On the relativistic supersymmetric quantum mechanics

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The present paper is devoted to the one-dimensional relativistic supersymmetric quantum mechanics (RSUSYQM). A short formulation of RSUSYQM is given. We show that RSUSYQM is a q-deformed non-relativistic SUSYQM. Two simple examples are given.

1 Introduction

The version of the relativistic quantum mechanics (RQM) which is a key concept of our consideration is based on the notion of non-commutative differential geometry. Referring the reader to review papers [1–5] for the comprehensive presentation of the subject and further references we shall outline here the essential aspects of the approach[1]).

Let us consider a very simple version of the one variable noncommutative differential calculus in which differential does not commute with the coordinate

$$[dx, x] = \frac{i}{2}\frac{\hbar}{mc}dx \tag{1}$$

where m is a mass of a particle. In the non-relativistic limit (1) goes over into usual relation $[x, dx] = 0$. In what follows we shall use the unit system in which $\hbar = c = m = 1$.

The momentum operator has the form [4, 5]

$$\hat{p} = -\frac{i}{2}\left(\underset{\rightarrow}{\partial} + \underset{\leftarrow}{\partial}\right) \tag{2}$$

where $\underset{\rightarrow}{\partial}$ and $\underset{\leftarrow}{\partial}$ are the right and left interior derivatives. The relativistic free Schrödinger equation has the "non-relativistic" form

$$(h_0 - e)\,\psi(x) = \left(\frac{\hat{p}^2}{2} - e\right)\psi\,(x) = 0, \tag{3}$$

where $e = E - 1$ is the relativistic kinetic energy.

It is important that in our particular case of the non-commutative differential calculus the interior derivatives $\underset{\rightarrow}{\partial}$ and $\underset{\leftarrow}{\partial}$ are equal to the finite-difference operators

$$\underset{\rightarrow}{\partial}\,\psi(x) = \frac{\psi(x+\frac{i}{2}) - \psi(x)}{\frac{i}{2}} \qquad \underset{\leftarrow}{\partial}\,\psi(x) = \frac{\psi(x) - \psi(x-\frac{i}{2})}{\frac{i}{2}} \tag{4}$$

[1]) The present paper is the fragment of the lecture delivered at PRAHA–SPIN–2001. It is a natural continuation of the lecture [6] given at "SYMMETRY AND SPIN" — PRAHA '98

correspondingly and $\hat{p}$ (formula (2)) takes the form

$$\hat{p} = -2\sinh\frac{i}{2}\frac{d}{dx} \tag{5}$$

The solution of the free Schrödinger equation (3), the relativistic plane wave is

$$\langle x|p\rangle = (E-p)^{-ix} = e^{ix\chi} \tag{6}$$

where χ is the rapidity $\chi = \ln(E+1)$. Plane waves (6) are the direct generalizations (the Gelfand-Graev kernels) of the standard plane waves for the case of the relativistic kinematics. We refer the reader to [1–5] where in particular it had been shown that the approach schematically described above represents an actual alternative to the standard RQM. For example the Poincare group generators are easily realized in terms of the non-commutative derivatives. The natural connection of RQM with q-deformations [8] is discussed in the articles [2, 3]. For the noncommutative differential calculus and finite-difference calculus see [7, 9].

2 RSUSYQM

The RSUSYQM is developed in analogy with the non-relativistic SUSYQM [10]-[13]. The factorization method for the Schrödinger equation plays important role for the non-relativistic SUSYQM [2, 3]. So we construct here the factorization method modified for the relativistic finite-difference Schrödinger equation

$$(h-e)\,\psi(x) = (h_0 + V(x) - e)\,\psi(x) = 0 \tag{7}$$

Let us introduce a couple of ladder operators [2, 3]

$$A^{\pm} = \pm i\sqrt{2}\cdot\alpha(x)\cdot e^{\pm\rho(x)}\sinh\frac{i}{2}\frac{d}{dx}\cdot e^{\mp\rho(x)} \tag{8}$$

or

$$A^{\pm} = -i\sqrt{2}\cdot\alpha(x)\cdot e^{\pm\xi(x)}\left[\sinh\rho_{\frac{s}{2}}(x)\cosh\frac{i}{2}\frac{d}{dx} \mp \cosh\rho_{\frac{s}{2}}(x)\sinh\frac{i}{2}\frac{d}{dx}\right] \tag{9}$$

where

$$\rho_{\frac{s}{2}}(x) = \sinh\frac{i}{2}\frac{d}{dx}\rho(x) \qquad\qquad \rho_{\frac{c}{2}}(x) = \cosh\frac{i}{2}\frac{d}{dx}\rho(x) \tag{10}$$

$$\xi(x) = \rho(x) - \rho_{\frac{c}{2}}(x) \tag{11}$$

and $\rho(x)$ is the logarithm of the ground state wave function of eq. (3)

$$\psi_0(x) = e^{-\rho(x)} \tag{12}$$

In the nonrelativistic limit the operators $A^{\pm}$ turn into the usual ladder operators (cf. [2, 3]). In our case we must take into account the modified Leibnitz rules for the non-commutative differential calculus, which complicate the calculations.

In addition to the finite-difference character of the operators (8, 9) there are two factors $\alpha(x)$ and $e^{\pm\xi(x)}$ whose analogs don't appear in the non-relativistic case. The factor $\alpha(x)$ is connected with a natural lattice variable and is also expressed in terms of $\rho(x)$ (see [2, 7]). Factors $e^{\pm\xi(x)}$ are connected with deformations [2–5, 8]. A deformation

$$q(x) = e^{a(x)} \tag{13}$$

must be introduced to cancel $e^{\pm\xi(x)}$. In the non-relativistic case $\xi(x) \to 1$ and there is no deformation : $q(x) \to 1$. Let us consider the $q(x)$-mutator

$$\begin{aligned} &[A^-, A^+]_{q(x)} = A^- \cdot q(x) \cdot A^+ - A^+ \cdot q^{-1}(x) \cdot A^- = \\ &= \frac{\alpha(x)}{2} \left\{ \begin{array}{c} e^{\frac{i}{2}\frac{d}{dx}} \sinh Z(x)\, \alpha(x)\, e^{\frac{i}{2}\frac{d}{dx}} + e^{-\frac{i}{2}\frac{d}{dx}} \sinh Z(x)\, \alpha(x)\, e^{-\frac{i}{2}\frac{d}{dx}} - \\ -e^{\frac{i}{2}\frac{d}{dx}} \sinh\left(Z(x) + 2\rho_{\frac{s}{2}}(x)\right) \alpha(x) e^{-\frac{i}{2}\frac{d}{dx}} - \\ -e^{-\frac{i}{2}\frac{d}{dx}} \sinh\left(Z(x) - 2\rho_{\frac{s}{2}}(x)\right) \alpha(x)\, e^{\frac{i}{2}\frac{d}{dx}} \end{array} \right\} \end{aligned} \tag{14}$$

where

$$Z(x) = 2\xi(x) + a(x) \tag{15}$$

Let us recall that the commutator of the non-relativistic ladder operators $a^{\pm}$ does not contain the differentiation operators

$$\left[a^-, a^+\right] = \frac{d^2\rho(x)}{dx^2} \tag{16}$$

By analogy with (16) we shall require that there are no non-commutative derivatives (4) in the r.h.s. of (14). The simplest way to achieve this is to put[2])

$$Z(x) = 0 \tag{17}$$

The last equation gives the relation connecting $\rho(x)$ and $a(x)$ (or $q(x)$):

$$a(x) = -2\xi(x) \qquad q(x) = e^{-2\xi(x)} \tag{18}$$

We have

$$\begin{aligned} [A^-, A^+]_{q(x)} &= -2\alpha(x) \sinh \tfrac{i}{2}\tfrac{d}{dx} \left[\alpha(x) \sinh 2\rho_{\frac{s}{2}}(x)\right] \to \\ &\to \tfrac{d^2\rho(x)}{dx^2} \end{aligned} \tag{19}$$

Now let us write down the basic relations of relativistic RSUSYQM, i.e., the relativistic quantum mechanical system whose Hamiltonian is constructed of anticommuting charges Q:

$$\hat{H} = \frac{1}{2} \cdot \left\{Q, Q^\dagger\right\}_{q(x)} = \frac{1}{2} \cdot \left(Q \cdot q(x)^{-1} \cdot Q^\dagger + Q^\dagger \cdot q(x) \cdot Q\right) \tag{20}$$

[2]) In the recent paper [14] the detailed analysis of all admissible solutions for $\alpha(x)$ and $Z(x)$ is given.

$$\{Q, Q\} = \{Q^{\dagger}, Q^{\dagger}\} = 0 \tag{21}$$

As in the non relativistic, case the supersymmetry property of Hamiltonian $\hat{H}$

$$\left[\hat{H}, Q\right] = \left[\hat{H}, Q^{\dagger}\right] = 0 \tag{22}$$

is provided by nilpotency of the charge operators (21). The Hamiltonian $\hat{H}$ contains coordinates that are quantized by $q(x)$-mutators and anticommutators. They are mixed by deformed supersymmetry transformations. The explicit realization of Q and $Q^{\dagger}$ is

$$Q = i\sqrt{2} \cdot A^{+} \cdot \overset{\wedge}{\psi}^{\dagger}, \qquad Q^{\dagger} = -i\sqrt{2} \cdot A^{-} \cdot \overset{\wedge}{\psi} \tag{23}$$

In the simplest case the bosonic degrees of freedom represented by the ladder operators $A^{\pm}$ are described by the momentum operator (2) and the position operator x with the commutation relation

$$\left[x, \hat{p}\right] = i - \frac{1}{4}\left(\underset{\rightarrow}{\partial} - \underset{\leftarrow}{\partial}\right) = i \cosh \frac{i}{2}\frac{d}{dx} \tag{24}$$

whereas $\overset{\wedge}{\psi}^{\dagger}$ and $\overset{\wedge}{\psi}$ are Fermi degrees of freedom with the corresponding anticommutation relations:

$$\left\{\overset{\wedge}{\psi}^{\dagger}, \overset{\wedge}{\psi}\right\} = 1, \quad \left\{\overset{\wedge}{\psi}, \overset{\wedge}{\psi}\right\} = \left\{\overset{\wedge}{\psi}^{\dagger}, \overset{\wedge}{\psi}^{\dagger}\right\} = 0 \tag{25}$$

This yields (21) and

$$\hat{H} = H - \frac{1}{2} \cdot \left[\overset{\wedge}{\psi}^{\dagger}, \overset{\wedge}{\psi}\right] \cdot \Delta V(x). \tag{26}$$

We introduce the operator

$$\begin{aligned} H &= \tfrac{1}{2} \cdot \{A^{-}, A^{+}\}_{q(x)} = \tfrac{1}{2} \cdot \left(A^{-} \cdot q(x) \cdot A^{+} + A^{+} \cdot q^{-1}(x) \cdot A^{-}\right) = \\ &= H_0 + \alpha(x)\, \alpha_{\frac{\varsigma}{2}}(x) - \alpha(x) \cdot \cosh \tfrac{i}{2}\tfrac{d}{dx}\left(\alpha(x) \cdot \cosh 2\rho_{\frac{s}{2}}(x)\right) \rightarrow \\ &\rightarrow -\tfrac{1}{2}\tfrac{d^2}{dx^2} + \tfrac{1}{2}\left(\rho'(x)\right)^2, \end{aligned} \tag{27}$$

where the operator

$$H_0 = \frac{1}{2}\left[\alpha(x) \cdot \hat{p}\right]^2 = \frac{\overset{\wedge}{\Pi}^2}{2} \rightarrow -\frac{1}{2}\frac{d^2}{dx^2} \tag{28}$$

plays the role of free hamiltonian with the finite-difference momentum operator

$$\overset{\wedge}{\Pi} = 2\alpha(x) \cdot \hat{p} \tag{29}$$

modified by the interaction

$$
\begin{aligned}
H_+ &= A^+ \cdot q^{-1}(x) \cdot A^- = H - \Delta V(x) \\
H_- &= A^- \cdot q(x) \cdot A^+ = H + \Delta V(x)
\end{aligned}
\tag{30}
$$

$$
\Delta V(x) = -\alpha(x) \sinh \frac{i}{2}\frac{d}{dx} \left[\alpha(x) \sinh 2\rho_{\frac{x}{2}}(x)\right]
\tag{31}
$$

In the (2×2)-representation

$$
\overset{\wedge}{\psi}{}^{\dagger} = \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \overset{\wedge}{\psi} = \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\tag{32}
$$

$$
\left[\overset{\wedge}{\psi}{}^{\dagger}, \overset{\wedge}{\psi}\right] = -\sigma_3
\tag{33}
$$

we find from (26)

$$
\begin{aligned}
\hat{H} &= H + \tfrac{1}{2} \cdot \Delta V(x) \cdot \sigma_3 = \\
&= \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} = \begin{pmatrix} A^- \cdot q(x) \cdot A^+ & 0 \\ 0 & A^+ \cdot q^{-1}(x) \cdot A^- \end{pmatrix}
\end{aligned}
\tag{34}
$$

3 Examples

3.1 Relativistic oscillator (q-oscillator)

In this case, we have [2, 3]:

$$
\rho(x) = \frac{m\omega x^2}{2\hbar}
\tag{35}
$$

The deformation parameter is a constant and we come to the q-oscillator with

$$
q(x) = const = q = e^{-\frac{\omega\hbar}{4mc^2}}
\tag{36}
$$

and

$$
\alpha(x) = \frac{1}{\cos\frac{\omega x}{2c}}.
\tag{37}
$$

The finite-difference ladder operators have the form

$$
A^{\pm} = \pm i\sqrt{2} \cdot e^{\pm\frac{\omega}{8}} \cdot \left(\sinh \frac{i}{2}\frac{d}{dx} \mp i \tan\frac{\omega x}{2} \cdot \cosh \frac{i}{2}\frac{d}{dx}\right).
\tag{38}
$$

SUSY Hamiltonian (34) becomes

$$
\begin{aligned}
\hat{H} &= \begin{pmatrix} e^{-\frac{\omega}{4}} A^- A^+ & 0 \\ 0 & e^{\frac{\omega}{4}} A^+ A^- \end{pmatrix} = \begin{pmatrix} h + e_0 & 0 \\ 0 & h - e_0 \end{pmatrix} \to \\
&\to \begin{pmatrix} -\frac{1}{2}\frac{d^2}{dx^2} + \frac{\omega^2 x^2}{2} + \frac{\omega}{2} & 0 \\ 0 & -\frac{1}{2}\frac{d^2}{dx^2} + \frac{\omega^2 x^2}{2} - \frac{\omega}{2} \end{pmatrix}
\end{aligned}
\tag{39}
$$

where

$$e_0 = 2\sinh\frac{\omega}{4} \to \frac{\omega}{2} \tag{40}$$

and

$$\begin{aligned} h = \{A^-, A^+\}_q = 2\left\{\left(\frac{1}{\cos\frac{\omega x}{2}}\cdot\cosh\frac{i}{2}\frac{d}{dx}\right)^2 - \cosh\frac{\omega}{4}\right\} = \\ = \frac{\hat{P}}{2} + V(x) \end{aligned} \tag{41}$$

$$\hat{P} = -\frac{2}{\cos\frac{\omega x}{2}}\cdot\sinh\frac{i}{2}\frac{d}{dx} \to -i\frac{d}{dx} \tag{42}$$

The relativistic oscillator potential is

$$V(x) = \frac{\cosh\frac{\omega}{4}\cdot\left[\sin^2\frac{\omega x}{2} - \sinh^2\frac{\omega}{4}\right]}{\left[\cos^2\frac{\omega x}{2} + \sinh^2\frac{\omega}{4}\right]} \to \frac{\omega^2 x^2}{2} \tag{43}$$

The spectrum is

$$\begin{aligned} &\begin{pmatrix} e_n^- = 2\left(e^{\frac{2n+1}{4}\omega} - e^{-\frac{\omega}{4}}\right) & 0 \\ 0 & e_n^+ = 2\left(e^{\frac{2n+1}{4}\omega} - e^{\frac{\omega}{4}}\right) \end{pmatrix} \to \\ &\to \begin{pmatrix} \left(n+\frac{1}{2}\right)\omega + \frac{\omega}{2} & 0 \\ 0 & \left(n+\frac{1}{2}\right)\omega - \frac{\omega}{2} \end{pmatrix} \end{aligned} \tag{44}$$

Thus, we have two q-oscillators with zero point of energy shifted by $\pm e_0$. In other words, the energies of q-supersymmetric partners are connected by

$$e_{n+1}^- = q^{-2}\cdot e_n^+ \tag{45}$$

3.2 Radial part of three-dimensional relativistic Schrödinger equation

This equation

$$\begin{aligned} H_l s_l(r,\chi) = \left\{2\sinh^2\frac{i}{2}\frac{d}{dr} + \frac{l(l+1)}{2r(r+i)}e^{i\frac{d}{dr}}\right\} s_l(r,\chi) = \\ = (\cosh\chi - 1)\, s_l(r,\chi) \end{aligned} \tag{46}$$

can be considered as one-dimensional with the potential $\frac{l(l+1)}{2r(r+i)}e^{i\frac{d}{dr}}$. Solutions of this equation, i.e., the free relativistic radial waves have the form

$$s_l(r,\chi) = \sqrt{\frac{\pi\sinh\chi}{2}}\cdot(-i)^{l+1}\cdot\frac{\Gamma(ir+l+1)}{\Gamma(ir)}\cdot P_{ir-\frac{1}{2}}^{-(l+\frac{1}{2})}(\cosh\chi) \tag{47}$$

In the nonrelativistic limit, these functions turn into free solutions of the Schrödinger equation

$$s_l(r,\chi) \to s_l(pr) = \sqrt{\frac{\pi r p}{2}}\cdot J_{l+\frac{1}{2}}(pr) \tag{48}$$

In this case the relativistic finite-difference ladder operators have the form

$$\lambda^{\pm} = \pm\frac{i}{\sqrt{2}}\left[\frac{ir \mp (l+1)}{ir} - e^{i\frac{d}{dr}}\right] \rightarrow a^{\pm} = \mp\left[\frac{d}{dr} \pm \frac{l+1}{r}\right] \tag{49}$$

Hence,

$$H_{+} = H_l = \lambda^{+}\cdot e^{i\frac{d}{dr}}\cdot\lambda^{-} \qquad H_{-} = H_{l+1} = \lambda^{-}\cdot e^{i\frac{d}{dr}}\cdot\lambda^{+} \tag{50}$$

In contrast with the nonrelativistic case, the rising and lowering operators $\Lambda^{\pm}$, which shift the value of the angular momentum

$$\Lambda^{+}s_{l+1}(r,\chi) = s_l(r,\chi) \qquad \Lambda^{-}s_l(r,\chi) = s_{l+1}(r,\chi) \tag{51}$$

and the ladder operators (49) factorizing the Hamiltonian are different:

$$\begin{aligned} \Lambda^{+} &= \tfrac{i}{\sinh\chi}\cdot\left[\cosh\chi - \tfrac{ir-l-2}{ir-1}\cdot e^{i\frac{d}{dr}}\right] \rightarrow a^{+} \\ \Lambda^{-} &= -\tfrac{i}{\sinh\chi}\cdot\left[\cosh\chi - \tfrac{ir+l}{ir-1}\cdot e^{i\frac{d}{dr}}\right] \rightarrow a^{-} \end{aligned} \tag{52}$$

Let us consider the identity

$$[-H_{l+1} + H_l]\cdot H_l - [H_{l+1} - H_l]\cdot H_l \equiv 0 \tag{53}$$

Using the relation

$$\lambda^{-}e^{i\frac{d}{dr}} - e^{i\frac{d}{dr}}\lambda^{-} = -\frac{i}{\sqrt{2}}\cdot[H_{l+1} - H_l] \tag{54}$$

we have

$$[-H_{l+1} + H_l]\cdot H_l - i\sqrt{2}\cdot\left[\lambda^{-}e^{i\frac{d}{dr}} - e^{i\frac{d}{dr}}\lambda^{-}\right]\cdot H_l = 0 \tag{55}$$

After acting on $s_l(r,\chi)$ and taking into account (46) and the relation

$$\Lambda^{-} = \frac{i}{\sinh\chi}\cdot\left[-(\cosh\chi - 1) + i\sqrt{2}\cdot e^{i\frac{d}{dr}}\cdot\lambda^{-}\right], \tag{56}$$

we come to the formula

$$H_{l+1}\cdot\Lambda^{-}s_l(r,\chi) = \Lambda^{-}\cdot H_l s_l(r,\chi) \tag{57}$$

which allows us to consider relativistic l and $l+1$ states as deformed supersymmetric partner states. If $s_l(r,\chi)$ is the eigenstate of H_l, then $(\Lambda^{-}s_l(r,\chi))$ is the eigenstate of H_{l+1} with the same eigenvalue

$$\begin{aligned} &H_l s_l(r,\chi) = (\cosh\chi - 1)\cdot s_l(r,\chi) \rightarrow \\ &\rightarrow H_{l+1}\cdot(\Lambda^{-}s_l(r,\chi)) = (\cosh\chi - 1)\cdot(\Lambda^{-}s_l(r,\chi)) \end{aligned} \tag{58}$$

The non-relativistic (non-deformed) analog of (57) is the relation

$$H_{l+1}\cdot a^{-}s_l(pr) = a^{-}\cdot H_l s_l(pr) \tag{59}$$

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