Proc. Indian Acad. Sci. (Math. Sci.) (2020) 130:11
https://doi.org/10.1007/s12044-019-0538-0

The group of invertible ideals of a Prüfer ring

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MS received 18 March 2019; revised 17 June 2019; accepted 24 July 2019;
published online 9 January 2020

Abstract. Let R be a commutative ring and $\mathcal{I}(R)$ denote the multiplicative group of all invertible fractional ideals of R, ordered by $A \leqslant B$ if and only if $B \subseteq A$. We investigate when there is an order homomorphism from $\mathcal{I}(R)$ into the cardinal direct sum $\coprod_{i\in I} G_i$, where G_i's are value groups, if R is a Marot Prüfer ring of finite character. Furthermore, over Prüfer rings with zero-divisors, we investigate the conditions that make this monomorphism *onto*.

Keywords. Prüfer ring; invertible ideals; H-local ring; Marot ring; weakly additively regular ring.

2010 Mathematics Subject Classification. 13A15, 13A18, 13F05, 13F30, 13F99, 06F20.

1. Introduction

Let R be a commutative ring with identity. We call an element of R *regular* if it is not a zero-divisor. Let $\mathrm{Reg}(R)$ denote the monoid of regular elements of R and $Q(R) = Q$ denote the total ring of fractions of R. We note that $Q = (\mathrm{Reg}(R))^{-1}R$. We say that an ideal I of R is *regular* if I contains a regular element of R. We call X a fractional ideal of R if X is an R-submodule of Q such that $rX \subseteq R$ for some regular element $r \in R$. Let $\mathcal{F}(R)$ be the semigroup of all fractional regular ideals of R under the usual ideal multiplication. The set of all invertible fractional ideals of R is a subgroup of $\mathcal{F}(R)$; this group is denoted by $\mathcal{I}(R)$. The group $\mathcal{I}(R)$ is partially ordered under the order $A \leqslant B$ if and only if $B \subseteq A$. The principal fractional regular ideals form a subgroup $\beta(R)$ in $\mathcal{I}(R)$. We note that every invertible fractional ideal of R is finitely generated and regular.

Let S be a multiplicatively closed subset of R. We set $R_{(S)} = \{q \in Q : qs \in R \text{ for some regular element } s \in S\}$ and $R_{[S]} = \{q \in Q : qs \in R \text{ for some element } s \in S\}$. It is clear that $R_{(S)} \subseteq R_{[S]} \subseteq Q$. Let I be an ideal of R. Then $I_{(S)} = IR_{(S)} = \{q \in R_{(S)} : qs \in I \text{ for some regular element } s \in S\}$ and $I_{[S]} = \{q \in R_{[S]} : qs \in I \text{ for some element } s \in S\}$. In the case $S = R - P$ for some prime ideal P of R, we use $R_{(P)}$ and $R_{[P]}$ in place of $R_{(S)}$ and $R_{[S]}$.

We recall that a valuation is a map ν from Q onto a totally ordered abelian group G and a symbol ∞, such that for all x and y in Q:

(1) $\nu(xy) = \nu(x) + \nu(y)$.
(2) $\nu(x + y) \geqslant \min\{\nu(x), \nu(y)\}$.
(3) $\nu(1) = 0$ and $\nu(0) = \infty$.

The ring $R_\nu = \{x \in Q | \nu(x) \geqslant 0\}$, together with the ideal $P_\nu = \{x \in Q | \nu(x) > 0\}$, denoted by (R_ν, P_ν) is called a *valuation pair* (*of* K). R_ν is called a valuation ring (of K), and G is called *the value group of* R_ν. We note that given a valuation pair (R, P), $R = R_{[P]}$.

A ring R is called a *Marot ring* if every regular ideal can be generated by a set of regular elements. Moreover, every overring of a Marot ring is Marot. Below we see some characterizations of a Marot valuation ring (see [5] for details).

PROPOSITION 1.1 [4, Proposition 4.1]

Let R be a Marot ring. Assume that $R \neq Q$. Then the following conditions are equivalent:

(1) *R is a valuation ring.*
(2) *For each regular element $x \in Q$, either $x \in R$ or $x^{-1} \in R$.*
(3) *R has only one regular maximal ideal and each of its finitely generated regular ideals is principal.*

Lemma 1.2 [1, *Lemma* 1.3]. *Let R be a Marot ring. Assume that $R \neq Q$. Then R is a valuation ring if and only if the set of R-submodules M of Q such that $M \cap \mathrm{Reg}(Q) \neq \emptyset$ is totally ordered by inclusion.*

Let R be a commutative ring. R is said to be *additively regular* if for each $z \in Q$, there exists a $u \in R$ such that $z + u$ is a regular element in Q, or, equivalently, for each $a \in R$ and each regular element $b \in R$, there exists a $u \in R$ such that $a + ub$ is regular in R. R is said to be a *weakly additively regular* if for $x, y \in R$ with $x \in \mathrm{Reg}(R)$, there is a pair of elements $s, t \in R$ such that $ys + xt \in \mathrm{Reg}(R)$ and $sR + xR = R$. Weakly additively regular rings sit properly between additively regular rings and Marot rings (see [7] for details).

We recall that a regular prime ideal P is called a *minimal regular prime ideal* of R if $Q \subset P$ and Q is a prime ideal, imply that Q consists of zero-divisors. We say that a ring R is of *finite character* if every regular element of R is contained in at most finitely many maximal ideals of R. Furthermore, if R is of finite character, and if every regular prime ideal of R is contained in at most one maximal ideal, then R is called an *h-local ring*.

If R is a Dedekind domain, then all nonzero fractional ideals of R are invertible, and the class group $\mathcal{I}(R)/\beta(R)$ is a measure of unique factorization of elements of R. If the class group is trivial, then R is a unique factorization domain, and hence a principal ideal domain. If R is a Dedekind domain with maximal ideals $\{M_i\}_{i\in I}$, then for a nonzero fractional ideal A, we have $A = M_1^{e_{i_1}} \dots M_n^{e_{i_n}}$, and the mapping $A \to (e_{i_1}, \dots, e_{i_n})$ is an order isomorphism from $\mathcal{I}(R)$ onto the cardinal sum $\coprod_{i\in I} \mathbb{Z}_i$, where $\mathbb{Z}_i \cong \mathbb{Z}$ for each i.

Let R be a Prüfer domain of finite character. In [2], the Brewer and Klingler showed that there is an order monomorphism from $\mathcal{I}(R)$ *into* the cardinal direct sum $\coprod_{i\in I} G_i$, where each G_i is a value group, and it is *onto* if R is an h-local Prüfer domain. We recall that a commutative ring R is called a Prüfer ring if every finitely generated regular ideal of R is invertible. In this paper, we aim to generalize their results for weakly additively regular Prüfer rings.

In section 2, it is shown that, over a certain Marot Prüfer ring R of finite character, $\mathcal{I}(R)$ maps isomorphically *into* the cardinal direct sum $\coprod_{i\in I} G_i$, where each G_i is a value group. Futhermore, we give some results on weakly additively regular rings. In section 3, for weakly additively regular Prüfer rings with zero divisors, we prove the 'very strong approximation theorem' for finitely generated regular ideals. Furthermore, given a weakly additively regular Prüfer ring R, we show that R is h-local if and only if there is a monomorphism from $\mathcal{I}(R)$ *onto* the cardinal sum $\coprod_{i\in I} G_i$, where each G_i is a value group.

2. Embedding $\mathcal{I}(R)$ into $\coprod_{i\in I} G_i$

Let R be a Marot ring of finite character and $\{M_i\}_{i\in I}$ the set of regular maximal ideals of R. In this section, we prove that $\mathcal{I}(R)$ maps isomorphically *into* the cardinal direct sum $\coprod_{i\in I} G_i$, where each G_i is a value group. Futhermore, we generalize [1, Proposition 2.4] for weakly additively regular rings. This helps us to prove that, given a weakly additively regular ring with only finitely many regular maximal ideals, it is a Prüfer ring if and only if it is a regular Bezout ring (a regular Bezout ring is a ring for which each finitely generated regular ideal is principal). These tools will be helpful to develop some *approximation* theorems for Prüfer rings in the next section.

PROPOSITION 2.1 [1, Proposition 2.2]

Let R be a Marot ring with $\{V_i\}_{i\in I}$ a collection of valuation overrings of R such that $R=\bigcap_{i\in I} V_i$. Denote by v_i the valuation associated with V_i, and by G_i the corresponding value group. Let $A=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ be an invertible fractional ideal of R. Then the mapping

$$\Phi:\mathcal{I}(R)\to\prod_{i\in I} G_i$$

defined by

$$\Phi(A)=(v_i(A))_{i\in I}=(\min\{v_i(\alpha_j)\}_{1\leqslant j\leqslant n})_{i\in I}$$

is an order-preserving monomorphism from $\mathcal{I}(R)$ into $\prod_{i\in I} G_i$.

Theorem 2.2. *Let R be a Marot Prüfer ring of finite character and $\{P_i\}_{i\in I}$ the set of regular maximal ideals of R. Denote by v_i the valuation associated with $R_{(P_i)}$, and by G_i the corresponding value group. Let Φ be the mapping defined in Proposition 2.1. Then*

(1) *The mapping Φ is an order-preserving monomorphism from $\mathcal{I}(R)$ into $\prod_{i\in I} G_i$, the cardinal product of the G_is.*
(2) *Φ maps $\mathcal{I}(R)$ into $\coprod_{i\in I} G_i$, the cardinal direct sum of the G_is.*

Proof.

(1) Since $R = \bigcap R_{(M)}$, over all regular maximal ideals M of R [5, Theorem 6.1], Φ is an order-preserving monomorphism from $\mathcal{I}(R)$ into $\prod_{i\in I} G_i$ by Proposition 2.1.
(2) Since R is of finite character, each of its regular elements is contained in at most finitely many P_i's. If A is an invertible fractional ideal, it is finitely generated and regular, and hence $\Phi(A)$ is finitely nonzero, that is, $\Phi(A) \in \coprod_{i\in I} G_i$. So, Φ maps $\mathcal{I}(R)$ into $\coprod_{i\in I} G_i$.

□

Next, our aim is to prove some generalizations of the results from [7] and [1]. We first give a generalization of [7, Theorem 2.1], and then we generalize [1, Proposition 2.4] which helps us to generalize [7, Theorem 3.4(2)].

PROPOSITION 2.3

Let R be a weakly additively regular ring, I a regular ideal and $P_1, \ldots, P_n$ a collection of regular prime ideals such that $I \nsubseteq P_i$ for any i. Then $\mathrm{Reg}(I) \nsubseteq \bigcup_{i=1}^n P_i$. *Furthermore, if $S = R - \bigcup_{i=1}^n P_i$, then $I_{(S)} = R_{(S)}$.*

Proof. We have that $I \nsubseteq P_i$ for any i. Then there is $x \in I - \bigcup_{i=1}^n P_i$. Since product of regular elements is regular, there exists a regular element $z \in I \cap P_1 \cap P_2 \cap \cdots \cap P_n$. Thus, there exist $u, v \in R$ such that $y = vx + uz$ is regular in R and $vR + zR = R$. Suppose $y \in \bigcup_{i=1}^n P_i$. Then $vx \in \bigcup_{i=1}^n P_i$ so that $vx \in P_i$ for some i. Since P_i is prime, either $v \in P_i$ or $x \in P_i$. But $z \in P_i$ for all i and $vR + zR = R$ so that $v \notin P_i$, and hence $x \in P_i$, which is a contradiction. Thus, y is a regular element of I such that $y \notin \bigcup_{i=1}^n P_i$. Furthermore, y is a unit in S, and hence $I_{(S)} = R_{(S)}$.

Theorem 2.4. *Let R be a weakly additively regular ring and $P_1, \ldots, P_n$ a collection of regular prime ideals of R. Let $S = R - \bigcup_{i=1}^n P_i$. The only regular ideals of R that survive in $R_{(S)}$ are contained in at least one P_i.*

Proof. Suppose I is a regular ideal of R such that $R \nsubseteq P_i$ for all i. Then $I \nsubseteq \bigcup_{i=1}^n P_i$. By Proposition 2.3, there is a regular element $y \in I$ which is contained in $R_{(S)}$ so that $yR_{(S)} = R_{(S)} = I_{(S)}$. So, the only regular ideals of R that survive in $R_{(S)}$ are contained in at least one P_i. □

Theorem 2.5 [7, Corollary 3.2]. *Let R be a weakly additively regular ring. If R has only finitely many regular maximal ideals, then each invertible ideal is principal.*

Proof. This immediately follows from Theorem 2.4.

We recall that a regular Bezout ring is a ring for which each finitely generated regular ideal is principal.

COROLLARY 2.6 [6, Corollary 3.6]

If R is a weakly additively regular ring with only finitely many regular maximal ideals, then it is a Prüfer ring if and only if it is a regular Bezout ring.

3. Approximation theorems for Prüfer rings

Let R be a weakly additively regular Prüfer ring. In this section, we study the 'approximation' and 'strong approximation' theorems for R.

First, we need the following definition. Two valuation rings V and W with the same total ring of fractions Q are said to be *independent* if and only if V and W generate Q if and only if there does not exist a valuation ring $U \subsetneq Q$ such that $V \subseteq U$ and $W \subseteq U$ (since any overring of a Marot valuation ring is a valuation ring [5, Corollary 7.8]).

We recall that if R is a Prüfer ring with zero divisors then $(R_{[M]}, [M]R_{[M]})$ is a valuation pair for each maximal ideal M of R. If, in addition, R is a Marot ring, then $R = \bigcap R_{(M_i)}$, where M_i's are all maximal ideals of R [5, Theorem 6.1] (see [5] for details).

Lemma 3.1. *Let R be a weakly additively Prüfer ring with $\{(M_i)\}_{i\in I}$ a collection of valuation overrings of R, where M_i's are regular maximal ideals of R. Let v_i be the valuation associated with $R_{(M_i)}$ and G_i the corresponding value group. Furthermore, suppose that each nonzero regular prime ideal of R is contained in at most one M_i. Then for every finite collection $M_1, M_2, \ldots, M_n$, with corresponding valuations $v_1, \ldots, v_n$, and given some nonnegative value g of v_1, there is a regular element $r \in R$ such that $v_1(r) > g$ and $v_i(r) = 0$, $2 \leqslant i \leqslant n$.*

Proof. Let $b \in Q$ such that $v_1(b) = g$. Since R is a Marot ring, each $R_{(M_i)}$ is a valuation ring by [5, Theorems 7.6 and 7.7], so [5, Theorem 7.9] implies that the regular elements of $R_{(M_i)}$ for each i, map onto the positive elements of the corresponding value group. Thus, we can take $b \in \mathrm{Reg}(Q)$, so let $b = \frac{s}{l}$, where $s, l \in \mathrm{Reg}(R)$. Hence, $v_1(s) \geqslant v_1(b) = g$.

Suppose that there is a minimal regular prime ideal L in M_1. Since $L \nsubseteq M_i$, for any $i \neq 1$, by Proposition 2.3, there is a regular element $c \in L$ such that $c \in L - \bigcup_{i=2}^{n} M_i$. Since $R_{(L)}$ is a rank one valuation ring, it follows from [3, Proposition II.2.1] that $R_{(L)}$ has an Archimedean value group. Suppose w is the valuation corresponding to L. Since $w(c) > 0$, there exists $z \in \mathbb{Z}$ such that $w(c^z) = z \cdot w(c) > w(s)$. Thus, $c^z R_{(M_1)} \subsetneq sR_{(M_1)}$ implies that $v_1(c^z) > v_1(s) \geqslant v_1(b) = g$. Since $c \notin M_i$ for each i such that $2 \leqslant i \leqslant n$, $c^z \notin M_i$. Hence, $v_i(c^z) = z \cdot v_i(c) = 0$ for $2 \leqslant i \leqslant n$. So, we can take $r = c^z$.

Suppose that there is no minimal regular prime ideal contained in M_1. Consider the regular prime ideals of $R_{(M_1)}$. Let I be their intersection. Note that they are totally ordered by inclusion, by Lemma 1.2, and hence I is a prime ideal. If I were a regular ideal, then I would become a minimal regular prime ideal. Thus, $s \notin I$, and hence there must be a prime regular ideal contained in M_1, say L, such that $s \notin L$. As in the first case, $L \nsubseteq \bigcup_{i=2}^{n} M_i$, and so, by Proposition 2.3, we can choose a regular element d of L such that $d \in L - \bigcup_{i=2}^{n} M_i$. Therefore, we have $v_1(d) > g$ and $v_i(d) = 0$ for $i \neq 1$.

Lemma 3.2. *Let R be a weakly additively regular Prüfer ring with $\{M_i\}_{i\in I}$ the collection of all regular maximal ideals of R. Denote by v_i the valuation associated with the valuation ring $R_{(M_i)}$ for each i and by G_i the associated value group. Let Φ be the mapping defined in Proposition 2.1. Then R is of finite character if and only if Φ maps $\mathcal{I}(R)$ into $\coprod_{i\in I} G_i$, the cardinal direct sum of the G_is.*

Proof. Since R is a Marot ring, by Theorem 2.2, one way is clear. Suppose that Φ maps $\mathcal{I}(R)$ into $\coprod_{i\in I} G_i$. Let A be a finitely generated regular ideal of R. Since R is a Prüfer ring, A is invertible. Then $\Phi(A) \in \coprod_{i\in I} G_i$. Hence, $\Phi(A)$ finitely nonzero implies that

I is contained in only finitely many regular maximal ideals of R. Thus, R is of finite character. □

Theorem 3.3. *Let R be a weakly additively regular Prüfer ring with $\{M_i\}$ the collection of all maximal ideals of R. Let v_i be the valuation associated with the valuation ring $R_{(M_i)}$ for each i and G_i the associated value group. Then the following are equivalent,*

1. *The valuation rings $\{R_{(M_i)}\}_{i\in I}$ are pairwise independent.*
2. *Each nonzero regular prime ideal of R is contained in a unique maximal ideal of R.*
3. *The 'strong approximation theorem' holds for regular elements in R; that is, for every finite collection of regular maximal ideals $M_1, \ldots, M_n$ of R and every choice of nonnegative elements $g_i \in G_i$, there is a regular element $d \in R$ such that $v_i(d) = g_i$ for $1 \leqslant i \leqslant n$.*

Proof.

(1) $\Rightarrow$ (2). Let P be a regular prime ideal of R such that $P \subseteq M_1, M_2$, where M_1 and M_2 are distinct regular maximal ideals. Then $R_{(M_1)} \subseteq R_{(P)}$ and $R_{(M_2)} \subseteq R_{(P)}$ so that $R_{(M_1)}$ and $R_{(M_2)}$ cannot be pairwise independent.
(2) $\Rightarrow$ (3). Let $g_1, g_2, \ldots, g_n$ be nonnegative elements of $G_1, G_2, \ldots, G_n$ respectively. By Lemma 3.1, we can choose, for each i, a regular element $r_i \in R$ such that $v_i(r_i) > g_i$ and $v_j(r_i) = 0$ for all $j \neq i$. Let $t_1, t_2, \ldots, t_n \in R$ be such that $v_i(t_i) = g_i$, $1 \leq i \leq n$, and set

$$s_i = t_i(r_1 \times \cdots \times r_{i-1} \times r_{i+1} \times \cdots \times r_n).$$

Then for $1 \leq i \leq n$, we get

$$v_i(s_i) = v_i(t_i) + v_j(r_1) + \cdots + v_i(r_{i-1}) + v_i(r_{i+1}) + \cdots + v_i(r_n) = g_i,$$

since $v_j(r_j) = 0$ for $j \neq i$. On the other hand, for $j \neq i$, we have that

$$\begin{aligned} v_j(s_i) &= v_j(t_i) + v_j(r_1) + \cdots + v_j(r_{i-1}) + v_j(r_{i-1}) + \cdots + v_j(r_n) \\ &= v_j(t_i) + v_j(r_j) \geq v_j(r_j) > g_j. \end{aligned}$$

Let $r = r_1 \times \cdots \times r_n$. Then $r \in \mathrm{Reg}(R)$. Set $s = s_1 + \cdots + s_n$ and $s' = xs + yr \in \mathrm{Reg}(R)$ for some $x, y \in R$ such that $xR + rR = R$. Since $xR + rR = R$, $v_i(x) = 0$ for each i, and also $v_i(y) \geq 0$ implies that $v_i(y) + v_i(r_1) + v_i(r_2) + \cdots + v_i(r_i) + \cdots + v_i(r_n) > g_i$. Thus, $v_i(s') = \min\{v_i(x) + v_i(s), v_i(y) + v_i(r_1) + v_i(r_2) + \cdots + v_i(r_n)\} = g_i$, for $1 \leqslant i \leqslant n$.
(3) $\Rightarrow$ (1). Let M_1 and M_2 be distinct regular maximal ideals of R. Let γ be a regular element of Q. If $v_1(\gamma) \geqslant 0$ or $v_2(\gamma) \geqslant 0$, then $\gamma \in R_{(M_1)}$ or $\gamma \in R_{(M_2)}$. Suppose that $v_1(\gamma) < 0$ and $v_2(\gamma) < 0$. Furthermore, there exists $r \in \mathrm{Reg}(R)$ such that $v_1(r) = -v_1(\gamma)$ and $v_2(r) = 0$. So, we write $\gamma = (\gamma r)r^{-1}$. Since $v_1(\gamma r) = v_1(\gamma) + v_1(r) = 0$, $\gamma r \in R_{(M_1)}$. Also, $v_2(r^{-1}) = -v_2(r) = 0$ implies that $r^{-1} \in R_{(M_2)}$. □

We say that the 'very strong approximation theorem' holds for finitely generated regular ideals of R if, for every finite collection of regular maximal ideals $M_1, \ldots, M_n$ of R, and every choice of nonnegative elements $g_i \in G_i$, there is a finitely generated regular ideal A

of R such that $v_i(A) = g_i$ for $1 \le i \le n$, and $v_j(A) = 0$ for all other M_j. In [2], it is shown that, given a Prüfer domain R, R is h-local if and only if the 'very strong approximation theorem' holds for nonzero finitely generated ideals of R. Here we prove, given a weakly additively regular Prüfer ring R, the 'very strong approximation theorem' holds for finitely generated regular ideals of R if and olnly if R is h-local.

Theorem 3.4. *Let R be a weakly additively regular Prüfer ring with $\{M_i\}$ the collection of all regular maximal ideals of R. Denote by v_i the valuation associated with the valuation ring $R_{(M_i)}$ for each i and by G_i the associated value group. The 'very strong approximation theorem' holds for finitely generated regular ideals of R if and only if R is h-local.*

Proof. Suppose that the 'very strong approximation theorem' holds for finitely generated regular ideals of R. Let P be a regular prime ideal of R. If P is maximal, then we are done. Suppose that $P \subset M_1, M_2$, where M_1 and M_2 are maximal ideals of R. Since R is a Marot ring, we can choose regular elements $a_1 \in M_1 - P$ and $a_2 \in M_2 - P$. Let b be a regular element of P. We note that $v_1(a_1) < v_1(b)$ and $v_2(a_2) < v_2(b)$. Suppose that there exists a finitely generated regular ideal A of R such that $v_1(A) = v_1(b)$ and $v_2(A) = v_2(a_2)$. Then $A_{(M_1)} = bR_{(M_1)} \subseteq P_{(M_1)}$. So, $A \subseteq A_{(M_1)} \cap R \subseteq P_{(P)} \cap R = P$, and hence $A_{(M_2)} \subseteq P_{(M_2)} \subsetneq a_2R_{(M_2)}$. Thus, $v_2(A) > v_2(a_2)$, which is a contradiction. Therefore, P is contained in a unique maximal ideal of R. Since weakly additively regular rings are Marot, to complete the proof, by [8, Proposition 5.4] and [9, Theorem 5.1.21], it is enough the show that for each regular prime P of R, there exists a finitely generated regular ideal contained in P which is contained in a unique maximal ideal. Let M_1 be the unique maximal ideal containing P and $0 \neq b$ be a regular element in P. By assumption, there exists a finitely generated regular ideal A of R such that $v_1(A) = v_1(b)$ and $v_j(A) = 0$ for all other maximal ideals M_j of R, implying that M_1 is the unique maximal ideal of R containing A. Since $A_{(M_1)} = bR_{(M_1)} \subseteq P_{(M_1)}$, we have $A \subseteq A_{(M_1)} \cap R \subseteq P_{(P)} \cap R = P$.

Suppose R is h-local. Then each nonzero regular prime ideal of R is contained in a unique maximal ideal. So, by the 'strong approximation theorem', there exists $r \in \mathrm{Reg}(R)$ such that $v_i(r) = g_i$ for $1 \leqslant i \leqslant n$. Since R is of finite character, there are at most finitely many other maximal ideals $N_1, \ldots, N_t$ with corresponding valuations $w_1, \ldots, w_t$ at which r is positive. By Theorem 3.3 again, there is $s \in \mathrm{Reg}(R)$ such that $v_i(s) = g_i$ for $1 \leqslant i \leqslant n$, and $w_j(s) = 0$ for $1 \leqslant j \leqslant t$. Then for the regular ideal $A = (r, s)$, $v_i(A) = g_i$ for $1 \le i \le n$, and $v_j(A) = 0$ for all other M_j. □

COROLLARY 3.5

Let R be a weakly additively regular Prüfer ring with only finitely many regular maximal ideals. If each nonzero regular prime ideal of R is contained in a unique maximal ideal of R, then the 'very strong approximation theorem' holds for regular elements of R; that is, for every finite collection of regular maximal ideals $M_1, M_2, \ldots, M_n$ of R, and every choice of nonnegative elements $g_i \in G_i$, there is a regular element $a \in R$ such that $v_i(a) = g_i$ for $1 \leqslant i \leqslant n$, and $v_j(a) = 0$ for all other regular maximal ideals M_j of R.

Proof. By Theorem 3.4, there exists a finitely generated regular ideal A such that $v_i(A) = g_i$ for $1 \le i \le n$, and $v_j(A) = 0$ for all other M_j. By Corollary 2.6, R is a regular Bezout ring, and hence A is principal. Therefore, the 'very strong approximation theorem' holds for regular elements of R. □

COROLLARY 3.6

Let R be a weakly additively regular Prüfer ring with $\{M_i\}$ the collection of all regular maximal ideals of R. Let Φ be the mapping defined in Proposition 2.1. Then R is h-local if and only if Φ is an isomorphism from the group $\mathcal{I}(R)$ of invertible fractional regular ideals of R onto the cardinal direct sum $\coprod_{i\in I} G_i$, where G_i's are the value groups.

Proof. R is of finite character if and only if the mapping Φ is an isomorphism from the group $\mathcal{I}(R)$ of invertible fractional ideals of R into the cardinal direct sum $\coprod_{i\in I} G_i$, by Lemma 3.2. So, R is h-local if and only if Φ maps $\mathcal{I}(R)$ onto $\coprod_{i\in I} G_i$, by Theorem 3.4. □

COROLLARY 3.7

Let R be a weakly additively regular Prüfer ring with only finitely many regular maximal ideals $\{M_i\}$. Let Φ be the mapping defined in Proposition 2.1. Then R is h-local if and only if Φ is an isomorphism from the group $\mathcal{B}(R)$ of principal fractional regular ideals of R onto the cardinal direct sum $\coprod_{i\in I} G_i$, where G_i's are the value groups.

Proof. By Corollary 2.6, R is a regular Bezout ring so that $\mathcal{I}(R) = \mathcal{B}(R)$. So, the corollary now follows from Corollary 3.6. □

Acknowledgements

The author would like to thank the anonymous referee for his/her careful reading and valuable suggestions.

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Communicating Editor: B Sury