$

where $\sigma(t) = |\epsilon(t)|$ satisfies the Ermakov–Pinney nonlinear differential equation,

$$\ddot{\sigma}(t) + \frac{\dot{\mu}}{\mu}\dot{\sigma}(t) + \left(\omega^2(t) - \left(\dot{B} + B^2 + \frac{\dot{\mu}}{\mu}B\right)\right)\sigma(t) = \frac{1}{\mu^2\sigma^3(t)}, \tag{46}$$

with the initial conditions

$$\sigma(t_0) = 1, \quad \dot{\sigma}(t_0) = B(t_0). \tag{47}$$

Then, the quadratic invariant becomes

$$\hat{I}_\theta(t) = \frac{1}{2\omega_0\hbar}\sum_{j=1}^{2}\left\{\frac{\omega_0^2}{|\epsilon(t)|^2}\hat{Q}_j^2 + \left[|\epsilon(t)|\mu(t)\left(B(t) - \frac{d}{dt}\ln|\epsilon(t)|\right)\hat{Q}_j + |\epsilon(t)|\hat{P}_j\right]^2\right\}, \tag{48}$$

and it is special in the sense that $|\epsilon(t)|$ is a particular solution of the Ermakov–Pinney equation (46) satisfying the initial conditions (47). Now, since the following commutation relations hold,

$$[\hat{A}_{\theta,i}(t), \hat{A}^{\dagger}_{\theta,j}(t)] = \delta_{ij}, \quad [\hat{I}_\theta(t), \hat{A}_{\theta,j}(t)] = -\hat{A}_{\theta,j}(t), \quad [\hat{I}_\theta(t), \hat{A}^{\dagger}_{\theta,j}(t)] = \hat{A}^{\dagger}_{\theta,j}(t), \quad j = 1,2,$$

then the eigenvalues and eigenstates of the invariant $\hat{I}_\theta(t)$ can be found by the same algebraic procedure as for the simple harmonic oscillator. Here, $\hat{H}_0\varphi_n(\mathbf{q}) = E_n\varphi_n(\mathbf{q})$ so that by construction of (45), we have $\hat{I}_\theta(t)\Psi_n^\theta(\mathbf{q},t) = E_n\Psi_n^\theta(\mathbf{q},t)$, showing that time-evolved wave function solutions of the Scrödinger equation found as $\Psi_n^\theta(\mathbf{q},t) = \hat{\mathcal{U}}_{gen}(t,t_0)\varphi_n(\mathbf{q})$ in (29) are eigenstates of the invariant $\hat{I}_\theta(t)$ corresponding to eigenvalues $E_n = \hbar\omega_0(n_1 + n_2 + 1)$.

In the work of Lewis and Riesenfeld,[3] for a quantum system described by an explicitly time-dependent Hamiltonian $\hat{H}(t)$, a dynamical invariant is defined to be an operator $\hat{I}(t)$ satisfying $i\hbar\partial_t\hat{I}(t) - [\hat{H}(t), \hat{I}(t)] = 0$. As known, the LR-approach for solving nonstationary quantum oscillators is based on finding Hermitian quadratic invariant of the form (48). Then, eigenstates of the quadratic invariant constructed by the LR-technique and solutions of the Scrödinger equation usually differ by a time-dependent phase factor. We note that in Ref. 3, the Hamiltonian describing a charged particle in a time-dependent electromagnetic field is a particular case of Hamiltonian $\hat{\mathcal{H}}_{gen}(t)$ given by (1). For more recent and related results based on linear and quadratic invariants, one can see also Ref. 34.

VI. CAUCHY–EULER TYPE QUANTUM OSCILLATOR IN TIME-VARIABLE MAGNETIC AND ELECTRIC FIELDS

Now, we introduce and discuss an exactly solvable quantum model described by the Hamiltonian,

$$\hat{\mathcal{H}}_{gen}(t) = \sum_{j=1}^{2}\left[\frac{1}{2t^\gamma}\hat{p}_j^2 + \frac{B(t)}{2}(\hat{q}_j\hat{p}_j + \hat{p}_j\hat{q}_j) + \frac{\omega_0^2 t^{\gamma-2}}{2}\hat{q}_j^2\right] + E_0\, t^\gamma\, \sin(\Omega_E\, \ln\, t)\hat{q}_1 + E_0\, t^\gamma\, \cos(\Omega_E\, \ln\, t)\hat{q}_2 + \frac{\lambda_0}{t}(\hat{q}_1\hat{p}_2 - \hat{q}_2\hat{p}_1). \tag{49}$$

In this model, for $t \geq t_0, t_0 = 1$, we have time-dependent increasing mass $\mu(t) = t^\gamma$ for the damping parameter $\gamma \geq 1$ and decreasing frequency $\omega^2(t) = \omega_0^2/t^2$, $\omega_0 > 0$. Then, to preserve the Cauchy–Euler structure of the oscillator, we take $B(t) = -\Omega_B\tan(\Omega_B\ln t)/t$, where $\Omega_B = \sqrt{\omega_B^2 - (\gamma-1)^2/4}$ and $\omega_B^2 > (\gamma-1)^2/4$. In addition, we consider external electric fields $E_1(t) = E_0\, t^\gamma\sin(\Omega_E\ln t)$, $E_2(t) = E_0\, t^\gamma\cos(\Omega_E\ln t)$ with E_0, Ω_E-real constants that are oscillating in time with increasing amplitude and decreasing frequency. The last term in (49) is the angular momentum with the Larmor type frequency $\lambda(t) = \lambda_0/t$, λ_0-real constant that depends on time and tends to zero when time increases.

In what follows, first we write the solutions to the corresponding coupled system of classical equations of motion. Then, we describe in detail the time-evolved eigenfunctions and coherent states.

A. The classical problem

For the quantum evolution problem with Hamiltonian (49), the corresponding coupled system of classical equations of motion is of the form

$$\begin{pmatrix}\ddot{X}_1\\ \ddot{X}_2\end{pmatrix}+\begin{pmatrix}\gamma/t & 2\lambda_0/t\\ -2\lambda_0/t & \gamma/t\end{pmatrix}\begin{pmatrix}\dot{X}_1\\ \dot{X}_2\end{pmatrix}+\begin{pmatrix}\dfrac{\omega_0^2+\omega_B^2-\lambda_0^2}{t^2} & \dfrac{\lambda_0(\gamma-1)}{t^2}\\ -\dfrac{\lambda_0(\gamma-1)}{t^2} & \dfrac{\omega_0^2+\omega_B^2-\lambda_0^2}{t^2}\end{pmatrix}\begin{pmatrix}X_1\\ X_2\end{pmatrix}=\begin{pmatrix}-E_0\sin(\Omega_E\ln t)\\ -E_0\cos(\Omega_E\ln t)\end{pmatrix}. \tag{50}$$

For $E_0 = 0$, system (50) with initial conditions (12) has homogeneous a solution $\mathbf{X}^{(h)}(t) \equiv R_\theta^T(t)\mathbf{x}^{(h)}(t)$, explicitly found as

$$\mathbf{X}^{(h)}(t) = R_\theta^T(t)\begin{pmatrix}\dfrac{\sqrt{\omega_0^2+\omega_B^2}}{\Omega_g}t^{-(\gamma-1)/2}\cos(\Omega_g\ln t-\delta_g)\\ \dfrac{1}{\Omega_g}t^{-(\gamma-1)/2}\sin(\Omega_g\ln t)\end{pmatrix},\quad t\ge 1, \tag{51}$$

where $\Omega_g = \sqrt{\omega_0^2+\omega_B^2-(\gamma-1)^2/4}$ is the oscillator frequency and $\delta_g = \arctan((\gamma-1)/2\Omega_g)$.

For $E_0 \neq 0$, a particular solution is $\mathbf{X}^{(p)}(t) \equiv R_\theta^T(t)\mathbf{x}^{(p)}(t)$, and explicitly, we have

$$\mathbf{X}^{(p)}(t) = R_\theta^T(t)\begin{pmatrix}A_1^{(h)}t^{-(\gamma-1)/2}\cos(\Omega_g\ln t-\delta_1^{(h)})-\dfrac{E_0}{\sqrt{a^2+b^2}}\cos((\Omega_E+\lambda_0)\ln t-\delta_p)\\ A_2^{(h)}t^{-(\gamma-1)/2}\sin(\Omega_g\ln t-\delta_2^{(h)})+\dfrac{E_0}{\sqrt{a^2+b^2}}\sin((\Omega_E+\lambda_0)\ln t-\delta_p)\end{pmatrix},\quad t\ge 1, \tag{52}$$

where $a = (\omega_0^2+\omega_B^2)-(\Omega_E+\lambda_0)^2$, $b = (1-\gamma)(\Omega_E+\lambda_0)$, $\delta_p = \operatorname{arccot}(b/a)$, and $A_j^{(h)}, \delta_j^{(h)}$, $j = 1,2$, are constants of the transient part such that $\mathbf{X}^{(p)}(t)$ satisfies the initial conditions (13). Here, the rotation angle is $\theta(t) = \lambda_0 \ln t$, and the rotation matrix becomes

$$R_\theta(t) = \begin{pmatrix}\cos(\lambda_0\ln t) & \sin(\lambda_0\ln t)\\ -\sin(\lambda_0\ln t) & \cos(\lambda_0\ln t)\end{pmatrix},\quad t\ge 1,$$

where the sign of λ_0 determines the direction of rotation.

B. Time-evolution of the wave functions $\Psi_n^\theta(\mathbf{q},t)$

For the wave functions $\Psi_n^\theta(\mathbf{q},t)$, the probability densities are given by Eq. (30), that is,

$$\rho_n^\theta(\mathbf{q},t) = \mathbf{N}_n^2\frac{1}{|\epsilon(t)|^2}\exp\left\{-\frac{\omega_0}{\hbar}\frac{\left|\mathbf{q}-\mathbf{X}^{(p)}(t)\right|^2}{|\epsilon(t)|^2}\right\}\mathbf{H}_n^2\left(\sqrt{\frac{\omega_0}{\hbar}}\frac{R_\theta(t)\left(\mathbf{q}-\mathbf{X}^{(p)}(t)\right)}{|\epsilon(t)|},t\right),\quad n=(n_1,n_2),$$

where $\mathbf{X}^{(p)}(t)$ is found in (52), and the squeezing coefficient is

$$|\epsilon(t)| = \frac{t^{-(\gamma-1)/2}}{\Omega_g}\sqrt{(\omega_0^2+\omega_B^2)\cos^2(\Omega_g\ln t-\delta_g)+\omega_0^2\sin^2(\Omega_g\ln t)}, \tag{53}$$

which is smooth and oscillatory for $t \ge 1$. Then, for given $\omega_0 > 0$ and $\gamma \ge 1$, the frequency $\Omega_g = \sqrt{\omega_0^2+\omega_B^2-(\gamma-1)^2/4}$ of oscillations in $|\epsilon(t)|$ can be increased by increasing the value of ω_B in parameter $B(t)$. When $\gamma = 1$, the amplitude is fixed, and one has $|\epsilon(t)| \to 1$ as $\omega_B \to 0$. However, when $\gamma > 1$, the amplitude of oscillations in $|\epsilon(t)|$ decreases and approaches zero as time increases.

In Fig. 1, we plot the probability density $\rho_n^\theta(\mathbf{q},t)$ with $n = (1,2)$ at three different times. For this, we take $\gamma = 2$, $B(t) = -3\sqrt{11}\tan(3\sqrt{11}\ln t)/t$, $\omega_B = \sqrt{397}/2$, Larmor type frequency $\lambda(t) = 7/t$, and $E_0 = 0$ so that there are no external electric fields. These plots show how the width and amplitude of the wave packets change with time and how they are rotated with angle $\theta(t) = 7\ln t$ under the influence of the magnetic field. Uncertainties of the position and momentum at time-evolved wave functions $\Psi_n^\theta(\mathbf{q},t)$ are found by Eqs. (32) and (33), respectively,

$$(\Delta\hat{\mathbf{q}})_n^\theta(t) = \sqrt{\frac{\hbar}{\omega_0}}|\epsilon(t)|\,\mathbf{\Lambda}(n_1,n_2,\theta(t)),\qquad (\Delta\hat{\mathbf{p}})_n^\theta(t) = \sqrt{\hbar\omega_0}\frac{\Sigma(t)}{|\epsilon(t)|}\mathbf{\Lambda}(n_1,n_2,\theta(t)),$$

where for this model we obtain the vector valued function

$$\mathbf{\Lambda}(n_1,n_2,\theta(t)) = \begin{pmatrix}\left(\cos^2(\lambda_0\ln t)n_1+\sin^2(\lambda_0\ln t)n_2+1/2\right)^{\frac12}\\ \left(\sin^2(\lambda_0\ln t)n_1+\cos^2(\lambda_0\ln t)n_2+1/2\right)^{\frac12}\end{pmatrix},$$

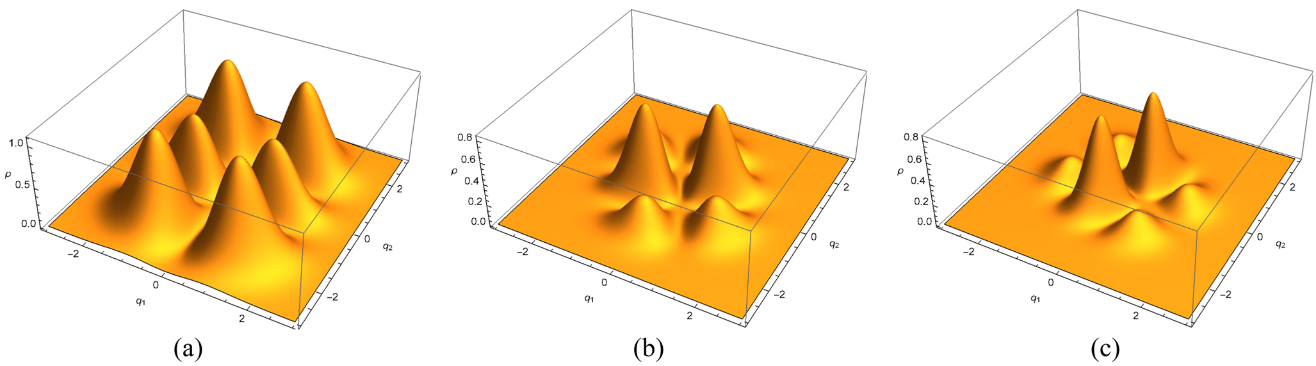

FIG. 1. Probability density $\rho_n^\theta(\mathbf{q},t)$ for $n=(1,2)$, $\gamma=2, \hbar=1$, $\omega_0=1$, $\theta(t)=7\ln t$, $E_0=0$, at times (a) $t_0=1$, (b) $t=1.07$, and (c) $t=1.85$.

and the coefficient

$$\Sigma(t)=\Big\{1+\frac{1}{4\omega_0^2t^2}\big[(2\Omega_B\tan(\Omega_B\ln t)-\gamma+1)\big((\omega_0^2+\omega_B^2)\cos^2(\Omega_g\ln t-\delta_g)+\omega_0^2\sin^2(\Omega_g\ln t)\big)$$
$$+\Omega_g\big(-(\omega_0^2+\omega_B^2)\sin(2(\Omega_g\ln t-\delta_g))+\omega_0^2\sin(2\Omega_g\ln t)\big)\big]^2\Big\}^{1/2}. \tag{54}$$

Clearly, $\mathbf{\Lambda}(n_1,n_2,\theta(t))$ carries the dependence of the uncertainties on the wave numbers n_1,n_2 and the rotation angle $\theta(t)$, while $|\epsilon(t)|$ and $\Sigma(t)$ depend only on parameters $\mu(t),\omega^2(t)$ and $B(t)$. We note that for $\gamma=1$ and $\omega_B\to 0$, one has $\Sigma(t)\to 1$. Otherwise, coefficient $\Sigma(t)$ has singularities due to the singularities in $B(t)$, and this affects the uncertainties in momentum. Using the same parameters as in Fig. 1, then in Fig. 2, we plot uncertainties in the position and momentum at wave function $\Psi_n^\theta(\mathbf{q},t)$ for $n=(1,2)$. As can be seen in Fig. 2(a), uncertainty in the position is smooth, oscillatory, and approaches to zero as time increases. However, singularities appear in uncertainties of momentum since they depend on the coefficient $\Sigma(t)$ found by (54).

Now, we discuss possible trajectories of the wave packets in the two-dimensional coordinate space, which are determined by the expectation values of the position at state $\Psi_n^\theta(\mathbf{q},t)$. According to the general results found in (31), if there are no external fields, wave packets are localized at $(q_1,q_2)=(0,0)$ in $\mathbb{R}^2$, as in Fig. 1. However, in the presence of external fields, wave packets will move along the trajectory $\langle\hat{\mathbf{q}}\rangle_n^\theta(t)=\mathbf{X}^{(p)}(t)$ in $\mathbb{R}^2$, which for this model is given by (52). Then, depending on parameter $\gamma\geq 1$ in (52), we consider the following two cases:

(i) For $\gamma=1$, we have the trajectory

$$\langle\hat{\mathbf{q}}\rangle_n^\theta(t)=R_\theta^T(t)\begin{pmatrix}\frac{E_0}{|a|\Omega_g}((\Omega_E+\lambda_0)\sin(\Omega_g\ln t)-\Omega_g\sin((\Omega_E+\lambda_0)\ln t))\\ \frac{E_0}{|a|}(\cos(\Omega_g\ln t)-\cos((\Omega_E+\lambda_0)\ln t))\end{pmatrix},\quad t\geq 1,$$

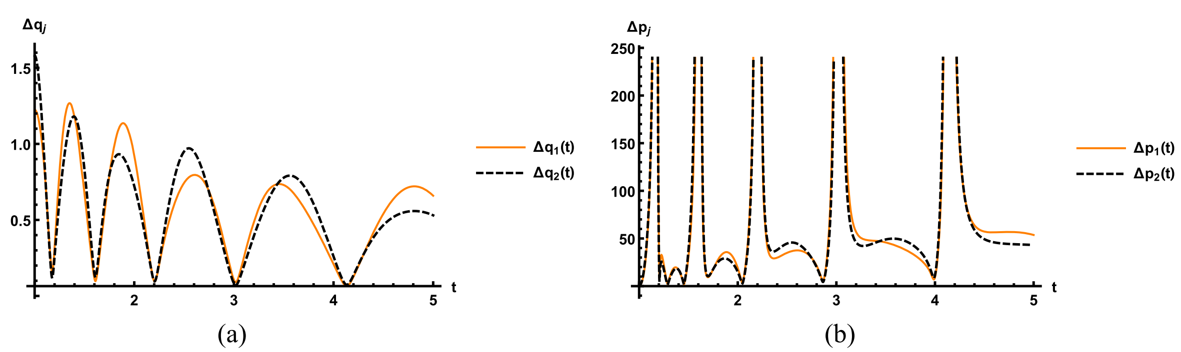

FIG. 2. Uncertainties in the position and momentum for $n=(1,2)$ and $\gamma=2,\hbar=1$, $\omega_0=1$: (a) $(\Delta\hat{q}_j)_{n_j}^\theta(t)$, $j=1,2$, for $t\in[1,5]$ and (b) $(\Delta\hat{p}_j)_{n_j}^\theta(t)$, $j=1,2$, for $t\in[1,5]$.

where $\Omega_g = \sqrt{\omega_0^2 + \omega_B^2}$ is the oscillator frequency and Ω_E, λ_0 are frequencies due to the external fields. When $(\Omega_E + \lambda_0) = \Omega_g$, then one has balance between frequencies and the particle is localized at the origin for any time. When $(\Omega_E + \lambda_0)/\Omega_g$ is a rational number, the trajectory $\langle \hat{\mathbf{q}} \rangle_n^\theta(t)$ is a closed plane curve. In this case, a particle moving along the trajectory returns to its starting point after some time, whatever the starting point is, and then retraces the same curve. On the other hand, when $(\Omega_E + \lambda_0)/\Omega_g$ is not rational, the curve will never close and the particle will pass through every point of a bounded region containing the origin in $\mathbb{R}^2$, eventually filling it. Clearly, we have non-uniform motion with smoothly decreasing speed.

(ii) For $\gamma > 1$, since transient part of $\mathbf{X}^{(p)}(t)$ quickly tends to zero, after some time, we have

$$\langle \hat{\mathbf{q}} \rangle_n^\theta(t) \approx R_\theta^T(t) \begin{pmatrix} -\dfrac{E_0}{\sqrt{a^2+b^2}} \cos((\Omega_E + \lambda_0) \ln t - \delta_p) \\ \dfrac{E_0}{\sqrt{a^2+b^2}} \sin((\Omega_E + \lambda_0) \ln t - \delta_p) \end{pmatrix} = \begin{pmatrix} -\dfrac{E_0}{\sqrt{a^2+b^2}} \cos(\Omega_E \ln t - \delta_p) \\ \dfrac{E_0}{\sqrt{a^2+b^2}} \sin(\Omega_E \ln t - \delta_p) \end{pmatrix}.$$

Then, the particle exhibits again a non-uniform motion with decreasing speed and with λ_0 contributing to the phase and radius of the orbit. In that case, the trajectory is not closed since usually it does not repeat, but in the long time limit, it converges to a circular orbit given by (55). □

As an example, for $\gamma = 1$ in Fig. 3, we plot the trajectory

$$\langle \hat{\mathbf{q}} \rangle_n^\theta(t) = \frac{E_0}{|a|} R_\theta^T(t) \begin{pmatrix} \dfrac{15+\lambda_0}{10} \sin(10 \ln t) - \sin((15+\lambda_0) \ln t) \\ \cos(10 \ln t) - \cos((15+\lambda_0) \ln t) \end{pmatrix}, \quad \theta(t) = \lambda_0 \ln t,$$

starting at the origin and with parameters $\Omega_g = 10$, $\Omega_E = 15, E_0 = 800$, $a = 100 - (15+\lambda_0)^2$. In Fig. 3(a), we see the plot for $\theta(t) = 0$, which is a closed curve since $\Omega_E/\Omega_g = 3/2$ is rational. In Fig. 3(b), we show this curve under rotation with the rotation angle $\theta(t) = 15 \ln t$.

Another example for $\gamma = 1$ is given in Fig. 4, where we plot the trajectory

$$\langle \hat{\mathbf{q}} \rangle_n^\theta(t) = \frac{E_0}{|a|} R_\theta^T(t) \begin{pmatrix} \dfrac{20\pi+\lambda_0}{10} \sin(10 \ln t) - \sin((20\pi+\lambda_0) \ln t) \\ \cos(10 \ln t) - \cos((20\pi+\lambda_0) \ln t) \end{pmatrix}, \quad \theta(t) = \lambda_0 \ln t,$$

with parameters $\Omega_g = 10, \Omega_E = 20\pi, E_0 = 5 \times 10^3$ and $a = 100 - (20\pi + \lambda_0)^2$. In Fig. 4(a), we have $\theta(t) = 0$, and note that $\Omega_E/\Omega_g = 2\pi$ is irrational so that the curve is not closed. The particle will start from the origin, and then it will pass through every point of a bounded region in $\mathbb{R}^2$ as $t \to \infty$. In Fig. 4(b), we take $\theta(t) = 20 \ln t$ and see particle motion along another open trajectory confined to a bounded region.

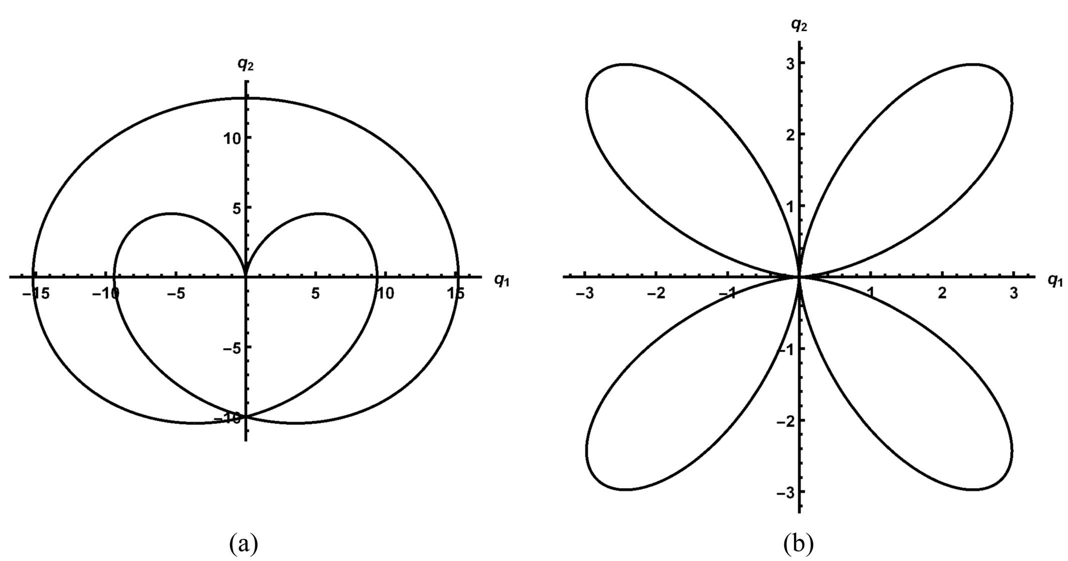

FIG. 3. Trajectory of the wave packets $|\Psi_n^\theta(\mathbf{q},t)|^2$ for $\forall n$ and $\gamma = 1$, $\omega_0 = 1, \hbar = 1$, $B(t) = -3\sqrt{11}\tan(3\sqrt{11}\ln t)/t$, $\omega_B = 3\sqrt{11}$, $E_1(t) = 800t\sin(15\ln t), E_2(t) = 800t\cos(15\ln t)$, $t \in [1, 4]$: (a) when $\lambda(t) = 0$ and (b) when $\lambda(t) = 15/t$.

C. Time-evolution of coherent states $\Phi_\alpha^\theta(\mathbf{q},t)$

At coherent states $\Phi_\alpha^\theta(\mathbf{q},t)$, probability densities are two-dimensional Gaussian wave packets given by (38), i.e.,

$$\rho_\alpha^\theta(\mathbf{q},t) = \left(\frac{\omega_0}{\pi\hbar}\right)\frac{1}{|\epsilon(t)|^2}\exp\left\{-\frac{\omega_0}{\hbar}\frac{\left|\mathbf{q}-\langle\hat{\mathbf{q}}\rangle_\alpha^\theta(t)\right|^2}{|\epsilon(t)|^2}\right\},\quad \alpha=(\alpha_1,\alpha_2)\in\mathbf{C}^2,$$

where for this model, the squeezing coefficient is explicitly given by (53). As an example, in Fig. 5, we plot the probability density for $\alpha = (20\sqrt{2}/\sqrt{401}, 10i)$, $\gamma = 2$, $\hbar = 1$, $\omega_0 = 1$, $\lambda(t) = 10/t$, and squeezing parameter $B(t) = -3\sqrt{11}\tan(3\sqrt{11}\ln t)/t$, $\omega_B = \sqrt{397}/2$ at different times $t = 1, 1.2, 2$. These plots show the changes in width and amplitude of the wave packet that follows a trajectory

$$\langle\hat{\mathbf{q}}\rangle_\alpha^\theta(t) = R_\theta^T(t)\begin{pmatrix} 2t^{-1/2}\cos(10\ln t - \arctan(1/20)) \\ \sqrt{2}t^{-1/2}\sin(10\ln t)\end{pmatrix}, \tag{55}$$

with the rotation angle $\theta(t) = 10\ln t$, and in the case $E_0 = 0$. Explicitly, the corresponding uncertainties are found according to (56), that is,

$$(\Delta\hat{q}_j)_\alpha^0(t) = \sqrt{\frac{\hbar}{2\omega_0}}|\epsilon(t)|,\quad (\Delta\hat{p}_j)_\alpha^0(t) = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{|\epsilon(t)|}\Sigma(t),\quad j=1,2,$$

where the coefficients $|\epsilon(t)|$ and $\Sigma(t)$ are given by (53) and (54), respectively. Clearly, uncertainties do not depend on α and $\theta(t)$, and they are equal in both directions. Figure 6 shows $(\Delta\hat{q}_j)_{\alpha_j}(t)$ and $(\Delta\hat{p}_j)_{\alpha_j}(t)$ for each $j = 1, 2$, where we take $B(t) = -3\sqrt{11}\tan(3\sqrt{11}\ln t)/t$, $\omega_B = \sqrt{397}/2$ as in Fig. 5. We note that uncertainties in position are smooth, oscillatory, and approach zero as $t\to\infty$, but uncertainties in momentum have singularities due to the singularities in $B(t)$, as we see in Fig. 6(b).

Now, we recall that the center of the wave packet $\rho_\alpha^\theta(\mathbf{q},t)$ in the two-dimensional coordinate space follows the classical trajectory

$$\langle\hat{\mathbf{q}}\rangle_\alpha^\theta(t) = R_\theta^T(t)\langle\hat{\mathbf{q}}\rangle_\alpha^0(t) = \sqrt{\frac{2\hbar}{\omega_0}}\mathbf{C}_\alpha^\theta(t)\mathbf{X}^{(h)}(t) + \mathbf{X}^{(p)}(t), \tag{56}$$

and for this model, $\mathbf{C}_\alpha^\theta(t)$ is defined by (39) with $\theta(t) = \lambda_0\ln t$, $\mathbf{X}^{(h)}(t)$ is given by (51), and $\mathbf{X}^{(p)}(t)$ is given by (52). In particular, when $\theta(t) = 0$ and there are no external electric fields ($E_0 = 0$), then the trajectory will be $\langle\hat{\mathbf{q}}\rangle_\alpha^0(t) = \sqrt{2\hbar/\omega_0}\,\mathbf{C}_\alpha^0\,\mathbf{x}^{(h)}(t)$, which can be written explicitly as

$$\langle\hat{\mathbf{q}}\rangle_\alpha^0(t) = \sqrt{\frac{2\hbar}{\omega_0}}\begin{pmatrix}\alpha_1^{(1)} & \omega_0\alpha_1^{(2)} \\ \alpha_2^{(1)} & \omega_0\alpha_2^{(2)}\end{pmatrix}\begin{pmatrix}\frac{\sqrt{\omega_0^2+\omega_B^2}}{\Omega_g}t^{-(\gamma-1)/2}\cos(\Omega_g\ln t - \delta_g) \\ \frac{1}{\Omega_g}t^{-(\gamma-1)/2}\sin(\Omega_g\ln t)\end{pmatrix}. \tag{57}$$

In Eq. (57), depending on the values of $\gamma\geq 1$ and $\alpha = (\alpha_1,\alpha_2)\in\mathbb{C}^2$, we note the following possibilities:

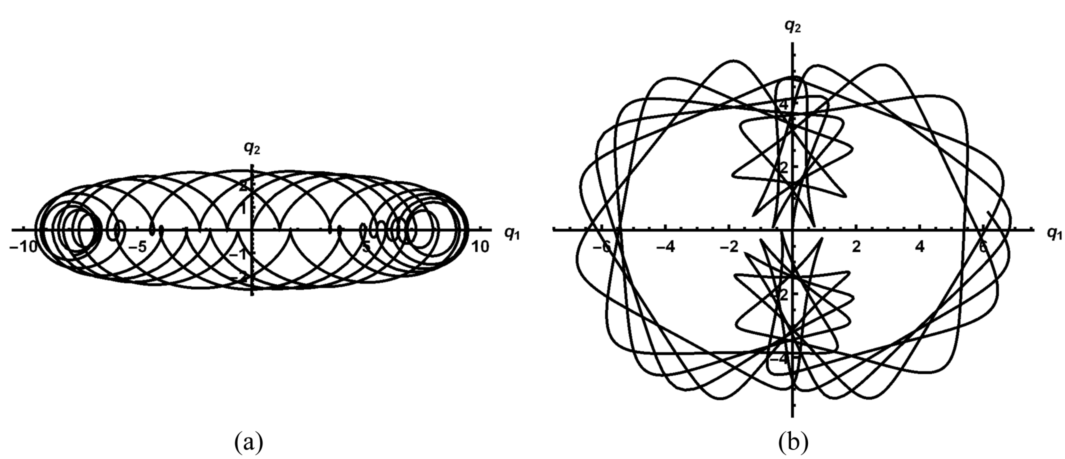

FIG. 4. Trajectory of the wave packets $|\Psi_n^\theta(\mathbf{q},t)|^2$ for any n, with $\gamma = 1$, $\omega_0 = 1$, $\hbar = 1$, $B(t) = -3\sqrt{11}\tan(3\sqrt{11}\ln t)/t$, $\omega_B = 3\sqrt{11}$, $E_1(t) = 5\times10^3 t\sin(20\pi\ln t)$, $E_2(t) = 5\times10^3 t\cos(20\pi\ln t)$, $t\in[1,20]$: (a) when $\lambda(t) = 0$ and (b) when $\lambda(t) = 20/t$.

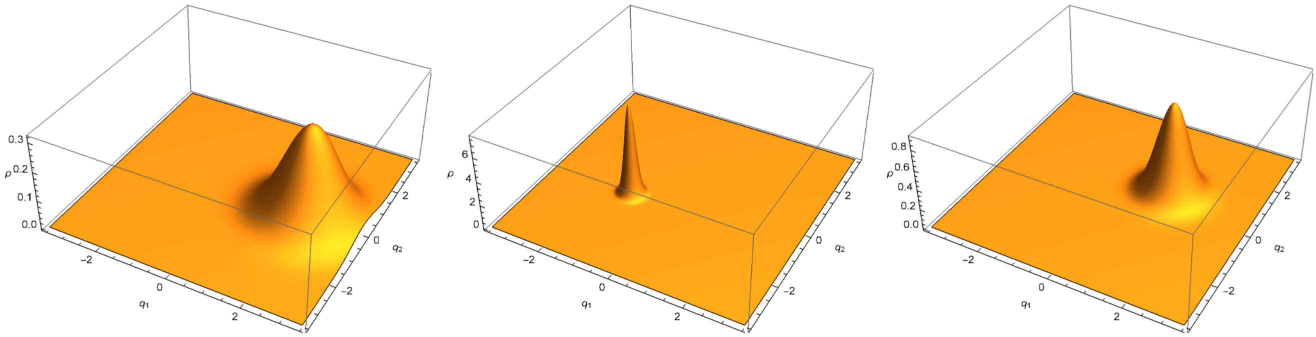

FIG. 5. Probability density $\rho_\alpha^\theta(\mathbf{q}, t)$ for $\alpha = (20\sqrt{2}/\sqrt{401}, 10i)$, $\lambda(t) = 10/t$, $\gamma = 2, \hbar = 1, \omega_0 = 1, E_0 = 0$ at times: (a) $t = t_0 = 1$, (b) $t = 1.2$, and (c) $t = 2$.

(a) For $\gamma = 1$, the trajectory could be a line segment, a circle, or an ellipse in $\mathbb{R}^2$, centered at the origin. In the case $det(C_\alpha^0) \equiv \omega_0(\alpha_1^{(1)}\alpha_2^{(2)} - \alpha_2^{(1)}\alpha_1^{(2)}) = 0$, the wave packet oscillates along a line segment. If $\alpha_1^{(2)} = \alpha_2^{(1)} = 0$ and $|\alpha_2^{(2)}| = \sqrt{1 + (\omega_B^2/\omega_0^2)}\ |\alpha_1^{(1)}|$ or similarly if $\alpha_1^{(1)} = \alpha_2^{(2)} = 0$ and $|\alpha_1^{(2)}| = \sqrt{1 + (\omega_B^2/\omega_0^2)}\ |\alpha_2^{(1)}|$, then we have a circular motion. Otherwise, the motion is along an ellipse, and in any case, it is a non-uniform motion.

(b) For $\gamma > 1$, the wave packet moves toward the origin due to damping effects. In the case $det(C_\alpha^0) = 0$, it oscillates forth and back along a line segment passing through the origin, with decreasing amplitude and approaching the origin. If $det(C_\alpha^0) \neq 0$, the wave packet moves inward usually along a spiral like trajectory as time increases.

It follows that when $\theta(t) \neq 0$, the rotated trajectories (except the circular ones) will be more complicated, as one can see in the following plots. For example, we consider the trajectory (56) for $\gamma = 1$,

$$\langle \hat{\mathbf{q}} \rangle_\alpha^\theta(t) = R_\theta^T(t) \begin{pmatrix} \cos(10 \ln t) \\ \frac{1}{2} \sin(10 \ln t) \end{pmatrix}, \qquad \theta(t) = \lambda_0 \ln(t),$$

which is an ellipse for $\theta = 0$ and $E_0 = 0$, as we see in Fig. 7(a). In Fig. 7(b), we see the rotated ellipse for $\theta(t) = 25 \ln t, \lambda_0 = 25$. Then, in Fig. 7(c), we plot the trajectory

$$\langle \hat{\mathbf{q}} \rangle_\alpha^\theta(t) = R_\theta^T(t) \left[\begin{pmatrix} \cos(10 \ln t) \\ \frac{1}{2} \sin(10 \ln t) \end{pmatrix} + \begin{pmatrix} 3 \sin(10 \ln t) - \sin(30 \ln t) \\ \cos(10 \ln t) - \cos(30 \ln t) \end{pmatrix} \right],$$

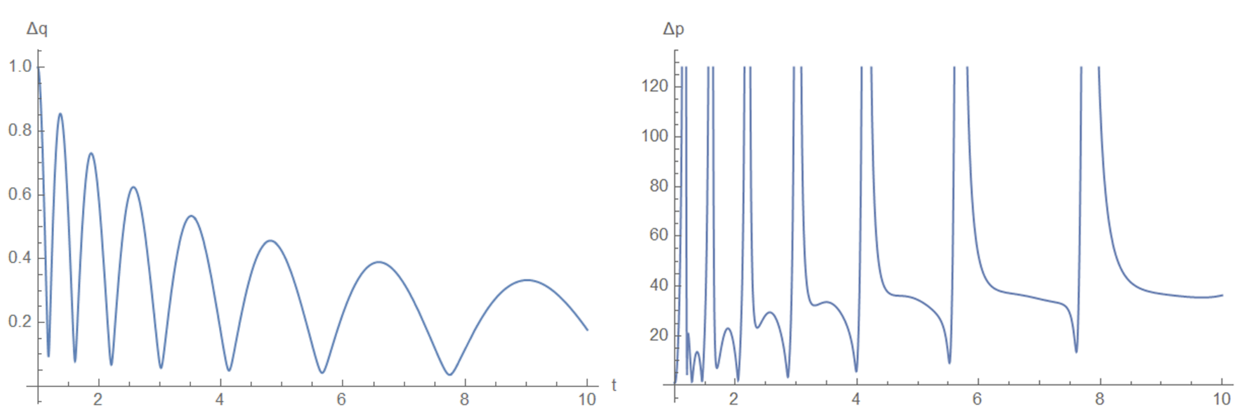

FIG. 6. Uncertainties for $\gamma = 2, \hbar = 1, \omega_0 = 1$: (a) $(\Delta\hat{q}_j)_{\alpha_j}(t),\ j = 1, 2$, and (b) $(\Delta\hat{p}_j)_{\alpha_j}(t),\ j = 1, 2, t \in [1, 10]$.

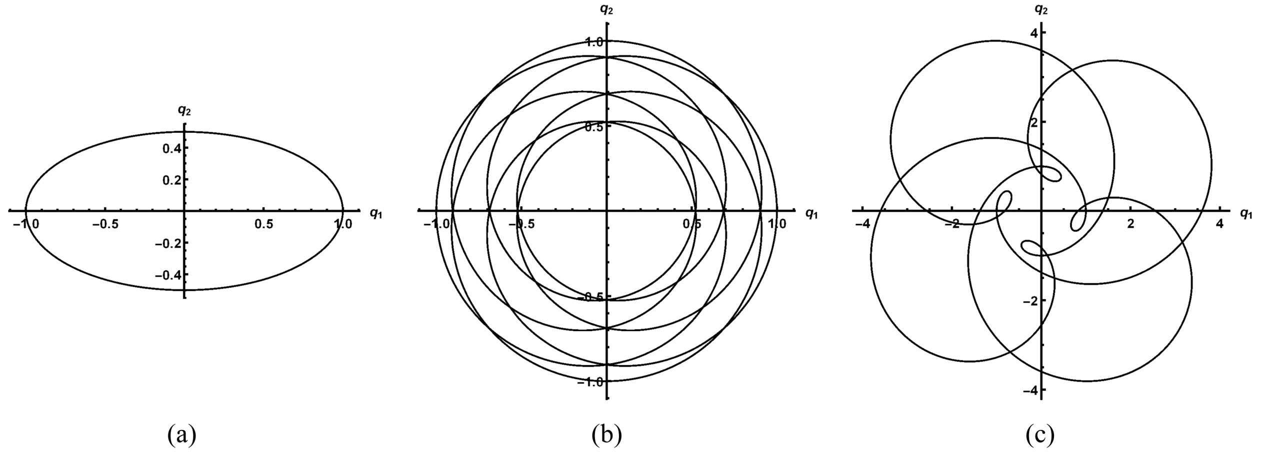

FIG. 7. Trajectories of $\rho_\alpha^\theta(\mathbf{q},t)$ with $\gamma = 1$, $\alpha = (\sqrt{2}/2, 5\sqrt{2}i/2)$, $B(t) = -3\sqrt{11}\tan(3\sqrt{11}\ln t)/t$, $\omega_B = 3\sqrt{11}$, $\hbar = \omega_0 = 1$: (a) $\lambda(t) = 0$, $E_0 = 0$, $t \in [1, 2]$, (b) $\lambda(t) = 25/t$, $E_0 = 0$, $t \in [1, 4]$, and (c) $\lambda(t) = 25/t$, $E_1(t) = 800t\sin(5\ln t)$, $E_2(t) = 800t\cos(5\ln t)$, $t \in [1, 4]$.

with the rotation angle $\theta(t) = 25\ln t$ and under the influence of electric fields. In that case, the trajectory depends also on the particular solution of the classical system, and since the ratio $(\Omega_E + \lambda_0)/\Omega_g = 3$ is a rational number, the trajectory is closed.

As another example, for $\gamma = 2$ in Fig. 8, we plot

$$\langle\hat{\mathbf{q}}\rangle_\alpha^\theta(t) = 2\sqrt{2}R_\theta^T(t)\begin{pmatrix} t^{-1/2}\ \sin(10\ \ln\ t) \\ t^{-1/2}\ \sin(10\ \ln\ t)\end{pmatrix}, \qquad \theta(t) = \lambda_0\ \ln(t), \tag{58}$$

where in Fig. 8(a) we have $\theta = 0$ and $det(C_\alpha^0) = 0$, so that the wave packet oscillates along a straight line and approaches the origin as time increases. Figure 8(b) shows the trajectory given by Eq. (58) with the rotation angle $\theta(t) = 20\ln t$. Then, in Fig. 8(c), we plot

$$\langle\hat{\mathbf{q}}\rangle_\alpha^\theta(t) = 2\sqrt{2}R_\theta^T(t)\begin{pmatrix} t^{-1/2}\ \sin(10\ \ln\ t) \\ t^{-1/2}\ \sin(10\ \ln\ t)\end{pmatrix} + \frac{E_0}{\sqrt{a^2+b^2}}\begin{pmatrix} -\cos(5\ \ln\ t - \text{arccot}(b/a)) \\ \sin(5\ \ln\ t - \text{arccot}(b/a))\end{pmatrix},$$

where $a = 401/4 - (5+\lambda_0)^2$, $b = -(5+\lambda_0)$, for $\theta(t) = 20\ln t$, and in the presence of electric fields.

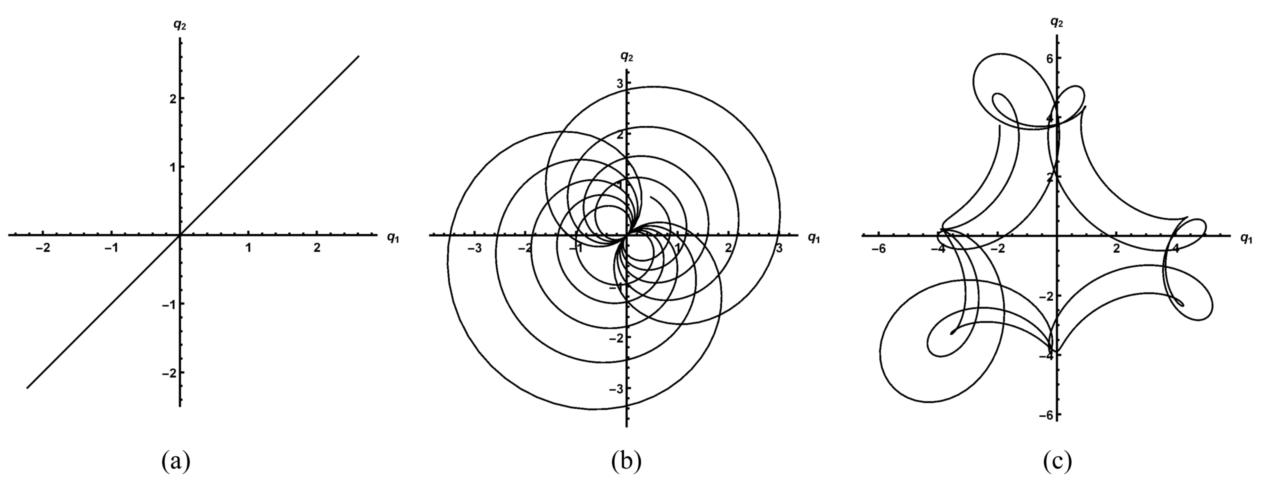

FIG. 8. Trajectories of $\rho_\alpha^\theta(\mathbf{q},t)$ with $\gamma = 2$, $\alpha = (20i, 20i)$, $B(t) = -3\sqrt{11}\tan(3\sqrt{11}\ln t)/t$, $\omega_B = \sqrt{397}/2$, $\hbar = \omega_0 = 1$, $t \in [1, 20]$: (a) $\lambda(t) = 0$, $E_0 = 0$, (b) $\lambda(t) = 20/t$, $E_0 = 0$, and (c) $\lambda(t) = 20/t$, $E_1(t) = 2\times 10^3 t\sin(5\ln t)$, $E_2(t) = 2\times 10^3 t\cos(5\ln t)$.

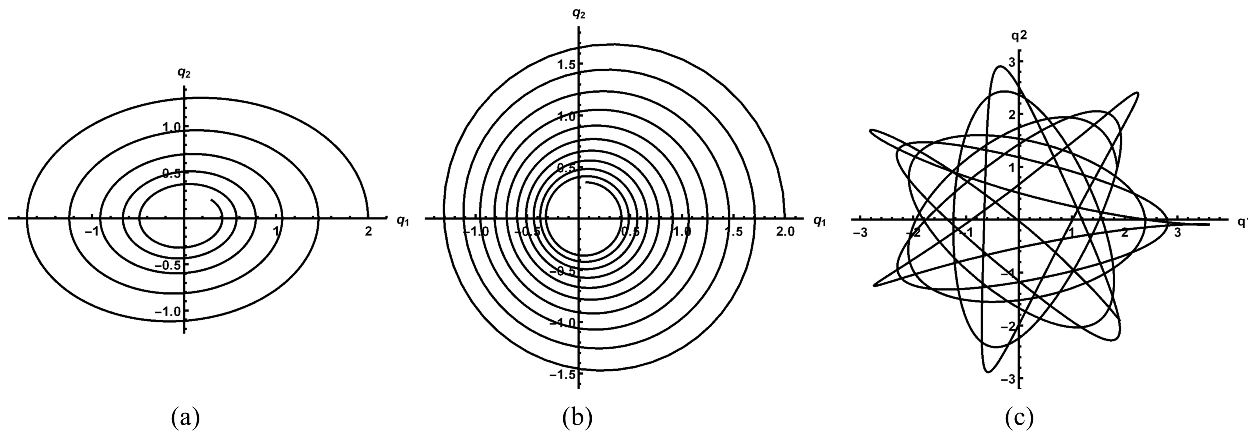

FIG. 9. Trajectories of $\rho_{\alpha}^{\theta}(\mathbf{q}, t)$ with $\gamma = 2$, $\alpha = (20\sqrt{2}/\sqrt{401}, 10i)$, $B(t) = -3\sqrt{11}\tan(3\sqrt{11}\ln t)/t$, $\omega_B = \sqrt{397}/2$, $\hbar = \omega_0 = 1$, $t \in [1, 25]$: (a) $\lambda(t) = 0, E_0 = 0$. (b) $\lambda(t) = 10/t, E_0 = 0$. (c) $\lambda(t) = 10/t, E_1(t) = 10^3 t\sin(15\ln t), E_2(t) = 10^3 t\cos(15\ln t)$.

Finally, for $\gamma = 2$ in Fig. 9, we show the trajectory given by Eq. (55). In Fig. 9(a), we take $\theta(t) = 0$, $E_0 = 0$, and since $det(C_{\alpha}^{0}) \neq 0$, the wave packet initially located at $(q_1, q_2) = (2\cos(\arctan(1/20)), 0)$ follows a spiral like path approaching the origin as time increases. In Fig. 9(b), we have $\theta(t) = 10\ln t$, $E_0 = 0$, and the wave packet again moves inward along a spiral. Then, in Fig. 9(c), we display the trajectory

$$\langle\hat{\mathbf{q}}\rangle_{\alpha}^{\theta}(t) = R_{\theta}^{T}(t)\begin{pmatrix} 2t^{-1/2}\cos(10\ln t - \arctan(1/20)) \\ \sqrt{2}t^{-1/2}\sin(10\ln t) \end{pmatrix} + \frac{E_0}{\sqrt{a^2+b^2}}\begin{pmatrix} -\cos(15\ln t - \operatorname{arccot}(b/a)) \\ \sin(15\ln t - \operatorname{arccot}(b/a)) \end{pmatrix}$$

of $\rho_{\alpha}^{\theta}(\mathbf{q}, t)$ for $\theta(t) = 10\ln t$, $a = 401/4 - (15+\lambda_0)^2$, $b = -(15+\lambda_0)$, and electric fields $E_1(t) = 10^3 t\sin(15\ln t)$ and $E_2(t) = 10^3 t\cos(15\ln t)$.

Briefly saying, we have discussed the squeezing properties of the wave packets due to influence of parameters $B(t)$ and $\gamma \geq 1$. Then, the trajectories of the wave packets in coordinate space were investigated according to the value of the damping parameter $\gamma \geq 1$. For coherent states, we have seen that their center follows the path of the classical particle in the two-dimensional configuration space and that the shape of the trajectory is closely related with the choice of $\alpha = (\alpha_1, \alpha_2)$. Finally, according to their presence, the effects of magnetic and electric fields were illustrated by considering three different cases: $(a)\lambda_0 = 0, E_0 = 0$, $(b)\lambda_0 \neq 0, E_0 = 0$, and $(c)\lambda_0 \neq 0, E_0 \neq 0$.

VII. CONCLUSION

A generalized two-dimensional quantum parametric oscillator in the presence of time-varying magnetic and electric fields was solved using the evolution operator method. The evolution operator and the propagator were found exactly in terms of solutions to the corresponding system of coupled classical equations of motion. Then, the evolution operator was applied to initial states, such as the eigenstates and coherent states of the simple two-dimensional harmonic oscillator, and propagation of the time-dependent wave functions was described explicitly.

In addition, by the evolution operator formalism, we constructed linear and quadratic invariants for the generalized two-dimensional quantum oscillator. These dynamical invariants can be used to find propagators and time-evolved wave function solutions by employing well-known techniques, such as the Lewis–Riesenfeld approach based on self-adjoint quadratic invariants and the Malkin–Man'ko–Trifonov approach based on dynamical symmetries of the Schrödinger equation. As known, a common point of the LR, MMT, and WN techniques is that the solution of the quantum problem reduces to that of solving an associated classical equation of motion. In the literature, usually quadratic invariants are found in terms of solutions to the nonlinear Ermakov–Pinney differential equation,[3] linear invariants are given in terms of complex-valued solutions to the linear classical equation of motion,[4] while using the Wei–Norman algebraic procedure, the evolution operator is determined in terms of real-valued solutions to the corresponding classical problem.[16] Then, using the well-known relations between the classical solutions and the results in the present work, one can show that under the same initial conditions, the solutions obtained by the LR, MMT, and WN techniques will eventually coincide.

Finally, as an exactly solvable model, we introduced a two-dimensional Cauchy–Euler type quantum parametric oscillator with smoothly decreasing Larmor type frequency in an oscillating external electric field. After solving the problem at the classical level, the probability densities, uncertainties, and expectations at time-evolved eigenstates and coherent states were evaluated explicitly and their behavior was studied in detail. That gave us more insight into how one can control the dynamics of the system by varying the parameters of damping and squeezing terms and by choosing proper external forces. Therefore, we can say that the compact and explicitly found formulas in the present work can provide a good basis for understanding the influence of all time-dependent parameters and external fields on the behavior of the quantum system. This can also contribute to the exact and approximate study of other two-dimensional quantum parametric oscillators in magnetic and electric fields. For example, such interesting models could be related with the classical orthogonal polynomials and hypergeometric functions[15] or associated with the Heun equation and, in particular, Mathieu, Lamé, and Coulomb equations.[43]

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Şirin A. Büyükaşık: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Supervision (equal); Writing – review & editing (equal). **Zehra Çayiç:** Conceptualization (equal); Investigation (equal); Methodology (equal); Validation (equal); Visualization (equal); Writing – original draft (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

REFERENCES

[1] R. P. Feynman, Phys. Rev. **84**, 108 (1951).
[2] K. Husimi, Prog. Theor. Phys. **9**(4), 381 (1953).
[3] H. R. Lewis, Jr. and W. B. Riesenfeld, J. Math. Phys. **10**, 1458 (1969).
[4] I. A. Malkin, V. I. Man'ko, and D. A. Trifonov, Phys. Rev. D **2**, 1371 (1970).
[5] I. A. Malkin, V. I. Man'ko, and D. A. Trifonov, Nuovo Cimento A **4**, 773 (1971).
[6] J. Wei and E. Norman, J. Math. Phys. **4**, 575 (1963).
[7] V. V. Dodonov and V. I. Man'ko, Phys. Rev. A **20**(2), 550 (1979).
[8] H. Dekker, Phys. Rep. **80**, 1 (1981).
[9] A. Perelomov, *Generalized Coherent States and Their Applications* (Springer-Verlag, 1986).
[10] R. K. Colegrave and M. S. Abdalla, Opt. Acta **28**, 495 (1981).
[11] I. A. Pedrosa, A. Rosas, and I. Guedes, J. Phys. A: Math. Gen. **38**, 7757 (2005).
[12] G. Dattoli, J. C. Gallardo, and A. Torre, "An algebraic view to the operatorial ordering and its applications to optics," Riv. Nuovo Cim. **11**, 1–79 (1988).
[13] A. B. Nassar, Phys. Lett. A **106**, 43 (1984).
[14] C. F. Lo, Phys. Rev. A **43**(1), 404 (1991).
[15] Ş. A. Büyükaşık, O. K. Pashaev, and E. Ulaş-Tigrak, J. Math. Phys. **50**, 072102 (2009).
[16] Ş. A. Büyükaşık and Z. Çayiç, J. Math. Phys. **57**, 122107 (2016).
[17] Ş. A. Büyükaşık, J. Math. Phys. **59**, 082104 (2018).
[18] Ş. A. Büyükaşık and Z. Çayiç, J. Math. Phys. **60**, 062104 (2019).
[19] I. A. Malkin, V. I. Man'ko, and D. A. Trifonov, J. Math. Phys. **14**(5), 576 (1973).
[20] J. R. Ray and J. G. Hartley, Phys. Lett. A **88**(3), 125 (1982).
[21] Y. F. Chen, Y. P. Lan, and K. F. Huang, Phys. Rev. A **68**, 043803 (2003).
[22] Y. F. Chen, J. C. Tung, P. H. Tuan, Y. T. Yu, H. C. Liang, and K. F. Huang, Phys. Rev. E **95**, 012217 (2017).
[23] D. Puertas-Centeno, I. Toranzo, and J. Dehesa, Entropy **19**, 164 (2017).
[24] L. M. Lawson, G. Y. H. Avossevou, and L. Gouba, J. Math. Phys. **59**, 112101 (2018).
[25] I. A. Malkin and V. I. Man'ko, Zh. Eksp. Teor. Fiz. **55**, 1014–1025 (1968) [Sov. Phys. JETP **28**, 527–532 (1969)].
[26] I. A. Malkin, V. I. Man'ko, and D. A. Trifonov, Phys. Lett. A **30**, 414 (1969).
[27] I. A. Malkin, V. I. Man'ko, and D. A. Trifonov, Zh. Eksp. Teor. Fiz. **58**, 721 (1970) [Sov. Phys. JETP **31**, 386 (1970)].
[28] I. A. Malkin and V. I. Man'ko, Zh. Eksp. Teor. Fiz. **59**, 1746 (1970) [Sov. Phys. JETP **32**, 949 (1971)].
[29] V. V. Dodonov, I. A. Malkin, and V. I. Man'ko, Physica **59**, 241 (1972).
[30] V. V. Dodonov, I. A. Malkin, and V. I. Man'ko, Int. J. Theor. Phys. **14**(1), 37 (1975).
[31] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[32] V. V. Dodonov, "Coherent states and their generalizations for a charged particle in a magnetic field," in *Coherent States and Their Applications*, Springer Proceedings in Physics Vol. 205, edited by J.-P. Antoine *et al.* (Springer, 2018), p. 311.
[33] M. Sebawe Abdalla, Nuovo Cimento B **101**, 267 (1988).
[34] M. S. Abdalla and J.-R. Choi, Ann. Phys. **322**, 2795 (2007).
[35] M. S. Abdalla and P. G. L. Leach, J. Math. Phys. **52**, 083504 (2011).
[36] W. Liu and J. Wang, J. Phys. A: Math. Theor. **40**, 1057 (2007).
[37] S. Menouar, M. Maamache, and J. R. Choi, Ann. Phys. **325**, 1708 (2010).
[38] V. G. Ibarra-Sierra *et al.*, Ann. Phys. **335**, 86 (2013).
[39] V. G. Ibarra-Sierra *et al.*, Ann. Phys. **362**, 83 (2015).

[40] J. C. Sandoval-Santana, V. G. Ibarra-Sierra, J. L. Cardoso, and A. Kunold, J. Math. Phys. **57**, 042104 (2016).
[41] A. Anzaldo-Meneses, Ann. Phys. **381**, 90 (2017).
[42] Ş. A. Büyükaşık and Z. Çayiç, J. Phys.: Conf. Ser. **766**, 012003 (2016).
[43] M. Hortaçsu, Adv. High Energy Phys. **2018**, 8621573.