Measure on Time Scales with Mathematica

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Abstract. In this paper we study the Lebesgue Δ-measure on time scales. We refer to [3, 4] for the main notions and facts from the general measure and Lebesgue Δ integral theory. The objective of this paper is to show how the main concepts of *Mathematica* can be applied to fundamentals of Lebesgue Δ- and Lebesgue ∇- measure on an arbitrary time scale and also on a discrete time scale whose rule is given by the reader. As the time scale theory is investigated in two parts, by means of σ and ρ operators, we named the measures on time scales by the set function **DMeasure** and **NMeasure** respectively for arbitrary time scales.

1 Introduction

Time Scales works can be found in [1, 2,5]. The software *Mathematica* is one of the most powerful tool in discrete and continuous analysis. Computational works on time scale calculus are collected in **TimeScale** package [6]. As probability theory is established on continuous and discrete analysis, we generalize **TimeScale** package in order to calculate the measure on time scales as a first step of probability theory and statistics.

In this paper first we give an introduction to the time scales, and we present the connection of the Lebesgue measure with the Lebesgue Δ-measure on an arbitrary bounded time scale T such that $\min \mathrm{T} = a$ and $\max \mathrm{T} = b$. We set out many basic concepts from measure theory to the Δ-measurable space. Finally we use *Mathematica* to illustrate the Lebesgue Δ- and ∇- measures to set out the differences between Lebesgue measure and Lebesgue Δ (∇)- measure.

2 Δ- and ∇- Measure on Time Scale

Let T be a time scale, $a < b$ be points in T, and $[a, b)$ be half closed bounded interval in T, σ and ρ be the forward and backward jump operators respectively on T. Let

$$\Im_1 = \{[a', b') \bigcap \mathrm{T} : a', b' \in \mathrm{T}, a' \leq b'\}$$

be the family of all left closed and right open intervals of T. Then $\Im_1$ is a semiring. Here $[a', a') = \emptyset$. $m_1 : \Im_1 \to [0, \infty]$ is a set function which assigns to each interval its length: $m_1([a', b')) = b' - a'$. So if $\{I_n\}$ is a sequence of disjoint intervals in $\Im_1$ then $m_1(\bigcup I_n) = \sum m_1(I_n)$.

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Let $E \subset \mathrm{T}$. By Carathéodory extension, outer measure of E is

$$m_1^*(E) = \inf_{E \subset \bigcup\limits_n I_n} \sum m_1(I_n)$$

where $I_n \in \Im_1$. If there is no such covering of E, then $m_1^*(E) = \infty$.

Definition 1. *A set $E \subset \mathrm{T}$ is said to be Δ- measurable if for each set A*

$$m_1^*(A) = m_1^*(A \cap E) + m_1^*(A \bigcap E^c)$$

where $E^c = \mathrm{T} - E$. If E is Δ-measurable then E^c is also Δ-measurable. Clearly $\emptyset$ and T are Δ- measurable.
Let $\mathfrak{M}(m_1^*) = \{E \subset \mathrm{T} : E \; is \; \Delta \; measurable\}$ be a family of Δ- measurable sets.

Corollary 1. *$\mathfrak{M}(m_1^*)$ is a σ algebra.*

Definition 2. *The restriction of m_1^* to $\mathfrak{M}(m_1^*)$ is called Lebesgue Δ- measure and denoted by μ_Δ.*

So $m_1^*(E) = \mu_\Delta(E)$ if $E \in \mathfrak{M}(m_1^*)$.

Similarly, if we take $\mathfrak{F}_2 = \{(a', b'] : a', b' \in \mathrm{T}, a' \leq b'\}$ where $(a', a']$ is understood as an empty set then $m_2 : \mathfrak{F}_2 \to [0, \infty]$ such that $m_2((a', b']) = b' - a'$ is a countably additive measure. Then $\mathfrak{M}(m_2^*)$ is the set of ∇- measurable sets and μ_∇ is Lebesgue ∇- measure on T.

Proposition 1. *Let $\{E_n\}$ be an infinite decreasing sequence of Δ- measurable sets, that is, a sequence $E_1 \supset E_2 \supset \cdots \supset E_n \supset \cdots$, $E_i \in \mathfrak{F}_1$ for each i, $\cap E_i \in \mathfrak{F}_1$ and $m_1^*(E_1) < \infty$. Then*

$$m_1^*(\bigcap_{n=1}^{\infty} E_i) = \lim_{n \to \infty} m_1^*(E_n).$$

3 ∇ and Δ Measures with Mathematica

Theorem 1. *For each $t_0 \in \mathrm{T} - \{\min \mathrm{T}\}$ the ∇- measure of the single point set $\{t_0\}$ is given by $\mu_\nabla(\{t_0\}) = t_0 - \rho(t_0)$.*

Proof. **Case 1.** Let t_0 be left scattered. Then $\{t_0\} = (\rho(t_0), t_0] \in \mathfrak{F}_2$. So $\{t_0\}$ is ∇ measurable and $\mu_\nabla(\{t_0\}) = t_0 - \rho(t_0)$.
Case 2. Let t_0 be left dense. Then there exists an increasing sequence $\{t_k\}$ of points of T such that $t_k \leq t_0$ and $t_k \uparrow t_0$. Since $\{t_0\} = \bigcap\limits_{k=1}^{\infty} (t_k, t_0] \in \mathfrak{F}_2$. Therefore $\{t_0\}$ is ∇ measurable. By continuity

$$\mu_\nabla(\{t_0\}) = \mu_\nabla(\bigcap_{k=1}^{\infty} (t_k, t_0]) = \lim_{n \to \infty} \mu_\nabla((t_n, t_n]) = \lim_{n \to \infty} t_0 - t_n = 0$$

which is the desired result since t_0 is left dense.

Theorem 2. *If* $a, b \in \mathrm{T}$ *and* $a \leq b$ *then* 1)$\mu_\nabla((a,b]) = b - a$, 2)$\mu_\nabla((a,b)) = \rho(b) - a$, 3) *If* $a, b \in \mathrm{T} - \min \mathrm{T}$ *then* $\mu_\nabla([a,b)) = \rho(b) - \rho(a)$ *and* $\mu_\nabla([a,b]) = b - \rho(a)$.

Proof. From the definition $\mu_\nabla((a,b]) = b - a$.

$$\mu_\nabla((a,b]) = \mu_\nabla((a,b) \bigcup \{b\}) = \mu_\nabla((a,b)) + \mu_\nabla(\{b\}) = \mu_\nabla((a,b)) + b - \rho(b)$$

$$b - a = \mu_\nabla((a,b)) + b - \rho(b)$$

So $\mu_\nabla((a,b)) = \rho(b) - a$.

iii) Let $a, b \in \mathrm{T} - \min \mathrm{T}$.

$$\mu_\nabla([a,b)) = \mu_\nabla(\{a\} \bigcup (a,b)) = \mu_\nabla(\{a\}) + \mu_\nabla((a,b)) = a - \rho(a) + \rho(b) - a = \rho(b) - \rho(a)$$

$$\mu_\nabla([a,b]) = \mu_\nabla([a,b) \bigcup \{b\}) = \mu_\nabla([a,b)) + \mu_\nabla(\{b\}) = \rho(b) - \rho(a) + b - \rho(b) = b - \rho(a)$$

Theorem 3. *For each* $t_0 \in \mathrm{T} - \{\max \mathrm{T}\}$ *the single point set* $\{t_0\}$ *is* Δ*- measurable and its* Δ*- measure is given by* $\mu_\Delta(\{t_0\}) = \sigma(t_0) - t_0$.

Proof. **Case 1.** Let t_0 be right scattered. Then $\{t_0\} = [t_0, \sigma(t_0)) \in \mathfrak{F}_1$. So $\{t_0\}$ is Δ- measurable and $\mu_\Delta(\{t_0\}) = \sigma(t_0) - t_0$.
Case 2. Let t_0 be right dense. Then there exists a decreasing sequence $\{t_k\}$ of points of T such that $t_0 \leq t_k$ and $t_k \downarrow t_0$. Since $\{t_0\} = \bigcap_{k=1}^{\infty} [t_0, t_k) \in \mathfrak{F}_1$. Therefore $\{t_0\}$ is Δ- measurable. By proposition 1

$$\mu_\Delta(\{t_0\}) = \mu_\Delta(\bigcap_{k=1}^{\infty} [t_0, t_k)) = \lim_{n \to \infty} \mu_\Delta([t_0, t_n)) = \lim_{n \to \infty} t_n - t_0 = 0$$

which is the desired result since t_0 is right dense.

Every kind of interval can be obtained from an interval of the form $[a, b)$ by adding or subtracting the end points a and b. Then each interval of T is Δ-measurable.

Theorem 4. *If* $a, b \in \mathrm{T}$ *and* $a \leq b$ *then*
1)$\mu_\Delta([a,b)) = b - a$, 2)$\mu_\Delta((a,b)) = b - \sigma(a)$, 3) *If* $a, b \in \mathrm{T} - \max \mathrm{T}$ *then* $\mu_\Delta((a,b]) = \sigma(b) - \sigma(a)$ *and* $\mu_\Delta([a,b]) = \sigma(b) - a$.

To illustrate these properties with mathematica, our **TimeScale** package must be loaded.
In[1]:= << TimeScale‘
Let the time scale is as follows
In[2]:= $T = \{5 <= x <= 7 || x == 15/2 || 9 <= x <= 11 || x == 12 || x == 18\}$
We must define set function with respect to the interval or a single point
In[3]:= ClosedSet={ closed,a,b,closed }; OpenSet={ open,a,b,open };

RSemiClosedSet={ open,a,b,closed };
LSemiClosedSet = { open,a,b,closed };
Spoint={ closed,a,closed };
DMeasure[ClosedSet[a_,b_]]:=sigma[b]-a;
NMeasure[OpenSet[a_,b_]]:=b-sigma[a];
DMeasure[LSemiClosedSet[a_,b_]]:=b-a;
DMeasure[RSemiClosedSet[a_,b_]]:=sigma[b]-sigma[a];
DMeasure[Spoint[a_]]:=sigma[a]-a;
Now, we would like to find the measures of $\{5\}$ and $(5, 11]$ **In[4]:= DMeasure[Spoint[7]]**
Out[4]:= $\frac{1}{2}$
In[5]:= DMeasure[RSemiClosedSet[5,11]]
Out[6]:=7
Mathematica applications of ∇-measure also can be done as Δ-measure. The sigma operator must be replaced by the r operator also.

4 Conclusion

In this work we worked on Lebesgue Δ-measure and Lebesgue ∇-measure on time scales with *Mathematica*. We investigated each of these two measures in two parts, arbitrary time scales and discrete time scales. To do these we improved the **TimeScale** package. In the future, we will work on generalizing the probability theory on Time Scales with *Mathematica*.

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