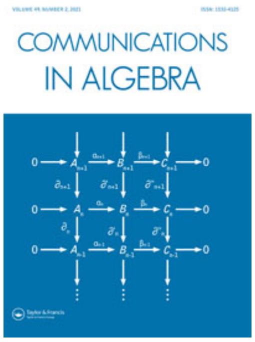

Taylor & Francis
Taylor & Francis Group

Communications in Algebra

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/lagb20

On simple-direct modules

Engin Büyükaşık, Özlem Demir & Müge Diril

To cite this article: Engin Büyükaşık, Özlem Demir & Müge Diril (2021) On simple-direct modules, Communications in Algebra, 49:2, 864-876, DOI: 10.1080/00927872.2020.1821207

To link to this article: https://doi.org/10.1080/00927872.2020.1821207

Published online: 23 Nov 2020.

COMMUNICATIONS IN ALGEBRA®
2021, VOL. 49, NO. 2, 864–876
https://doi.org/10.1080/00927872.2020.1821207

On simple-direct modules

Engin Büyükaşık, Özlem Demir, and Müge Diril

Department of Mathematics, Izmir Institute of Technology, Urla, İzmir, Turkey

ABSTRACT

Recently, in a series of papers "simple" versions of direct-injective and direct-projective modules have been investigated. These modules are termed as "simple-direct-injective" and "simple-direct-projective," respectively. In this paper, we give a complete characterization of the aforementioned modules over the ring of integers and over semilocal rings. The ring is semilocal if and only if every right module with zero Jacobson radical is simple-direct-projective. The rings whose simple-direct-injective right modules are simple-direct-projective are fully characterized. These are exactly the left perfect right *H*-rings. The rings whose simple-direct-projective right modules are simple-direct-injective are right max-rings. For a commutative Noetherian ring, we prove that simple-direct-projective modules are simple-direct-injective if and only if simple-direct-injective modules are simple-direct-projective if and only if the ring is Artinian. Various closure properties and some classes of modules that are simple-direct-injective (resp. projective) are given.

ARTICLE HISTORY
Received 25 March 2020
Revised 2 September 2020
Communicated by Alberto Facchini

KEYWORDS
Simple-direct-projective; Artinian rings; *H*-rings; perfect rings; simple-direct-injective; torsion submodule

2010 MATHEMATICS SUBJECT CLASSIFICATION
16D50; 16D60; 18G25

1. Introduction

In Ref. [16], a right module is called *direct-injective* if every submodule isomorphic to a direct summand is a direct summand. Direct-injective modules are also known as $C2$-modules. A right module is a $C3$-module if the sum of any two direct summands with zero intersection is again a direct summand. These modules and several generalizations are studied extensively in the literature. Recently, the "simple" version of $C2$-modules and $C3$-modules are studied in [4]. Namely, a right module is called *simple-direct-injective* if every simple submodule isomorphic to direct summand is itself a direct summand, or equivalently if the sum of any two simple direct summands with zero intersection is again a direct summand (see [4]).

Dual to direct-injective modules, a right module M is called *direct-projective, or a D2-module* if, for every submodule $A \subseteq M$ with $\frac{M}{A}$ isomorphic to a direct summand of M, then A is a direct summand of M (see [16]). In Refs. [11, 12] the authors investigate and study a dual notion of simple-direct-injective modules. A right module M is called *simple-direct-projective* if, whenever A and B are submodules of M with B simple and $\frac{M}{A} \cong B \subseteq^{\oplus} M$, then $A \subseteq^{\oplus} M$. Some well-known classes of rings and modules are characterized in terms of simple-direct-injective and simple-direct-projective modules (see [4, 11, 12]).

In this paper, we characterize simple-direct-injective and simple-direct-projective modules over the ring of integers and over semilocal rings. We show that, the ring is semilocal iff every right module with zero Jacobson radical is simple-direct-projective. We prove that the rings whose simple-direct-injective right modules are simple-direct-projective are exactly the left perfect right

CONTACT Engin Büyükaşık enginbuyukasik@iyte.edu.tr Izmir Institute of Technology, Department of Mathematics, 35430 Urla, İzmir, Turkey.

H-rings. We show that, the rings whose simple-direct-projective modules are simple-direct-injective are right max-rings. For a commutative Noetherian ring, we prove that, simple-direct-projective modules are simple-direct-injective iff simple-direct-injective modules are simple-direct-projective iff the ring is Artinian.

The paper is organized as follows.

In Section 2, we characterize simple-direct-projective abelian groups (Theorem 1). As a byproduct, a characterization of simple-direct-projective modules over local and local perfect rings is obtained. We prove that the ring is semilocal if and only if every right module with zero Jacobson radical is simple-direct-projective.

In Section 3, a complete characterization of simple-direct-injective abelian groups is given (Theorem 2). Motivated by the fact that nonsingular right modules are simple-direct-projective over any ring, we prove the corresponding result for simple-direct-injective modules. We show that, nonsingular right modules are simple-direct-injective iff projective simple right modules are injective. We also give a characterization of simple-direct-injective modules over semilocal rings. We show that simple-direct-injective modules are closed under coclosed submodules over any ring, and closed under pure submodules provided the ring is commutative. Partial converses of these results are given.

Following [18, sec. 4.4], we say R is a *right H-ring* if for nonisomorphic simple right R-modules S_1 and S_2, $\mathrm{Hom}_R(E(S_1), E(S_2)) = 0$. Commutative Noetherian rings, and commutative semiartinian rings are H-ring by Sharpe and Vamos [18, Proposition 4.21] and Camillo [3, Proposition 2], respectively. Right Artinian rings that are right H-rings are characterized in [17, Theorem 9]. Some classes of noncommutative H-rings are also studied in [10]. A ring R is called *right max-ring* if every nonzero right R-module has a maximal submodule.

In Ref. [4, Theorem 3.4.], the authors characterize the rings over which simple-direct-injective right modules are $C3$-modules. They prove that these rings are exactly the Artinian serial rings with $J^2(R) = 0$. In Ref. [11, Theorem 4.9.], the authors prove that every simple-direct-injective right R-module is $D3$-module iff every simple-direct-projective right R-module is $C3$-module iff R is uniserial with $J^2(R) = 0$.

At this point, it is natural to consider the rings whose simple-direct-injective modules are simple-direct-projective, and the rings whose simple-direct-projective modules are simple-direct-injective. Right $C3$-modules and right $D3$-modules are simple-direct-injective and simple-direct-projective respectively. Thus, uniserial rings with $J^2(R) = 0$ are examples of such rings.

In section 4, we prove that, every simple-direct-injective right module is simple-direct-projective iff the ring is left perfect right H-ring (Theorem 3). As a consequence, we show that, commutative perfect rings are examples of such rings. For a commutative Noetherian ring, we obtain that, simple-direct-injective modules are simple-direct-projective iff the ring is Artinian (Corollary 9). We show that, the rings whose simple-direct-projective right modules are simple-direct-injective are right max-rings (Proposition 8). For a commutative Noetherian ring, we prove that, simple-direct-projective modules are simple-direct-injective iff simple-direct-injective modules are simple-direct-projective iff the ring is Artinian (Corollary 10).

Throughout, rings are associative with unity and modules are unitary. For a module M, we denote by $rad(M), soc(M)$, $Z(M)$ and $E(M)$ the Jacobson radical, the socle, the singular submodule and the injective hull of M, respectively. The Jacobson radical of a ring R will be denoted by $J(R)$. We write $L \subseteq M$ if N is a submodule of M, and $L \subseteq^{\oplus} M$ if L is a direct summand of M. For a module M over a commutative domain R, we denote the torsion submodule of M by $T(M)$. Over the ring of integers, we denote by Ω the set of prime integers. It is well known that $T(M) = \oplus_{p\in\Omega} T_p(M)$, where $T_p(M)$ is the *p-primary component* of $T(M)$, i.e., the set of all $m \in T(M)$ such that $p^n.m = 0$ for some positive integer n. An abelian group G is *bounded* if $nG = 0$ for some positive integer n. For $p \in \Omega$, the simple $\mathbb{Z}$-module of order p will be denoted by $\mathbb{Z}_p$.

A monomorphism $f : M \to N$ of right modules is called *pure-monomorphism* if the induced map $f \otimes 1_L : M \otimes L \to N \otimes L$ is a monomorphism for each left module L. Let B be a right module and $A \subseteq B$. A is called a pure submodule of B if the inclusion map $i : A \to B$ is a pure-monomorphism. A subgroup A of an abelian group B is pure iff $nA = A \cap nB$ for each integer n (see [8]). A right module E is called *pure-injective* if, for every pure-monomorphism $f : M \to N$ of right modules, any homomorphism $g : M \to E$ can be extended to a homomorphism $h : N \to E$ (see, [9]).

2. Simple-direct-projective modules

In this section, we give a complete characterization of simple-direct-projective modules over the ring of integers. As a byproduct, we obtain a characterization of simple-direct-projective modules over local, and local right perfect rings. We also prove that, the ring is semilocal iff every right module with zero Jacobson radical is simple-direct-projective.

Following Ref. [11], a right R-module M is called *simple-direct-projective* if, whenever A and B are submodules of M with B simple and $\frac{M}{A} \cong B \subseteq^{\oplus} M$, then $A \subseteq^{\oplus} M$.

A submodule K of a module M is *small*, denoted as $K \ll M$, if $K + N = M$ implies $N = M$ for each $N \subseteq M$. A submodule L of a module M is *coclosed* in M if $\frac{L}{K} \ll \frac{M}{K}$ implies $K = L$ for each submodule K of L (see [5]).

Let S be a simple submodule of a module M. It is easy to see that, $S \ll M$ or $S \subseteq^{\oplus} M$. Thus any simple coclosed submodule is a direct summand. This fact will be used in the sequel.

In order to characterize simple-direct-projective abelian groups, we need several lemmas. We begin with the following.

Lemma 1. *Let M be a simple-direct-projective right module and L a coclosed submodule of M. If $soc(M) \subseteq L$, then L is simple-direct-projective.*

Proof. Let L be a coclosed submodule of M. Suppose $\frac{L}{K} \cong S \subseteq^{\oplus} L$, where S is a simple submodule of L. Then S is a coclosed submodule of M as well by Clark et al. [5, 3.7.(1)]. As S is a coclosed submodule of M, S is not small in M. Thus, $S \subseteq^{\oplus} M$. Since L is a coclosed submodule of M, $\frac{L}{K}$ is a coclosed submodule of $\frac{M}{K}$ by Clark et al. [5, 3.7.(1)]. Thus $\frac{L}{K}$ is not small in $\frac{M}{K}$, and so $\frac{L}{K} \oplus \frac{N}{K} = \frac{M}{K}$, for some submodule N of M. Clearly, $L \cap N = K$ and $\frac{M}{N} \cong S \subseteq^{\oplus} M$. Since M is simple-direct-projective, $M = N \oplus B$ for some simple submodule B of M. Using the fact that $soc(M) \subseteq L$ we get, by modular law, that $L = L \cap N \oplus B$, i.e., $L \cap N = K \subseteq^{\oplus} L$. Hence L is simple-direct-projective. □

Lemma 2. *Let G be an abelian group and T(G) the torsion submodule of G. Then T(G) is a coclosed submodule of G.*

Proof. Set $T = T(G)$. By Fuchs and Salce [9, Proposition 8.12], T is a pure submodule of G. In order to show that T is a coclosed submodule of G, suppose $\frac{T}{A}$ is small in $\frac{G}{A}$ for some proper submodule A of G, and let us obtain a contradiction. If $\frac{T}{A}$ has no maximal submodules, then $\frac{T}{A}$ is injective by Fuchs [8, pg. 99 Ex.1 and Theorem 21.1]. Being small and injective implies $\frac{T}{A} = 0$, i.e., $T = A$, a contradiction. Now, suppose there is a maximal submodule L of T such that $A \subseteq L \subseteq T$. By Anderson and Fuller [1, Lemma 5.18] homomorphic images of small submodules are small, and hence $\frac{T}{L}$ is small in $\frac{G}{L}$. By Fuchs [8, Lemma 26.1(ii)] pure subgroups are closed under factor modules, so $\frac{T}{L}$ is pure in $\frac{G}{L}$. On the other hand, $\frac{T}{L}$ is simple, and so it is bounded. Then $\frac{T}{L}$ is a direct summand of $\frac{G}{L}$ by Fuchs [8, Theorem 27.5]. Now, $\frac{T}{L}$ is both small and a direct summand in $\frac{G}{L}$, which is a contradiction. In conclusion $\frac{T}{A}$ is not small in $\frac{G}{A}$ for any proper subgroup $A \subseteq T$, that is, T is a coclosed subgroup of G. □

Corollary 1. *If M is a simple-direct-projective abelian group, then the torsion submodule $T(M)$ of M is simple-direct-projective.*

Proof. Let M be a simple-direct-projective abelian group. Since simple abelian groups are torsion, $soc(M) \subseteq T(M)$. Hence the proof is clear by Lemmas 1 and 2. □

The right modules with no simple summands, and the right modules whose maximal submodules are direct summands are trivial examples of simple-direct-projective modules. We include the following lemma for easy reference.

Lemma 3. *Let M be a right module. Suppose $soc(M) \subseteq rad(M)$ or $\frac{M}{soc(M)}$ has no maximal submodules. Then M is simple-direct-projective.*

Proof. If $soc(M) \subseteq rad(M)$, then M has no simple summands and so it is simple-direct-projective. Now, assume that $\frac{M}{soc(M)}$ has no maximal submodules, and let K be a maximal submodule of M. Then $K + soc(M) = M$. Thus, there is a simple submodule S of M such that $K+S=M$. By simplicity of S, $K \cap S = 0$, and so $K \subseteq^{\oplus} M$. Hence M is simple-direct-projective. □

First, we give a characterization of simple-direct-projective torsion abelian groups.

Proposition 1. *Let M be a torsion abelian group. The following statements are equivalent.*

1. *M is simple-direct-projective.*
2. *$T_p(M)$ is simple-direct-projective for every $p \in \Omega$.*
3. *For every $p \in \Omega$,*
   i. *$soc(T_p(M)) \subseteq rad(T_p(M))$, or*
   ii. *$\frac{T_p(M)}{soc(T_p(M))}$ has no maximal submodules.*

Proof. (1) ⇒ (2) Since M is torsion, $M = \oplus_{p\in\Omega} T_p(M)$. Then, by [11, Proposition 2.4], $T_p(M)$ is simple-direct-projective for every $p \in \Omega$.

(2) ⇒ (3) Suppose (i) does not hold. Then there is a simple subgroup S of $T_p(M)$ such that S is not contained in $rad(T_p(M))$. Thus S is not small in $T_p(M)$, and so $S \subseteq^{\oplus} T_p(M)$. Note that, all simple subgroups and simple factors of $T_p(M)$ are isomorphic to S. Assume that A is a maximal submodule of $T_p(M)$ such that $soc(T_p(M)) \subseteq A \subseteq T_p(M)$. Therefore, $\frac{T_p(M)}{A} \cong S \subseteq^{\oplus} T_p(M)$. Then, as $T_p(M)$ is simple-direct-projective, $T_p(M) = A \oplus S'$ for some simple submodule S' of $T_p(M)$. Consequently, $S' \subseteq soc(T_p(M)) \subseteq A$, which is a contradiction. Hence $\frac{T_p(M)}{soc(T_p(M))}$ has no maximal submodules, i.e., (ii) holds.

(3) ⇒ (2) By Lemma 3.

(2) ⇒ (1) Let A and B be subgroups of M with B simple and $\frac{M}{A} \cong B \subseteq^{\oplus} M$. As B is simple, there is a $p \in \Omega$ such that $B \subseteq T_p(M)$ and $pB=0$. As $B \subseteq^{\oplus} M, B \subseteq^{\oplus} T_p(M)$. Since $pB=0$ and $\frac{M}{A} \cong B$, we have $p(\frac{M}{A}) = 0$, i.e., $pM \subseteq A$. For any prime $q \neq p$, it is easy to see that, $T_q(M) = pT_q(M) \subseteq pM$. Thus for all primes $q \neq p, T_q(M) \subseteq pM \subseteq A$. Since A is a maximal subgroup, $T_p(M)$ is not contained in A. Otherwise we would have $M = \oplus_{q\in\Omega} T_q(M) \subseteq A$, which is not the case as A is a maximal subgroup of M. Thus, by the maximality of A, we have $A + T_p(M) = M$. Then

$$\frac{T_p(M)}{A \cap T_p(M)} \cong \frac{T_p(M) + A}{A} = \frac{M}{A} \cong B \subseteq^{\oplus} T_p(M).$$

Since $T_p(M)$ is simple-direct-projective, $A \cap T_p(M) \oplus C = T_p(M)$ for some simple subgroup C of $T_p(M)$. Then we get $M = A + T_p(M) = A + [A \cap T_p(M) \oplus C] = A \oplus C$. Hence M is simple-direct-projective. □

Theorem 1. *Let M be an abelian group. The following statements are equivalent.*

1. *M is simple-direct-projective.*
2. a. *$T(M)$ is simple-direct-projective, and*
   b. *for each $p \in \Omega$ such that $pM + T(M) \neq M, soc(T_p(M)) \subseteq rad(T_p(M))$.*

Proof. (1) ⇒ (2) By Corollary 1, $T(M)$ is simple-direct-projective. Now, let $p \in \Omega$ be such that $pM + T(M) \neq M$. Then, as $\frac{M}{pM}$ is a homogeneous semisimple with each simple subgroup isomorphic to $\mathbb{Z}_p$ and $\frac{pM+T(M)}{pM} \neq \frac{M}{pM}$, there is a maximal subgroup A of M such that $T(M) \subseteq pM + T(M) \subseteq A$ and $\frac{M}{A} \cong \mathbb{Z}_p$.

We need to show that $soc(T_p(M)) \subseteq rad(T_p(M))$. Suppose the contrary that $soc(T_p(M)) \nsubseteq rad(T_p(M))$. Then there is a simple subgroup S of $T_p(M)$ which is not contained in $rad(T_p(M))$. Then $S \subseteq^{\oplus} T_p(M)$, and since $T_p(M)$ is a direct summand of $T(M)$, $S \subseteq^{\oplus} T(M)$ as well. Then as S is a pure subgroup of $T(M)$ and $T(M)$ is pure subgroup of M, S is a pure subgroup of M. Thus S is a pure and bounded subgroup of M, and so S is a direct summand of M by Fuchs [8, Theorem 27.5]. Since $S \cong \mathbb{Z}_p$ and $\frac{M}{A} \cong \mathbb{Z}_p \cong S \subseteq^{\oplus} M$, simple-direct-projectivity of M implies that $A \subseteq^{\oplus} M$, i.e., $M = A \oplus D$ for some simple subgroup D of M. Then $D \subseteq T(M) \subseteq A$, which is a contradiction. Hence we must have $soc(T_p(M)) \subseteq rad(T_p(M))$, and this proves (2).

(2) ⇒ (1) Let A and B be subgroups of M with B simple and $\frac{M}{A} \cong B \subseteq^{\oplus} M$. Since B is simple, $B \cong \mathbb{Z}_p$ for some $p \in \Omega$, in particular $B \subseteq soc(T_p(M))$ and $p(\frac{M}{A}) \cong pB = 0$, i.e., $pM \subseteq A$. As $B \subseteq^{\oplus} M$, B is not contained in $rad(T_p(M))$. Thus $soc(T_p(M)) \nsubseteq rad(T_p(M))$. Then $pM + T(M) = M$ by (2). Thus, $A + T(M) = M$. By similar arguments as in the proof of [Proposition 1, (2) ⇒ (1)], we obtain that A is a direct summand of M. Hence M is simple-direct-projective. □

Corollary 2. *Let M be an abelian group. Suppose $\frac{M}{T(M)}$ has no maximal subgroups. Then M is simple-direct-projective iff every maximal submodule of M is a direct summand.*

Proof. Sufficiency is clear. To prove the necessity, let A be a maximal subgroup of M. Suppose $\frac{M}{A} \cong \mathbb{Z}_p$, where $p \in \Omega$. Then $pM \subseteq A$. Since $\frac{M}{T(M)}$ has no maximal subgroups and A is maximal, $A + T(M) = M$. Now, by the proof of [Theorem 1, (2) ⇒ (1)], $A \subseteq^{\oplus} M$. This completes the proof. □

Over local rings, simple-direct-projective modules are exactly the modules given in Lemma 3.

Proposition 2. *Let R be a local ring. A right module M is simple-direct-projective iff*

i. $soc(M) \subseteq rad(M)$, or
ii. $\frac{M}{soc(M)}$ *has no maximal submodules.*

Proof. Suppose (i) does not hold. Then there is a simple submodule S of M such that $M = N \oplus S$. Let K be a maximal submodule of M. Since R is a local ring, R has a unique simple module up to isomorphism. Thus $\frac{M}{K} \cong S \subseteq^{\oplus} M$. Hence simple-direct projectivity of M implies that $K \subseteq^{\oplus} M$. Thus any maximal submodule of M is a direct summand. Now, if L is a maximal submodule of M, such that $soc(M) \subseteq L \subseteq M$, then $M = L \oplus S'$ with S' a simple submodule of M. Then $S' \subseteq soc(M) \subseteq L$, a contradiction. Hence $\frac{M}{soc(M)}$ has no maximal submodules. This proves the necessity.

Sufficiency is clear by Lemma 3. □

Over a right perfect ring, every module has a maximal submodule [1, Theorem 28.4]. Hence the following is a consequence of Proposition 2.

Corollary 3. *Let R be a local right perfect ring. A right module M is simple-direct-projective iff M is semisimple or $soc(M) \subseteq rad(M)$.*

It is easy to see that every module M with $rad(M) = 0$ is simple-direct-injective (see [11, Remark 4.5]). The following is the corresponding result for simple-direct-projective modules. Note that, a finitely generated module M is semisimple iff every maximal submodule of M is a direct summand. Recall that, a ring R is semilocal if $\frac{R}{J(R)}$ is semisimple Artinian.

Proposition 3. *The following statements are equivalent for a ring R.*

1. *R is semilocal.*
2. *Every right R-module M with $rad(M) \subseteq^{\oplus} M$ is simple-direct-projective.*
3. *Every right R-module with $rad(M) = 0$ is simple-direct-projective.*
4. *Every 2-generated right R-module M with $rad(M) = 0$ is simple-direct-projective.*

In particular, the conditions (2)–(4) are left-right symmetric.

Proof. (1) $\Rightarrow$ (2) Write $M = rad(M) \oplus N$ for some submodule N of M. Since R is semilocal, $\frac{M}{rad(M)}$ is semisimple and thus N is semisimple. Now, we claim that every maximal submodule of M is a direct summand of M. For, let A be a maximal submodule of M. Clearly, $N \not\subseteq A$ and so there exists a simple submodule K of N with $K \not\subseteq A$. Then $M = K + A$ and since $K \not\subseteq A, K \cap A = 0$. Therefore, $M = K \oplus A$ and $A \subseteq^{\oplus} M$, proving the claim. Inasmuch as every maximal submodule of M is a direct summand of M, we infer that M is simple-direct-projective.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (1) Let $\bar{R} := \frac{R}{J(R)}$. We show that every simple right $\bar{R}$-module K is projective. Now, viewing K as an R-module, there exists an epimorphism $f : \bar{R} \to K$. By the hypothesis, the 2-generated right module $M_R := K \oplus \bar{R}$, as a right R-module, is simple-direct-projective and so f splits by Ibrahim et al. [11, Proposition 2.1]. Thus, K is isomorphic to a summand of $\bar{R}$ and so K, as an $\bar{R}$-module, is projective. Hence $\bar{R} := \frac{R}{J(R)}$ is semisimple; that is, R is semilocal.

The last statement comes from the fact that being semilocal is left-right symmetric. □

3. Simple-direct-injective modules

In this section, we give a characterization of simple-direct-injective modules over the ring of integers and over semilocal rings. Nonsingular right modules are simple-direct-projective over any ring [11, Example 2.5(2)]. Motivated by this fact, we obtain a characterization of the rings whose nonsingular right modules are simple-direct-injective.

Following [4], a right module M is called *simple-direct-injective* if, whenever A and B are simple submodules of M with $A \cong B$ and $B \subseteq^{\oplus} M$ we have $A \subseteq^{\oplus} M$.

The following lemma is well-known. We do not know a proper reference, we include the proof for completeness.

Lemma 4. *Let R be a ring and I a two sided ideal of R. Then any pure-injective right $\frac{R}{I}$-module is pure-injective as an R-module.*

Proof. Let M be a pure-injective right $\frac{R}{I}$-module. Let B be a right R-module, and A a pure submodule of B. Let $i : A \to B$ be the inclusion map. Then by Lam [14, Corollary 4.92] $AI = A \cap BI$. Thus the natural map $j : \frac{A}{AI} \to \frac{B}{BI}$ given by $j(a + AI) = a + BI$ is a pure monomorphism. In order to show that M is a pure-injective R-module, let $f : A \to M$ be an R-homomorphism. Then $f(AI) = f(A)I \subseteq MI = 0$. Thus $AI \subseteq ker(f)$, and so $f = \bar{f}\pi$, where $\pi : A \to \frac{A}{AI}$ is the natural epimorphism, and $\bar{f} : \frac{A}{AI} \to M$ is the homomorphism induced by f, i.e., $\bar{f}(a + AI) = f(a)$ for each $a \in A$. Since M is a pure-injective $\frac{R}{I}$-module, there is homomorphism $g : \frac{B}{BI} \to M$ such that $\bar{f} = gj$. Let $\pi' : B \to \frac{B}{BI}$ be the natural epimorphism. For $\phi = g\pi'$, it is straightforward to check that, $\phi i = f$, i.e., ϕ extends f and so M is a pure-injective R-module. □

Lemma 5. *Let R be a commutative ring. Let M be an R-module and N a pure submodule of M. If M is simple-direct-injective, then N is simple-direct-injective. The converse is true if $soc(M) \subseteq N$.*

Proof. Suppose M is a simple-direct-injective module and N a pure submodule of M. Let $S_1 \cong S_2$ with S_1, S_2 simple submodules of N and $S_1 \subseteq^{\oplus} N$. Now, S_1 is pure in N, and N is pure in M. Then S_1 is pure in M by Fuchs and Salce [9, pages: 39 and 43]. Since R is commutative, simple modules are pure-injective by Cheatham and Smith [6, Corollary 4]. Being pure and pure-injective implies $S_1 \subseteq^{\oplus} M$. Therefore $S_2 \subseteq^{\oplus} M$, because M is simple-direct-injective. Hence $S_2 \subseteq^{\oplus} N$, and so N is simple-direct-injective.

Now, assume that N is a pure submodule of M, and $soc(M) \subseteq N$. Let $S_1 \cong S_2$ be two simple submodules of M and $S_1 \subseteq^{\oplus} M$. Then $S_1 \subseteq N, S_2 \subseteq N$ and $S_1 \subseteq^{\oplus} N$. Since N is simple-direct-injective, $S_2 \subseteq^{\oplus} N$. As S_2 is pure in N and N is pure in M, S_2 is pure in M. Then $S_2 \subseteq^{\oplus} M$, because S_2 is both pure-injective and pure in M. Hence M is simple-direct-injective. □

A right module M is called *absolutely pure* if it is pure in every module containing it as a submodule.

Corollary 4. *Let R be a commutative ring and M be an absolutely pure module. Then each module K such that $M \subseteq K \subseteq E(M)$ is simple-direct-injective. In particular, absolutely pure modules are simple-direct-injective.*

Proof. Since M is a pure submodule of $E(M)$ and $E(M)$ is simple-direct-injective, M is simple-direct-injective by Lemma 5. As M is essential in $E(M)$, $soc(M) = soc(K)$ for each module K such that $M \subseteq K \subseteq E(M)$. Hence K is simple-direct-injective, again by Lemma 5. □

A commutative domain R is called Prüfer domain if each finitely generated ideal of R is projective.

Corollary 5. *Let R be a Prüfer domain. A module M is simple-direct-injective iff the torsion submodule $T(M)$ of M is simple-direct-injective.*

Proof. Let M be an R-module. Then $T(M)$ is pure in M by [9, Proposition 8.12]. Since simple modules are torsion, $soc(M) \subseteq T(M)$. Now, the proof is clear by Lemma 5. □

Lemma 6. *Let M be an R-module and N a coclosed submodule of M. If M is simple-direct-injective, then N is simple-direct-injective. The converse is true if $soc(M) \subseteq N$.*

Proof. Suppose M is simple-direct-injective and N is a coclosed submodule of M. Suppose $S_1 \cong S_2$ are simple submodules of N and $S_1 \subseteq^{\oplus} N$. Then S_1 is a coclosed submodule of M by Clark et al. [5, 3.7. (6)]. Thus S_1 is not small in M, and so $S_1 \subseteq^{\oplus} M$. By simple-direct-injectivity of M, $S_2 \subseteq^{\oplus} M$. Therefore, $S_2 \subseteq^{\oplus} N$, and N is simple-direct-injective.

Now, assume that N is a coclosed submodule of M, and $soc(M) \subseteq N$. Let $S_1 \cong S_2$ be two simple submodules of M and $S_1 \subseteq^{\oplus} M$. Then $S_1 \subseteq N, S_2 \subseteq N$ and $S_1 \subseteq^{\oplus} N$. Since N is simple-direct-injective, $S_2 \subseteq^{\oplus} N$. As S_2 is coclosed in N and N is coclosed in M, S_2 is coclosed in M. Then $S_2 \subseteq^{\oplus} M$, and so M is simple-direct-injective. □

Theorem 2. *Let M be an abelian group. The following statements are equivalent.*

1. *M is simple-direct-injective.*
2. *$T(M)$ is simple-direct-injective.*
3. *$T_p(M)$ is simple-direct-injective for each $p \in \Omega$.*
4. *For each $p \in \Omega, T_p(M)$ is semisimple, or $soc(T_p(M)) \subseteq rad(T_p(M))$.*

Proof. (1) $\Longleftrightarrow$ (2) By Corollary 5.

(2) $\Rightarrow$ (3) is clear, since $T(M) = \oplus_{p\in\Omega} T_p(M)$ and simple-direct-injective modules are closed under direct summands.

(3) $\Rightarrow$ (4) Assume that $soc(T_p(M)) \not\subseteq rad(T_p(M))$ for some $p \in \Omega$. Then there is a simple subgroup S of $T_p(M)$ such that $S \subseteq^{\oplus} T_p(M)$. Let A be the sum of all simple summands of $T_p(M)$. Then any finitely generated submodule of A is a direct summand (hence pure subgroup) of $T_p(M)$ by Camillo et al. [4, Lemma 2.4 (1)]. Since A is a direct limit of its finitely generated subgroups and direct limit of pure subgroups is pure, A is pure in $T_p(M)$. As A is semisimple and $A \subseteq T_p(M)$, $pA = 0$, i.e., A is bounded. Then $A \subseteq^{\oplus} T_p(M)$ by Fuchs [8, Theorem 27.5]. Let $T_p(M) = A \oplus B$. We claim that $B = 0$. For, if $B \neq 0$, then $soc(B) \neq 0$. Let U be a simple subgroup of B. Since $T_p(M)$ is a p-group, $soc(T_p(M))$ is homogeneous, i.e., all simple subgroups of $T_p(M)$ are isomorphic. Thus $U \subseteq^{\oplus} T_p(M)$. Then $U \subseteq A$, which is a contradiction. Therefore $B = 0$, and so $T_p(M) = A$ is semisimple. This proves (4).

(4) $\Rightarrow$ (2) Let U and V be simple submodules of $T(M)$ such that $U \cong V$ and $U \subseteq^{\oplus} T(M)$. Then there is a $p \in \Omega$ such that $U \subseteq^{\oplus} T_p(M)$. Thus $T_p(M)$ must be semisimple by (4). Since $V \cong U$, $V \subseteq^{\oplus} T_p(M)$. Hence $V \subseteq^{\oplus} T(M)$, and so $T(M)$ is simple-direct-injective. □

Proposition 4. *Let R be a semilocal ring. For a right R-module M, let S' be the sum of all simple direct summands of M. The following are equivalent.*

1. *M is simple-direct-injective.*
2. *S' is fully invariant and pure submodule of M.*
3. *$M = S' \oplus N$, and S' is a fully invariant submodule of M.*

Proof. (1) $\Rightarrow$ (2) By Camillo et al. [4, Lemma 2.4(2)], S' is a fully invariant submodule of M. Let $S' = \oplus_{i\in I} V_i$, where V_i are simple for each $i \in I$. Then for each finite subset $F \subseteq I, N_F = \oplus_{i\in F} V_i$ is a direct summand of M by Camillo et al. [4, Lemma 2.4(1)], and so N_F is a pure submodule of M. By Lam [14, 4.84.] direct limit of pure submodules is pure, and so $S' = \oplus_{i\in I} V_i = \lim_F N_F$ is a pure submodule of M. This proves (2).

(2) $\Rightarrow$ (3) Since R is a semilocal ring, $\frac{R}{J(R)}$ is semisimple. Thus every right $\frac{R}{J(R)}$-module is pure-injective. As S' is semisimple, $S'.J(R) = 0$. Thus S' is a pure-injective right R-module by Lemma 4. Being pure and pure-injective implies that $S' \subseteq^{\oplus} M$.

(3) $\Rightarrow$ (1) Let A and B be two simple submodules of M such that $A \cong B$ and $A \subseteq^{\oplus} M$. Then $A \subseteq S'$. Since S' is a fully invariant submodule of M, $B \subseteq S'$ and so $B \subseteq^{\oplus} M$. Hence M is simple-direct-injective. □

Simple submodules of nonsingular modules are projective. Thus nonsingular right modules are simple-direct-projective over any ring (see [11, Example 2.5]). The corresponding result for simple-direct-injective modules follows.

Proposition 5. *Let R be a ring. The following statements are equivalent.*

1. *Every projective simple right module is injective.*
2. *Every nonsingular right module is simple-direct-injective.*

Proof. (1) $\Rightarrow$ (2) Nonsingular simple right modules are projective, and so injective by (1). Thus, (2) follows.

(2) $\Rightarrow$ (1) Let S be projective simple right module. Then $E(S)$ and $S \oplus E(S)$ are nonsingular, and so $S \oplus E(S)$ is simple-direct-injective by (2). Since $S \oplus 0 \cong 0 \oplus S$ and $S \oplus 0 \subseteq^{\oplus} S \oplus E(S), S \subseteq^{\oplus} E(S)$. Hence S is injective. □

Corollary 6. *Let R be a commutative ring. Then every nonsingular module is simple-direct-injective.*

Proof. Let S be a projective simple module. Since S is projective, it is flat. Then S is injective by Ware [20, Lemma 2.6.]. Now, the conclusion follows by Proposition 5. □

Let M be a right module and $N \subseteq M$. N is called *coneat submodule* of M if for every simple right module S, any homomorphism $f : N \to S$ can be extended to a homomorphism $g : M \to S$ (see, [2, 7]). A right module M is called *absolutely coneat* if M is coneat in every module containing it as a submodule, equivalently M is coneat in $E(M)$. It is easy to see that absolutely coneat modules are closed under direct summands, and that a simple right module is absolutely coneat iff it is injective.

Proposition 6. *Absolutely coneat right modules are simple-direct-injective.*

Proof. Let M be an absolutely coneat right module. Suppose A and B are simple submodules of M with $A \cong B$ and $B \subseteq^{\oplus} M$. Then B is absolutely coneat as a direct summand of M. Thus, B is injective, and so A is injective too. Then $A \subseteq^{\oplus} M$, and hence M is simple-direct-injective. □

4. When simple-direct-injective (projective) modules are simple-direct-projective (injective)

In Ref. [4, Theorem 3.4.], the authors characterize the rings over which simple-direct-injective right modules are $C3$. They prove that these rings are exactly the Artinian serial rings with $J^2(R) = 0$. In Ref. [11, Theorem 4.9.], the authors prove that every simple-direct-injective right R-module is a $D3$-module iff every simple-direct-projective right R-module is a $C3$-module iff R is uniserial with $J^2(R) = 0$.

At this point it is natural to consider the rings whose simple-direct-injective (resp. projective) right modules are simple-direct-projective (resp. injective). Since $C3$-modules and $D3$-modules are simple-direct-injective and simple-direct-projective, respectively, uniserial rings with $J^2(R) = 0$ are examples of the aforementioned rings.

In this section, we prove that every simple-direct-injective right module is simple-direct-projective iff the ring is left perfect and right H-ring. As a consequence, we show that, commutative perfect rings are examples of such rings. We prove that the rings whose simple-direct-projective right modules are simple-direct-injective are right max-ring. For a commutative Noetherian ring, we prove that, simple-direct-projective modules are simple-direct-injective iff simple-direct-injective modules are simple-direct-projective iff the ring is Artinian.

Recall that, a ring R is called *right semiartinian* if every nonzero right R-module has nonzero socle. A right module M is called semiartinian (or Loewy) module if every nonzero factor of M has a nonzero socle. First, we give a characterization of the rings over which every simple-direct-injective right module is simple-direct-projective. We begin with the following.

Proposition 7. *Let R be a ring. Suppose every simple-direct-injective right R-module is simple-direct-projective. Then R is semilocal and right semiartinian, i.e., R is left perfect.*

Proof. Every right module M with $rad(M) = 0$ is simple-direct-injective (see, [11, Remark 4.5.]). Thus, by Proposition 3, R is semilocal. Suppose R is not right semiartinian. Then there is a nonzero finitely generated right module N with $soc(N) = 0$. As the ring is semilocal, there are only finitely many, say $S_1, S_2, \ldots, S_n$ simple right modules up to isomorphism. Let $K = S_1 \oplus S_2 \oplus \cdots \oplus S_n \oplus N$. Then every simple submodule of K is a direct summand, and so K is simple-direct-injective. Let us show that K is not simple-direct-projective, and get a contradiction. Let L be a maximal submodule of N. Since $soc(N) = 0$, L is not a direct summand of N, and hence not a direct summand of K too. Let $L' = S_1 \oplus S_2 \oplus \cdots \oplus S_n \oplus L$. Then L' is a maximal submodule of K and $\frac{K}{L'} \cong S_i \subseteq^{\oplus} K$, for some $i = 1, ..., n$. As L is not a direct summand of K, L' is not a direct summand of K too. Thus K is not simple-direct-projective, which is a contradiction. Therefore R must be right semiartinian. Hence R is left perfect by Lam [15, Theorem 23.20]. □

Theorem 3. *The following statements are equivalent for a ring R.*

1. *R is left perfect and right H-ring.*
2. *Every simple-direct-injective right module is simple-direct-projective.*

Proof. (1) ⇒ (2) Let M be a simple-direct-injective module. Let A be the sum of all simple summands of M. Then A is fully invariant and $M = A \oplus B$ by Proposition 4. Since A is a fully invariant submodule of M, $soc(B) \subseteq rad(M)$ and $\mathrm{Hom}(A, soc(B)) = 0$. By (1) the ring is right semiartinian, and so $soc(B)$ is an essential submodule of B. In order to prove that M is simple-direct-projective, suppose that $\frac{M}{K} \cong S \subseteq^{\oplus} M$ for some simple submodule S of M. Then as $S \subseteq^{\oplus} M, S \subseteq A$. We claim that, $A + K = M$. Suppose the contrary that, $A + K$ is properly contained in M, and let us find a contradiction. Then, by maximality of K, we have $A \subseteq K$. Thus from $M = A \oplus B$ and by modular law, we get $K = A \oplus K \cap B$, and

$$\frac{M}{K} = \frac{A \oplus B}{K} = \frac{A \oplus B}{A \oplus K \cap B} \cong \frac{B}{K \cap B} \cong S.$$

Thus, $K \cap B$ is a maximal submodule of B. Set $N := K \cap B$. Since the ring is semilocal, there are only finitely many simple right modules up to isomorphism. Thus, $soc(B) = U_1^{(I_1)} \oplus U_2^{(I_2)} \oplus \cdots \oplus U_k^{(I_k)}$, for some simple right modules $U_1, U_2, ..., U_k$ and index sets $I_1, I_2, ..., I_k$. Since $soc(B)$ is an essential submodule of B, the injective hull of B is $E(B) = \oplus_{i=1}^{k} E(U_i^{(I_i)})$. As $\frac{B}{N} \cong S$, there is an epimorphism $f : B \to S$. Let $e : S \to E(S)$ be the inclusion homomorphism. Then the homomorphism ef extends to a (nonzero) homomorphism $g : E(B) \to E(S)$. Since $E(B) = \oplus_{i=1}^{k} E(U_i^{(I_i)})$ and g is nonzero, there is a nonzero homomorphism $h : E(U_j^{(I_j)}) \to E(S)$, for some $j \in \{1, 2, ..., k\}$. It is clear that, $E(U_j^{(I_j)})$ can be embedded in $E(U_j)^{I_j}$. Thus, as h is nonzero, there is a nonzero homomorphism from $E(U_j)^{I_j}$ to $E(S)$. This leads to a nonzero homomorphism $t : E(U_j) \to E(S)$. So that, by the right H-ring assumption, we must have $S \cong U_j$. Then $\mathrm{Hom}(A, soc(B)) \neq 0$, which is a contradiction. Hence the case $A + K = M$ must hold. Therefore, as A is semisimple, there is a simple submodule U of A such that $U + K = M$ and $U \cap K = 0$, i.e., $K \subseteq^{\oplus} M$. Hence M is simple-direct-projective. This proves (2).

(2) ⇒ (1) The ring R is left perfect by Proposition 7. Suppose R is not right H-ring. Then there are nonisomorphic simple right modules S_1 and S_2 such that $\mathrm{Hom}(E(S_1), E(S_2)) \neq 0$. Let $0 \neq f : E(S_1) \to E(S_2)$, and $A = ker(f)$. Since $\frac{E(S_1)}{A} \cong f(E(S_1)) \subseteq E(S_2)$, there is a submodule $B \subseteq E(S_1)$ such that $\frac{B}{A} \cong S_2$. Then it is clear that $B \oplus S_2$ is a simple-direct-injective right module. On the other hand, $\frac{B \oplus S_2}{A \oplus S_2} \cong 0 \oplus S_2 \subseteq^{\oplus} B \oplus S_2$. But $A \oplus S_2$ is not a direct summand of $B \oplus S_2$. Thus $B \oplus S_2$ is not simple-direct-projective. This contradicts (2). Thus R must be right H-ring. □

Now, we give some consequences of Theorem 3.

Corollary 7. *Let R be a commutative ring. The following statements are equivalent.*

1. *R is a perfect ring.*
2. *Every simple-direct-injective module is simple-direct-projective.*

Proof. Commutative perfect rings are semiartinian (see, [15, Theorem 23.20]). Thus commutative perfect rings are H-ring by Camillo [3, Proposition 2]. Now, the proof is clear by Theorem 3. □

A right Noetherian right semiartian ring is right Artinian (see [19]). Left perfect rings are right semiartinian by Lam [15, Theorem 23.20]. Thus, the following is clear by Theorem 3.

Corollary 8. *Let R be a right Noetherian ring. The following statements are equivalent.*

1. *R is right Artinian right H-ring.*
2. *Every simple-direct-injective right module is simple-direct-projective.*

By Sharpe and Vamos [18, Proposition 4.21], commutative Noetherian rings are H-rings.

Corollary 9. *Let R be a commutative Noetherian ring. The following statements are equivalent.*

1. *R is Artinian ring.*
2. *Every simple-direct-injective module is simple-direct-projective.*

A ring R is called *right V-ring* if every simple right R-module is injective. By Camillo et al. [4, Theorem 4.1], R is right V-ring iff every right R-module is simple-direct-injective. A ring R is called *right max-ring* if every nonzero right R-module has a maximal submodule. Right V-rings are right max-rings (see [14, Theorem 3.75]). Clearly, over right V-rings simple-direct-projective right modules are simple-direct-injective.

Now, we consider the rings whose simple-direct-projective right modules are simple-direct-injective.

Proposition 8. *Let R be a ring. If each simple-direct-projective right R-module is simple-direct-injective, then R is a right max-ring.*

Proof. Suppose the ring is not right max-ring. Then there is a nonzero right module M such that $M = rad(M)$. Let $0 \neq m \in M$, and let K be a maximal submodule of mR. Let $h = i\pi : mR \to E\left(\frac{mR}{K}\right)$, where $\pi : mR \to \frac{mR}{K}$ is the natural epimorphism and $i : \frac{mR}{K} \to E\left(\frac{mR}{K}\right)$ is the inclusion homomorphism. By injectivity of $E\left(\frac{mR}{K}\right)$, there is a (nonzero) homomorphism $g : M \to E\left(\frac{mR}{K}\right)$ which extends h. Let $L := g(M)$. Since $\frac{M}{ker(g)} \cong L$ and $rad(M) = M, L = rad(L)$. Note that L has an essential socle isomorphic to $\frac{mR}{K}$. Consider the right module $N = \frac{mR}{K} \oplus L$. Then $0 \oplus L$ is the unique maximal submodule of N and $0 \oplus L \subseteq^{\oplus} N$. Thus N is simple-direct-projective. On the other hand, $0 \oplus soc(L) \cong \frac{mR}{K} \oplus 0 \subseteq^{\oplus} N$, but $0 \oplus soc(L)$ is not a direct summand of N. Therefore N is not simple-direct-injective. This contradicts with our assumption that simple-direct-projective modules are simple-direct-injective. Hence R must be right max-ring. □

A subfactor of a right module M, is a submodule of some factor module of M. The following lemma can be easily derived from the definition of H-ring. We include it for an easy reference.

Lemma 7. *R is a right H-ring iff for every simple right R-module S, every simple subfactor of E(S) is isomorphic to S.*

Proof. Suppose R is a right H-ring and S a simple right R-module. Let $\frac{A}{B}$ be a simple subfactor of $E(S)$. Assume that $\frac{A}{B}$ is not isomorphic to S. Let $i_1 : \frac{A}{B} \to \frac{E(S)}{B}$ and $i_2 : \frac{A}{B} \to E\left(\frac{A}{B}\right)$ be the corresponding inclusions. Then there is a nonzero homomorphism $f : \frac{E(S)}{B} \to E\left(\frac{A}{B}\right)$. Thus, $f\pi : E(S) \to E\left(\frac{A}{B}\right)$ is a nonzero homomorphism, where $\pi : E(S) \to \frac{E(S)}{B}$ is the canonical epimorphism. This contradicts with the assumption that R is right H-ring. Therefore every simple subfactor of $E(S)$ is isomorphic to S. This proves the necessity.

Conversely, let S_1 and S_2 be simple right R-modules and $0 \neq f \in \mathrm{Hom}_R(E(S_1), E(S_2))$. Then $\frac{E(S_1)}{ker(f)}$ has a simple subfactor isomorphic to S_2. Thus, by our assumption, we must have $S_1 \cong S_2$. Hence R is a right H-ring. □

Proposition 9. *Let R be a commutative Noetherian ring. The following statements are equivalent.*

1. *R is Artinian.*
2. *Every simple-direct-projective module is simple-direct-injective.*

Proof. (2) ⇒ (1) By Proposition 8, R is a max-ring. Commutative Noetherian max-rings are Artinian by Hamsher [13, Theorem 1].

(1) ⇒ (2) Let M be a simple-direct-projective R-module. Let S' be the sum of simple summands of M. Then, by the same arguments in the proof of [Proposition 4, (2) ⇒ (3)], S' is a pure and a pure-injective submodule of M, and so $S' \subseteq^{\oplus} M$. Let $M = S' \oplus N$. Clearly, by the construction of S', N has no simple (or maximal) submodule which is a direct summand. Now, in order to prove that M is simple-direct-injective, by Proposition 4, it is enough to see that S' is a fully invariant submodule of M. Suppose the contrary that there are simple submodules A, B of M such that $A \subseteq S', B \subseteq N$ and $A \cong B$. Since $B \subseteq N$, there is a nonzero homomorphism $g : N \to E(B)$. Then for $K = ker(g)$, the module $\frac{N}{K}$ has a maximal submodule say $\frac{L}{K}$ by the Artinianity of R. Since R is commutative and Noetherian, R is an H-ring. Thus, every simple subfactor of $E(B)$ is isomorphic to B by Lemma 7. Therefore, $\frac{N}{L} \cong B$. Now,

$$\frac{M}{S' \oplus L} = \frac{S' \oplus N}{S' \oplus L} \cong B \cong A \subseteq^{\oplus} M.$$

Then by simple-direct-projectivity of M, $S' \oplus L \subseteq^{\oplus} M$ and, by modular law, $L \subseteq^{\oplus} N$. This contradicts the fact that, N has no maximal summands. Hence S' is a fully invariant submodule of M, and so M is simple-direct-injective by Proposition 4. This proves (2). □

Proposition 10. *Let R be a commutative semilocal ring. The following statements are equivalent.*

1. *R is perfect.*
2. *Every simple-direct-projective module is simple-direct-injective.*

Proof. (2) ⇒ (1) R is a max-ring by Proposition 8. Semilocal max-rings are perfect by Anderson and Fuller [1, Theorem 28.4].

(1) ⇒ (2) Note that, commutative perfect rings are H-rings and max-rings. Now, replacing Artinian by perfect the same proof of [Proposition 9 (1) ⇒ (2)] holds. □

Remark 1. Over a right V-ring all right modules, in particular, simple-direct-projective right modules are simple-direct-injective (see [4, Theorem 4.1]). Since commutative perfect V-rings are semisimple, there is a simple-direct-injective R-module which is not simple-direct-projective over nonsemisimple commutative V-rings by Corollary 7. Therefore, nonsemisimple commutative V-rings are examples of rings such that simple-direct-projective modules are simple-direct-injective, and admit a simple-direct-injective module that is not simple-direct-projective.

Summing up, Corollaries 7, 9 and Propositions 9, 10 we obtain the following.

Corollary 10. *Let R be a commutative Noetherian ring. Then the following statements are equivalent.*

1. *R is Artinian.*

2. *Every simple-direct-injective module is simple-direct-projective.*
3. *Every simple-direct-projective module is simple-direct-injective.*

Corollary 11. *Let R be a commutative semilocal ring. Then the following statements are equivalent.*

1. *R is perfect.*
2. *Every simple-direct-injective module is simple-direct-projective.*
3. *Every simple-direct-projective module is simple-direct-injective.*

Acknowledgments

The authors would like to thank the referee for his/her careful reading of the manuscript and for his/her valuable comments and suggestions.

Funding

The third author is supported by TÜBITAK project number 119F176.

References

[1] Anderson, F. W., K. R., Fuller, K. R. (1992). *Rings and Categories of Modules*. New York, NY: Springer.
[2] Buyukasik, E., Durgun, Y. (2014). Coneat submodules and coneat-flat modules. *J. Korean Math. Soc.* 51(6): 1305–1319. DOI: 10.4134/JKMS.2014.51.6.1305.
[3] Camillo, V. (1978). Homological independence of injective hulls of simple modules over commutative rings. *Commun. Algebra* 6(14):1459–1469. DOI: 10.1080/00927877808822299.
[4] Camillo, V., Ibrahim, Y., Yousif, M., Zhou, Y. (2014). Simple-direct-injective modules. *J. Algebra* 420: 39–53. DOI: 10.1016/j.jalgebra.2014.07.033.
[5] Clark, J., Lomp, C., Vanaja, N., Wisbauer, R. (2006). *Lifting Modules*. Basel: Birkhäuser Verlag.
[6] Cheatham, T. J., Smith, J. R. (1976). Regular and semisimple modules. *Pacific J. Math.* 65(2):315–323. DOI: 10.2140/pjm.1976.65.315.
[7] Crivei, S. (2014). Neat and coneat submodules over commutative rings. *Bull. Aust. Math. Soc.* 89(2): 343–352. DOI: 10.1017/S0004972713000622.
[8] Fuchs, L. (1970). *Infinite Abelian Groups*. New York, NY: Academic Press.
[9] Fuchs, L., Salce, L. (2000). *Modules Over Non-Noetherian Domains. Math. Surveys Monogr*. Vol. 84, Providence: American Mathematical Society.
[10] Golan, J. S. (1981). Homological independence of injective hulls of simple modules over certain noncommutative rings. *Proc. Am. Math. Soc.* 81(3):377–381. DOI: 10.1090/S0002-9939-1981-0597644-1.
[11] Ibrahim, Y., Koşan, M. T., Quynh, T. C., Yousif, M. (2016). Simple direct-projective modules. *Commun. Algebra* 44(12):5163–5178. DOI: 10.1080/00927872.2016.1147574.
[12] Ibrahim, Y., Koşan, M. T., Quynh, T. C., Yousif, M. (2017). Simple-direct-modules. *Commun. Algebra* 45(8):3643–3652. DOI: 10.1080/00927872.2016.1243697.
[13] Hamsher, R. M. (1966). Commutative noetherian rings over which every module has a maximal submodule. *Proc. Am. Math. Soc.* 17(6):1471–1472. DOI: 10.2307/2035767.
[14] Lam, T. Y. (1999). *Lectures on Modules and Rings*. New York, NY: Springer.
[15] Lam, T. Y. (2001). *A First Course in Noncommutative Rings*. New York, NY: Springer.
[16] Nicholson, W. K. (1976). Semiregular modules and rings. *Can. J. Math.* 28(5):1105–1120. DOI: 10.4153/CJM-1976-109-2.
[17] Papp, Z. (1975). On stable Noetherian rings. *Trans. Am. Math. Soc.* 213:107–114. DOI: 10.1090/S0002-9947-1975-0393120-1.
[18] Sharpe, D. W., Vamos, P. (1972). *Injective Modules*. New York, NY: Cambridge University Press.
[19] Shock, R. C. (1974). Dual generalizations of the Artinian and Noetherian conditions. *Pacific J. Math.* 54(2): 227–235. DOI: 10.2140/pjm.1974.54.227.
[20] Ware, R. (1971). Endomorphism rings of projective modules. *Trans. Am. Math. Soc.* 155(1):233–256. DOI: 10.1090/S0002-9947-1971-0274511-2.