CHARACTERIZATIONS OF SIMPLE-DIRECT MODULES

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ABSTRACT

CHARACTERIZATIONS OF SIMPLE-DIRECT MODULES

In this thesis, we study on simple-direct-injective and simple-direct-projective modules. We give a complete characterization of the aforementioned modules simple-direct-injective and simple-direct projective modules over the ring of integers. The rings whose simple-direct-injective right modules are simple-direct-projective are fully characterized. These are exactly the left perfect right H-rings. The rings whose simple-direct-projective right modules are simple-direct-injective are right max-rings. For a commutative Noetherian ring, we prove that simple-direct-projective modules are simple-direct-injective if and only if simple-direct-injective modules are simple-direct-projective if and only if the ring is Artinian. In addition, various closure properties and some classes of modules that are simple-direct-injective (resp. projective) are given.

ÖZET

BASİT-DOĞRUDAN MODÜLLERİN KARAKTERİZASYONLARI

Bu tezde basit-doğrudan-injektif ve basit-doğrudan-projektif modüller üzerine çalışılmıştır. Bu modüllerin tam sayılar halkası üzerinde tam karakterizasyonları verilmiştir. Basit-doğrudan-injektif sağ modüllerin basit-doğrudan-projektif olduğu halkalar tam olarak karakterize edilmiştir. Bu halkalar tam olarak sol mükemmel ve sağ H-halkalar olmaktadır. Basit-doğrudan-projektif sağ modüllerin basit-doğrudan-injektif olduğu halkalar max-halka olmaktadır. Değişmeli Noether bir halka için, basit-doğrudan-projektif modüllerin basit-doğrudan-injektif olması ile basit-doğrudan-injektif modüllerin basit-doğrudan-projektif olmasının denk olduğu ve bu halkaların tam olarak Artin halkalar olduğu gösterilmiştir. Bunun yanında söz konusu bu modüllerin bazı özellikleri ve basit-doğrudan-injektif (projektif) olan bazı modül sınıfları verilmiştir.

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LIST OF ABBREVIATIONS

R	an associative ring with unit unless otherwise stated
$\mathbb{Z}$, $\mathbb{Z}^+$	the ring of integers, the ring of positive integers
$\mathbb{Q}$	the field of rational numbers
Ω	the set of prime integers
$\mathrm{Hom}_R(M, N)$	all R-module homomorphisms from M to N
$Mod - R$	the category of *right* R-modules
$\oplus_{i\in I} M_i$	direct sum of R-modules M_i
$\sum_{i\in I} M_i$	direct product of R-modules M_i
$M \otimes_R N$	the tensor product of the *right* R-module M and the *left* R-module N
$E(M)$	the injective envelope (hull) of a module M
$Ker(f)$	the kernel of the map f
$Im(f)$	the image of the map f
$soc(M)$	the socle of the R-module M
$rad(M)$	the radical of the R-module M
$T(M)$	the torsion submodule of a module M
$Z(M)$	the singular submodule of a module M
$\cong$	isomorphic
$\subseteq$	submodule
$\trianglelefteq$	essential submodule
$\subseteq^{\oplus}$	direct summand
$\ll$	small submodule

CHAPTER 1

INTRODUCTION

In (Nicholson, 1976), a right module is called *direct-injective* if every submodule isomorphic to a direct summand is a direct summand. Direct-injective modules are also known as $C2$-modules. A right module is a $C3$-module if the sum of any two direct summands with zero intersection is again a direct summand. These modules and several generalizations are studied extensively in the literature. Recently, the "simple" version of $C2$-modules and $C3$-modules are studied in (Camillo et al., 2014). Namely, a right module is called *simple-direct-injective* if every simple submodule isomorphic to direct summand is itself a direct summand, or equivalently if the sum of any two simple direct summands with zero intersection is again a direct summand (see (Camillo et al., 2014)).

Dual to direct-injective modules, a right module M is called *direct-projective, or a $D2$-module* if, for every submodule $A \subseteq M$ with $\frac{M}{A}$ isomorphic to a direct summand of M, then A is a direct summand of M (see (Nicholson, 1976)). In (Ibrahim et al., 2016) and (Ibrahim et al., 2017) the authors investigate and study a dual notion of simple-direct-injective modules. A right module M is called *simple-direct-projective* if, whenever A and B are submodules of M with B simple and $\frac{M}{A} \cong B \subseteq^{\oplus} M$, then $A \subseteq^{\oplus} M$. Some well known classes of rings and modules are characterized in terms of simple-direct-injective and simple-direct-projective modules (see (Camillo et al., 2014), (Ibrahim et al., 2016), (Ibrahim et al., 2017)).

In this thesis, we characterize simple-direct-injective and simple-direct-projective modules over the ring of integers and over semilocal rings. We show that, the ring is semilocal if and only if every right module with zero Jacobson radical is simple-direct-projective. We prove that the rings whose simple-direct-injective right modules are simple-direct-projective are exactly the left perfect right H-rings. We show that, the rings whose simple-direct-projective modules are simple-direct-injective are right max-rings. For a commutative Noetherian ring, we prove that, simple-direct-projective modules are simple-direct-injective if and only if simple-direct-injective modules are simple-direct-projective if and only if the ring is Artinian.

In chapter 3, we characterize simple-direct-projective abelian groups (Theorem 3.1). As a byproduct, a characterization of simple-direct-projective modules over local and local perfect rings is obtained. We prove that the ring is semilocal if and only if every right module with zero Jacobson radical is simple-direct-projective.

In chapter 4, a complete characterization of simple-direct-injective abelian groups is given (Theorem 4.1). Motivated by the fact that nonsingular right modules are simple-direct-projective over any ring, we prove the corresponding result for simple-direct-injective modules. We show that, nonsingular right modules are simple-direct-injective if and only if projective simple right modules are injective. We also give a characterization of simple-direct-injective modules over semilocal rings. We show that simple-direct-injective modules are closed under coclosed submodules over any ring, and closed under pure submodules provided the ring is commutative. Partial converses of these results are given.

Following ((Sharpe and Vamos, 1972), sec. 4.4), we say R is a *right H-ring* if for nonisomorphic simple right R-modules S_1 and S_2, $\mathrm{Hom}_R(E(S_1), E(S_2)) = 0$. Commutative Noetherian rings, and commutative semiartinian rings are H-ring by ((Sharpe and Vamos, 1972), Proposition 4.21) and ((Camillo, 1978), Proposition 2), respectively. Right Artinian rings that are right H-rings are characterized in ((Papp, 1975), Theorem 9). Some classes of noncommutative H-rings are also studied in (Golan, 1981). A ring R is called *right max-ring* if every nonzero right R-module has a maximal submodule.

In ((Camillo et al., 2014), Theorem 3.4.), the authors characterize the rings over which simple-direct-injective right modules are $C3$-modules. They prove that these rings are exactly the Artinian serial rings with $J^2(R) = 0$. In ((Ibrahim et al., 2016), Theorem 4.9.), the authors prove that every simple-direct-injective right R-module is $D3$-module if and only if every simple-direct-projective right R-module is $C3$-module if and only if R is uniserial with $J^2(R) = 0$.

At this point, it is natural to consider the rings whose simple-direct-injective modules are simple-direct-projective, and the rings whose simple-direct-projective modules are simple-direct-injective. Right $C3$-modules and right $D3$-modules are simple-direct-injective and simple-direct-projective respectively. Thus, uniserial rings with $J^2(R) = 0$ are examples of such rings.

In chapter 5, we prove that, every simple-direct-injective right module is simple-direct-projective if and only if the ring is left perfect right H-ring (Theorem 5.1). As a consequence, we show that, commutative perfect rings are examples of such rings. For a commutative Noetherian ring, we obtain that, simple-direct-injective modules are simple-direct-projective if and only if the ring is Artinian (Corollary 5.3). We show that, the rings whose simple-direct-projective right modules are simple-direct-injective are right max-rings (Proposition 5.2). For a commutative Noetherian ring, we prove that, simple-direct-projective modules are simple-direct-injective if and only if simple-direct-injective modules are simple-direct-projective if and only if the ring is Artinian (Corollary 5.4).

CHAPTER 2

PRELIMINARIES

In this chapter, we give some known definitions and results about rings and modules that are used in this thesis.

2.1. Rings, Modules and Module Homomorphisms

Definition 2.1 *A ring is defined as a non-empty set R with two binary operations $+, \cdot : R \times R \longrightarrow R$ with the properties:*

(i) $(R, +)$ *is an abelian group with zero element;*

(ii) $(R, \cdot)$ *is a semigroup;*

(iii) for all $a, b, c \in R$ the distributivity laws are valid

$$(a + b)\cdot c = a\cdot c + b\cdot c,\ a \cdot(b + c) = a\cdot b + a\cdot c.$$

A ring R is said to be **commutative** if the multiplication is commutative in R; that is, $a \cdot b = b \cdot a$ for all $a, b \in R$. We say that R is a **domain (or integral domain)** if R has no zero divisor.

Definition 2.2 *Let R be a ring. A subset I of R is called a* ***right ideal*** *in case the following conditions are satisfied:*

(i) For all $a, b \in I$, $a + b \in I$,

(ii) For all $a \in I$ and all $r \in R$, $ar \in I$.

Similarly, I is called a **left ideal**, whenever, for all $a \in I$ and all $r \in R$, $ra \in I$. In addition, if I is a left and right ideal of R, then I is called an **ideal**.

Definition 2.3 *An ideal P in a ring R is said to be a* ***prime ideal*** *if $P \neq R$ and, for ideals $I, J \subset R$, $IJ \subset P$ implies that $I \subset P$ or $J \subset P$.*

In particular, an ideal P of a commutative ring R is said to be a **prime ideal** if, for all $a, b \in R$, $a \cdot b \in I$ implies that either $a \in I$ or $b \in I$.

Definition 2.4 *Let R be a ring and $(M, +)$ be an abelian group with a function $f : M \times R \longrightarrow M$ defined by $f(m, r) = m \cdot r$. M is called a* ***right R-module****, denoted by M_R, if the following properties are satisfied for all $r, s \in R$ and all $m, n \in M$;*

(i) $(m + n) \cdot r = m \cdot r + n \cdot r$*;*

(ii) $m \cdot (r + s) = m \cdot r + m \cdot s$*;*

(iii) $m \cdot (r \cdot s) = (m \cdot r) \cdot s$*;*

(iv) $m \cdot 1_R = m$ *where* 1_R *is the identity of* R*.*

Throughout this thesis, rings are associative with unity (1_R) and modules are unitary. We generally use the right R-module, so a module and an R-module both will mean a right R-module.

Definition 2.5 *Let M be a right R-module. N is called a* ***submodule*** *of M if N is a subgroup of $(M, +)$ closed under scalar multiplication by R, that is, $nr \in R$ for all $r \in R, n \in N$, and denoted by $N \subseteq M$. Then N is also an R-module by the operation induced from M.*

Definition 2.6 *Let M and N be two right R-modules. The function $f : M \longrightarrow N$ is called a* ***right R-homomorphism (R-module homomorphism)*** *in case, for all $m, k \in M$ and all $r \in R$*

$$f(m + k) = f(m) + f(k),$$
$$f(m)r = f(m)r.$$

The set of R-homomorphisms from M to N is denoted by $Hom_R(M, N)$ or $Hom(M_R, N_R)$ or simply $Hom(M, N)$. For $f \in Hom(M_R, N_R)$, the **kernel** and the **image** of f are defined as follows;

$$Ker(f) = \{m \in M \mid f(m) = 0\}$$
$$Im(f) = \{f(m) \in N \mid m \in M\}.$$

Let M be a right R-module. For a submodule K of M, the set

$$\frac{M}{K} = \{x + K \mid x \in M\}$$

is called a **factor (or quotient) module** of M by K. $\dfrac{M}{K}$ is a right R-module by defining the operations of R on this factor module;

$$(m + K)r = mr + K,$$
$$(m + K) + (n + K) = (m + n) + K.$$

Lemma 2.1 (The Isomorphism Theorems) *((Anderson and Fuller, 1992), Corollary 3.7) Let M and N be a right R-modules.*

(1) If $f : M \longrightarrow N$ is an epimorphism with $Ker(f) = K$, then there is a unique isomorphism $\rho : \frac{M}{K} \longrightarrow N$ such that $\rho(m + K) = f(m)$ for all $m \in M$.

(2) If $K \subseteq L \subseteq M$, then
$$\frac{M}{L} \cong \frac{(\frac{M}{K})}{(\frac{L}{K})}.$$

(3) If $H \subseteq M$ and $K \subseteq M$, then

$$\frac{(H + K)}{K} \cong \frac{H}{(H \cap K)}.$$

2.2. Injective and Projective Modules

In this section, we give the definitions and some characterizations of injective and projective modules.

Definition 2.7 *A right R-module E is said to be **injective** if, for each right module B and for every submodule A of B, any homomorphism $f : A \longrightarrow E$ can be extended to a homomorphism $g : B \longrightarrow E$; such that $g \circ i = f$, that is, there exists a homomorphism*

$g : B \longrightarrow E$ making the following diagram commute;

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[dl, dashed, "g"] \\ & E & \end{tikzcd}$$

We give a characterization of injective modules in the following proposition.

Proposition 2.1 *((Anderson and Fuller, 1992), Proposition 18.1) The following statements about a right R-module E are equivalent:*

(1) E is injective;

(2) For each monomorphism $f : K_R \longrightarrow M_R$, the map

$$Hom(f,E) : Hom_R(M,E) \longrightarrow Hom_R(K,E)$$

is an epimorphism;

(3) For each bimodule structure ${}_RE_S$, the functor

$$Hom_R(-, {}_RE) : M_R \longrightarrow {}_SM$$

is exact;

(4) For every exact sequence

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

in M_R the sequence

$$Hom_R(M'',E) \xrightarrow{g^*} Hom_R(M,E) \xrightarrow{f^*} Hom_R(M',E)$$

is exact.

Proposition 2.2 *((Anderson and Fuller, 1992), Proposition 18.7) A right R-module E is injective if and only if every monomorphism*

$$0 \longrightarrow E_R \longrightarrow M_R$$

is splitting.

A group G is said to be **divisible** if, the equation $nx = a$ is solvable in G for all element a of G and for all positive integer n. Equivalently, G is divisible if and only if $nG = G$ for every positive integer n.

Lemma 2.2 *((Fuchs, 1970), p.99 Ex.1) A group is divisible if and only if it has no maximal subgroup.*

Theorem 2.1 *((Fuchs, 1970), Theorem 21.1) Divisible abelian groups are injective.*

Definition 2.8 *Let P be a right R-module. P is called* ***projective*** *if, for all right module homomorphism $g : P \longrightarrow B$ and for all epimorphism $f : A \longrightarrow B$, there exists a homomorphism $h : P \longrightarrow A$ such that $g = f \circ h$, that is, there exists a homomorphism $h : P \longrightarrow A$ making the following diagram commute;*

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r] & 0 \\
 & P \arrow[u, "g"'] \arrow[ul, dashed, "h"] &
\end{tikzcd}$$

Proposition 2.3 *((Anderson and Fuller, 1992), Proposition 17.2.) The following statements about a right R-module are equivalent:*

(1) P is projective;

(2) Every epimorphism $M_R \longrightarrow P_R \longrightarrow 0$ splits;

(3) P is isomorphic to a direct summand of a free right R-module.

2.3. Special Submodules

In this section, we recall the definitions of some special submodules, and give some examples of them. In addition to this, we represent their characterizations.

Definition 2.9 *A submodule L is called a **direct summand** of M, denoted by $L \subseteq^{\oplus} M$, if the following conditions are satisfied:*

(i) $L + K = M$ *for some submodule K of M, and*

(ii) $L \cap K = 0$.

Proposition 2.4 (Modular Law) *((Wisbauer, 1991), p.39) If $A,\ B,\ C$ are submodules of a module M and $B \subseteq A$, then we have*

$$A \cap (B + C) = B + (A \cap C).$$

2.3.1. Simple and Small Submodules

Definition 2.10 *(1) If M has no nontrivial proper submodule, that is, M has two submodules which are zero submodule and itself, then we say that M is a **simple** module.*

*(2) A submodule K of a right R-module M is called **small** if, whenever $K + L = M$ for some $L \subseteq M$, then $L = M$. In this case, we write $K \ll M$.*

*(3) Let M be an R-module and $N \subseteq M$. We call N is an **essential submodule** of M, denoted by $N \trianglelefteq M$, if $(N \cap L) \neq 0$ for each nonzero submodule L of M.*

Lemma 2.3 *((Anderson and Fuller, 1992), Corollary 2.10) A factor module $\frac{M}{L}$ is simple if and only if L is a maximal submodule of M.*

Definition 2.11 *Let M be an R-module.*

*(1) The sum of all simple submodules of M is called the **socle** of M, denoted by $soc(M)$.*

*(2) The intersection of all maximal submodules of M is called the **radical** of M, denoted by $rad(M)$.*

Proposition 2.5 *((Anderson and Fuller, 1992), Proposition 9.7) Let M be a right R-module, then*

$$\begin{aligned} soc(M) &= \sum\{K \subseteq M \mid K \text{ is simple in } M\} \\ &= \bigcap\{L \subseteq M \mid L \text{ is essential in } M\}. \end{aligned}$$

A module M is said to be **semisimple** if, $M = soc(M)$.

Proposition 2.6 *((Anderson and Fuller, 1992), Proposition 9.13) Let M be a right R-module. Then*

$$\begin{aligned} rad(M) &= \bigcap\{K \subseteq M \mid K \text{ is maximal in } M\} \\ &= \sum\{L \subseteq M \mid L \text{ is small in } M\}. \end{aligned}$$

2.3.2. Close, Coclosed and Pure Submodules

Definition 2.12 *A submodule N of a right R-module M is said to be a **closed submodule** of M whenever $N \trianglelefteq K$ for some submodule K of M, then $K = N$.*

Proposition 2.7 *Let A, B, C be submodules of a module M with $A \subseteq B \subseteq C$.*

(1) If A is closed in C, then A is closed in B.

(2) If A is closed in M, then $\frac{K}{A}$ is essential in $\frac{M}{A}$ where $K \trianglelefteq M$ and $A \subseteq K$.

(3) If B is closed in M, then $\frac{B}{A}$ is closed in $\frac{M}{A}$.

Definition 2.13 *A submodule L of a module M is said to be **coclosed** if $\frac{L}{K} \ll \frac{M}{K}$ for $K \subseteq L$, then $L = K$.*

A module L is said to be **hollow**, if every proper submodule of L is small.

Proposition 2.8 *((Clark et al., 2006), 3.7 Properties of coclosed submodule) Let M be a right R-module and $K \subseteq L \subseteq M$.*

(1) If L is a coclosed submodule of M, then $\frac{L}{K}$ is a coclosed submodule of $\frac{M}{K}$.

(2) If $K \ll L$ and $\frac{L}{K}$ is coclosed in $\frac{M}{K}$, then L is a coclosed submodule of M.

(3) If $L \subset M$ is coclosed, then $K \ll M \implies K \ll L$;
hence $rad(L) = L \cap rad(M)$.

(4) If L is hollow, then either L is closed in M or $L \ll M$.

(5) If $f : M \longrightarrow N$ is small epimorphism and L is coclosed in M, then $f(L)$ is coclosed in N.

(6) If K is coclosed in M, then K is coclosed in L. Converse is also true if L is coclosed in M.

Proposition 2.9 *Let M be a right R-module. If K is a coclosed and simple submodule of M, then K is a direct summand.*

Proof Let K be a coclosed and simple submodule of M. Then K is not small in M. Thus there is a maximal submodule L of M such that $L + K = M$. Since K is simple, we have $K \cap L = 0$. Then $M = L \oplus K$, that is, $K \subseteq^{\oplus} M$. □

Definition 2.14 *A monomorphism $f : M \longrightarrow N$ of right modules is called* ***pure-monomorphism*** *if the induced map*

$$f \otimes 1_L : M \otimes L \longrightarrow N \otimes L$$

is a monomorphism for each left module L.

Definition 2.15 *Let M be a right R-module. A submodule L of M is said to be* ***pure submodule*** *of M if the map*

$$i \otimes_R 1_N := L \otimes_R N \longrightarrow M \otimes_R N \text{ such that } i \otimes_R 1_N(a \otimes_R b) = a \otimes_R b$$

is a monomorphism for every left R-module N, where $i : L \longrightarrow M$ is the inclusion map and $1_N : N \longrightarrow N$ is the identity map. This means that, the inclusion map $i : L \longrightarrow M$ is a pure-monomorphism.

In particular, a subgroup A of an abelian group B is called **pure subgroup** of B if and only if

$$nA = A \cap nB$$

for each integer n (see (Fuchs, 1970)).

Lemma 2.4 *((Fuchs, 1970), Lemma 26.1) Let $K,\ L$ be submodules of M such that $K \subseteq L \subseteq M$. Then;*

(1) If K is pure in L and L is pure in M, then K is pure in M.

(2) If L is pure in M, then $\frac{L}{K}$ is pure in $\frac{M}{K}$.

(3) If K is pure in M and $\frac{L}{K}$ is pure in $\frac{M}{K}$, then L is pure in M.

Now, we give a characterization of pure submodule.

Definition 2.16 *A right module M is said to be **finitely presented** if $M \cong \frac{R^k}{L}$ for some finitely generated $L \subseteq R^k$.*

Definition 2.17 *A sequence of right R-modules, and homomorphisms f_i for all $i \in \mathbb{Z}^+$,*

$$\cdots \xrightarrow{f_{i-2}} A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} \cdots$$

*is called an **exact sequence** provided that $Im(f_i) = Ker(f_{i+1})$ for each positive integer $i \in \mathbb{Z}^+$.*

Proposition 2.10 *Let A be a submodule of a right R-module B. The following conditions are equivalent;*

(1) A is a pure submodule of B.

(2) $0 \longrightarrow A \otimes N \longrightarrow B \otimes N$ is monic for every left R-module N.

(3) For every finitely presented module N,

$$Hom(N, B) \longrightarrow Hom(N, \frac{B}{A}) \longrightarrow 0$$

is exact.

(4) Every system of m linear equations

$$\sum_{i=1}^{n} x_i r_{ij} = a_j, \ \ j = 1, \cdots m$$

with $r_{ij} \in R$ and $a_j \in A$ ($i = 1, \cdots, n$, $j = 1, \cdots, m$) which has a solution in B^n, also has a solution in A^n.

A sequence of right modules $0 \to A \to B \to C \to 0$ is called a **pure-exact** sequence if the sequence $0 \to A \otimes N \to B \otimes N \to C \otimes N \to 0$ is exact for each left R-module N.

Definition 2.18 *A right R-module M is called* ***flat module*** *if every short exact sequence of the form*

$$0 \longrightarrow A \xrightarrow{i} B \longrightarrow M \longrightarrow 0$$

is pure exact, that is $i(A)$ is a pure submodule of B, where $i : A \longrightarrow B$ is the inclusion map.

An abelian group G is called **bounded** if $nG = 0$ for some positive integer n.

Theorem 2.2 *((Fuchs, 1970), Theorem 27.5) A bounded pure subgroup is a direct summand.*

Lemma 2.5 *((Lam, 1999), Corollary 4.92) If A is a pure submodule of a right R-module B, then*

$$A \cap BI = AI$$

for any left ideal I of a ring R.

2.3.3. Torsion Submodule

Let G be an abelian group. The set

$$T(G) = \{g \in G \mid ng = 0 \; for \; some \; positive \; integer \; n\}$$

is a subgroup of G. $T(G)$ is called the **torsion subgroup** of G. We say that a group G is *torsion*, if $T(G) = G$. A group G is called *torsion free* if G has no nonzero element which has finite order, that is, $T(G) = 0$. A group G is said to be *mixed group* if G is neither torsion nor torsion free group.

Let Ω be the set of prime integers. An abelian group G is called *p-group* if, the orders of elements of G are powers of a fixed prime $p \in \Omega$. Let the subset $T_p(G)$ of G consist of all elements $g \in G$ whose orders are a power of the prime $p \in \Omega$; that is, $p^n g = 0$ for some $n \in \mathbb{Z}^+$, that is,

$$T_p(G) = \{g \in G \mid p^n g = 0 \; for \; some \; positive \; integer \; n\}.$$

$T_p(G)$ is a subgroup of G which is called the **p-primary component** of G.

Theorem 2.3 *((Fuchs, 1970), Theorem 8.4) A torsion group G is direct sum of p-groups $T_p(G)$ belonging to different primes p, that is, $G = \oplus_{p \in \Omega} T_p(G)$. The $T_p(G)$ are uniquely determined by G.*

In particular, $T(G) = \oplus_{p \in \Omega} T_p(G)$ for every group G.

Definition 2.19 *A commutative domain R is called **Prüfer** domain if each finitely generated ideal of R is projective.*

Proposition 2.11 *((Fuchs and Salce, 2000), Proposition 8.12) A domain R has the property that the torsion submodules of all mixed R-modules are pure if and only if R is Prüfer.*

Lemma 2.6 *Let G be a $\mathbb{Z}$-module and $T(G)$ the torsion submodule of G. Then $T(G)$ is a coclosed submodule of G.*

Proof Set $T = T(G)$. By Proposition 2.11, T is a pure submodule of G. In order to show that T is a coclosed submodule of G, suppose $\frac{T}{A}$ is small in $\frac{G}{A}$ for some proper submodule A of G, and let us obtain a contradiction. If $\frac{T}{A}$ has no maximal submodules, then $\frac{T}{A}$ is injective by Lemma 2.2 and Theorem 2.1. Being small and injective implies $\frac{T}{A} = 0$, that is, $T = A$, a contradiction. Now, suppose there is a maximal submodule L of T such that $A \subseteq L \subseteq T$. By ((Anderson and Fuller, 1992), Lemma 5.18), homomorphic images of small submodules are small, and hence $\frac{T}{L}$ is small in $\frac{G}{L}$. By Lemma 2.4 (2) pure submodules are closed under factor modules, so $\frac{T}{L}$ is pure in $\frac{G}{L}$. On the other hand, $\frac{T}{L}$ is simple, and so it is bounded. Then $\frac{T}{L}$ is a direct summand of $\frac{G}{L}$ by Theorem 2.2. Now, $\frac{T}{L}$ is both small and a direct summand in $\frac{G}{L}$, which is a contradiction. In conclusion $\frac{T}{A}$ is not small in $\frac{G}{A}$ for any proper subgroup $A \subseteq T$, that is, T is a coclosed subgroup of G. □

2.3.4. Singular and Nonsingular Modules and Submodules

Now, we recall the concepts of the singular and nonsingular modules and submodules.

Definition 2.20 *Let $\mathfrak{P}(R)$ be the set of all essential right ideals of the ring R. Given any R-module M, we set*

$$Z(M) = \{x \in M \mid xI = 0 \text{ for some } I \in \mathfrak{P}(R)\}.$$

Then $Z(M)$ is a submodule of M. $Z(M)$ is said to be the ***singular submodule****.*

Definition 2.21 *Let M be an R-module. If $Z(M) = M$, then M is called* ***singular****. If $Z(M) = 0$, then M is called* ***nonsingular****.*

Example 2.1 *Let R be a commutative domain and M be an R-module. Then the singular submodule $Z(M)$ of M is equal to the torsion submodule $T(M)$ of M.*

Definition 2.22 *Let $\mathcal{A}$ be any class of modules.*

(1) Let A be a submodule of an R-module M. M is called an ***essential extension*** *of A if, every nonzero submodule of M has nonzero intersection with A. We say that $\mathcal{A}$*

is closed under essential extension provided $B \in \mathcal{A}$ whenever $A \in \mathcal{A}$ and $A \trianglelefteq B$.

(2) *Given any short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of modules, the middle term B is called a ***module extension (or just extension)*** *of A by C. We say that $\mathcal{A}$ is closed under module extension provided $B \in \mathcal{A}$ whenever B is an extension of a module $A \in \mathcal{A}$ by $C \in \mathcal{A}$.*

Proposition 2.12 *((Goodearl, 1976), Proposition 1.22)*

(1) *The class of nonsingular right R-modules are closed under submodules, direct product, essential extensions and module extensions.*

(2) *The class of singular right R-modules are closed under submodules, factor modules and direct sum.*

Proposition 2.13 *((Goodearl, 1976), Proposition 1.24) A simple right R-module S is either singular or projective, but not both.*

Lemma 2.7 *((Goodearl, 1976), Corollary 1.25) Every nonsingular semisimple right R-module is projective.*

In particular, every nonsingular simple right R-module is projective.

2.3.5. Neat, Coneat and Absolutely Coneat Submodules

Now, we give the definitions of neat, coneat and absolutely coneat submodules. The classical notion of purity for abelian groups is generalized to the notion of neatness. Thus, for a subgroup A of an abelian group B is called **neat** if $pA = A \cap pB$, for every prime integer $p \in \Omega$ which is similar with pure subgroup of an abelian group.

Definition 2.23 *A submodule N of a right R-module is said to be* ***coneat*** *in M if*

$$Hom(M, S) \longrightarrow Hom(N, S) \longrightarrow 0$$

is epic for every simple right R-module S.

Definition 2.24 *A right module M is called **absolutely-coneat** if, M is coneat in every module containing it as a submodule.*

Proposition 2.14 *((Crivei, 2014), Theorem 3.2) The following are equivalent for an R-module M.*

(1) M is absolutely coneat.

(2) M is absolutely coneat submodule of an injective module.

(3) M is absolutely coneat submodule of an absolutely coneat module.

2.4. Noetherian and Artinian Rings and Modules

Definition 2.25 *(1) A module M is said to be **Noetherian** if, for every ascending chain*

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq ...$$

of submodules of M, there is an integer $n \in \mathbb{Z}^+$ such that $M_n = M_{n+k}$ for each positive integer k.

*(2) A module M is said to be **Artinian** if, for every descending chain*

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq ...$$

of submodules of M, there is an integer $n \in \mathbb{Z}^+$ such that $M_n = M_{n+k}$ for each positive integer k.

*(3) A right R-module M is called **semiartinian** if, every nonzero factor of M has a nonzero socle.*

Proposition 2.15 *((Anderson and Fuller, 1992), Corollary 10.11) Let M be a nonzero module.*

(1) If M is Artinian, then M has simple submodule; in fact, $soc(M)$ is an essential submodule.

(2) If M is Noetherian, then M has a maximal submodule; in fact, $rad(M)$ is a small submodule.

Definition 2.26 *(1) A ring R is said to be **right Noetherian** if all nonempty set of right ideals of R contains a maximal member.*

*(2) A ring R is said to be **right Artinian** if every nonempty set of right ideals of R contains a minimal member.*

*(3) A ring R is called **semiartinian** if every nonzero right R-module has a nonzero socle.*

Proposition 2.16 *((Sharpe and Vamos, 1972), Proposition 1.19) The following statements are equivalent:*

(1) R is right Noetherian (resp. right Artinian).

(2) Every finitely generated right R-module is right Noetherian (resp. right Artinian).

2.5. Some Special Rings

In this section, we recall the definitions of local, semilocal, perfect and H-rings. Also, we give some characterizations of these special rings.

2.5.1. Local and Semilocal Rings

Definition 2.27 *A ring R is called **local** if it has a unique maximal right ideal.*

Definition 2.28 *A ring R is called **semilocal** if $\frac{R}{J(R)}$ is a semisimple ring.*

Proposition 2.17 *((Anderson and Fuller, 1992), Proposition 15.15) For a ring R, the following statements are equivalent:*

(1) R is a local ring,

(2) R has a unique maximal left ideal,

(3) $J(R)$ is a maximal left ideal,

(4) The set of elements of R without left inverse is closed under addition,

(5) $J(R) = \{x \in R \mid Rx \neq R\}$,

(6) $\frac{R}{J(R)}$ is a division ring,

(7) $J(R) = \{x \in R \mid x$ is not invertible$\}$,

(8) If $x \in R$ then either x or $1 - x$ is invertible.

Lemma 2.8 *((Anderson and Fuller, 1992), Corollary 15.18) Let R be a ring with radical $J = J(R)$. Then for every left R-module M,*

$$JM \subseteq Rad(M).$$

If R is semisimple modulo its radical, then for every left R-module M,

$$JM = Rad(M)$$

and $\frac{M}{JM}$ is semisimple.

2.5.2. Perfect Rings

In this part, our aim is to remind the notion of right (left) perfect rings. This depends on a new notion called T-nilpotency.

Definition 2.29 *A subset A of a ring R is called* ***right (resp. left) T-nilpotent*** *if, for any sequence of elemets of $\{a_1, a_2, a_3, ...\} \subseteq A$, there exists an integer $n \in \mathbb{Z}^+$ such that $a_n...a_2a_1 = 0$ (resp. $a_1a_2...a_n = 0$).*

Definition 2.30 *A ring R is called* ***right (resp. left) perfect*** *if $\frac{R}{J(R)}$ is semisimple and $J(R)$ is right (resp. left) T-nilpotent.*

Theorem 2.4 *((Anderson and Fuller, 1992), Theorem 28.4) Let R be a ring with radical* $J = J(R)$. *Then the following statements are equivalent:*

(1) R is right (left) perfect,

(2) $\frac{R}{J}$ *is semisimple and every nonzero right (left) R-module contains a maximal submodule,*

(3) Every flat right (left) R-module is projective;

(4) R satisfies the minimum condition for principal left (right) ideals,

(5) R contains no infinite orthagonal set of idempotents and every nonzero left (right) R-module contains a simple submodule.

2.5.3. H-Rings

Now, we give the definition and characterization of H-rings.

Definition 2.31 *A ring R is called* ***right H-ring*** *if,* $\mathrm{Hom}_R(E(S_1), E(S_2)) = 0$ *for nonisomorphic simple right R-modules* S_1 *and* S_2.

Proposition 2.18 *((Sharpe and Vamos, 1972), Proposition 4.21) Let R be a Noetherian ring and* P_1, P_2 *are prime ideals of R. Then the following statements are equivalent:*

(1) $P_2 \subseteq P_1$,

(2) $\mathrm{Hom}_R(E(\frac{R}{P_2}), E(\frac{R}{P_1})) \neq 0$.

For a commutative ring R, it is known that every maximal ideal of R is prime. Hence commutative Noetherian rings are H-rings by Proposition 2.18.

CHAPTER 3

SIMPLE-DIRECT-PROJECTIVE MODULES

In this chapter, we give some closure properties of simple-direct-projective modules. Then we give a complete characterization of simple-direct-projective modules over the ring of integers and over semilocal rings.

Definition 3.1 *A right R-module M is called* ***simple-direct-projective*** *if, whenever A and B are submodules of M with B simple, and* $\frac{M}{A} \cong B \subseteq^{\oplus} M$*, then* $A \subseteq^{\oplus} M$*. This module is simple version of* $D3$ *module.*

Lemma 3.1 *Let M be a simple-direct-projective right module and L a coclosed submodule of* M*. If* $soc(M) \subseteq L$*, then L is simple-direct-projective.*

Proof Let L be a coclosed submodule of M. Suppose $\frac{L}{K} \cong S \subseteq^{\oplus} L$, where S is a simple submodule of L. Then S is a coclosed submodule of M as well by Proposition 2.8 (1). As S is a coclosed submodule of M, S is not small in M. Thus $S \subseteq^{\oplus} M$ by Proposition 2.9. Since L is a coclosed submodule of M, $\frac{L}{K}$ is a coclosed submodule of $\frac{M}{K}$ by Proposition 2.8 (1). Thus $\frac{L}{K}$ is not small in $\frac{M}{K}$, and so $\frac{L}{K} \oplus \frac{N}{K} = \frac{M}{K}$, for some submodule N of M. Clearly, $L \cap N = K$ and $\frac{M}{N} \cong S \subseteq^{\oplus} M$. Since M is simple-direct-projective, $M = N \oplus B$ for some simple submodule B of M. Using the fact that $soc(M) \subseteq L$ we get, by modular law, that $L = L \cap N \oplus B$, that is, $L \cap N = K \subseteq^{\oplus} L$. Hence L is simple-direct-projective. □

The followings are trivial examples of simple-direct-projective modules.

Example 3.1 *(1) The right R-modules with no simple summands are simple-direct-projective.*

(2) The right R-modules with no maximal submodules are simple-direct-projective.

(3) The right R-modules whose maximal submodules are direct summands are simple-direct-projective.

Lemma 3.2 *Let M be a right module. Suppose* $soc(M) \subseteq rad(M)$ *or* $\dfrac{M}{soc(M)}$ *has no maximal submodules. Then M is simple-direct-projective.*

Proof If $soc(M) \subseteq rad(M)$, then M has no simple summands and so it is simple-direct-projective. Now, assume that $\dfrac{M}{soc(M)}$ has no maximal submodules, and let K be a maximal submodule of M. Then $K + soc(M) = M$. Thus there is a simple submodule S of M such that $K + S = M$. By simplicity of S, $K \cap S = 0$, and so $K \subseteq^{\oplus} M$. Hence M is simple-direct-projective. □

Proposition 3.1 *((Ibrahim et al., 2016), Proposition 2.4) A direct summand of a simple-direct-projective module is again simple-direct-projective.*

Proof Let $K \subseteq^{\oplus} M$, L be a maximal submodule of K and $\frac{K}{L} \cong B \subseteq^{\oplus} K$ with B simple. Our aim is to show that $L \subseteq^{\oplus} K$. Since K is a direct summand, $M = K \oplus T$ for some submodule T of M. Using the modular law and the isomorphism theorem (3), we get

$$\frac{M}{L+T} = \frac{L+T+K}{L+T} \cong \frac{K}{(L+T)\cap K} = \frac{K}{L+(T\cap K)} = \frac{K}{L}.$$

Then $\frac{M}{L+T} \cong B$. Since $\frac{M}{L+T}$ is simple, $L \oplus T$ is a maximal submodule of M. On the other hand, $K = X \oplus B$ for some $X \subseteq K$. Then $M = X \oplus B \oplus T$, so $B \subseteq^{\oplus} M$. Since M is simple-direct-projective, $L \oplus T \subseteq^{\oplus} M$. Thus $M = L \oplus T \oplus S$ for some $S \subseteq M$. Then, we have

$$M = K \oplus T = L \oplus T \oplus S \implies K = L \oplus S.$$

Hence, $L \subseteq^{\oplus} K$, and so K is simple-direct-projective. □

3.1. Simple-Direct-Projective Abelian Group

Now, by using the results of the previous section, we are able to give a characterization of simple-direct-projective abelian groups. For torsion groups we have the following.

Proposition 3.2 *If M is a simple-direct-projective abelian group, then the torsion subgroup $T(M)$ of M is simple-direct-projective.*

Proof Let M be a simple-direct-projective abelian group. Since simple abelian groups are torsion, $soc(M) \subseteq T(M)$. Hence the proof is clear by Lemma 3.1 and Lemma 2.6. □

Proposition 3.3 *Let M be a torsion abelian group. The following statements are equivalent.*

(1) M is simple-direct-projective.

(2) $T_p(M)$ is simple-direct-projective for every $p \in \Omega$.

(3) For every $p \in \Omega$,

(i) *$soc(T_p(M)) \subseteq rad(T_p(M))$, or*

(ii) $\dfrac{T_p(M)}{soc(T_p(M))}$ *has no maximal subgroup.*

Proof

(1) $\Rightarrow$ (2) Since M is torsion, $M = \oplus_{p\in\Omega} T_p(M)$ by Theorem 2.3. Then, by Proposition 3.1, $T_p(M)$ is simple-direct-projective for every $p \in \Omega$.

(2) $\Rightarrow$ (3) Suppose that (i) does not hold. Then there is a simple subgroup S of $T_p(M)$ such that S is not contained in $rad(T_p(M))$. Thus S is not small in $T_p(M)$, and so, $S \subseteq^{\oplus} T_p(M)$. Note that, all simple subgroups and simple factors of $T_p(M)$ are isomorphic to S. Assume that A is a maximal subgroup of $T_p(M)$ such that

$$soc(T_p(M)) \subseteq A \subseteq T_p(M).$$

Therefore, $\frac{T_p(M)}{A} \cong S \subseteq^{\oplus} T_p(M)$. Then, as $T_p(M)$ is simple-direct-projective, $T_p(M) = A \oplus S'$ for some simple submodule S' of $T_p(M)$. Consequently,

$$S' \subseteq soc(T_p(M)) \subseteq A,$$

which is a contradiction. Hence $\dfrac{T_p(M)}{soc(T_p(M))}$ has no maximal subgroup, that is, (ii) holds.

(3) $\Rightarrow$ (2) It follows by Lemma 3.2.

(2) $\Rightarrow$ (1) Let A and B be subgroups of M with B simple and $\frac{M}{A} \cong B \subseteq^{\oplus} M$. As B is simple, there is a $p \in \Omega$ such that $B \subseteq T_p(M)$ and $pB = 0$. As $B \subseteq^{\oplus} M$, $B \subseteq^{\oplus} T_p(M)$. Since $pB = 0$ and $\frac{M}{A} \cong B$, we have $p(\frac{M}{A}) = 0$, that is, $pM \subseteq A$. For any prime $q \neq p$, it is easy to see that, $T_q(M) = pT_q(M) \subseteq pM$. Thus, for all primes $q \neq p$, $T_q(M) \subseteq pM \subseteq A$.

Since A is a maximal subgroup, $T_p(M)$ is not contained in A. Otherwise we would have $M = \oplus_{q\in\Omega}T_q(M) \subseteq A$, which is not the case as A is a maximal subgroup of M. Thus, by the maximality of A, we have $A + T_p(M) = M$. Then,

$$\frac{T_p(M)}{A \cap T_p(M)} \cong \frac{T_p(M) + A}{A} = \frac{M}{A} \cong B \subseteq^{\oplus} T_p(M).$$

Since $T_p(M)$ is simple-direct-projective, $A \cap T_p(M) \oplus C = T_p(M)$ for some simple subgroup C of $T_p(M)$. Then we get

$$M = A + T_p(M) = A + [A \cap T_p(M) \oplus C] = A \oplus C.$$

Hence M is simple-direct-projective. □

Theorem 3.1 *Let M be an abelian group. The following statements are equivalent.*

(1) M is simple-direct-projective.

(2) (i) $T(M)$ is simple-direct-projective, and

(ii) for each $p \in \Omega$ such that $pM + T(M) \neq M$, $soc(T_p(M)) \subseteq rad(T_p(M))$.

Proof (1) ⇒ (2) By Proposition 3.2, $T(M)$ is simple-direct-projective. Now, let $p \in \Omega$ be such that $pM + T(M) \neq M$. Then, as $\frac{M}{pM}$ is a homogoneous semisimple with each simple subgroup isomorphic to $\mathbb{Z}_p$ and

$$\frac{pM + T(M)}{pM} \neq \frac{M}{pM},$$

there is a maximal subgroup A of M such that $T(M) \subseteq pM + T(M) \subseteq A$ and $\frac{M}{A} \cong \mathbb{Z}_p$.

We need to show that $soc(T_p(M)) \subseteq rad(T_p(M))$. Suppose the contrary that $soc(T_p(M)) \nsubseteq rad(T_p(M))$. Then there is a simple subgroup S of $T_p(M)$ which is not contained in $rad(T_p(M))$. Then $S \subseteq^{\oplus} T_p(M)$, and since $T_p(M)$ is a direct summand of $T(M)$, $S \subseteq^{\oplus} T(M)$ as well. Then as S is a pure subgroup of $T(M)$ and $T(M)$ is pure subgroup of M, S is a pure subgroup of M. Thus S is a pure and bounded subgroup of M, and so S is a direct summand of M by Theorem 2.2. Since $S \cong \mathbb{Z}_p$ and $\frac{M}{A} \cong \mathbb{Z}_p \cong S \subseteq^{\oplus} M$, simple-direct-projectivity of M implies that $A \subseteq^{\oplus} M$, that is, $M = A \oplus D$ for some simple subgroup D of M. Then $D \subseteq T(M) \subseteq A$, which is a contradiction. Hence we must have $soc(T_p(M)) \subseteq rad(T_p(M))$, and this proves (2).

(2) $\Rightarrow$ (1) Let A and B be subgroups of M with B simple and $\frac{M}{A} \cong B \subseteq^{\oplus} M$. Since B is simple, $B \cong \mathbb{Z}_p$ for some $p \in \Omega$, in particular $B \subseteq soc(T_p(M))$ and $p(\frac{M}{A}) \cong pB = 0$, that is, $pM \subseteq A$. As $B \subseteq^{\oplus} M$, B is not contained in $rad(T_p(M))$. Thus $soc(T_p(M)) \nsubseteq rad(T_p(M))$. Then $pM + T(M) = M$ by (2). Thus $A + T(M) = M$. By similar arguments as in the proof of [Proposition 3.3, (2) $\Rightarrow$ (1)], we obtain that A is a direct summand of M. Hence M is simple-direct-projective. □

Corollary 3.1 *Let M be an abelian group. Suppose $\frac{M}{T(M)}$ has no maximal subgroups. Then M is simple-direct-projective if and only if every maximal subgroup of M is a direct summand.*

Proof Sufficiency is clear. To prove the necessity, let A be a maximal subgroup of M. Suppose $\frac{M}{A} \cong \mathbb{Z}_p$, where $p \in \Omega$. Then $pM \subseteq A$. Since $\frac{M}{T(M)}$ has no maximal subgroup and A is maximal, $A + T(M) = M$. Now, by the proof of [Proposition 3.3, (2) $\Rightarrow$ (1)], $A \subseteq^{\oplus} M$. This completes the proof. □

3.2. Simple-Direct-Projective Modules Over Semilocal Rings

Over local rings, simple-direct-projective modules are exactly the modules given in Lemma 3.2.

Proposition 3.4 *Let R be a local ring. A right module M is simple-direct-projective if and only if*

(i) $soc(M) \subseteq rad(M)$

(ii) $\dfrac{M}{soc(M)}$ *has no maximal submodules.*

Proof

Suppose (i) does not hold. Then there is a simple submodule S of M such that $M = N \oplus S$. Let K be a maximal submodule of M. Since R is a local ring, R has a unique simple module up to isomorphism. Thus $\frac{M}{K} \cong S \subseteq^{\oplus} M$. Hence simple-direct projectivity of M implies that $K \subseteq^{\oplus} M$. Thus any maximal submodule of M is a direct summand.

Now, if L is a maximal submodule of M, such that $soc(M) \subseteq L \subseteq M$, then $M = L \oplus S'$ with S' a simple submodule of M. Then $S' \subseteq soc(M) \subseteq L$, a contradiction. Hence $\frac{M}{soc(M)}$ has no maximal submodules. This proves the necessity.

Sufficiency is clear by Lemma 3.2. □

Over a right perfect ring, every module has a maximal submodule by Theorem 2.4. Hence the following is a consequence of Proposition 3.4.

Corollary 3.2 *Let R be a local right perfect ring. A right module M is simple-direct-projective if and only if M is semisimple or $soc(M) \subseteq rad(M)$.*

It is easy to see that every module M with $rad(M) = 0$ is simple-direct-injective (see (Ibrahim et al., 2016), Remark 4.5). The following is the corresponding result for simple-direct-projective modules. Note that, a finitely generated module M is semisimple if and only if every maximal submodule of M is a direct summand. Recall that a ring R is *semilocal* if $\frac{R}{J(R)}$ is semisimple Artinian.

Proposition 3.5 *The following statements are equivalent for a ring R.*

(1) R is semilocal.

(2) Every right R-module M with $rad(M) \subseteq^{\oplus} M$ is simple-direct-projective.

(3) Every right R-module with $rad(M) = 0$ is simple-direct-projective.

(4) Every 2-generated right R-module M with $rad(M) = 0$ is simple-direct-projective.

In particular, the conditions (2)-(4) are left-right symmetric.

Proof (1) ⇒ (2) Write $M = rad(M) \oplus N$ for some submodule N of M. Since R is semilocal, $\dfrac{M}{rad(M)}$ is semisimple and thus N is semisimple. Now, we claim that every maximal submodule of M is a direct summand of M. For, let A be a maximal submodule of M. Clearly, $N \nsubseteq A$ and so there exists a simple submodule K of N with $K \nsubseteq A$. Then $M = K + A$ and since $K \nsubseteq A$, $K \cap A = 0$. Therefore, $M = K \oplus A$ and $A \subseteq^{\oplus} M$, proving the claim. Inasmuch as every maximal submodule of M is a direct summand of M, we infer that M is simple-direct-projective.

(2) ⇒ (3) ⇒ (4) Clear.

(4) ⇒ (1) Let $\bar{R} := \frac{R}{J(R)}$. We show that every simple right $\bar{R}$-module K is projective. Now, viewing K as an R-module, there exists an epimorphism $f : \bar{R} \to K$. By the

hypothesis, the 2-generated right module $M_R := K \oplus \bar{R}$, as a right R-module, is simple-direct-projective and so f splits by ((Ibrahim et al., 2016), Proposition 2.1). Thus K is isomorphic to a summand of $\bar{R}$ and so K, as an $\bar{R}$-module, is projective. Hence $\bar{R} := \frac{R}{J(R)}$ is semisimple; that is, R is semilocal.

The last statement comes from the fact that being semilocal is left-right symmetric.

□

CHAPTER 4

SIMPLE-DIRECT-INJECTIVE MODULES

In this chapter, we give a characterization of simple-direct-injective modules over the ring of integers and over semilocal rings. Nonsingular right modules are simple-direct-projective over any ring ((Ibrahim et al., 2016), Example 2.5(2)). Motivated by this fact, we also obtain a characterization of the rings whose nonsingular right modules are simple-direct-injective.

Definition 4.1 *A right R-module M is called **simple-direct-injective** if, whenever A and B are simple submodules of M with $A \cong B \subseteq^{\oplus} M$, then $A \subseteq^{\oplus} M$, that is, every simple submodule of M isomorphic to a direct summand is itself a direct summand.*

Lemma 4.1 *Let K be a direct summand of an R-module M. If M is simple-direct-injective, then K is also simple-direct-injective.*

Proof Let A and B be simple submodules of K with $A \cong B \subseteq^{\oplus} K$. Then we get $K = B \oplus X$ for some $X \subseteq K$. On the other hand, $M = K \oplus T$ for some submodule T of M, since K is a direct summand of M. Then $M = B \oplus X \oplus T$,and so $B \subseteq^{\oplus} M$. By simple-direct-injectivity of M, A is a direct summand of M. Then A is also a direct summand of K, so K is simple-direct-injective. □

Definition 4.2 *Let A be an R-module. A is called **pure-injective** if, for all pure monomorphism $f : M \longrightarrow N$ of right modules, any homomorphism $g : M \longrightarrow A$ can be extended to a homomorphism $h : N \longrightarrow A$ such that $g = hf$.*

Lemma 4.2 *Let R be a ring and I an ideal of R. Then any pure-injective right $\frac{R}{I}$-module is pure-injective as an R-module.*

Proof Let M be a pure-injective right $\frac{R}{I}$-module. Let B be a right R-module, and A a pure submodule of B. Let $i : A \rightarrow B$ be the inclusion map. Then by Lemma 2.5 $AI = A \cap BI$. Thus the natural map

$$j : \frac{A}{AI} \longrightarrow \frac{B}{BI}$$

given by $j(a + AI) = a + BI$ is a pure monomorphism. In order to show that M is a pure-injective R-module, let $f : A \to M$ be an R-homomorphism. Then

$$f(AI) = f(A)I \subseteq MI = 0.$$

Thus $AI \subseteq Ker(f)$, and so $f = \overline{f}\pi$, where $\pi : A \to \frac{A}{AI}$ is the natural epimorphism, and $\overline{f} : \frac{A}{AI} \to M$ is the homomorphism induced by f, that is, $\overline{f}(a + AI) = f(a)$ for each $a \in A$. Since M is a pure-injective $\frac{R}{I}$-module, there is a homomorphism

$$g : \frac{B}{BI} \longrightarrow M$$

such that $\overline{f} = gj$. Let $\pi' : B \to \frac{B}{BI}$ be the natural epimorphism. For $\phi = g\pi'$, it is straightforward to check that, $\phi i = f$, that is, ϕ extends f, and so M is a pure-injective R-module. □

Lemma 4.3 *Let R be a commutative ring. Let M be an R-module and N a pure submodule of M. If M is simple-direct-injective, then N is simple-direct-injective. The converse is true if soc(M) $\subseteq$ N.*

Proof Suppose M is a simple-direct-injective module and N a pure submodule of M. Let $S_1 \cong S_2$ with S_1, S_2 simple submodules of N and $S_1 \subseteq^{\oplus} N$. Now, S_1 is pure in N, and N is pure in M. Then S_1 is pure in M by Lemma 2.4. Since R is commutative, simple modules are pure-injective by ((Cheatham and Smith, 1976), Corollary 4). Being pure and pure-injective implies $S_1 \subseteq^{\oplus} M$. Therefore $S_2 \subseteq^{\oplus} M$, because M is simple-direct-injective. Using the modular law, for some submodule X of M, we get

$$N = N \cap M = N \cap (S_2 \oplus X) = S_2 \oplus (N \cap X).$$

Hence $S_2 \subseteq^{\oplus} N$, and so N is simple-direct-injective.

Now, assume that N is a pure submodule of M, and $soc(M) \subseteq N$. Let $S_1 \cong S_2$ be two simple submodules of M and $S_1 \subseteq^{\oplus} M$. Then $S_1 \subseteq N$, $S_2 \subseteq N$ and $S_1 \subseteq^{\oplus} N$. Since N is simple-direct-injective, $S_2 \subseteq^{\oplus} N$. As S_2 is pure in N and N is pure in M, S_2 is pure in M. Then $S_2 \subseteq^{\oplus} M$, because S_2 is both pure-injective and pure in M. Hence M is simple-direct-injective. □

Definition 4.3 *A right R-module M is said to be* ***absolutely pure*** *if it is pure in all module containing it as a submodule.*

Definition 4.4 *An R-module E' is called an* ***injective envelope (or injective hull)*** *of an R-module M if, it is both an injective module and essential extension of M, and denoted by $E' = E(M)$.*

Corollary 4.1 *Let R be a commutative ring and M be an absolutely pure module. Then each module K such that $M \subseteq K \subseteq E(M)$ is simple-direct-injective.*

In particular, absolutely pure modules are simple-direct-injective.

Proof Since M is a pure submodule of $E(M)$ and $E(M)$ is simple-direct-injective, M is simple-direct-injective by Lemma 4.3. As M is essential in $E(M)$, $soc(M) = soc(K)$ for each module K such that $M \subseteq K \subseteq E(M)$. Hence K is simple-direct-injective, again by Lemma 4.3. □

Corollary 4.2 *Let R be a Prüfer domain. A module M is simple-direct-injective if and only if the torsion submodule $T(M)$ of M is simple-direct-injective.*

Proof Let M be an R-module. Then $T(M)$ is pure in M by Proposition 2.11. Since simple modules are torsion, $soc(M) \subseteq T(M)$. Now, the proof is clear by Lemma 4.3. □

Lemma 4.4 *Let M be an R-module and N a coclosed submodule of M. If M is simple-direct-injective, then N is simple-direct-injective. The converse is true if $soc(M) \subseteq N$.*

Proof Suppose M is simple-direct-injective and N is a coclosed submodule of M. Suppose $S_1 \cong S_2$ are simple submodules of N and $S_1 \subseteq^{\oplus} N$. Then S_1 is a coclosed submodule of M by Proposition 2.8 (6). Thus S_1 is not small in M, and so $S_1 \subseteq^{\oplus} M$. By simple-direct-injectivity of M, $S_2 \subseteq^{\oplus} M$. Therefore $S_2 \subseteq^{\oplus} N$, and N is simple-direct-injective.

Now, assume that N is a coclosed submodule of M, and $soc(M) \subseteq N$. Let $S_1 \cong S_2$ be two simple submodules of M and $S_1 \subseteq^{\oplus} M$. Then $S_1 \subseteq N$, $S_2 \subseteq N$ and $S_1 \subseteq^{\oplus} N$. Since N is simple-direct-injective, $S_2 \subseteq^{\oplus} N$. As S_2 is coclosed in N and N is coclosed in M, S_2 is coclosed in M. Then $S_2 \subseteq^{\oplus} M$, and so M is simple-direct-injective. □

Now, we will mention about the fully invariant submodule and the direct limit. Aforementioned notions will be used in the following Theorem 4.1 and Proposition 4.1.

Definition 4.5 *A submodule A of an R-module M is called a **fully invariant** in M, if $f(A) \subseteq A$ for each $f \in End_R(M)$.*

For an R-module M socle, radical and singular submodule of M are trivial examples of fully invariant submodule.

Lemma 4.5 *((Camillo et al., 2014), Lemma 2.4) Let M be a simple-direct-injective module. Then,*

(1) For any finite set $\{X_1, X_2, ..., X_k\}$ of simple summands of M, $\sum_{i=1}^{k} X_i \subseteq^{\oplus} M$.

(2) The sum of all simple summands of M is fully invariant in M.

To remind the direct limit, we mention about the direct system. A *direct system of morphisms* from $(M_i, f_{ij})_\sigma$ into an R-module L is a family of morphism

$$\{u_i : M_i \longrightarrow L\}_\sigma \textit{ with } f_{ij}u_j = u_i \textit{ whenever } i \leq j,$$

where $(\sigma, \leq)$ is a quasi ordered directed set and $f_{ij} : M_i \longrightarrow M_j$ for all (i, j) with $i \leq j$ is a family of morphism, satisfying $f_{ii} = id_{M_i}$, $f_{ij}f_{jk} = f_{ik}$ for $i \leq j \leq k$.

Definition 4.6 *Let $(M_i, f_{ij})_\sigma$ be a direct system of R-modules and M an R-module. A direct system of morphisms $\{f_i : M_i \longrightarrow M\}_\sigma$ is called **direct limit of** $(M_i, f_{ij})_\sigma$ if, for every direct system of morphisms $\{u_i : M_i \longrightarrow L\}_\sigma$, $L \in Mod - R$, there is a unique morphism $u : M \longrightarrow L$ which makes the following diagram commutative for every $i \in \sigma$*

$$\begin{tikzcd} M_i \arrow[r, "f_i"] \arrow[d, "u_i"'] & M \arrow[dl, "u"] \\ L & \end{tikzcd}$$

Now, we give a characterization of simple-direct-injective abelian groups.

Theorem 4.1 *Let M be an abelian group. The following statements are equivalent.*

(1) M is simple-direct-injective.

(2) $T(M)$ is simple-direct-injective.

(3) $T_p(M)$ is simple-direct-injective for each $p \in \Omega$.

(4) For each $p \in \Omega$,
$T_p(M)$ is semisimple, or $soc(T_p(M)) \subseteq rad(T_p(M))$.

Proof (1) $\Leftrightarrow$ (2) Torsion subgroup of M is simple-direct-injective by Corollary 4.2.

(2) $\Rightarrow$ (3) is clear, since $T(M) = \oplus_{p\in\Omega} T_p(M)$ and simple-direct-injective modules are closed under direct summands by Lemma 4.1.

(3) $\Rightarrow$ (4) Assume that $soc(T_p(M)) \nsubseteq rad(T_p(M))$ for some $p \in \Omega$. Then there is a simple subgroup S of $T_p(M)$ such that $S \subseteq^{\oplus} T_p(M)$. Let A be the sum of all simple summands of $T_p(M)$. Then any finitely generated subgroup of A is a direct summand (hence pure subgroup) of $T_p(M)$ by Lemma 4.5. Since A is a direct limit of its finitely generated subgroups and direct limit of pure subgroups is pure (see resp. (Wisbauer, 1991), 24.7, 33.8.), A is pure in $T_p(M)$. As A is semisimple and $A \subseteq T_p(M)$, $pA = 0$, that is, A is bounded. Then $A \subseteq^{\oplus} T_p(M)$ by Theorem 2.2. Let $T_p(M) = A \oplus B$. We claim that $B = 0$. For, if $B \neq 0$, then $soc(B) \neq 0$. Let U be a simple subgroup of B. Since $T_p(M)$ is a p-group, $soc(T_p(M))$ is homogeneous, that is, all simple subgroups of $T_p(M)$ are isomorphic. Thus $U \subseteq^{\oplus} T_p(M)$. Then $U \subseteq A$, which is a contradiction. Therefore $B = 0$, and so $T_p(M) = A$ is semisimple. This proves (4).

(4) $\Rightarrow$ (2) Let U and V be simple subgroups of $T(M)$ such that $U \cong V$ and $U \subseteq^{\oplus} T(M)$. Then there is a $p \in \Omega$ such that $U \subseteq^{\oplus} T_p(M)$. Thus $T_p(M)$ must be semisimple by (4). Since $V \cong U$, $V \subseteq^{\oplus} T_p(M)$. Hence $V \subseteq^{\oplus} T(M)$, and so $T(M)$ is simple-direct-injective. □

Proposition 4.1 *Let R be a semilocal ring. For a right R-module M, let S' be the sum of all simple direct summands of M. The following are equivalent.*

(1) M is simple-direct-injective.

(2) S' is fully invariant and pure submodule of M.

(3) $M = S' \oplus N$, and S' is a fully invariant submodule of M.

Proof (1) $\Rightarrow$ (2) By Lemma 4.5 (2), S' is a fully invariant submodule of M. Let $S' = \oplus_{i \in I} V_i$, where V_i are simple for each $i \in I$. Then for each finite subset $F \subseteq I$, $N_F = \oplus_{i \in F} V_i$ is a direct summand of M by Lemma 4.5 (1), and so N_F is a pure submodule of M. By ((Lam, 1999), 4.84 (c)) direct limit of pure submodules is pure, and so $S' = \oplus_{i \in I} V_i = \lim_F N_F$ is a pure submodule of M. This proves (2).

(2) $\Rightarrow$ (3) Since R is a semilocal ring, $\frac{R}{J(R)}$ is semisimple. Thus every right $\frac{R}{J(R)}$-module is pure-injective. As S' is semisimple, $S'.J(R) = 0$. Thus S' is a pure-injective right R-module by Lemma 4.2. Being pure and pure-injective implies that $S' \subseteq^{\oplus} M$.

(3) $\Rightarrow$ (1) Let A and B be two simple submodules of M such that $A \cong B$ and $A \subseteq^{\oplus} M$. Then $A \subseteq S'$. Since S' is a fully invariant submodule of M, $B \subseteq S'$ and so $B \subseteq^{\oplus} M$. Hence M is simple-direct-injective. □

Simple submodules of nonsingular modules are projective. Thus nonsingular right modules are simple-direct-projective over any ring (see (Ibrahim et al., 2017) 2.5 (2)). The corresponding result for simple-direct-injective modules follows.

Proposition 4.2 *Let R be a ring. The following statements are equivalent.*

(1) Every projective simple right module is injective.

(2) Every nonsingular right module is simple-direct-injective.

Proof (1) $\Rightarrow$ (2) Nonsingular simple right modules are projective, and so injective by (1). Thus (2) follows.

(2) $\Rightarrow$ (1) Let S be a projective simple right module. Then $E(S)$ and $S \oplus E(S)$ are nonsingular, and so $S \oplus E(S)$ is simple-direct-injective by (2). Since $S \oplus 0 \cong 0 \oplus S$ and $S \oplus 0 \subseteq^{\oplus} S \oplus E(S)$, $S \subseteq^{\oplus} E(S)$. Hence S is injective. □

Lemma 4.6 *((Ware, 1971), Lemma 2.6) Let R be a commutative ring and S a simple R-module. Then S is flat if and only if S is injective.*

Corollary 4.3 *Let R be a commutative ring. Then every nonsingular module is simple-direct-injective.*

Proof Let S be a projective simple module. Since S is projective, it is flat. Then S is injective by Lemma 4.6. Now, the conclusion follows by Proposition 4.2. □

Proposition 4.3 *Absolutely-coneat right modules are simple-direct-injective.*

Proof Let M be an absolutely-coneat right module. Suppose A and B are simple submodules of M with $A \cong B$ and $B \subseteq^{\oplus} M$. Then B is absolutely-coneat as a direct summand of M. Thus B is injective, and so A is injective too. Then $A \subseteq^{\oplus} M$, and hence M is simple-direct-injective. $\square$

We close this chapter by recalling a characterization of right V-rings by simple-direct-injective right modules that is proved in (Camillo et al., 2014). First we give the definition right V-rings and a well-known characterizations of these rings.

Definition 4.7 *A ring R is called **right V-ring** if every simple right R-module is injective.*

Theorem 4.2 *((Lam, 1999), Theorem 3.75) For a ring R, the following are equivalent:*

(1) R is a right V-ring.

(2) Any proper right ideal I of R is an intersection of maximal right ideals.

(3) For any right R-module M, $rad(M) = 0$.

Proposition 4.4 *((Camillo et al., 2014), Theorem 4.1) The following conditions are equivalent for a ring R:*

(1) R is a right V-ring.

(2) Every right R-module is simple-direct-injective.

(3) Every finitely cogenerated right R-module is simple-direct-injective.

(4) Direct sum of simple-direct-injective modules is simple-direct-injective.

(5) Every 2-generated right R-module is simple-direct-injective.

CHAPTER 5

WHEN SIMPLE-DIRECT-INJECTIVE (PROJECTIVE) MODULES ARE SIMPLE-DIRECT-PROJECTIVE (INJECTIVE)

In the last chapter, we prove that every simple-direct-injective right module is simple-direct-projective if and only if the ring is left perfect and right H-ring. As a consequence, we show that, commutative perfect rings are examples of such rings. We prove that the rings whose simple-direct-projective right modules are simple-direct-injective are right max-ring. For a commutative Noetherian ring, we prove that, simple-direct-projective modules are simple-direct-injective if and only if simple-direct-injective modules are simple-direct-projective if and only if the ring is Artinian.

Definition 5.1 *(1) A module M is said to be **uniserial**, if its lattice of submodules is linearly ordered by inclusion.*

*(2) A module M is said to be **serial**, if it can be written as a direct sum of uniserial modules.*

*(3) The ring R is called **right (left) uniserial (serial)** provided R has the corresponding properties as right (left) R-module.*

Now, we give a characterization of the rings over which every simple-direct-injective right module is simple-direct-projective. We begin with the following.

Proposition 5.1 *Let R be a ring. Suppose every simple-direct-injective right R-module is simple-direct-projective. Then R is semilocal and right semiartinian, that is, R is left perfect.*

Proof Every right module M with $rad(M) = 0$ is simple-direct-injective (see, (Ibrahim et al., 2016), Remark 4.5). Thus, by Proposition 3.5, R is semilocal. Suppose R is not right semiartinian. Then there is a nonzero finitely generated right module N with $soc(N) = 0$.

As the ring is semilocal, there are only finitely many, say $S_1, S_2, \cdots, S_n$ simple right modules up to isomorphism. Let

$$K = S_1 \oplus S_2 \oplus \cdots \oplus S_n \oplus N.$$

Then every simple submodule of K is a direct summand, and so K is simple-direct-injective. Let us show that K is not simple-direct-projective, and get a contradiction. Let L be a maximal submodule of N. Since $soc(N) = 0$, L is not a direct summand of N, and hence not a direct summand of K too. Let

$$L' = S_1 \oplus S_2 \oplus \cdots \oplus S_n \oplus L.$$

Then L' is a maximal submodule of K and $\frac{K}{L'} \cong S_i \subseteq^{\oplus} K$, for some $i = 1, \cdots, n$. As L is not a direct summand of K, L' is not a direct summand of K too. Thus K is not simple-direct-projective, which is a contradiction. Therefore R must be right semiartinian. Hence R is left perfect by Theorem 2.4. □

Theorem 5.1 *The following statements are equivalent for a ring R.*

(1) R is left perfect and right H-ring.

(2) Every simple-direct-injective right module is simple-direct-projective.

Proof (1) $\Rightarrow$ (2) Let M be a simple-direct-injective module. Let A be the sum of all simple summands of M. Then A is fully invariant and $M = A \oplus B$ by Proposition 4.1. Since A is a fully invariant submodule of M, $soc(B) \subseteq rad(M)$ and $\mathrm{Hom}(A, soc(B)) = 0$. By (1) the ring is right semiartinian, and so $soc(B)$ is an essential submodule of B. In order to prove that M is simple-direct-projective, suppose that $\frac{M}{K} \cong S \subseteq^{\oplus} M$ for some simple submodule S of M. Then as $S \subseteq^{\oplus} M$, $S \subseteq A$. We claim that, $A + K = M$. Suppose the contrary that, $A + K$ is properly contained in M, and let us find a contradiction. Then, by maximality of K, we have $A \subseteq K$. Thus from $M = A \oplus B$ and by modular law, we get

$K = A \oplus K \cap B$, and

$$\frac{M}{K} = \frac{A \oplus B}{K} = \frac{A \oplus B}{A \oplus K \cap B} \cong \frac{B}{K \cap B} \cong S.$$

Thus $K \cap B$ is a maximal submodule of B. Set $N := K \cap B$. Since the ring is semilocal, there are only finitely many simple right modules up to isomorphism. Thus

$$soc(B) = U_1^{(I_1)} \oplus U_2^{(I_2)} \oplus \cdots \oplus U_k^{(I_k)}$$

for some simple right modules $U_1, U_2, \cdots, U_k$ and index sets $I_1, I_2, \cdots, I_k$. Since $soc(B)$ is an essential submodule of B, the injective hull of B is $E(B) = \oplus_{i=1}^k E(U_i^{(I_i)})$. As $\frac{B}{N} \cong S$, there is an epimorphism $f : B \to S$. Let $e : S \to E(S)$ be the inclusion homomorphism. Then the homomorphism ef extends to a (nonzero) homomorphism $g : E(B) \to E(S)$. Since $E(B) = \oplus_{i=1}^k E(U_i^{(I_i)})$ and g is nonzero, there is a nonzero homomorphism $h : E(U_j^{(I_j)}) \to E(S)$, for some $j \in \{1, 2, \cdots, k\}$. It is clear that, $E(U_j^{(I_j)})$ can be embedded in $E(U_j)^{I_j}$. Thus, as h is nonzero, there is a nonzero homomorphism from $E(U_j)^{I_j}$ to $E(S)$. This leads to a nonzero homomorphism $t : E(U_j) \to E(S)$. So that, by the right H-ring assumption, we must have $S \cong U_j$. Then $\mathrm{Hom}(A, soc(B)) \neq 0$, which is a contradiction. Hence the case $A + K = M$ must hold. Therefore, as A is semisimple, there is a simple submodule U of A such that $U + K = M$ and $U \cap K = 0$, that is, $K \subseteq^{\oplus} M$. Hence M is simple-direct-projective. This proves (2).

(2) $\Rightarrow$ (1) The ring R is left perfect by Proposition 5.1. Suppose R is not right H-ring. Then there are nonisomorphic simple right modules S_1 and S_2 such that

$$\mathrm{Hom}(E(S_1), E(S_2)) \neq 0.$$

Let $0 \neq f : E(S_1) \to E(S_2)$, and $A = Ker(f)$. Since $\frac{E(S_1)}{A} \cong f(E(S_1)) \subseteq E(S_2)$, there is a submodule $B \subseteq E(S_1)$ such that $\frac{B}{A} \cong S_2$. Then it is clear that $B \oplus S_2$ is a simple-direct-injective right module. On the other hand,

$$\frac{B \oplus S_2}{A \oplus S_2} \cong 0 \oplus S_2 \subseteq^{\oplus} B \oplus S_2.$$

But $A \oplus S_2$ is not a direct summand of $B \oplus S_2$. Thus $B \oplus S_2$ is not simple-direct-projective. This contradicts with (2). Thus R must be right H-ring. □

Now, we give some consequences of Theorem 5.1.

Corollary 5.1 *Let R be a commutative ring. The following statements are equivalent.*

(1) R is a perfect ring.

(2) Every simple-direct-injective module is simple-direct-projective.

Proof Commutative perfect rings are semiartinian by Theorem 2.4. Thus commutative perfect rings are H-ring by ((Camillo, 1978), Proposition 2). Now, the proof is clear by Theorem 5.1. □

A right Noetherian right semiartian ring is right Artinian (see, (Shock, 1974)). Left perfect rings are right semiartinian by Theorem 2.4. Thus the following is clear by Theorem 5.1.

Corollary 5.2 *Let R be a right Noetherian ring. The following statements are equivalent.*

(1) R is right Artinian right H-ring.

(2) Every simple-direct-injective right module is simple-direct-projective.

Since commutative Noetherian rings are H-rings, we obtain the following corollary.

Corollary 5.3 *Let R be a commutative Noetherian ring. The following statements are equivalent.*

(1) R is Artinian ring.

(2) Every simple-direct-injective module is simple-direct-projective.

By Proposition 4.4, R is right V-ring if and only if every right R-module is simple-direct-injective. Right V-rings are right max-rings (see, Theorem 4.2). Clearly, over right V-rings simple-direct-projective right modules are simple-direct-injective.

Now, we consider the rings whose simple-direct-projective right modules are simple-direct-injective.

Definition 5.2 *A ring R is said to be a* ***right max-ring*** *if every nonzero right R-module has a maximal submodule. In particular, $rad(M) \ll M$ for every right R-module M.*

Proposition 5.2 *Let R be a ring. If each simple-direct-projective right R-module is simple-direct-injective, then R is a right max-ring.*

Proof Suppose the ring is not right max-ring. Then there is a nonzero right module M such that $M = rad(M)$. Let $0 \neq m \in M$, and let K be a maximal submodule of mR. Let

$$h = i\pi : mR \longrightarrow E(\frac{mR}{K}),$$

where $\pi : mR \longrightarrow \frac{mR}{K}$ and $i : \frac{mR}{K} \longrightarrow E(\frac{mR}{K})$ are the natural epimorphism and the inclusion homomorphism, respectively. By injectivity of $E(\frac{mR}{K})$, there is a (nonzero) homomorphism $g : M \to E(\frac{mR}{K})$ which extends h. Let $L := g(M)$. Since $\frac{M}{Ker(g)} \cong L$ and $rad(M) = M$, $L = rad(L)$. Note that L has an essential socle isomorphic to $\frac{mR}{K}$. Consider the right module $N = \frac{mR}{K} \oplus L$. Then $0 \oplus L$ is the unique maximal submodule of N and $0 \oplus L \subseteq^{\oplus} N$. Thus N is simple-direct-projective. On the other hand,

$$0 \oplus soc(L) \cong \frac{mR}{K} \oplus 0 \subseteq^{\oplus} N,$$

but $0 \oplus soc(L)$ is not a direct summand of N. Therefore N is not simple-direct-injective. This contradicts with our assumption that simple-direct-projective modules are simple-direct-injective. Hence R must be right max-ring. □

A subfactor of a right module M is a submodule of some factor module of M. The following lemma can be easily derived from the definition of H-ring. We include it for an easy reference.

Lemma 5.1 *R is a right H-ring if and only if for every simple right R-module S, every simple subfactor of $E(S)$ is isomorphic to S.*

Proof Suppose R is a right H-ring and S a simple right R-module. Let $\frac{A}{B}$ be a simple subfactor of $E(S)$. Assume that $\frac{A}{B}$ is not isomorphic to S. Let $i_1 : \frac{A}{B} \longrightarrow \frac{E(S)}{B}$ and $i_2 : \frac{A}{B} \longrightarrow E(\frac{A}{B})$ be the corresponding inclusions. Then there is a nonzero homomorphism $f : \frac{E(S)}{B} \longrightarrow E(\frac{A}{B})$. Thus, $f\pi : E(S) \longrightarrow E(\frac{A}{B})$ is a nonzero homomorphism, where $\pi : E(S) \longrightarrow \frac{E(S)}{B}$ is the canonical epimorphism. This contradicts with the assumption that R is right H-ring. Therefore every simple subfactor of $E(S)$ is isomorphic to S. This proves the necessity.

Conversely, let S_1 and S_2 be simple right R-modules, $0 \neq f \in \mathrm{Hom}_R(E(S_1), E(S_2))$. Then $\frac{E(S_1)}{Ker(f)}$ has a simple subfactor isomorphic to S_2. Thus, by our assumption, we must have $S_1 \cong S_2$. Hence R is a right H-ring. □

Proposition 5.3 *Let R be a commutative Noetherian ring. The following statements are equivalent.*

(1) R is Artinian.

(2) Every simple-direct-projective module is simple-direct-injective.

Proof (2) $\Rightarrow$ (1) By Proposition 5.2, R is a max-ring. Commutative Noetherian max-rings are Artinian by ((Hamsher, 1966), Theorem 1).

(1) $\Rightarrow$ (2) Let M be a simple-direct-projective R-module. Let S' be the sum of simple summands of M. Then, by the same arguments in the proof of [Proposition 4.1, (2) $\Rightarrow$ (3)], S' is a pure and a pure-injective submodule of M, and so $S' \subseteq^{\oplus} M$. Let $M = S' \oplus N$. Clearly, by the construction of S', N has no simple (or maximal) submodule which is a direct summand. Now, in order to prove that M is simple-direct-injective, by Proposition 4.1, it is enough to see that S' is a fully invariant submodule of M. Suppose the contrary that there are simple submodules A, B of M such that $A \subseteq S'$, $B \subseteq N$ and $A \cong B$. Since $B \subseteq N$, there is a nonzero homomorphism $g : N \longrightarrow E(B)$. Then for $K = Ker(g)$, the module $\frac{N}{K}$ has a maximal submodule say $\frac{L}{K}$ by the Artinianity of R. Since R is commutative and Noetherian, R is an H-ring. Thus every simple subfactor of $E(B)$ is isomorphic to B by Lemma 5.1. Therefore $\frac{N}{L} \cong B$. Now,

$$\frac{M}{S' \oplus L} = \frac{S' \oplus N}{S' \oplus L} \cong B \cong A \subseteq^{\oplus} M.$$

Then by simple-direct-projectivity of M, $S' \oplus L \subseteq^{\oplus} M$ and, by modular law, $L \subseteq^{\oplus} N$. This contradicts the fact that, N has no maximal summands. Hence S' is a fully invariant submodule of M, and so M is simple-direct-injective by Proposition 4.1. This proves (2).

□

Proposition 5.4 *Let R be a commutative semilocal ring. The following statements are equivalent.*

(1) R is perfect.

(2) Every simple-direct-projective module is simple-direct-injective.

Proof (2) ⇒ (1) R is a max-ring by Proposition 5.2. Semilocal max-rings are perfect by Theorem 2.4 (3).

(1) ⇒ (2) Note that, commutative perfect rings are H-rings and max-rings. Now, replacing Artinian by perfect the same proof of (Proposition 5.3 (1) ⇒ (2)) holds. □

Remark 5.1 *Over a right V-ring all right modules, in particular, simple-direct-projective right modules are simple-direct-injective (see, ((Camillo et al., 2014), Theorem 4.1)). Since commutative perfect V-rings are semisimple, there is a simple-direct-injective R-module which is not simple-direct-projective over nonsemisimple commutative V-rings by Corollary 5.1. Therefore nonsemisimple commutative V-rings are examples of rings such that simple-direct-projective modules are simple-direct-injective, and admit a simple-direct-injective module that is not simple-direct-projective.*

Summing up, Corollary 5.1, Corollary 5.3, Proposition 5.3 and Proposition 5.4 we obtain the following.

Corollary 5.4 *Let R be a commutative Noetherian ring. Then the following statements are equivalent.*

(1) R is Artinian.

(2) Every simple-direct-injective module is simple-direct-projective.

(3) Every simple-direct-projective module is simple-direct-injective.

Corollary 5.5 *Let R be a commutative semilocal ring. Then the following statements are equivalent.*

(1) R is perfect.

(2) Every simple-direct-injective module is simple-direct-projective.

(3) Every simple-direct-projective module is simple-direct-injective.

CHAPTER 6

CONCLUSION

Simple-direct-injective and simple-direct-projective modules are investigated and studied in (Camillo et al., 2014), (Ibrahim et al., 2016) and (Ibrahim et al., 2017). In this thesis, these modules are studied further and some open problems about these modules are addressed. The structure of simple-direct-projective and simple-direct-injective modules are completely characterized in Theorem 3.1 and Theorem 4.1, respectively. We prove that the rings whose simple-direct-injective right modules are simple-direct-projective are exactly the left perfect right H-rings in Theorem 5.1. We also consider the rings over which simple-direct-projective right modules are simple-direct-injective. These rings are right max-rings (see, Proposition 5.2). For a commutative Noetherian ring, we prove that simple-direct-projective modules are simple-direct-injective if and only if simple-direct-injective modules are simple-direct-projective if and only if the ring is Artinian. The results obtained in the thesis are published in (Büyükaşık et al., 2020).

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