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Relaxation of Conditions of Lyapunov Functions

Gökhan ŞAHAN[*1]

[1]Izmir Institute of Technology, Department of Mathematics, 35437, Izmir, Turkey

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Abstract: In this study, stability conditions are given for nonlinear time varying systems using the classical Lyapunov 2nd Method and its arguments. A novel approach is utilized and so that uniform stability can also be proved by using an unclassical Lyapunov Function. In contrast with the studies in the literature, Lyapunov Function is allowed to be negative definite and increasing through the system. To construct a classical Lyapunov Function, we use a reverse time approach methodology for the intervals where the unclassical one is increasing. So we prove the stability using a new Lyapunov Function construction methodology. The main result shows that the existence of such a function guarantees the stability of the origin. Some numerical examples are also given to demonstrate the efficiency of the method we use.

Lyapunov Fonksiyonun Koşullarının Gevşetilmesi

Anahtar Kelimeler
Doğrusal olmayan sistemler,
Kararlılık analizi,
Lyapunov'un 2. metodu,
Lyapunov fonksiyonu,
Zamanla değişen sistemler,
Düzgün kararlılık

Özet: Bu çalışmada, klasik Lyapunov 2. Metodu ve bu metoda dair argümanlar kullanılarak, zamanla değişen yapıdaki Doğrusal Olmayan Sistemler için kararlı olma koşulları verilmektedir. Özgün bir yaklaşım kullanılmış ve böylece düzgün kararlılık, klasik olmayan bir Lyapunov Fonksiyonu kullanılarak da ayrıca ispatlanabilmiştir. Literatürdeki çalışmaların aksine, kullandığımız klasik olmayan Lyapunov Fonksiyonunun bazı aralıklar için sistem boyunca artan ve negatif tanımlı olmasına izin verilmiştir. Klasik Lyapunov Fonksiyonu'nu inşaa etmek için, klasik olmayan Lyapunov Fonksiyonu'nun artan olduğu aralıklarda ters zaman yaklaşımını kullanıyoruz. Böylece yeni bir Lyapunov Fonksiyonu inşa etme yaklaşımı kullanarak kararlığı ispatlamış oluyoruz. Ana sonuç böyle bir fonksiyonun varlığının, orjinin kararlılığını garantilediğini gösterir. Yaklaşımın efektif olduğunu göstermek için ayrıca bir takım nümerik örnekler de verilmiştir.

1. Introduction

Consider the following system

$$\dot{x} = f(t,x), \tag{1}$$

where $f: J \times D \to R^m$ is piecewise continuous in t and locally Lipschitz in x on $J \times D$ and $0 \in D \subset R^m$ is the domain of the system where $J \coloneqq [0,\infty)$.

One of the most popular method to show the stability of a nonlinear system is Lyapunov's Direct Method, [1], [2] and [3]. It's known that the time derivative of the scalar valued function $V(t,x): J \times D \to R$ must be nonpositive through the system, that is, we must have

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq 0. \tag{2}$$

Since $V(t,x)$ is nonincreasing and is bounded by $V(t_0,x_0)$, the corresponding system (1) exhibits a stable behaviour. However, the existing literature shows that the use of a Lyapunov Function (LF) that satisfies (2) is not the only way to show the stability of the system. The condition (2) can be relaxed or changed with some other mild assumptions.

To this end, one of the main attempts is the use of higher order derivatives. In [4], it is proposed to use $\dot{V}, \ddot{V}$ and $\dddot{V}$ for the stability of system (1) instead of (2). This result was convexified in [5] and generalized using the weighted average of $V^{(i)}(t,x)$ (several order derivatives) in [6] and [7]. After that, in [8] the authors adapted the results of [6] to Linear Time Invariant (LTI) systems to compute the stability, and in [9] these results were used to show the robust stability of uncertain LTI systems. On the other hand, in [5] the authors used the same combination given in [6] to state a converse result: "If we have a globally asymptotically stable (GAS) system and a nonmonotonic LF $V(x)$ that satisfies the condition

*Corresponding author: gsahan@iyte.edu.tr

given for higher order derivatives, then we have another LF

$$W(x) = \sum_{n=0}^{m} V^{(n-1)}(x)\tau_i,$$

$\tau_i \in R$, in a classical fashion.". That is, $W(0) = 0$, $W(x) > 0$ for $x \neq 0$ and $\dot{W}(x) < 0$. This method, to construct a new LF using the given one, inspired us to prove our main results. However, we use neither the GAS assumption nor the inequality in [6].

Revision of monotonicity condition has also been studied to prove the stability of fuzzy systems. For this purpose, in [10] and [11] the authors introduced "Fuzzy LF ". A Fuzzy LF is actually a multiple LF with a new condition instead of (2). In [12], this concept is revised by redefining the Fuzzy LF. Here, the authors defined the Fuzzy LF as the work done from the origin to the current state in the fuzzy vector field. But the condition they put implies the condition (2) again. These works led to many challenges in the fuzzy systems literature, [13], [14] and [15].

Studies on removing monotonicity have also been done using joint spectral radius and some convex optimization techniques like the sum of squares method, and semidefinite programming, [5] and [16]. The monotonicity condition was replaced by some other conditions, [17]. Some other different numerical techniques are also used instead of the monotonicity requirement.

There exist some special approaches to remove monotonicity as well. In [18], the authors introduced the Almost LF concept and related the volume of the region where the LF is increased with a region where the trajectories converged, and they improved it in [19]. Nondecreasing LFs were also used for switched or hybrid systems in [20], for linear time-varying systems in [21], for arbitrary discrete time systems in [22] and for model predictive control in [23]. To our best knowledge, there does not exists any work that relaxes both monotonicity and positivity of LFs.

In this paper, we firstly remove the monotonicity condition of the classical LF and after that the positivity. We state that "When we have a scalar valued function $V(t,x)$ and allow it to be nondecreasing and nonpositive for some intervals, a classical LF $W(t,x)$ can be constructed, and so uniform stability of the corresponding system can be proved.". The only restriction we put for V is to be positive definite (PD) and nonincreasing for $t > a \in R_{\geq 0}$ or to be negative definite and nondecreasing for $t > a \in R_{\geq 0}$. Then we construct a classical LF $W(t,x)$ using the given unclassical one $V(t,x)$. We mainly need $W(t,x)$ just to show how stability is possible. Thus, to show the uniform stability, it is enough just to show the existence of such a function $V(t,x)$. This already implies uniform stability according to the result we prove. Also, if we find a suitable $V(t,x)$ as determined in the results below, then again there is no need to solve the differential system as in Lyapunov's Direct Method. By relaxing the decreasing and positivity features of the classical LF, now, it is easier to find an appropriate LF.

The paper consists of three main parts. Section 2 gives some basic definitions and results for nonlinear systems. Section 3 includes our main results and some examples that show the efficiency of our results.

Nomenclature: We define the set of continuous function whose derivatives are continuous up to nth order as C^n. For a function α, the representation $\alpha \in K$ means it's a class K function.

2. Background

We start by defining some fundamentals on nonlinear systems, [1] and [2].

Definition 1 (Class K and Class L functions): Consider the continuous function $\alpha: J \to J$. If it is strictly increasing and $\alpha(0) = 0$, then we call it class K function; if it is continuous, strictly decreasing, $\alpha(0) < \infty$ and $\alpha(r) \to 0$ as $r \to \infty$, then it is called a Class L function.

Definition 2 (Uniform stability): The equilibrium point $x = 0$ of the system (1) is uniformly stable (US) iff there exist a function $\alpha \in K$ and a constant $c > 0$, independent of t_0, such that

$$\|x(t)\| \leq \alpha(\|x(t_0)\|), \;\; \forall t \geq t_0, \; \forall \|x(t_0)\| < c \, .$$

We now give some conclusions on class K functions for the sake of completeness, [1].

Lemma 2.1: Consider the class K functions α_1 and α_2 on $[0,a)$ and let α_1^{-1} define the inverse function of α_1. Then both $\alpha_1 \circ \alpha_2$ and α_1^{-1} are class K functions, too.

Lemma 2.2: Consider the continuous, PD function $V: D \to R$ where $0 \in D \subset R^m$ and a subset $B_r \subset D$ for some $r > 0$. Then there exists class K functions α_1 and α_2 defined on $[0,r]$ such that

$$\alpha_1(\|x\|) \leq \mathrm{V(x)} \leq \alpha_2(\|x\|).$$

Let us also give the extension of the classical result of Lyapunov's 2nd method to nonautonomous systems, (Theorem 4.8 of [1]).

Theorem 2.1. Assume that $x = 0$ be the equilibrium point for (1) and $0 \in D \subset R^m$ is its domain. Let us define $V(t,x): J \times D \to R$, $V \in C^1$ such that

$$W_1(x) \leq V(t,x) \leq W_2(x), \tag{3}$$

$$\dot{V}(t,x) \leq 0. \tag{4}$$

$\forall t \geq t_0 \geq 0$ and $\forall x \in D$, where $W_1, W_2 \in C$ and PD on D. Then $x = 0$ is US.

Note that Theorem 2.1 is given for $t \geq 0$ and means a Lyapunov candidate Function must satisfy the positivity and monotonicity requirement for all $t \geq 0$. But there exist many examples in the literature, especially on relaxing monotonicity, [4], [5], [9], [16], [21], and [22]. That is, there are some PD continuously differentible Lyapunov-like Functions which are not defined exactly by Lyapunov, [1] and [2]. However, they can still prove the stability with the help of some other conditions, [4] and [21] or using equivalent conditions to the monotonicity requirement, [10], [11] and [12]

3. Main Results

Motivated by these points, we generalize Theorem 2.1 and give the restriction (2) just necessarily for $t \geq a \in R_{\geq 0}$. So, the Lyapunov-like Function $V(t,x)$ can be increasing for the whole $(0,a)$ or just for some parts of it. But after $t = a$, it must have a decreasing behaviour, i.e. $\dot{V}(t,x) \leq 0$ for $t \geq a$. This enables us to find a Lyapunov candidate Function much more easily. Now we state our first main result. We give a simple case first to give an idea about our method

Theorem 3.1: Consider (1) and its equilibrium point $x = 0$. Let $0 \in D \subset R^m$ be its domain and $V(t,x): J \times D \to R, V \in C^1$ be a PD function such that

$$\begin{aligned} \dot{V}(t,x) \leq 0, \quad t \in [0,a_1] \cup [a_2,\infty), \\ \dot{V}(t,x) \geq 0, \quad t \in [a_1,a_2]. \end{aligned} \tag{5}$$

Define the classical LF

$$W(t,x) := \begin{cases} V(t,x) + 2\Delta & if\ t \in [0,a_1] \\ V\left(\tau(t), x\left(\tau(t)\right)\right) + \Delta & if\ t \in [a_1,a_2] \\ V(t,x) & if\ t \in [a_2,\infty) \end{cases} \tag{6}$$

where $\tau(t) := a_1 + a_2 - t,\ x = x(t)$ and $\Delta := V(a_2, x(a_2))$- $V(a_1, x(a_1))$ such that

$$W_1(t,x) \leq V(t,x) \leq W(t,x) \leq W_2(t,x), \tag{7}$$

where $W_1, W_2 \in C$ are PD functions on D. Then $x = 0$ is US.

Proof: Assume all the hypotheses of Theorem 3.1 hold. $W(t,x)$ is nonincreasing for $t \in [0,a_1]$ and $t \in [a_2,\infty]$. It is guaranteed by the definition (6) and the hypothesis of the theorem. Now we prove that it is also true for $t \in [a_1,a_2]$.

Let's define the moments $a_1 < \tau_1 < \tau_2 < a_2$. As the given nonmonotonic LF $V(t,x)$ is increasing there, we have

$$\begin{aligned} V\big((a_1, x(a_1)\big) < V\big((\tau_1, x(\tau_1)\big) < V\big((\tau_2, x(\tau_2)\big) \\ < V\big((a_2, x(a_2)\big). \end{aligned}$$

With a simple manipulation on moments τ_1 and τ_2,

$$a_1 < a_1 + a_2 - \tau_2 < a_1 + a_2 - \tau_1 < a_2.$$

Now define $t_1 := a_1 + a_2 - \tau_1$ and $t_2 := a_1 + a_2 - \tau_2$. Then we have $a_1 < t_2 < t_1 < a_2$.

As $V(t,x)$ is increasing there, we have

$$\begin{aligned} V\big((a_1, x(a_1)\big) < V\big((t_2, x(t_2)\big) < V\big((t_1, x(t_1)\big) \\ < V\big((a_2, x(a_2)\big). \end{aligned}$$

So we introduced a reverse time approach and assigned the values of the function $V(t,x)$ in a reverse time order when it is increasing. Then we'll use them to construct a decreasing function for that interval. Adding Δ to each side of this inequality which helps to achieve continuity, we get

$$\begin{aligned} V\big((a_1, x(a_1)\big) + \Delta = W\big(a_2, x(a_2)\big) < W\big(\tau_2, x(\tau_2)\big) \\ < W\big(\tau_1, x(\tau_1)\big) < W\big(a_1, x(a_1)\big) = V\big(a_2, x(a_2)\big) + \Delta. \end{aligned}$$

Consequently, for the interval $[a_1,a_2]$, we matched the increasing values of function $V(t,x)$ with the values of $W(t,x)$ in a reverse time order and added the difference Δ to imply continuity. So we obtain that

$$a_1 < \tau_1 < \tau_2 < a_2 \Longrightarrow W\big(\tau_2, x(\tau_2)\big) < W\big(\tau_1, x(\tau_1)\big)$$

which means $W(t,x)$is decreasing for $t \in (a_1,a_2)$ with the values of $V(t,x)$ in reverse order. As a result we have

$$\dot{W}(t,x) \leq 0$$

for $t \in (0,\infty)$. We assumed that the function $V(t,x)$ is PD. Note that to construct $W(t,x)$, we just added some positive constants Δ and 2Δ to $V(t,x)$ for some intervals. As a result, the function $W(t,x)$ is also continuous and PD. So $W(t,x)$ is a classical LF, while $V(t,x)$ is not.

The remaining part follows lines similar to the proof of Theorem 4.8 in [1]. We choose $c, r > 0$ as $B_r \subset D$, $c < min_{\|x(t)\|=r} W_1(x)$ and define

$$\begin{aligned} \Omega_{t,c} &:= \{x \in B_r \ :\ W\big(t, x(t)\big) \leq c\}, \\ \Omega_{W_1} &:= \{x \in B_r \ :\ W_1(x) \leq c\}, \\ \Omega_{W_2} &:= \{x \in B_r \ :\ W_2(x) \leq c\}. \end{aligned}$$

Then we have the following chain between the sets,

$$\Omega_{W_2} \subset \Omega_{t,c} \subset \Omega_{W_1} \subset B_r \subset D.$$

As we have $\dot{W}(t,x) \leq 0$ on D, any solution starting from the time dependent set $\Omega_{t,c}$ remains in the same set irrespective of $t_0 \in (a_1, a_2)$ and goes on so for all $t \geq t_0$. Note also that, depending on the nature of the $W(t,x)$, this valid for any initial conditions x_0. Hence, a solution starting from Ω_{w_2} stays in $\Omega_{t,c}$ and according to the chain also in Ω_{w_1}. As a result, we have a bounded solution and it's defined for all $t \geq t_0$.

Although $\dot{V}(t,x)$ is not negative for all $t \geq 0$, we have $\dot{W}(t,x) \leq 0$. So we get

$$W\big(t, x(t)\big) \leq W\big(t_0, x(t_0)\big) = V\big(t_0, x(t_0)\big) + 2\Delta, \quad \forall t \geq t_0.$$

Here, a question should come to mind on the differentiablity of the classical Lyapunov Function $W(t,x)$. It is already clear except for the moments $t = a_1$ and a_2. But W is differentiable for $t = a_1$ and a_2 as well. Because, when t approaches to the points a_1 and a_2 the derivative of the unclassical Lyapunov function $V(t,x)$ already goes to zero. We only added constants Δ or 2Δ of which its derivative is zero. Thus, the resulting function $W(t,x)$ is also differentiable for the points $t = a_1$ and $t = a_2$.

According to Lemma 4.3 of [1], it is possible to bound $W(t, x(t))$ with some $\alpha_1, \alpha_2 \in K$ defined on $[0, r]$ as follows

$$\begin{aligned} \alpha_1(\|x\|) &\leq W_1(t,x) \leq W(t,x) \leq W_2(t,x) \leq \alpha_2(\|x\|) \\ \Longrightarrow \|x(t)\| &\leq \alpha_1^{-1}(W(t,x)) \leq \alpha_1^{-1}(W(t_0,x_0)) \\ &\leq \alpha_1^{-1}\big(\alpha_2(\|x_0\|)\big). \end{aligned}$$

We know using Lemma 2.1 (which is Lemma 4.2 of [1]) that $\alpha_1^{-1}\alpha_2$ is also a Class K function. Using this fact,

$$\|x(t)\| \leq \alpha_1^{-1}\big(\alpha_2(\|x(t_0)\|)\big)$$

shows that the origin is US.

Remark 1: Note that Theorem 3.1 is a generalization of the classical result Theorem 2.1. It seems that we proved Theorem 3.1 with just LF $W(t,x)$. But the existence of the unclassical LF $V(t,x)$ that satisfies requirement (5) makes the construction of $W(t,x)$ possible.

Remark 2: In order to construct $W(t,x)$ for $t \in (a_1, a_2)$, we first use a reverse time approach for the interval (a_1, a_2). More clearly, we take the symmetry of $V(t,x)$ with respect to the $t = \frac{a_1 + a_2}{2}$ line when $t \in (a_1, a_2)$. After that, to preserve the continuity of $W(t,x)$, we add constant Δ. The rest of $W(t,x)$ is quite similar with $V(t,x)$.

Theorem 3.1 is given for a LF which is increasing just for $t \in (a_1, a_2)$. But using a similar methodology, a LF $W(t,x)$ can be constructed and uniform stability can be proved in case of more than one intervals that $V(t,x)$ increases. Therefore, we should give the following corollary which generalizes Theorem 3.1.

Corollary 3.2: Let $x = 0$ be an equilibrium point for (1), $D \subset R^m$ m be its domain, $0 \in D, V : J \times D \to R,\ V \in C^1$ be a PD function such that

$$\dot{V}(t,x) \leq 0, t \in \mathbb{I}_1, \ \ \dot{V}(t,x) \geq 0, t \in \mathbb{I}_2, \tag{8}$$

where $\mathbb{I}_1 := \bigcup_{k=0}^{\frac{n-2}{2}} [a_{2k}, a_{2k+1}] \cup [a_n, \infty)$, $\mathbb{I}_2 := \bigcup_{k=0}^{\frac{n-2}{2}} [a_{2k+1}, a_{2k+2}]$, $\mathbb{I}_1, \ \mathbb{I}_2 \subset [0, \infty)$, $a_0 = 0$ assuming n is even and positive without loss of generality. Define the classical LF as follows.

$$W(t,x) = \begin{cases} V((t), x(t)) & if\, t \in [a_n, \infty] \\ V(\tau_1, x(\tau_1)) + \Delta_1 & if\, t \in [a_{n-1}, a_n] \\ V(t, x(t)) + 2\Delta_1 & if\, t \in [a_{n-2}, a_{n-1}] \\ V(\tau_2, x(\tau_2)) + \Delta_2 & if\, t \in [a_{n-3}, a_{n-2}] \\ V(t, x(t)) + 2\Delta_2 & if\, t \in [a_{n-4}, a_{n-3}] \\ \ldots & \\ V(\tau_{\frac{n}{2}}, x(\tau_{\frac{n}{2}})) + \Delta_{\frac{n}{2}} & if\, t \in [a_1, a_2] \\ V(t, x(t)) + 2\Delta_{\frac{n}{2}} & if\, t \in [a_0, a_1] \end{cases} \tag{9}$$

where $\tau_i(t) := a_{n-2i+1} + a_{n-2i+2} - t$, $\Delta_i := V(a_{n-2i+2}, x(a_{n-2i+2}) - V(a_{n-2i+1}, x(a_{n-2i+1}), i \in I = \{1,2, \ldots \frac{n}{2}\}$ and $x = x(t)$ such that (7) holds. Then $x = 0$ is US.

Proof: The proof can be done following a path similar to the proof of Theorem 3.1.

Remark 3: Note that Corollary 3.2 is given for the unclassical LF which is decreasing for $t \in [a_0, a_1]$. But a similar conclusion can be given also for an increasing one for $[a_0, a_1]$. For the construction of $W(t,x)$, we assume then $a_1 = 0$ and define the function for $[a_1, \infty)$. So the last row of $W(t,x)$ in (9) is deleted. Then the rest is the same.

We now give an example of given methodology.

Example 3.3: Consider $\dot{x} = f'(t)g(x)$. Assume that $g(x)$ is a Class K, $f(t)$ is an arbitrary differentiable function and is a Class L function for (a, ∞); where $a \in R$ is a fixed number, $f'(t) = \frac{df}{dt}$. Define $V(t,x) = \frac{g(x)^2}{2} = W_1(x) = W_2(x)$. (So we use the terminology which is also used in the literature, Lemma 3.14.1 of [22].)

Then $V(t,x)$ is PD and $V(t,0) = 0$. We also have $\dot{V}(t,x) = g(x)^2 \frac{dg}{dx} f'(t)$and its sign is determined by the sign of $f'(t)$. Note that $f(t)$ has a decreasing fashion just for $t > a$. So it may be increasing or decreasing for the interval $t \in (0, a)$. As a result $V(t,x)$ is not a classical LF. However the system is US with Theorem 3.1 and we give this decision again without solving the system.

Now let us specify $f(t)$ and $g(x)$ as follows.

Example 3.4: Consider $\dot{x} = -(t^2 - 3t + 2)x$, $t_0 = 0$, $x(0) = x_0$, and $V(t, x) = \frac{x^2}{2}$. Note that $\dot{V}(t, x) = -(t^2 - 3t + 2)x^2$ and

$$\dot{V}(t,x) \leq 0 \ \text{ for } t \in [0,1] \cup [2,\infty)$$
$$\dot{V}(t,x) \geq 0 \ \text{ for } t \in [1,2].$$

According to Theorem 3.1, the system is US. We obtain this without solving the system as we have in Lyapunov's 2nd Method. On the other hand, the solution of the system for an initial condition, say $x_0 = 1$, is $x(t) = exp(-\frac{1}{3}t^3 + \frac{3}{2}t^2 - 2t)$ so it is stable for $t \to \infty$ (Fig.1) and it reinforces our claim. It can also be seen by the geometrical interpretations. Construct

$$W(t,x) := \begin{cases} \frac{x^2}{2} + 2\triangle & \text{if } t \in [0,1] \\ V(3-t, x(3-t)) + \triangle & \text{if } t \in [1,2] \\ \frac{x^2}{2} & \text{if } t \in [2,\infty] \end{cases}$$

where $x = x(t)$, $\Delta := V(2, x(2)) - V(1, x(1))$. So unclassical LF $V(t, x)$ is nonmonotonic (see Fig.2), but the classical one $W(t, x)$ is monotonic-nonincreasing for $t \in [0, \infty)$, (see Fig.3).

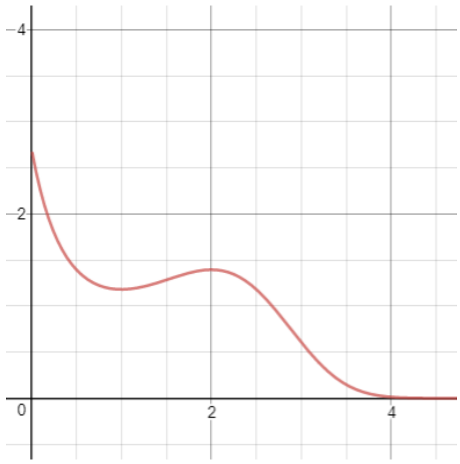

Figure 1. Graph of the solution $x(t)$

Figure 2. Graph of unclassical LF $V(x, t)$

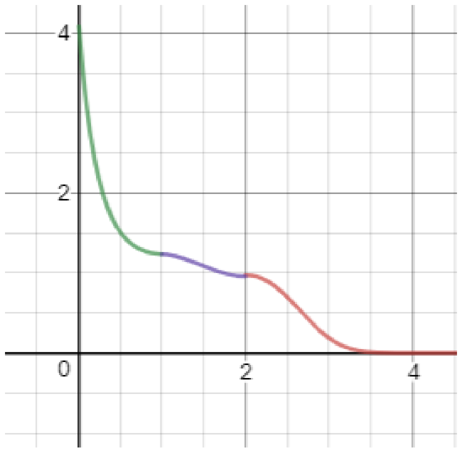

Figure 3. Graph of classical LF $W(x, t)$

Now we also relax the positivity assumption of classical LF.

Corollary 3.5: Consider (1) and its equilibrium point $x = 0$. Let $D \subset R^m$ be its domain, $0 \in D$. Let $V : J \times D \to R$, $V \in C^1$ and monotonic for $t \in [a, \infty)$, $a \in R_{\geq 0}$ such that one of the followings hold:

1) $V(t, x) > 0$ and $\dot{V}(t, x) \leq 0$ for $t \in [a, \infty)$,

2) $V(t, x) < 0$ and $\dot{V}(t, x) \geq 0$ for $t \in [a, \infty)$.

Define the classical LF $W(t, x)$ of the items (1) and (2) as follows:

1) Same as in (9) or Remark 3

2) Define $W(t, x) = -V(t, x)$ for $t \in [a, \infty)$ and use (9) or Remark 3 for $W(t, x)$ such that (7) holds.

Let $[a, \infty)$ be the largest interval that satisfy the corresponding condition. Then $x = 0$ is US.

Proof: The proof of the condition 1 and 2 is clear by Corollary 3.2. and the proof of Theorem 3.1. Because of the nature of the classical LF $W(t, x)$, it is already PD even if we have a negative definite $V(t, x)$.

Example 3.6: Consider $\dot{x} = (1 - t)x$, and $V(t, x) = x^2(t - 2)$. Then $\dot{V}(t, x) = x^2(-2t^2 + 6t - 3)$.

The function $V(t, x)$ is neither PD nor monotonic for $t \geq 0$ and thus cannot be used as a classical LF. However, it satisfies the hypothesis of Corollary 3.5 above and thus can be used for stability analysis. Define $a = \frac{(3+\sqrt{3})}{2}$. Then $V(t, x) > 0$ and $\dot{V}(t, x) < 0$ for $t \in (a, \infty)$ and as a result the system is US. Note again that we give this decision without solving the nonlinear system.

Remark 4. Corollary 3.5 is another relaxation of the classical Lyapunov Theorem. We put monotonicity assumption just for $t \geq a$ by also relaxing positivity for one case and giving a conclusion on stability without solving the system. As a result, Corollary 3.5 is an improvement of the Lyapunov Direct Method. On the other hand, the relaxation is given for some fixed t values and thus, finding an appropriate LF which holds the hypothesis of Corollary 3.5 may be difficult for some kind of systems.

4. Conclusion

This work includes some novel Lyapunov-type stability conditions for general nonlinear systems. The novelty lies in the relaxation of monotonicity and positivity of the classical LF, which is commonly required by classical approaches.

Unlike classical Lyapunov Theory, a negative definite function or a function which has a positive derivative through the system can be used for analysis purposes but again without solving the system. The main results show that this more suitable Lyapunov like function can be used for the stability of the origin. Thus, it enlarges the family of appropriate LFs and makes nonlinear system analysis easier.

For this approach, dependency on t can be considered as the weakness of the given results. In addition, the corresponding conclusions that we have given can be regarded as a result of continuation of the solution. So the proofs of the main results can also be shown via this way and without constructing $W(t,x)$. However, we give a methodology to construct a LF as well. There always exist difficulties to construct an appropriate LF and thus, any advances on this subject is valuable, [24] and [25].

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Etik Beyanı/Declaration of Ethical Code

In this study, we undertake that all the rules required to be followed within the scope of the "Higher Education Institutions Scientific Research and Publication Ethics Directive" are complied with, and that none of the actions stated under the heading "Actions Against Scientific Research and Publication Ethics" are not carried out.

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