Applicable Algebra in Engineering, Communication and Computing (2021) 32:195–215
https://doi.org/10.1007/s00200-020-00482-4

ORIGINAL PAPER

On max-flat and max-cotorsion modules

Yusuf Alagöz[1,2] · Engin Büyükaşık[1]

Received: 21 January 2020 / Accepted: 11 December 2020 / Published online: 6 January 2021

Abstract
In this paper, we continue to study and investigate the homological objects related to s-pure and neat exact sequences of modules and module homomorphisms. A right module A is called *max-flat* if $\mathrm{Tor}_1^R(A, R/I) = 0$ for any maximal left ideal I of R. A right module B is said to be *max-cotorsion* if $\mathrm{Ext}_R^1(A, B) = 0$ for any max-flat right module A. We characterize some classes of rings such as perfect rings, max-injective rings, SF rings and max-hereditary rings by max-flat and max-cotorsion modules. We prove that every right module has a max-flat cover and max-cotorsion envelope. We show that a left perfect right max-injective ring R is QF if and only if maximal right ideals of R are finitely generated. The max-flat dimensions of modules and rings are studied in terms of right derived functors of $- \otimes -$. Finally, we study the modules that are injective and flat relative to s-pure exact sequences.

Keywords (Max-)flat modules · Max-cotorsion modules · (s-)pure submodule · SP-flat modules · Max-hereditary rings · Quasi-Frobenius rings

Mathematics Subject Classification 16D40 · 16E10 · 16E30

1 Introduction

Throughout, R will denote an associative ring with identity, and modules will be unital R-modules, unless otherwise stated. As usual, we denote by $\mathfrak{M}_R$ ($_R\mathfrak{M}$) the category of right (left) R-modules. For a module A, $E(A)$, $\mathrm{Rad}(A)$, $l(A)$ and A^+ denote the injective hull, Jacobson radical, left annihilator and the character module $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ of A, respectively.

✉ Yusuf Alagöz
yusuf.alagoz@siirt.edu.tr

Engin Büyükaşık
enginbuyukasik@iyte.edu.tr

1 Department of Mathematics, Izmir Institute of Technology, İzmir, Turkey

2 Present Address: Department of Mathematics, Siirt University, Siirt, Turkey

Given a class $\mathfrak{C}$ of R-modules, we denote by $\mathfrak{C}^{\perp} = \{X : \mathrm{Ext}_R^1(C, X) = 0$ for all $C \in \mathfrak{C}\}$ the right orthogonal class of $\mathfrak{C}$. Let A be a right R-module. A homomorphism $f : C \to A$ with $C \in \mathfrak{C}$ is called a *$\mathfrak{C}$-precover* of A [9] if for any homomorphism $g : D \to A$ with $D \in \mathfrak{C}$, there is a homomorphism $h : D \to C$ such that $fh = g$. Moreover, if the only such h are automorphisms of C when $C = D$ and $g = f$, the $\mathfrak{C}$-precover is called a $\mathfrak{C}$-cover of A. Following [9], an epimorphism $\alpha : C \to A$ with $C \in \mathfrak{C}$ is said to be a special $\mathfrak{C}$-precover of A if $ker(\alpha) \in \mathfrak{C}^{\perp}$. Dually, we have the definitions of a (special) $\mathfrak{C}$-preenvelope and a $\mathfrak{C}$-envelope. $\mathfrak{C}$-envelopes ($\mathfrak{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

Since its development, the Cohn purity plays a significant role in module theory and homological algebra. One of the main reasons is that, some significant homological objects such as, flat modules, cotorsion modules, absolutely pure modules and pure-injective modules arose from this notion of purity. Recall that, a submodule B of a right module A is a pure submodule of A if $i \otimes 1_F : B \otimes F \to A \otimes F$ is a monomorphism for every finitely presented left module F, or equivalently $\mathrm{Hom}(F', A) \to \mathrm{Hom}(F', A/B)$ is an epimorphism for every finitely presented right module F' (see, [11, Theorem 1.27]). In the same manner, instead of finitely presented modules one can consider different classes of modules to obtain different purities. Let A be a submodule of a right module B and $i : A \to B$ and $\pi : B \to B/A$ be the inclusion and the natural epimorphism, respectively. In [5], the submodule A of B is called *s-pure submodule of* B if $i \otimes 1_S : A \otimes S \to B \otimes S$ is a monomorphism for every simple left module S. Similarly, the submodule A of B is called *neat submodule of* B if $\mathrm{Hom}(S, B) \to \mathrm{Hom}(S, B/A)$ is an epimorphism for every simple right module S.

Unlike the generation of pure submodules, the notions that are obtained by replacing finitely presented modules with simple modules are not the same, in general. Moreover, the notions of s-pure and neat submodules are not only inequivalent, they are also incomparable. The commutative domains for which the notions of s-pure and neat submodules are equivalent were considered in [12]. These are the commutative domains whose maximal ideals are invertible, and these domains termed as N-domains. In [6], Crivei proved that if the ring is commutative and the maximal ideals are principal, then the notions of s-pure and neat submodules coincide. Recently, the commutative rings with this property were completely characterized in [18, Theorem 3.7]. These are exactly the commutative rings whose maximal ideals are finitely generated and locally principal.

A left R-module A is called *max-injective* if for the inclusion map $i : I \to R$ with I maximal left ideal, and any homomorphism $f : I \to A$ there exist a homomorphism $g : R \to A$ such that $gi = f$, or equivalently $\mathrm{Ext}_R^1(R/I, A) = 0$ for any maximal left ideal I. A ring R is said to be left *max-injective* if R is max-injective as a left R-module [24]. As observed by Crivei in [6, Theorem 3.4], a left R-module A is max-injective if and only if A is a neat submodule of every module containing it. A right R-module A is called *max-flat* if $\mathrm{Tor}_1^R(A, R/I) = 0$ for any maximal left ideal I of R (see [23]). A right R-module A is max-flat if and only if A^+ is max-injective by the isomorphism $\mathrm{Ext}_R^1(R/I, A^+) \cong \mathrm{Tor}_1^R(A, R/I)^+$ for any maximal left ideal I of R.

Indeed, we show in Proposition 5 that, a right R-module A is max-flat if and only if any short exact sequence ending with A is s-pure.

So far, s-pure and neat submodules and homological objects related to s-pure and neat-exact sequences are studied by many authors (see, [3, 5, 6, 8, 12, 14, 15, 18, 24, 25]).

In this paper, we continue the study and investigation of the homological objects related to s-pure and neat short exact sequences. Namely, we study max-injective, max-flat, max-cotorsion and SP-flat modules.

The concept of max-cotorsion modules is first introduced in Sect. 2. A right module A is said to be *max-cotorsion* if $\mathrm{Ext}^1_R(B, A) = 0$ for any max-flat right R-module B. Several elementary properties of max-flat, max-injective and max-cotorsion modules are obtained in this section. From now on, for the class of all max-injective left, all max-flat right and all max-cotorsion right R-modules we write $\mathfrak{m}\text{-}\mathfrak{in}$, $\mathfrak{m}\text{-}\mathfrak{fl}$ and $\mathfrak{m}\text{-}\mathfrak{cot}$, respectively. We prove, in this section, that every right module has a max-flat cover and max-cotorsion envelope. For a left N-ring R, we prove that R is left max-injective if and only if all injective right R-modules are max-flat if and only if all flat left R-modules are max-injective if and only if F^+ is max-flat for every free left R-module F. In [10], Faith conjectured that every left (or right) perfect right self-injective ring is QF. Recently this conjecture was considered in many papers and has been proved under some restricted conditions. In [24], the authors considered equivalent form of Faith's conjecture: Any left perfect, right max-injective ring is QF and they gave a partial affirmative answer to Faith's conjecture (see [24, Theorem 3.6]). We extend the partial affirmative answer of [24, Theorem 3.6] to Faith's conjecture by further using the property of N-rings. We prove that a left perfect right max-injective ring R is QF if and only if R is a right N-ring.

In Sect. 3, we study max-flat dimensions of modules and rings in terms of right derived functors of $-\otimes-$. For a left N-ring R, we prove that R is left max-hereditary (i.e. if every maximal left ideal is projective) if and only if every factor of a max-injective left R-module is max-injective if and only if every submodule of a max-flat right R-module is max-flat if and only if all left R-modules have a monic max-injective cover if and only if kernel of epimorphisms between max-flat modules are max-flat and gl left max-fd$(\mathfrak{M}_R) \leq 1$ (gl right max-id$({}_R\mathfrak{M}) \leq 1$). For a left N-ring R, it is also shown that, R is a left SF-ring if and only if gl right max-id$({}_R\mathfrak{M}) = 0$ if and only if all cotorsion right R-modules are max-flat if and only if R has (P) and all max-cotorsion right R-modules are max-flat. Indeed, we consider the projectivity of max-flat modules. We prove that R is right perfect if and only if all max-flat right R-modules are projective.

A left R-module N is *s-pure injective* (*SP-injective*, for short) [3],(in [14] it is called coneat injective) if it is injective relative to s-pure short exact sequences. Clearly, every SP-injective module is pure-injective. In Section 4, the concept of SP-flat module is introduced. We call a right R-module A *SP-flat* if for every s-pure exact sequence $0 \to K \to L \to M \to 0$ of left R-modules, the sequence $0 \to A \otimes K \to A \otimes L \to A \otimes M \to 0$ is exact. Flat modules and simple modules are SP-flat. We obtain some preliminary properties of SP-injective and SP-flat modules. We then give several characterizations of s-purity and max-flat modules in terms of SP-injective modules. For a commutative ring R, we show that a module A is max-flat if and only if its localization A_m is max-flat R_m-module for all maximal ideals m of R. Finally we prove that a ring R left SF if and only if all max-cotorsion right (SP-injective right) R-modules are injective if and only if all SP-flat left R-modules are flat if and only if gl left max-fd$(\mathfrak{M}_R) = 0$.

2 Max-flat and max-cotorsion modules

Recall that a ring R is called *left coherent* if every finitely generated left ideal of R is finitely presented. A ring R is called *left max-coherent* if every maximal left ideal of R is finitely presented. Following [3], R is called a *left N-ring* if every maximal left ideal is finitely generated. While left max-coherent rings are left N-ring, left coherent rings need not be left N-ring (see Example in [25, Remark 2.2(3)]). The following lemma is proved for left max-coherent rings in [25]. Using similar arguments in [25], one can prove the following lemma over left N-rings.

Lemma 1 *For a left N-ring R, the following are true.*

1. *A left module A is max-injective if and only if A^+ is max-flat.*
2. *A right module A is max-flat if and only if A^{++} is max-flat.*
3. $\mathfrak{m}$-$\mathfrak{in}$ *is closed under pure submodules, pure quotients, direct sums and direct limits.*
4. *Any direct product of max-flat right R-modules is max-flat.*
5. *All left R-modules have an* $\mathfrak{m}$-$\mathfrak{in}$*-cover.*

Recall that an exact sequence of right R-modules $0 \to A \to B \to C \to 0$ is called s-pure exact provided that $0 \to A \otimes_R S \to B \otimes_R S \to C \otimes_R S \to 0$ is exact for any simple left R-module S, [6]. In this case, C is said to be an s-pure quotient of B.

Lemma 2 $\mathfrak{m}$-$\mathfrak{fl}$ *is closed under extensions, direct sums, direct summands, pure submodules and (s-)pure quotients.*

Proof The class $\mathfrak{m}$-$\mathfrak{fl}$ is closed under extensions, direct sums, direct summands by [25, Proposition 2.4(2)].

Consider the pure exact sequence of right R-modules $0 \to B \to A \to A/B \to 0$ with A is max-flat. Then $\mathrm{Tor}_1^R(A, R/I)^+ = 0 = \mathrm{Ext}_R^1(R/I, A^+)$ for any maximal left ideal I of R. Since $0 \to (A/B)^+ \to A^+ \to B^+ \to 0$ splits and A^+ is max-injective, B^+ and $(A/B)^+$ are max-injective. Hence B and A/B are max-flat (see, [2, Lemma 2.3(1)]).

Let B be an s-pure submodule of a max-flat right R-module A. For any maximal left ideal I of R, we have the exact sequence $0 = \mathrm{Tor}_1^R(A, R/I) \to \mathrm{Tor}_1^R(A/B, R/I) \to B \otimes R/I \to A \otimes R/I$. Since, $0 \to B \otimes R/I \to A \otimes R/I$ is exact, $\mathrm{Tor}_1^R(A/B, R/I) = 0$. So A/B is max-flat. □

Definition 1 A right R-module A is said to be *max-cotorsion* if $\mathrm{Ext}_R^1(B, A) = 0$ for any max-flat right R-module B. The left version can be defined similarly.

Remark 1 By the definition, any SP-injective right module is max-cotorsion. Moreover, any max-cotorsion right module is cotorsion. (a right module C is called cotorsion provided that $\mathrm{Ext}^1_R(F, C) = 0$ for any flat right module F [9]).

It is well known that all modules have a cotorsion envelope and a flat cover. The corresponding results are also true if we consider max-cotorsion and max-flat modules.

Lemma 3 *All right modules have* $\mathfrak{m}$-$\mathfrak{fl}$-*covers and* $\mathfrak{m}$-$\mathfrak{cot}$-*envelopes. In particular, all right modules have special* $\mathfrak{m}$-$\mathfrak{fl}$-*precovers and special* $\mathfrak{m}$-$\mathfrak{cot}$-*preenvelopes.*

Proof All right modules have $\mathfrak{m}$-$\mathfrak{fl}$-covers and $\mathfrak{m}$-$\mathfrak{cot}$-envelopes by Lemma 2 and [16, Theorem 3.4]. The rest follows by Wakamatsu's Lemmas [26, §2.1]. □

Corollary 1 *Let R be a left N-ring. Then the following are equivalent.*

1. *All max-flat right R-modules are flat.*
2. *All max-injective left R-modules are FP-injective.*

In this case, R is a left coherent ring.

Proof (1) ⇒ (2) Let A be any max-injective left R-module. Then A^+ is max-flat by Lemma 1(1), and so A^+ is flat by (1). Moreover, $0 = \mathrm{Tor}^R_1(A^+, B) \cong (\mathrm{Ext}^1_R(B, A))^+$ for any finitely presented left R-module B. Thus A is FP-injective.

(2) ⇒ (1) Let A be any max-flat right R-module. Then A^+ is max-injective, and so A^+ is FP-injective by (2). Hence A is flat.

To prove the last statement, let M be an FP-injective left R-module with N a pure submodule. Then M/N is m-injective by Lemma 1(3) since R is a left N-ring. Therefore M/N is FP-injective by (2), and hence R is a left coherent ring by [19, Theorem 3.7]. □

Recall that a ring R is said to be a left *C-ring* if $\mathrm{Soc}(R/I) \neq 0$ for every essential left ideal I of R. Right perfect rings, left semiartinian rings are well known examples of left C-rings [4, 10.10]. It is shown in [22, Lemma4] that R is a left C-ring if and only if every max-injective left R-module is injective.

Corollary 2 *Consider the following statements for a ring R*:

1. *R is a left C-ring.*
2. *All max-flat right R-modules are flat.*
3. *All cotorsion right R-modules are max-cotorsion.*

Then (1) ⇒ (2) ⇔ (3). *If R is a left Noetherian ring, then* (2) ⇒ (1).

Proof (2) ⇔ (3) is clear.

(1) ⇒ (2) Let A be any max-flat right R-module. Then A^+ is max-injective, and so A^+ is injective by (1). Thus A is flat.

(2) ⇒ (1) Let A be any max-injective left R-module. Then A^+ is max-flat, and so A^+ is flat by (2). Thus A is injective by the Noetherianity of R. Hence R is a left C-ring by [22, Lemma 4]. □

In the following theorem, we give some new characterizations of left max-injective rings over a left N-ring.

Theorem 1 *Let R be a left N-ring. Then the following are equivalent.*

1. *R is left max-injective.*
2. *All right R-modules have a monic $\mathfrak{m}$-$\mathfrak{fl}$-preenvelope.*
3. *All injective right R-modules are max-flat.*
4. *All flat left R-modules are max-injective.*
5. *All right R-modules have $\mathfrak{m}$-$\mathfrak{in}$-covers and $\mathfrak{m}$-$\mathfrak{in}^{\perp}$-envelopes.*
6. *For every free left R-module F, F^+ is max-flat.*

Proof (1) ⇔ (2) using similar arguments of [25, Theorem 2.5 and Theorem 2.11].

(2) ⇒ (3) is clear since by (2), every injective right R-module can be embedded in a max-flat right R-module.

(3) ⇒ (4) Let A be a flat left R-module. Then A^+ is injective, so A^+ is max-flat by (3). Thus A is max-injective by Lemma 1(1).

(4) ⇒ (5) Note that the class $\mathfrak{m}$-$\mathfrak{in}$ is closed under extensions and by Lemma 1(3) is closed under pure submodules, pure quotients and direct sums over a left N-ring. Hence (5) follows by (4) and [16, Theorem 3.4].

(5) ⇒ (1) is clear.

(3) ⇒ (6) Let F be a free left R-module. Then F^+ is injective, and so F^+ is max-flat by (3).

(6) ⇒ (3) For any injective right R-module E, there is an epimorphism $F \to E^+$ with F a free left R-module. So there exists a monomorphism $E^{++} \to F^+$ with $E \subseteq E^{++}$. Since E is injective, E is a direct summand of F^+, and so E is max-flat.

□

In [10], Faith conjectured that every left (or right) perfect right self-injective ring is QF. In [24], the authors considered equivalent form of Faith's conjecture: Any left perfect, right max-injective ring is QF. Regarding this conjecture we obtain the following partial affirmative answer.

Proposition 1 *Let R be a left perfect right max-injective ring. Then R is QF if and only if R is right N-ring.*

Proof Necessity is clear. To prove the sufficiency, let A be an injective left R-module. Since R is right max-injective and right N-ring, A is max-flat by Theorem 1. Being left perfect implies R is right C-ring. Then A is flat by Corollary 2. Hence A is projective by the left perfectness of R, and so R is QF. □

We conclude this section with the following theorem.

Theorem 2 *Let R be a ring. Then the following are equivalent.*

1. *Every factor of a max-cotorsion right R-module is max-cotorsion.*
2. *All max-flat right R-modules are of projective dimension* ≤ 1.
3. *For any s-pure exact sequence* $0 \to B \to A \to C \to 0$ *with A projective right R-module, B is projective.*

Proof (1) $\Rightarrow$ (3) Let $0 \to B \to A \to C \to 0$ be an s-pure exact sequence with A projective right R-module. Then C is max-flat by Lemma 2. For any right R-module M, there exists an exact sequence $0 \to M \to E \to N \to 0$ with E injective. Note that N is max-cotorsion by (1), and hence $\mathrm{Ext}_R^2(C,M) = \mathrm{Ext}_R^1(C,N) = 0$. Thus, $pd(C) \leq 1$, so B is projective.

(3) $\Rightarrow$ (2) Let A be any max-flat right R-module. There exists an exact sequence $0 \to B \to P \to A \to 0$ with P projective. Since $0 = \mathrm{Tor}_1^R(A, R/I) \to B \otimes R/I \to P \otimes R/I \to A \otimes R/I \to 0$ is exact for any maximal left ideal I, this sequence is s-pure, so B is projective by (3). It follows that $pd(A) \leq 1$.

(2) $\Rightarrow$ (1) Let A be any max-cotorsion right R-module and C a submodule of A. For any max-flat right R-module B, the exactness of the sequence $0 \to C \to A \to A/C \to 0$ induces the exact sequence $0 = \mathrm{Ext}_R^1(B,A) \to \mathrm{Ext}_R^1(B,A/C) \to \mathrm{Ext}_R^2(B,C)$. By (2), $\mathrm{Ext}_R^2(B,C) = 0$, so $\mathrm{Ext}_R^1(B,A/C) = 0$. □

3 Max-flat dimensions

In this section we investigate the max-flat dimension of modules. We begin with the following lemma.

Lemma 4 *Let R be a ring. Then the following are equivalent.*

1. *For any max-flat right modules* B, C *and epimorphism* $f : B \to C$, $\mathrm{Ker}(f)$ *is max-flat.*
2. *If* $0 \to A \to M \to B \to 0$ *is an exact sequence of right R-modules with M and B max-flat, A is max-flat.*
3. $\mathrm{Tor}_i^R(A, R/I) = 0$ *for every max-flat right R-module* A, *every maximal left ideal I of R and every* $i \geq 1$.

Proof (1) $\Leftrightarrow$ (2) is clear.

(2) ⇒ (3) Let A be a max-flat right R-module. Then there is an exact sequence $0 \to B \to F \to A \to 0$ with F projective, so B is max-flat by (2). Thus, $\mathrm{Tor}_2^R(A, R/I) \cong \mathrm{Tor}_1^R(B, R/I) = 0$ for every maximal left ideal I of R, hence (3) holds by induction.

(3) ⇒ (2) is easy. □

For convenience, we will define the following condition for a ring R:
(P): R satisfies the equivalent conditions of Lemma 4.

Remark 2

(a) If R is a left SF-ring (i.e, every simple left R-module is flat), then clearly it satisfies (P).
(b) If R is a left C-ring, then every max-flat right module is flat by Corollary 2. So every left C-ring ring has (P) by Lemma 4 and [17, Corollary 4.86(2)]. Left semiartinian rings and right perfect rings are left C-rings, and so these rings have the property (P).

Lemma 5 *Let R be a left N-ring. Then the following are equivalent.*

1. *R has (P).*
2. *If* $0 \to A \to B \to C \to 0$ *is an exact sequence of left R-modules with A and B max-injective, then C is max-injective.*
3. $\mathrm{Ext}_R^i(R/I, A) = 0$ *for every max-injective left R-module A, every maximal left ideal I of R and every* $i \geq 1$.

Proof (1) ⇒ (2) Let $0 \to A \to B \to C \to 0$ be an exact sequence of left R-modules with A and B max-injective. Then we get an exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$. Note that A^+ and B^+ are max-flat by Lemma 1(1). Thus C^+ is max-flat by (1), so C is max-injective by Lemma 1(1).

(2) ⇒ (1) Let $0 \to A \to B \to C \to 0$ be an exact sequence of right R-modules with B and C max-flat. Then we get an exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$. Since C^+ and B^+ is max-injective, so is A^+ by (2). So, A is max-flat. Hence R has (P) by Lemma 4.

(2) ⇔ (3) The proof is dual to that of Lemma 4. □

Note that every right R-module over any ring R has a max-flat cover by Lemma 3. So A has a left max-flat resolution, that is, there is a $Hom(\mathfrak{m}-\mathfrak{fl}, -)$ exact complex $\cdots \to B_1 \to B_0 \to A \to 0$ with each B_i max-flat. Obviously, this complex is exact. The left max-flat dimension of a right R-module A, denoted by left max-fd(A), is defined as $\inf\{n$: there is a left max-flat resolution of A of the form $0 \to B_n \to \cdots \to B_1 \to B_0 \to A \to 0\}$. If no such n exists, set left max-fd(A)= ∞. The global left max-flat dimension of $\mathfrak{M}_R$, denoted by gl left max-fd($\mathfrak{M}_R$), is defined to be $\sup\{$left max-fd(A): $A \in \mathfrak{M}_R\}$ and is infinite otherwise.

Proposition 2 *Let R be a ring, n a nonnegative integer and A a right R-module. Consider the following conditions:*

1. *left max-fd(A)$\leq n$.*
2. $\mathrm{Tor}^R_{n+k}(A, R/I) = 0$ *for every maximal left ideal I of R and every $k \geq 1$.*
3. $\mathrm{Tor}^R_{n+1}(A, R/I) = 0$ *for every maximal left ideal I of R.*
4. *If* $0 \to C \to B_{n-1} \to \cdots \to B_1 \to B_0 \to A \to 0$ *is exact with each B_i max-flat, then C is max-flat.*

(2) ⇒ (3) ⇒ (4) ⇒ (1). *If R has (P), then* (1) ⇒ (2).

Proof (2) ⇒ (3) is trivial.

(3) ⇒ (4) Let $0 \to C \to B_{n-1} \to \cdots \to B_1 \to B_0 \to A \to 0$ be an exact sequence with each B_i max-flat. Then for every maximal left ideal I of R, by (3), $\mathrm{Tor}^R_{n+1}(A, R/I) \cong \mathrm{Tor}^R_1(C, R/I) = 0$. So C is max-flat.

(4) ⇒ (1) Let $\cdots \to B_{n-1} \to \cdots \to B_1 \to B_0 \to A \to 0$ be a partial left max-flat resolution of A. Then we get an exact sequence $0 \to C \to B_{n-1} \to \cdots \to B_1 \to B_0 \to A \to 0$. By (4), C is max-flat. Thus left max-fd(A)$\leq n$.

(1) ⇒ (2) Since left max-fd(A)$\leq n$, there exists a left max-flat resolution $0 \to B^n \to B^{n-1} \to \cdots \to B^1 \to B^0 \to A \to 0$. So, for every maximal left ideal I of R and every $k \geq 1$, $\mathrm{Tor}^R_n(B^n, R/I) \cong \mathrm{Tor}^R_{n+k}(A, R/I) = 0$ by Lemma 4. □

Recall that over a left max-coherent ring, every left module has a right max-injective resolution which is exact (see [25]). This fact is also true for left N-rings by Lemma 1(5). As an analogous to that of Proposition 2, we have the following.

Proposition 3 *Let R be a left N-ring, n a nonnegative integer and A a right R-module. Consider the following conditions:*

1. *right max-id(A)$\leq n$.*
2. $\mathrm{Ext}^{n+k}_R(R/I, A) = 0$ *for every maximal left ideal I of R and every $k \geq 1$.*
3. $\mathrm{Ext}^{n+1}_R(R/I, A) = 0$ *for every maximal left ideal I of R.*
4. *If* $0 \to A \to B^0 \to B^1 \to \cdots \to B^{n-1} \to C \to 0$ *is exact with each B^i max-injective, then C is max-injective.*

(2) ⇒ (3) ⇒ (4) ⇒ (1). *If R has (P), then* (1) ⇒ (2).

Proof The proof is analogous to that of Proposition 2 by Lemma 5. □

Theorem 3 *Let R be a left N-ring satisfying the condition (P), n a nonnegative integer. The following are equivalent.*

1. *gl right max-id*$({}_R\mathfrak{M}) \leq n$.
2. *gl left max-fd*$(\mathfrak{M}_R) \leq n$.
3. *left max-fd(A)$\leq n$ for every max-cotorsion right R-module A.*
4. $\mathrm{Ext}^{n+1}_R(R/I, B) = 0$ *for every maximal left ideal of R and every left R-module B.*

5. $\mathrm{Tor}^R_{n+1}(A, R/I) = 0$ *for every maximal left ideal of R and every right R-module A.*
6. *All simple left R-modules have projective dimension* $\leq n$.
7. *All simple left R-modules have flat dimension* $\leq n$.

In this case, all max-cotorsion right R-modules have injective dimension $\leq n$.

Proof (2) ⇔ (5) and (1) ⇔ (4) follows from Propositions 2 and 3, respectively.

(2) ⇒ (3), (4) ⇔ (6) and (5) ⇔ (7) are obvious.

(3) ⇒ (2) Let A be any right R-module. Then, by Lemma 3, there is an exact sequence $0 \to A \to B \to C \to 0$, where B is max-cotorsion and C is max-flat. Thus we get an induced exact sequence $0 = \mathrm{Tor}^R_{n+2}(C, R/I) \to \mathrm{Tor}^R_{n+1}(A, R/I) \to \mathrm{Tor}^R_{n+1}(B, R/I) = 0$ for every maximal left ideal I of R by (3) and Proposition 2. So, left max-fd(A)$\leq n$ and (2) follows.

(4) ⇒ (5) holds because $\mathrm{Ext}^{n+1}_R(R/I, A^+) \cong \mathrm{Tor}^R_{n+1}(A, R/I)^+$ for every maximal left ideal I of R and every right R-module A.

(5) ⇒ (4) holds because $\mathrm{Tor}^R_{n+1}(B^+, R/I) \cong \mathrm{Ext}^{n+1}_R(R/I, B)^+$ for every maximal left ideal I of R and every left R-module B.

For the last statement let A be a max-cotorsion right R-module and B any right R-module. Then, by (5), there is an exact sequence $0 \to C \to B_{n-1} \to \cdots \to B_1 \to B_0 \to B \to 0$ with C max-flat and each B_i projective, and so $\mathrm{Ext}^{n+1}_R(B, A) \cong \mathrm{Ext}^1_R(C, A) = 0$. Thus A has injective dimension $\leq n$. □

Let $\mathfrak{C}$ be a class of left R-modules and A a left R-module. Recall that a $\mathfrak{C}$-cover $g : C \to A$ is said to have the unique mapping property if for any homomorphism $f : D \to A$ with $D \in \mathfrak{C}$, there is a unique homomorphism $h : D \to C$ such that $gh = f$, [7].

Corollary 3 *Let R be a left N-ring satisfying the condition (P). The following are equivalent.*

1. *gl right max-id*$({}_R\mathfrak{M}) \leq 2$.
2. *gl left max-fd*$(\mathfrak{M}_R) \leq 2$.
3. *All left R-modules have* $\mathfrak{m}$-$\mathfrak{in}$*-covers with the unique mapping property.*

Proof (1) ⇔ (2) holds by Theorem 3. (3) ⇒ (1) by [25, Theorem 4.6].

(1) ⇒ (3) Let A be a left R-module. Then A has an $\mathfrak{m}$-$\mathfrak{in}$-cover $g : D \to A$ by Lemma 1(5). It is enough to show that, for any max-injective left R-module B and any homomorphism $f : B \to D$ such that $gf = 0$, we have $f = 0$. In fact, there exists $\beta : D/Im(f) \to A$ such that $\beta\pi = g$ since $Im(f) \subseteq \ker(g)$, where $\pi : D \to D/Im(f)$ is the natural map. Consider the exact sequence $0 \to \mathrm{Ker}(f) \to G \to D \to D/Im(f) \to 0$. Note that $D/Im(f)$ is max-injective by (1) and Proposition 3. Thus there exists $\alpha : D/Im(f) \to D$ such that $\beta = g\alpha$, and so $g\alpha\pi = \beta\pi = g$. Hence $\alpha\pi$ is an isomorphism since g is a cover. Therefore π is monic, and so $f = 0$. □

Recall that a ring R is called left *max-hereditary* [1] if every maximal left ideal is projective. This is equivalent to saying that every factor of a max-injective left R-module is max-injective (see [1, Proposition 1.2]). Now we have the following characterizations of left max-hereditary rings.

Theorem 4 *Let R be a left N-ring. The following are equivalent.*

1. *R is left max-hereditary.*
2. *R has (P) and gl right max-id$({}_R\mathfrak{M}) \leq 1$.*
3. *R has (P) and gl left max-fd$(\mathfrak{M}_R) \leq 1$.*
4. *R has (P) and left max-fd$(M) \leq 1$ for every max-cotorsion right R-module A.*
5. *All submodules of max-flat right R-modules are max-flat.*
6. *All left R-modules have a monic max-injective cover.*

Proof (2) ⇔ (3) ⇔ (4) follows from Theorem 3.

(1) ⇒ (5) Let A be a submodule of a max-flat right R-module B. Then the inclusion $i : A \to B$ induces the epimorphism $\pi : B^+ \to A^+$. Note that B^+ is max-injective, so A^+ is max-injective by (1), and hence A is max-flat.

(5) ⇒ (1) Let B be a factor module of a max-injective left R-module A. Then the exact sequence $0 \to C \to A \to B \to 0$ implies the exactness of $0 \to B^+ \to A^+ \to C^+ \to 0$. Since A^+ is max-flat, B^+ is max-flat by (5) and so B is max-injective. Hence by [1, Proposition 1.2], R is left max-hereditary.

(1) ⇒ (2) is clear by [1, Proposition 1.2].

(2) ⇒ (1) Let A be any max-injective left R-module and B a submodule of A. By (2), there is a right max-injective resolution $0 \to B \to C \to D \to 0$. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & B & \longrightarrow & A & \longrightarrow & A/B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & C & \longrightarrow & E & \longrightarrow & A/B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & D & = & D & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since A and D are max-injective, E is max-injective by [25, Proposition 2.4(1)]. So A/B is max-injective by Lemma 5. Hence R is a left max-hereditary ring by [1, Proposition 1.2].

(1) ⇔ (6) holds by [13, Proposition4] since $\mathfrak{m}\text{-}\mathfrak{in}$ is closed under direct sums by Lemma 1(3). □

Now, we give some new characterizations of left SF-rings. Recall that a ring R is called a left *SF-ring* [20] if every simple left R-module is flat, or equivalently every right R-module is max-flat.

Corollary 4 *Let R be a left N-ring. The following are equivalent.*

1. *R is left SF-ring.*
2. *gl right max-id$({}_R\mathfrak{M}) = 0$.*
3. *All cotorsion right R-modules are max-flat.*
4. *R has (P) and all max-cotorsion right R-modules are max-flat.*

Proof (1) $\Leftrightarrow$ (2) comes from [1, Theorem 1.2].

(2) $\Leftrightarrow$ (4) comes from Theorem 3 and Lemma 5.

(1) $\Rightarrow$ (3) is clear since over a left SF-ring, every right R-module is max-flat.

(3) $\Rightarrow$ (2) Let A be any left R-module. Then A^+ is max-flat by (3). Thus A^{++} is max-injective. Note that A is a pure submodule of A^{++}, so A is max-injective by Lemma 1(3). □

Corollary 5 *The following are equivalent for a ring R.*

1. *R is right perfect.*
2. *R has (P) and all max-flat right R-modules are max-cotorsion.*
3. *All max-flat right R-modules are projective.*

Proof (1) $\Rightarrow$ (2) Since R is right perfect, R is a left C-ring. So R has (P) by Remark 2(b). Let A be a max-flat right R-module. Then by Lemma 3, there exists an exact sequence $0 \to A \to C \to B \to 0$ with C max-cotorsion and B max-flat. Since R is a left C-ring, B is flat, and so is projective by the hypothesis. Thus A is isomorphic to a direct summand of C, whence A is max-cotorsion.

(2) $\Rightarrow$ (3) For any max-flat right R-module A, we have an exact sequence $0 \to B \to F \to A \to 0$ with F projective. Since R has (P), B is max-flat by Lemma 4. By (2), B is max-cotorsion, and so $\mathrm{Ext}^1_R(A, B) = 0$. This means that $0 \to B \to F \to A \to 0$ splits, whence A is projective.

(3) $\Rightarrow$ (1) is clear since every flat module is max-flat. □

4 SP-flat modules

In [3], the authors introduced that a left R-module B is s-pure injective (in short SP-injective), (in [14] is called coneat injective) if it is injective with respect to s-pure short exact sequences. Clearly, every SP-injective module is pure-injective. Motivated by this, we first introduce the concept of SP-flat modules.

Definition 2 Let R be a ring. A right R-module A is called *SP-flat* if for every s-pure exact sequence $0 \to K \to L \to M \to 0$ of left R-modules, the sequence $0 \to A \otimes K \to A \otimes L \to A \otimes M \to 0$ is exact.

Remark 3

(1) By the definition, any simple module is SP-flat.
(2) Flat right modules are SP-flat. But the converse is not true in general. For example, $\mathbb{Z}_p$ is an SP-flat $\mathbb{Z}$-module for a prime integer p since $\mathbb{Z}_p$ is a simple $\mathbb{Z}$-module. But it is not a flat $\mathbb{Z}$-module.

Lemma 6 *Let R be a ring. Then*

1. *A right R-module A is SP-flat if and only if A^+ is SP-injective.*
2. *The class of SP-flat right R-modules is closed under pure submodules and pure quotient modules.*

Proof

(1) Let A be a right R-module and $0 \to K \to L \to M \to 0$ an s-pure exact sequence of left R-modules. Then the sequence $0 \to A \otimes K \to A \otimes L \to A \otimes M \to 0$ is exact if and only if the sequence $0 \to (A \otimes M)^+ \to (A \otimes L)^+ \to (A \otimes K)^+ \to 0$ is exact if and only if $0 \to \mathrm{Hom}(M, A^+) \to \mathrm{Hom}(L, A^+) \to \mathrm{Hom}(K, A^+) \to 0$ is exact. So A is SP-flat if and only if A^+ is SP-injective.
(2) Let $0 \to K \to L \to M \to 0$ be a pure exact sequence of right R-modules with L SP-flat. Then we get the split exact sequence $0 \to M^+ \to L^+ \to K^+ \to 0$. Since L^+ is SP-injective by (1), K^+ and M^+ are SP-injective. So K and M are SP-flat.

□

Remark 4

(1) All modules can be embedded as an s-pure submodule in an SP-injective module by [14, Corollary 2.4].
(2) All right modules have an SP-flat cover by Lemma 6 and [16, Theorem 2.5].
(3) If R is a left N-ring, then every SP-injective right modules has an injective cover. In fact let M be an SP-injective left R-module. By [3, Proposition 5.1], M has an absolutely s-pure cover $f : A \to M$. Hence by [3, Proposition 5.2], A is injective.

Corollary 6 *Let R be a ring. The following are equivalent:*

1. *All right R-modules are SP-flat.*
2. *All s-pure exact sequences $0 \to K \to L \to M \to 0$ of left R-modules are pure.*
3. *All pure-injective left R-modules are SP-injective.*

Proof (1) ⇒ (2) is clear. (2) ⇔ (3) by [14, Proposition 3.15].

(3) ⇒ (1) Let A be a right R-module. Then A^+ is pure-injective and so SP-injective by (3). Thus A is SP-flat by Lemma 6(1). □

The following lemma gives further characterizations of s-pure exact sequences.

Lemma 7 *The following are equivalent for an exact sequence* $0 \to K \to L \to M \to 0$ *of left R-modules.*

1. $0 \to K \to L \to M \to 0$ *is s-pure.*
2. *The sequence* $0 \to \mathrm{Hom}(M, B) \to \mathrm{Hom}(L, B) \to \mathrm{Hom}(K, B) \to 0$ *is exact for any SP-injective left R-module B.*
3. *Every simple right R-module is projective with respect to the exact sequence* $0 \to M^+ \to L^+ \to K^+ \to 0$.
4. *The sequence* $0 \to A \otimes K \to A \otimes L \to A \otimes M \to 0$ *is exact for any SP-flat right R-module A.*

Proof (1) ⇒ (2) and (1) ⇒ (4) are clear by the definition.

(4) ⇒ (1) is clear since every simple right R-module is SP-flat.

(2) ⇒ (1) Let S be a simple right R-module. Then S^+ is SP-injective. Thus by (2), $0 \to \mathrm{Hom}(M, S^+) \to \mathrm{Hom}(L, S^+) \to \mathrm{Hom}(K, S^+) \to 0$ is exact. Hence $0 \to (S \otimes M)^+ \to (S \otimes L)^+ \to (S \otimes K)^+ \to 0$ is exact. So we get the exact sequence $0 \to S \otimes K \to S \otimes L \to S \otimes M \to 0$ and (1) follows.

(1) ⇔ (3) Let S be a simple right R-module. Then the exact sequence $0 \to S \otimes K \to S \otimes L \to S \otimes M \to 0$ is exact if and only if $0 \to (S \otimes M)^+ \to (S \otimes L)^+ \to (S \otimes K)^+ \to 0$ is exact if and only if $0 \to \mathrm{Hom}(S, M^+) \to \mathrm{Hom}(S, L^+) \to \mathrm{Hom}(S, K^+) \to 0$ is exact. So (1) ⇔ (3) holds. □

Proposition 4 *The following are equivalent for a left R-module A*:

1. *A is absolutely s-pure.*
2. *Every exact sequence* $0 \to A \to B \to C \to 0$ *is s-pure.*
3. *There exists an s-pure exact sequence* $0 \to A \to E \to C \to 0$ *with E absolutely s-pure.*
4. *For every SP-injective left R-module B, every homomorphism* $f : A \to B$ *factors through an injective left R-module.*

Proof (1) ⇔ (2) ⇔ (3) by [3, Lemma 3.3].

(2) ⇒ (4) is easy since A can be embedded in an injective left R-module.

(4) ⇒ (2) Let $0 \to A \overset{i}{\longrightarrow} B \to C \to 0$ be an exact sequence. For any SP-injective left module D and any homomorphism $g : A \to D$, there are an injective left module E, $f : A \to E$ and $h : E \to D$ such that $g = hf$ by (4). Since E is injective, there is $\alpha : B \to E$ such that $\alpha i = f$. Thus $g = h\alpha i$. So the sequence $0 \to A \to B \to C \to 0$ is s-pure by Lemma 7. □

The following proposition gives some characterizations of max-flat modules in terms of s-purity.

Proposition 5 *The following are equivalent for a right R-module A:*

1. *A is max-flat.*
2. *Every exact sequence $0 \to C \to B \to A \to 0$ is s-pure.*
3. $\mathrm{Ext}_R^1(A, B) = 0$ *for any SP-injective right R-module B.*
4. *There exists an s-pure exact sequence $0 \to B \to F \to A \to 0$ with F max-flat.*

Proof (1) $\Rightarrow$ (2) Let $0 \to C \to B \to A \to 0$ be an exact sequence. Since A is max-flat, we have the exact sequence $0 = \mathrm{Tor}_1^R(A, R/I) \to C \otimes R/I \to B \otimes R/I \to A \otimes R/I \to 0$ for any maximal left ideal I of R. So the exact sequence $0 \to C \to B \to A \to 0$ is s-pure by [2, Lemma 4.1].

(2) $\Rightarrow$ (3) There is an s-pure exact sequence $0 \to C \to F \to A \to 0$ with F projective by (2). Thus, by Lemma 7, $\mathrm{Hom}(F, B) \to \mathrm{Hom}(C, B) \to 0$ is exact for any SP-injective left R-module B. Consider the induced exact sequence: $\mathrm{Hom}(F, B) \to \mathrm{Hom}(C, B) \to \mathrm{Ext}_R^1(A, B) \to \mathrm{Ext}_R^1(F, B) = 0$. So $\mathrm{Ext}_R^1(A, B) = 0$.

(3) $\Rightarrow$ (4) Let $0 \to C \to F \to A \to 0$ be an exact sequence with F (max-)flat. For any SP-injective right R-module B, by (3), we have the exact sequence $0 \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(F, B) \to \mathrm{Hom}(C, B) \to \mathrm{Ext}_R^1(A, B) = 0$. Thus, $0 \to C \to F \to A \to 0$ is s-pure by Lemma 7.

(4) $\Rightarrow$ (1) Let $0 \to B \to F \to A \to 0$ be an s-pure exact sequence with F max-flat. For any maximal left ideal I of R, we have the exact sequence $0 = \mathrm{Tor}_1^R(F, R/I) \to \mathrm{Tor}_1^R(A, R/I) \to B \otimes R/I \to F \otimes R/I$. Since by (4), $B \otimes R/I \to F \otimes R/I$ is monic, $\mathrm{Tor}_1^R(A, R/I) = 0$. Hence, A is max-flat. □

In [18, Lemma 3.6.], it is shown that, over a commutative ring R, a short exact sequence $0 \to C \to B \to A \to 0$ is s-pure if and only if $0 \to C_m \to B_m \to A_m \to 0$ is s-pure for each maximal ideal m of R. By using this result, we have the following.

Corollary 7 *Let R be a commutative ring. A module A is max-flat if and only if A_m is a max-flat R_m-module for all maximal ideals m of R.*

A right module A is called *neat-flat* if for any epimorphism $f : B \to A$, the induced map $\mathrm{Hom}(S, B) \to \mathrm{Hom}(S, A)$ is epic for any simple right module S, equivalently any short exact sequence ending with A is neat-exact (see [3]). In [18, Theorem 3.7], it is shown that, over a commutative ring R, every maximal ideal m of R is finitely generated and locally principal if and only if s-pure short exact sequences coincide with neat short exact sequences. As a consequences of [18, Theorem 3.7] and [2, Corollary 4.2], we obtain the following.

Corollary 8 *Let R be a commutative ring and A an R-module. Suppose every maximal ideal m of R is finitely generated and locally principal. Then A is max-flat if and only if A is neat-flat.*

A left module B is said to be *absolutely s-pure* if it is s-pure in every extension of it (see [3]). The following gives the relationship between SP-injective (resp. SP-flat) modules and injective (resp. flat) modules.

Corollary 9 *The following are true for any ring R:*

1. *Any absolutely s-pure SP-injective left R-module is injective.*
2. *If R is a left N-ring, any neat flat SP-flat right R-module is flat.*

Proof (1) Let A be any absolutely s-pure SP-injective left R-module. By Proposition 4, there exists an s-pure exact sequence $0 \to A \to E \to B \to 0$ with E injective. So the exact sequence splits, and hence A is injective.

(2) Let A be any neat flat SP-flat right R-module. Then A^+ is absolutely s-pure by [3, Proposition 4.3] and SP-injective by Lemma 6, and so is injective by (1). Thus A is flat. □

Theorem 5 *The following are equivalent for a ring R and integer $n \geq 0$:*

1. *gl left max-fd*$(\mathfrak{M}_R) \leq n$
2. *All max-cotorsion right R-modules have injective dimension $\leq n$.*
3. *All SP-injective right R-modules have injective dimension $\leq n$.*
4. *All SP-flat left R-modules have flat dimension $\leq n$.*

Proof (1) $\Rightarrow$ (2) Let A be a max-cotorsion right R-module and B any right R-module. Since left max-fd(B)$\leq n$, there is an exact sequence $0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to B \to 0$ with each C_i max-flat. So $\mathrm{Ext}_R^{n+1}(B, A) = \mathrm{Ext}_R^1(C_n, A) = 0$. It follows that A has injective dimension $\leq n$.

(2) $\Rightarrow$ (3) is trivial by Proposition 5.

(3) $\Rightarrow$ (4) For any SP-flat left R-module A, A^+ is SP-injective. By (3), for every left R-module B, we have $\mathrm{Tor}_{n+1}^R(B, A)^+ \cong \mathrm{Ext}_R^{n+1}(B, A^+) = 0$. So, $\mathrm{Tor}_{n+1}^R(A, B) = 0$, and hence A has flat dimension $\leq n$.

(4) $\Rightarrow$ (1) Let $\cdots \to B_{n-1} \to \cdots \to B_1 \to B_0 \to A \to 0$ be a partial left max-flat resolution of A. Then we get an exact sequence $0 \to C \to B_{n-1} \to \cdots \to B_1 \to B_0 \to A \to 0$. Since every simple left R-module is SP-flat, by (4), $\mathrm{Tor}_1^R(C, R/I) = \mathrm{Tor}_{n+1}^R(A, R/I) = 0$ for any maximal left ideal I of R. Hence C is max-flat. □

As a consequences of Theorem 5 and [14, Theorem 3.16], we obtain a new characterization of left SF-rings.

Corollary 10 *Let R be a ring. Then the following are equivalent.*

1. *R is left SF-ring.*
2. *gl left max-fd*$(\mathfrak{M}_R) = 0$.

3. *All max-cotorsion right R-modules are injective.*
4. *All SP-injective right R-modules are injective.*
5. *All SP-injective right R-modules are absolutely s-pure.*
6. *All SP-flat left R-modules are flat.*
7. *All exact sequences of right R-modules are s-pure.*
8. *All right R-modules are absolutely s-pure.*

Remark 5 The class of SP-injective modules need not be closed under extensions. Note that for each simple right R-module S, S^+ is an SP-injective left R-module by the standard adjoint isomorphism. Consider the short exact sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$. The simple $\mathbb{Z}$-modules $\mathbb{Z}_2$ are SP-injective, but $\mathbb{Z}_4$ is not SP-injective.

Proposition 6 *Let R be a ring. Then the following are equivalent.*

1. *The class of SP-injective left R-module is closed under extensions.*
2. *All max-cotorsion left R-modules are SP-injective.*

In this case, the class of SP-flat right R-modules is closed under extensions.

Proof (1) ⇒ (2) Let A be a max-cotorsion left R-module. By Remark 4(1), we have an s-pure exact sequence $0 \to A \to B \to C \to 0$ with B is SP-injective. By (1) and [26, Lemma 2.1.2] $\mathrm{Ext}^1_R(C, D) = 0$ for every SP-injective left R-module D, and so C is max-flat by Proposition 5. Therefore $\mathrm{Ext}^1_R(C, A) = 0$, and hence the sequence $0 \to A \to B \to C \to 0$ is split. Thus A is isomorphic to a direct summand of B and so is SP-injective.

(2) ⇒ (1) is obvious since max-cotorsion modules are closed under extensions.

In this case, if $0 \to A \to B \to C \to 0$ is an exact sequence of right R-modules with A and C SP-flat, then we get the exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$. By Lemma 6(1), C^+ and A^+ are SP-injective. Thus B^+ is SP-injective, and hence B is SP-flat by Lemma 6(1). □

Recall that all R-modules have max-flat covers and all R-modules have max-cotorsion envelopes for an arbitrary ring R by Lemma 3. In [21], Rothmaler considered the pure-injective cotorsion envelopes of flat R-modules. Motivated by this, we next study when the max-cotorsion envelope of every max-flat R-module is SP-injective.

Theorem 6 *Let R be a ring. Then the following are equivalent.*

1. *All max-flat max-cotorsion left R-modules are SP-injective.*
2. *If $0 \to K \to L \to M \to 0$ is an exact sequence of left R-modules, where K is SP-injective and M is an SP-injective envelope of a max-flat left R-module, then L is SP-injective.*
3. *The max-flat cover of every max-cotorsion left R-module is SP-injective.*
4. *The max-flat cover of every SP-injective left R-module is SP-injective.*
5. *The SP-injective envelope of every max-flat left R-module is max-flat.*

6. *The max-cotorsion envelope of every max-flat left R-module is SP-injective.*

Proof (3) ⇒ (4) and (6) ⇒ (1) are trivial.

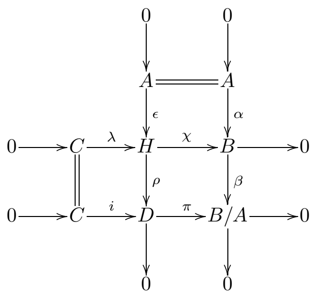
(1) ⇒ (6) Let $g : A \to Y$ be a max-cotorsion envelope of a max-flat module A. Since max-cotorsion modules are closed under extensions, *coker*(g) is max-flat by [26, Lemma 2.1.2]. Hence, Y is max-flat implies that Y is SP-injective by (1).

(1) ⇒ (3) Let $g : Y \to A$ be a max-flat cover of a max-cotorsion module A. Since max-flat modules are closed under extensions, ker(g) is max-cotorsion by [26, Lemma 2.1.1]. Hence, Y is max-cotorsion implies that Y is SP-injective by (1).

(4) ⇒ (5) Let A be a max-flat left R-module, $g : A \to B$ an SP-injective envelope, and $f : D \to B$ a max-flat cover of B. Then there exists $h : A \to D$ such that $fh = g$. On the other hand, since D is SP-injective by (4), there exists $\beta : B \to D$ such that $\beta g = h$. Thus $(f\beta)g = fh = g$, and so $f\beta$ is an isomorphism since g is an envelope. It follows that B is max-flat.

(5) ⇒ (1) Let A be a max-flat max-cotorsion left R-module. By Remark 4(1), we have an exact sequence $0 \to A \xrightarrow{i} B \to C \to 0$ where $i : A \to B$ is a SP-injective envelope of A, and the sequence is s-pure. By (5), B is max-flat, and so C is max-flat by Proposition 5. Therefore $\mathrm{Ext}^1_R(C, A) = 0$, and hence the sequence $0 \to A \xrightarrow{i} B \to C \to 0$ splits. Thus A is SP-injective.

(2) ⇒ (5) Let $\alpha : A \to B$ be an SP-injective envelope of a max-flat left R-module A. We need to show that $\mathrm{Ext}^1_R(B/A, C) = 0$ for any SP-injective left R-module C. In fact, let $0 \to C \to D \to B/A \to 0$ be any exact sequence. Then we have the following pullback diagram:



By (2), H is SP-injective. So there exists $\psi : B \to H$ such that $\epsilon = \psi\alpha$. Note that $\alpha = \chi\epsilon = \chi\psi\alpha$, thus $\chi\psi$ is an isomorphism since α is an envelope. So $(\chi\psi)^{-1}\alpha = \alpha$. It follows that $\rho\psi(\chi\psi)^{-1}(A) = \rho\psi(\chi\psi)^{-1}\alpha(A) = \rho\psi\alpha(A) = \rho\epsilon(A) = 0$. Thus we get an induced map $\theta : B/A \to D$ such that $\theta\beta = \rho\psi(\chi\psi)^{-1}$. Hence $\pi\theta\beta = \pi\rho\psi(\chi\psi)^{-1} = \beta\chi\psi(\chi\psi)^{-1}$. So $\pi\theta = 1$ since β is epic. Thus the sequence $0 \to C \to D \to B/A \to 0$ splits, and so $\mathrm{Ext}^1_R(B/A, C) = 0$. By Proposition 5, B/A is max-flat. Hence B is max-flat by Lemma 2.

(5) ⇒ (2) If $0 \to A \to B \to C \to 0$ is an exact sequence of left R-modules, where A is SP-injective and C is an SP-injective envelope of a max-flat left R-module, then C is max-flat by (5). So the sequence $0 \to A \to B \to C \to 0$ splits. Thus B is SP-injective. □

Next we characterize SP-flat and SP-injective modules in terms of s-pure exact sequences.

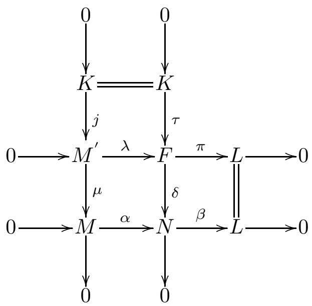
Proposition 7 *Let R be a ring. The following are equivalent for a left R-module A.*

1. *A is SP-injective.*
2. *Every s-pure exact sequence $0 \to A \to B \to C \to 0$ of left R-modules splits.*
3. *A is injective with respect to every s-pure exact sequence $0 \to M \to N \to L \to 0$ of left R-modules with N pure-projective.*

Proof (1) ⇒ (2) and (1) ⇒ (3) are obvious.

(2) ⇒ (1) By [14, Corollary 2.4], there is an s-pure exact sequence $0 \to A \to B \to C \to 0$ with B SP-injective. So A is SP-injective by (2).

(3) ⇒ (1) Let $0 \to M \to N \to L \to 0$ be an s-pure exact sequence of left R-modules. By [9, Example 8.3.2], there is an (s-)pure exact sequence $0 \to K \to F \to N \to 0$ with F pure-projective. Then we have the following pullback diagram:



Thus, $\tau = \lambda j$ and $\pi = \beta\delta$. $\pi = \beta\delta$ is an s-pure epimorphism since β and δ are s-pure epimorphisms. Hence, $0 \to M' \to F \to L \to 0$ is s-pure. Let $g : M \to A$ be any homomorphism. By (3), there exists $f : F \to A$ such that $f\lambda = g\mu$. Since $f\lambda j = g\mu j = 0$, we have $\ker(\delta) = Im(\tau) = Im(\lambda j) \subseteq \ker(f)$. So there exists an induced map $h : N \to A$ such that $h\delta = f$. Thus, $g\mu = h\delta\lambda = h\alpha\mu$. Since μ is epic, $g = h\alpha$. Hence A is SP-injective. □

Proposition 8 *Let R be a ring. The following are equivalent for a right R-module A.*

1. *A is an SP-flat right R-module.*
2. *For every s-pure exact sequence $0 \to M \to N \to L \to 0$ of left R-modules with N pure-projective, the sequence $0 \to A \otimes M \to A \otimes N \to A \otimes L \to 0$ is exact.*

Proof (1) ⇒ (2) is clear.

(2) ⇒ (1) Let $0 \to M \to N \to L \to 0$ be any s-pure exact sequence of left R-modules with N pure-projective. By (2), we get the exact sequence $0 \to A \otimes M \to A \otimes N \to A \otimes L \to 0$, which induces the exact sequence

$0 \to \mathrm{Hom}(L, A^+) \to \mathrm{Hom}(N, A^+) \to \mathrm{Hom}(M, A^+) \to 0$. So A^+ is SP-injective by Proposition 7. Thus A is SP-flat by Lemma 6(1). □

In [6], a submodule B of a right R-module A is called *coneat* in A if $\mathrm{Hom}(A, S) \to \mathrm{Hom}(B, S)$ is epic for every simple right R-module S. In [8, Definition3.1], a right R-module A is called *coneat-injective* if it is injective with respect to the coneat monomorphisms. If R is commutative, then s-pure short exact sequences coincide with coneat short exact sequences, [12, Proposition 3.1].

Proposition 9 *Let R be a commutative ring. The following are equivalent for an R-module M.*

1. *A is an SP-injective R-module.*
2. *A is a coneat-injective R-module.*
3. *$Hom(F, A)$ is an SP-injective R-module for any flat R-module F.*

Proof (1) ⇔ (2) is clear. (3) ⇒ (1) is clear by letting $F = R$.

(1) ⇒ (3) Let $0 \to M \to N \to L \to 0$ be an s-pure exact sequence of left R-modules. For any simple R-module S, we get the exact sequence $0 \to S \otimes M \to S \otimes N \to S \otimes L \to 0$. It follows that, for any flat R-module F, we get the exact sequence $0 \to F \otimes S \otimes M \to F \otimes S \otimes N \to F \otimes S \otimes L \to 0$. Hence the sequence $0 \to S \otimes (F \otimes M) \to S \otimes (F \otimes N) \to S \otimes (F \otimes L) \to 0$ is exact. So the exact sequence $0 \to F \otimes M \to F \otimes N \to F \otimes L \to 0$ is s-pure. Since A is SP-injective, we obtain an exact sequence $0 \to \mathrm{Hom}(F \otimes L, A) \to \mathrm{Hom}(F \otimes N, A) \to \mathrm{Hom}(F \otimes M, A) \to 0$ which gives the exactness of the sequence $0 \to \mathrm{Hom}(L, \mathrm{Hom}(F, A)) \to \mathrm{Hom}(N, \mathrm{Hom}(F, A)) \to \mathrm{Hom}(M, \mathrm{Hom}(F, A)) \to 0$. Thus, $\mathrm{Hom}(F, A)$ is an SP-injective R-module. □

Proposition 10 *Let R be a commutative ring. The following are equivalent for an R-module A.*

1. *A is an SP-flat R-module.*
2. $\mathrm{Hom}(A, E)$ *is an SP-injective R-module for any injective R-module E.*
3. *$A \otimes F$ is an SP-flat R-module for any flat R-module F.*

Proof (1) ⇒ (2) Let E be an injective R-module. Then there is a splitting exact sequence $0 \to E \to \prod R^+$. So, we get the splitting exact sequence $0 \to \mathrm{Hom}(A, E) \to \mathrm{Hom}(A, \prod R^+) \cong \prod(\mathrm{Hom}(A, R^+)) \cong \prod A^+$. By (1), A^+ is SP-injective, and so $\prod A^+$ is SP-injective. Thus, $\mathrm{Hom}(A, E)$ is SP-injective.

(2) ⇒ (3) Let F be any flat module. Then F^+ is injective. So, $\mathrm{Hom}(A, F^+) \cong (A \otimes F)^+$ is SP-injective by (2). Thus, $A \otimes F$ is SP-flat.

(3) ⇒ (1) is clear by letting $F = R$. □

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