DARK ENERGY MECHANISMS IN THE CONTEXT OF EXTRA DIMENSIONAL MODELS

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ABSTRACT

DARK ENERGY MECHANISMS IN THE CONTEXT OF EXTRA DIMENSIONAL MODELS

Dark energy is the simplest and the most standart explanation to account for the observed accelerated expansion of the universe. In this thesis we use the term 'dark energy' in its standart meaning i.e. a field or fluid that is responsible for the cosmic acceleration in the frame work of general relativity. Meanwhile extra dimensions is an attractive framework to understand many otherwise unexplained physical phenomena in a clear, simple formulation. Therefore the study of extra dimensional cosmological models is an attractive area of study. In this thesis we have considered viability of extra dimensional cosmological models in the light of the accelerated expansion of the universe. We have confirmed the results of studies that have shown the incompatibility of a broad class of extra dimensional cosmological models with a dark energy of an equation of state close to that of cosmological constant. We have discussed also possible theoritical and observational ways to avoid the no-go theorems for extra dimensional cosmological models as well.

ÖZET

EK BOYUTLU MODELLER ÇERÇEVESİNDE KARANLIK ENERJİ MEKANİZMALARI

Karanlık enerji evrenin ivmelenerek genişlemesi gözlemlerini açıklayan en basit ve en standart yoldur. Bu tezde karanlık enerji terimi en standart anlamında yani genel görelilik çerçevesindeki kozmik ivmelenmenin sorumlusu olan bir alan yada akışkan anlamında kullanılmıştır. Öte yandan, ek boyutlar diğer birçok açıklanmamış fiziksel fenomenleri açık ve basit formülasyonda anlamaya yarayan bir çerçevedir. Bu yüzden ek boyutlu kozmolojik modellerin araştırılması ilgi çekici bir çalışma alanıdır. Bu tezde evrenin ivmelenerek genişlemesi gerçeğinin ışığı altında ek boyutlu kozmolojik modeller incelendi. Ek boyutlu kozmolojik modeller çerçevesinde durum eşitliği kozmolojik sabite yakın olan karanlık enerji modelleri elde etmenin zorluğuna ilişkin daha önce yapılmış çalışmalar gözden geçirildi ve elde edilen sonuçlar doğrulandı. Ek boyutlu kozmolojik modellere ilişkin elde edilmiş olan sınırlayıcı teoremlerden kaçınmanın teorik ve gözlemsel yolları da tartışıldı.

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CHAPTER 1

INTRODUCTION

Since 1929 it had been known that the universe is expanding [1, 2]. Researchers were expecting that the expansion should be slowing because the universe's own gravity tugs against the expansion. The cosmic acceleration rate can be observed by measuring the distances to exploding stars known as supernovae. In 1998 in a quite unexpected way the observations of high redshift supernovae revealed that the universe is accelerating at present [3, 4]. There must exist something to make the universe expand at an accelerating rate. There are some frameworks to explain this acceleration; some form of energy (called dark energy) [5–9], modification of gravity [6, 10], inhomogeneity in the matter distribution [11, 12] are the main of these frameworks. The framework we adopt in this thesis is the most standard of these, known as 'Dark Energy', some kind of fluid or matter field that causes accelerated expansion of universe [13] in the context of Einstein's theory of general relativity [14]. In the following sections we consider some basic candidates for dark energy.

Dimension is a natural concept to humans in everyday life. A dimension is a parameter or measurement used to describe some relevant characteristic of a place or object. The time and space are known examples of dimensions. Einstein's theory of relativity is formulated in 4-dimensions. The question is how it is possible to have more than 4 dimensions because we do not see the effect of extra dimensions. The possibility of existence of extra dimensions although we do not see them in everyday life may be seen through an example. Let us suppose an ant which is moving on a cord. When we look from a distance we see the cord as one dimensional. But when we zoom onto the cord, we see one dimension is not enough to describe the exact position of the ant. Therefore we need a second dimension which takes the form of a small compact circle having the thickness of the cord. The ant can also move along this circle. As we see from this example a one dimensional picture from a distance could in fact contain two dimensions. In fact this (i.e. taking the extra dimension small and compact) is one way to explain why it is not observed at low energies. Another way is to make matter be confined to a four dimensional wall i.e. a brane in extra dimensions [15, 16].

We may ask what the physical effect of extra dimensions would be here. Let us start with the gravitational force between two objects. This force has a magnitude proportional to $\frac{1}{r^2}$. When we suppose in addition N extra dimensions, we will see this force changing to $\frac{1}{r^{2+N}}$. Hence we see the number of extra dimensions change the nature of the physical law of this force. This is why the question of existance of extra dimensions becomes an experimental question. Another reason for extra dimensions is also related to gravity. We do not know the behavior of gravity at distances shorter than 10^{-4} cm and at distances larger than 10^{28} cm . All what we know about the gravity is within this range. In addition to gravity, electromagnetic interactions which obey inverse square law also are known down to distances of 10^{-16} cm but below this scale there might exist a change in the behaviour of it. Therefore there is a possibility that they can change with the laws of extra dimensional space if extra dimensions exist. One can ask what are the benefits of a world with extra dimensions? I will give few titles related to this questions, unification of gravity, quantization of gravitational interactions, Higgs mass hierarchy problem, cosmological constant problem, etc.

As we have discussed above, extra dimension is an attractive framework to explain some phenomena or relations in nature that seem unaddressed in a simple way. Therefore the use of extra dimensions to account for accelerated expansion of the universe (cosmic acceleration) is quite natural and is discussed in many studies and models [17–19].

In this thesis we questioned if extra dimensional models that include cosmic acceleration may be realized in a way consistent with observations and within standart theoretical framework. To this end first we reviewed dark energy and extra dimensions. Then, we reconsidered the constraints on broad class of models that are derived from energy conditions [20, 21]. We have confirmed their conclusion and discussed possible routes to avoid these constraints.

Note that we take the signature of the four dimensional metric be (-,+,+,+) unless otherwise stated.

CHAPTER 2

BASICS OF COSMIC EXPANSION

2.1. Hubble's Law

In this chapter I will write the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric and the corresponding equations to explain the rate of expansion of the universe. But firstly we start with Hubble law.

Hubble law [1] is a natural consequence of homogeneity and isotropy in an expanding universe (here homogeneity tells us that universe looks the same when it is observed from any point whereas isotropy means that the universe looks same in any direction). Now let us start with a coordinate system with origin O at which matter is at rest. The velocity field that is the relative average velocity of matter at any two points depends on the radius r (i.e. the distance between the points) and time t (i.e. the time that takes light to travel between these points). We can denote this velocity field by $v(r,t)$ and write down as;

$$v(r,t) = H(t)r(t) \tag{2.1}$$

where $H(t)$ is known as the Hubble parameter. Hubble's law tells us how the average velocity between any two points in space changes with time. Therefore one names $H(t)$ as the expansion rate of the universe. From equation (2.1) one may write;

$$\frac{dr}{dt} = H(t)r \tag{2.2}$$

$$\frac{dr}{r} = H(t)dt \tag{2.3}$$

$$\int_{r_0}^{r_t} \frac{dr}{r} = \int_0^t H(t)dt \tag{2.4}$$

Now integrating both sides we have;

$$r(t) = r(0)\exp\int_o^t Hdt \tag{2.5}$$

Here we will introduce $a(t)$ which is known as scale factor :

$$a(t) = \frac{r(t)}{r(0)} \tag{2.6}$$

then we may write

$$a(t) = \exp \int_0^t H(t')dt' \tag{2.7}$$

taking the natural logarithm of both sides and taking the derivatives with respect to time we have;

$$H(t) = \frac{\dot{a}}{a} \tag{2.8}$$

As it can be seen from (2.8) the expansion rate $H(t)$ is a function of scale factor $a(t)$. This scale factor is so important that it contains complete information about the dynamics of homogeneous and isotropic universe.

According to the Hubble's law, if the universe is isotropic and homogeneous, particles move radially from an observer which may be supposed as located at the origin of a sphere. We can see this directly from the relation (2.1) which tells us that the velocity of a moving particle is associated with the radial distance of the particle from the observer. This motion is known as 'Hubble Flow'.

2.2. Newtonian Cosmology

The picture in Friedmann-Lemaitre-Robertson-Walker (FLRW) space may be made more plausible through a naive Newtonian analysis [22, 23]. One may write the evolution equation for Hubble's parameter in (2.8) by using a naive Newtonian approach. Let us start with a particle which is located at a point on a sphere and the particle is at a distance $r(t)$ from the origin. Here because of the isotropy we take spherical symmetry. We consider that at a given time t, there exists matter with density $\rho_b(t)$ at the origin. For given particle of mass m we can write the gravitational force and then by calculating the potential energy of it we may write the total energy of this particle. The gravitational force on this mass m is,

$$\overrightarrow{F} = -\frac{GMm}{r^2}\widehat{r} = -\overrightarrow{\nabla}U \tag{2.9}$$

where M is the mass located at the origin of the sphere. From equation (2.9) we may calculate the potential energy of this mass m which is;

$$U(r) = -\frac{GMm}{r} \tag{2.10}$$

The total energy of the mass m is;

$$E_{tot} = \frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} \tag{2.11}$$

Since we are on the sphere, we may write (2.11) in terms of the volume and density of matter located at the origin as;

$$\rho_b(t) = \frac{M}{V} \tag{2.12}$$

$$V = \frac{4}{3}\pi r^3(t) \tag{2.13}$$

$$E_{tot} = \frac{1}{2}m\dot{r}^2 - \frac{4\pi}{3}Gm\rho_b r^2 \tag{2.14}$$

Now from the total energy equality if we divide each term by $\frac{mr^2}{2}$, we will have;

$$\frac{\dot{r}^2}{r^2} = \frac{8\pi}{3}G\rho_b + \frac{2E_{tot}}{mr^2} \tag{2.15}$$

Here the term on the left hand side of equation (2.15) may be written as $\frac{\dot{a}^2}{a^2}$ which is also equal to the square of Hubble's parameter H. Then (2.15) may be written as;

$$H^2 = \frac{8\pi}{3}G\rho_b + \frac{2E_{tot}}{mr^2} = \frac{\dot{a}^2}{a^2} \tag{2.16}$$

This equation is known as the Friedmann equation. In this equation we will write the second term on the right side in terms of a new parameter K. Then it may be written as $K = -\frac{2E_{tot}}{r^2(0)m}$. Here the sign of K depends on the energies of the mass m that is comparison of kinetic energy and potential energy because if $E_k > E_p$ in (2.14) then we see K is negative but if $E_k < E_p$, K is positive. Also it can be zero when two energies are equal. Now one may relate the results of this analysis to the general relativity concepts. As we will seee in the next section the sign of K shows the geometry of space. If K is be zero, then we say universe is flat, for K is negative it is an open universe which is also called hyperbolic like and finally for K is positive it is a closed universe that is spherical like.

2.3. FLRW Metric and the Corresponding Einstein Equations

Friedmann-Lemaitre-Robertson-Walker (FLRW) metric is an exact solution to the Einstein's field equations of general relativity. The form of the metric describes the universe as homogeneous, isotropic and expanding. Since the scale factor is written in the metric, the solutions of field equations must give the size of universe as a function of time. This metric may be written in 4-D as;

$$ds^2 = -dt^2 + a^2(t)[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + sin^2(\theta)d\phi^2)] \tag{2.17}$$

where $a(t)$ is the scale factor and $K = 0, \pm 1$. As we mentioned before, the sign of K shows the geometry of space. If K is zero, then we say the universe is flat, if K is negative it is an open universe which is also called hyperbolic like, and finally if K is positive it is a closed universe that is spherical like. The coordinates (r,θ,ϕ) are co-moving coordinates in which a freely moving particle comes to rest. This form of the metric is written in spherical coordinates. The spatially flat case i.e. $K = 0$ may be written in Cartesian coordinates as;

$$ds^2 = -dt^2 + a^2(t)[(dx_1)^2 + (dx_2)^2 + (dx_3)^2] \tag{2.18}$$

To have information about the dynamics of this metric we should solve Einstein's equations for the scale factor $a(t)$. Now to have a differential equation for the scale factor we should consider Einstein equations which are written as;

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{2.19}$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor and $T_{\mu\nu}$ denotes the energy momentum tensor. Ricci tensor may be written as;

$$R_{\mu\nu} = \Gamma^{\rho}_{\mu\nu},_{\rho} - \Gamma^{\rho}_{\mu\rho},_{\nu} + \Gamma^{\rho}_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu} \tag{2.20}$$

where $\Gamma^{\rho}_{\mu\nu},_{\rho}$ denotes $\dfrac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\rho}}$ and $\Gamma^{\rho}_{\mu\nu}$ is called 'Affine connection' in general. When it is symmetric i.e. when it is the metric compatible; $\nabla_{\mu} g_{\rho\tau} = 0$ it is called the 'Christoffel symbol'. Christoffel symbols have the form written as ;

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} [g_{\mu\sigma},_{\nu} + g_{\sigma\nu},_{\mu} - g_{\mu\nu},_{\sigma}] \tag{2.21}$$

Now after giving these information we may calculate Einstein tensor for FLRW metric given in (2.17). The elements of our metric may be written as; $g_{00} = -1, \;\; g_{11} = \dfrac{a^2(t)}{1 - Kr^2}, \;\; g_{22} = a^2(t) r^2, \;\; g_{33} = a^2(t) r^2 sin^2(\theta)$. The explicit calculations of $G_{\mu\nu}$ are given in appendix A.

The elements of the Einstein tensor are;

$$G_{00} = \frac{3}{a^2}(\dot{a}^2 + K), \quad G_{11} = (Kr^2 - 1)(2\ddot{a}a + \dot{a}^2 + K)$$
$$G_{22} = -r^2(2a\ddot{a} + \dot{a}^2 + K), \quad G_{33} = -r^2 sin^2(\theta)(2a\ddot{a} + \dot{a}^2 + K)$$

We may also write the elements of energy momentum tensor using these equations and the relation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ as;

$$T_{00} = \frac{1}{(8\pi G)} G_{00} = \frac{1}{(8\pi G)} \frac{3}{a^2}(\dot{a}^2 + K)$$
$$T_{11} = \frac{1}{(8\pi G)} G_{11} = \frac{1}{(8\pi G)} (Kr^2 - 1)(2\ddot{a}a + \dot{a}^2 + K)$$
$$T_{22} = \frac{1}{(8\pi G)} G_{22} = \frac{1}{(8\pi G)} r^2 (2a\ddot{a} + \dot{a}^2 + K)$$
$$T_{33} = \frac{1}{(8\pi G)} G_{33} = \frac{1}{(8\pi G)} r^2 sin^2(\theta)(2a\ddot{a} + \dot{a}^2 + K)$$

In all our calculations i, j range from 1 to 3, G is gravitational constant and R is Ricci curvature scalar. When we suppose an ideal perfect fluid, because of homogeneity and isotropy, the energy momentum tensor takes the form;

$$T^{\mu\nu} = (\rho + P)U^{\mu}U^{\nu} + Pg^{\mu\nu} \tag{2.22}$$

where ρ is the energy density, P is the pressure, U^{μ} is the velocity vector field. If the three dimensional space is flat then (2.22) in co-moving coordinates becomes ;

$$T_{\mu\nu} = Diag(\rho, P, P, P) \tag{2.23}$$

where $U^{\mu} = (1, 0, 0, 0)$ in co-moving coordinates.

We have written Einstein equations and the elements of Einstein tensor $G_{\mu\nu}$. From the four Einstein equations, we are able to write two independent equations. For the 00 component we have;

$$\begin{aligned} G_{00} &= 8\pi G T_{00} \\ 8\pi G\rho &= \frac{3}{a^2}(\dot{a}^2 + K) \Rightarrow H^2 = \frac{(8\pi G)}{3}\rho - \frac{K}{a^2} \end{aligned} \tag{2.24}$$

where $H = \dfrac{\dot{a}}{a}$ is the Hubble parameter. For another element of Einstein tensor we have ;

$$\begin{aligned} G_{11} &= 8\pi G T_{11} \\ -2\frac{\ddot{a}}{a} - (\frac{\dot{a}}{a})^2 - \frac{K}{a^2} &= 8\pi G P \end{aligned} \tag{2.25}$$

After multiplying (2.25) by three and then adding the resulting equation to (2.24) one gets;

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{2.26}$$

We remind that ρ denotes the total energy density of all the fluid components present in the universe. In this equation we want $\ddot{a}$ to be positive to have an accelerating universe, so the parenthesis on right side of equation must be negative.

2.4. Observational Evidence for Cosmic Expansion and Acceleration

In 1929 Edwin Hubble, who is an astronomer, was working at the Carnegie Observatories in Pasadena, California. He made some observation about the expansion of universe and he measured the redshifts of a number of distant galaxies. He also measured the relative distances of these galaxies by measuring the apparent brightness of a class of stars in each galaxy. When he plotted redshift against relative distance, he found that the average redshift of distant galaxies increased as a linear function of their distance as we mentioned in the first section of this chapter. But there must have existed an explanation for this relation and the only explanation is that the universe was expanding. This may be shown by using mathematical explanation of redshift.

In the picture of Hubble, ratio of the wavelength of an observed light to the that of source is given as;

$$\frac{\lambda_{obs}}{\lambda_{source}} = \frac{a(t_0)}{a(t)}$$

where λ_{obs} is the wavelength of observed light and λ_{source} is the wavelength of the source. The times t and t_0 are the time when light emitted and observation (the present) time respectively. One may write this equation as;

$$\lambda_{obs} = \lambda_{source}\frac{a(t_0)}{a(t)}$$

As wee see there is a difference between wavelengths with the fraction of scale factors. It is known that in the case of redshift we have $\lambda_{obs} > \lambda_{source}$, then one may conclude $\frac{a(t_0)}{a(t)} = 1 + z > 1$ where z is the redshift. This shows us that the scale factor of present time is grater than that of the time of the light emission which means that there must exist an expansion.

So what about the expansion rate? Is it accelerating, decelerating or at a constant rate? These questions found their answers in 1998 when there were some observations of type Ia supernovae [3, 4] suggested that the expansion of the universe has been accelerating. This of course may be seen mathematically by looking at the luminosity distance which is defined as the relationship between the absolute magnitude M and apparent magnitude m of an astronomical object. In general luminosity distance, D_L, is defined by the formula;

$$D_L = \sqrt{\frac{L}{4\pi F}}$$

where F is the observed flux and L is the intrinsic luminosity of the source. Also the luminosity distance in cosmology is known in another form which depends on redshift, z, that may be derived from (2.24);

$$D_L = \frac{1+z}{H_0} \int_0^z \frac{dz'}{\sqrt{\Omega_M (1+z')^3 + \Omega_\Lambda (1+z')^{3(1+w)}}}$$

where $\Omega_x = \frac{\rho_x}{\rho_c}$ which is the ratio of density of any source to the critical density which is the energy density at $K = 0$. Here Ω_M stands for matter while Ω_Λ stands for cosmological constant. When we perform this integration for matter and for cosmological constant seperately we have the relations;

$$D_L = \frac{2}{H_0}(1 + z - \sqrt{1+z}), \quad for \quad \Omega_M = 1$$
$$D_L = \frac{z}{H_0}(1 + z), \quad for \quad \Omega_\Lambda = 1$$

One may see from these two relations that the luminosity distance for cosmological constant is larger than that of matter. In terms of the the absolute magnitude M and apparent magnitude m, luminosity distance in Mpc is given in [24, 25] as;

$$m - M = 5 \log D_L + 25$$

If one sketches the graph of $m - M$ to z (we can see the relations by putting the found values of D_L into this equation), the graph shows that there is deflection in the line. When this graph is performed for matter dominated universe, we expect a line which must curve to the axis of z but found graph is like linear. This only can be explained as there must occur an acceleration to cause this deflection.

CHAPTER 3

DARK ENERGY

3.1. Cosmological Constant

The question of evolution of universe begins with Einstein and his belief that the universe should be static. But when he wrote down the equations of the general relativity for a static universe, he realised that the universe was not static as he thought. Therefore he needed to modify his equation by introducing a term which is called 'Cosmological constant'[26–30] (see Appendix B). This may be supported by mathematical tools. We know that Einstein tensor and the energy momentum tensor satisfy the Bianchi identity. Here we mean that their covariant derivatives are zero. Also we know that the covariant derivative of metric is also zero. Therefore there is a freedom to add a term like $\Lambda g_{\mu\nu}$ because it also satisfies the Bianchi identity. Although Einstein has added Λ, cosmological constant, he has seen that the static universe he obtained is not stable and in fact it stands for universe that expands or contracts depending on the sign of Λ. In fact the cosmological observations of Hubble as we have mentioned in chapter2 suggest that the universe is expanding.

In the presence of cosmological constant Einstein equations read;

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{3.1}$$

where Λ is called the 'Cosmological constant'. It should be noted that the effect of including Λ in the equations may be observed more prominently in large distance scales at which contributions from higher order derivatives of the metric tensor tend to fall.

Now what about the equation of state of cosmological constant? The equation of state is defined as the ratio of pressure to the total energy density and denoted by w. In the case of cosmological constant we have $w_\Lambda = -1$. This is the simplest candidate for the dark energy. But as we mentioned in early sections there are also some scalar fields which are slowly varying with time to describe dark energy similiar to cosmological constant [6, 7, 27]. We require all dark energy candidates mimic cosmological constant since a positive cosmological constant fits observational data very well.

3.2. Slowly Varying Scalar Fields

A field which is invariant under Lorentz transformations is called a ”scalar field”. In cosmology, as we mentioned before, scalar fields that homogeneous and weakly coupled to ordinary matter are alternative ways to describe the dark energy. If the scalar field is slowly varying and if the potential of this scalar field slowly decreased towards zero for large potential, the energy density associated with it could act like cosmological constant that varying with time less rapidly than the mass densities of matter and radiation. These fields are known as quintessence, tachyon and phantom. Now we will look at these fields in details.

3.2.1. Quintessence

Quintessence is a hypothetical form of dark energy postulated as an explanation of observations of an accelerating universe [3]. Also it may be defined as a time-varying form of vacuum energy. Quintessence is a standart scalar field that is minimally coupled to gravity [6, 7, 27, 31]. We may write the action which is related to the quintessence as;

$$S = \int \mathcal{L}\sqrt{-g}d^4x = -\int [\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)]\sqrt{-g}d^4x \tag{3.2}$$

where g denotes the determinant of FLRW metric and $\mathcal{L}$ is the Lagrangian density of quintessence. Now by using this action we may find the related energy momentum tensor for this scalar field as;

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi + g_{\mu\nu}\mathcal{L} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)] \tag{3.3}$$

Here the scalar field ϕ is considered as a function of time only because of the homogeniety and isotropy of the universe. It does not depend on space part so we will have only time derivative of it. We may now calculate energy density and pressure in FLRW background. As we know the energy density is equal to T_{00}. Then it is found as;

$$\rho_{00} = T_{00} = \partial_0\phi\partial_0\phi - g_{00}[\frac{1}{2}(g^{00}\partial_0\phi\partial_0\phi + g^{ij}\partial_i\phi\partial_j\phi) + V(\phi)] = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{3.4}$$

Also we may calculate the pressure by the space component of energy momentum tensor.

$$T_{ij} = \partial_i \phi \partial_j \phi - g_{ij} [\frac{1}{2} (g^{00} \partial_0 \phi \partial_0 \phi + g^{kr} \partial_k \phi \partial_r \phi) + V(\phi)] = [\frac{1}{2} \dot{\phi}^2 - V(\phi)] \delta_{ij} \quad (3.5)$$

$$P_\phi = T_{11} = T_{22} = T_{33} = \frac{1}{2} \dot{\phi}^2 - V(\phi) \quad (3.6)$$

After finding energy density and pressure, we may write the equation of state which is the ratio of pressure to the total energy density as introduced before is $w_\phi = \frac{P_\phi}{\rho_\phi}$. When we put values of energy density and pressure we get;

$$w_\phi = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \quad (3.7)$$

As we said before if we want to approach to the cosmological constant from a scalar field, it must vary slowly with time that is $\dot{\phi} \ll V(\phi)$. Under this condition equation of state approaches to -1 which is the value for cosmological constant. When we have this limit we mean that $V(\phi)$ is a flat potential. If $\ddot{a} > 0$ this requires that $\rho + 3P < 0$, this term is written for quintessence as;

$$\rho + 3P = \frac{1}{2}\dot{\phi}^2 + V(\phi) + 3[\frac{1}{2}\dot{\phi}^2 - V(\phi)] = 2\dot{\phi}^2 - 2V(\phi)$$
$$\dot{\phi}^2 < V(\phi)$$

this means that for accelerated expansion we need a nearly flat potential in time.

3.2.2. Tachyon Field

Tachyon is a particle with 4-momentum and imaginary proper time, moving faster than light i.e has imaginary proper time. As we mentioned before tachyon field can be considered phenomenologically as a suitable candidate for a viable model of dark energy. The tachyon is an unstable field [8, 32–34], its state parameter in the equation of state varies smoothly between -1 and 0.

Tachyons we consider are string theory type of tachyons whose action is;

$$S \;\; = \int -V(\phi) \sqrt{1 - \partial^\mu \phi \partial_\mu \phi} \sqrt{-g} d^4 x \quad (3.8)$$

where the signature of the metric is taken to be (+,-,-,-), this is called the 'Dirac-Born-Infel(DBI)' type action. And here we shall consider potential that when $\phi \rightarrow \infty$ then $V(\phi) \rightarrow 0$. Now we are ready to calculate corresponding energy momentum tensor as;

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi - g_{\mu\nu}\mathcal{L} = \frac{V(\phi)\partial_\mu\phi\partial_\nu\phi}{\sqrt{1-\partial^\mu\phi\partial_\mu\phi}} + V(\phi)\sqrt{1-\partial^\mu\phi\partial_\mu\phi} \tag{3.9}$$

We assume that ϕ is spatially constant (i.e. it only depends on time).

$$\rho_\phi = T_{00} = \frac{V(\phi)}{\sqrt{1-\dot{\phi}^2}} \tag{3.10}$$

and also the pressure may be found from T_{ij} as

$$T_{ij} = -V(\phi)\sqrt{1-\dot{\phi}^2}\delta_{ij} \tag{3.11}$$

$$P_\phi = -V(\phi)\sqrt{1-\dot{\phi}^2} \tag{3.12}$$

The corresponding equation of state may be written as;

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \dot{\phi}^2 - 1 \tag{3.13}$$

Since the pressure and energy density must be real then we may set the condition for $\dot{\phi}^2$: $1-\dot{\phi}^2 \geq 0 \Rightarrow \dot{\phi}^2 \leq 1$. From this relation we may find the range of w_ϕ. We have $-1 \leq w_\phi \leq 0$.

3.2.3. Phantom Field

The scalar field models as we gave in previous sections lead to $w_\phi \geq -1$. But now we want to talk about phantom field [35–38] whose equation of state is $w_\phi < -1$. The simplest way by which we may get a phantom field is to have a scalar field with a negative kinetic energy term (i.e. a ghost field). The action of the standart phantom field may be written as;

$$S = \int \mathcal{L}\sqrt{-g}d^4x = \int[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)]\sqrt{-g}d^4x \tag{3.14}$$

Again as we performed for quintessence and tachyon fields, the corresponding energy density and pressure of phantom field may be found from energy-momentum tensor so we have;

$$P_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{3.15}$$

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{3.16}$$

$$w_\phi = \frac{\frac{1}{2}\dot{\phi}^2 + V(\phi)}{\frac{1}{2}\dot{\phi}^2 - V(\phi)} \tag{3.17}$$

As we see from the equation of state, when $\frac{1}{2}\dot{\phi}^2 < V(\phi)$ then we will get the condition that we need for phantom field. Therefore we see that the equation of state is less than -1 i.e $w_\phi < -1$. One may say this equation of state of phantom field is the same as that of ordinary scalar field with inverted potential.

CHAPTER 4

EXTRA DIMENSIONS

The concepts of extra dimensions are discussed in Introduction. As we mentioned there are several models associated with extra dimensions. Let us consider some basic models of extra dimensions.

4.1. Kaluza-Klein Theory

Kaluza-Klein Theory was an idea which was developed in the 1920's as an attempt to unify the forces of electromagnetism and gravity. This theory was first published in 1921 by Theodor Kaluza who suggested that in extending Einstein's theory of general relativity to a five dimensional space-time the first part of resulting equations is Maxwell's equations for electromagnetism, the second part is Einstein equations, and the final part is an extra scalar field now termed the "radiation". But in Kaluza-Klein approach [39–41] extra spatial dimensions are not similar to the three dimensions. In this theory the extra dimensions form a compact space with a scale L. For one extra dimension we have a circle with radius L and for higher dimensions we have sphere, torus, or any other manifold. From now on I will denote the name of this theory by KK. In general, the D-dimensional space-time in the KK approach has a geometry of $M^4 \times X^{D-4}$. Here M^4 denotes four dimensional (4-D) Minkowski space-time and X^{D-4} denotes manifold of extra dimensions. It is also called 'internal manifold'.

Kaluza introduced a condition that is called 'cylinder condition' in order to explain absence of the evidence of the extra dimension. This means the all partial derivatives with respect to the fifth dimension are zero. Then in 1926 Oscar Klein showed that 'cylinder condition' may be explained if the fifth dimension is circular, that means the fifth dimension is periodic. Under this assumption Kaluza's cylinder condition arises naturally (see Appendix C).

We consider (4+1) dimensional gravity i.e. Kaluza-Klein theory itself and see how 4D gravity may be unified with electromagnetism in 5-D. The corresponding 5D action is;

$$\widetilde{S} = -\frac{1}{16\pi\widetilde{G}}\int d^5x\sqrt{-\widetilde{g}}\widetilde{R} \tag{4.1}$$

where the tilde notation denotes the 5D variables, $\widetilde{R}$ Ricci scalar in 5D and $\sqrt{-\widetilde{g}}$ is the determinant of metric in 5D. One may decompose $\tilde{g}_{AB}$ (A,B = 0,1,2,3,4) into its Kaluza-Klein models as given in (C.12) and may take the zero modes as the usual 4-dimensional fields. This explains the rationale Kaluza's assumption that the fields in the Kaluza-Klein expansion of $\widetilde{g}_{\mu\nu}$ depend only on 4-dimensions. For our present purpose it is enough to consider the zero mode, and take the non-zero elements of $\widetilde{g}_{\mu\nu}$.

$\widetilde{g}_{\mu\nu} = e^{\frac{\phi}{\sqrt{3}}}[g_{\mu\nu}(x) + e^{-\sqrt{3}\phi}A_\mu A_\nu]$
$\widetilde{g}_{5\mu} = \widetilde{g}_{\mu 5} = e^{\frac{-2\phi}{\sqrt{3}}}A_\mu$
$\widetilde{g}_{55} = e^{\frac{-2\phi}{\sqrt{3}}}$

i.e.;

$$\widetilde{g}_{AB} = \begin{bmatrix} e^{\frac{\phi}{\sqrt{3}}}(g_{\mu\nu} + e^{-\sqrt{3}\phi}A_\mu A_\nu & e^{\frac{-2\phi}{\sqrt{3}}}A_\mu \\ e^{\frac{-2\phi}{\sqrt{3}}}A_\nu & e^{\frac{-2\phi}{\sqrt{3}}} \end{bmatrix}$$

and the inverse of this matrix may be written as;

$$\widetilde{g}^{AB} = \begin{bmatrix} e^{-\frac{\phi}{\sqrt{3}}}g^{\mu\nu} & -e^{-\frac{\phi}{\sqrt{3}}}A^\mu \\ -e^{-\frac{\phi}{\sqrt{3}}}A^\nu & e^{-\frac{\phi}{\sqrt{3}}}(A^2 + e^{\frac{3\phi}{\sqrt{3}}}) \end{bmatrix}$$

The variation of (4.1) with respect to $\widetilde{g}_{AB}$ results in the Einstein equations which have the same form of that in 4D.

$$\widetilde{G}_{AB} = \widetilde{R}_{AB} - \frac{1}{2}\widetilde{g}_{AB}\widetilde{R} \tag{4.2}$$

where $\widetilde{R}_{AB}$ is Ricci tensor in 5D. It may be written in terms of 5D Christoffel symbols are defined by;

$$\widetilde{R}_{AB} = \widetilde{\Gamma}^C_{AB,C} - \widetilde{\Gamma}^C_{AC,B} + \widetilde{\Gamma}^C_{DC}\widetilde{\Gamma}^D_{AB} - \widetilde{\Gamma}^C_{DB}\widetilde{\Gamma}^D_{AC} \tag{4.3}$$

where the Christoffel symbols are defined by;

$$\widetilde{\Gamma}^C_{AB} = \frac{1}{2}\widetilde{g}^{CD}(\widetilde{g}_{BD,A} + \widetilde{g}_{DA,B} - \widetilde{g}_{AB,D}) \tag{4.4}$$

Here one should not forget that there is no dependence on the extra dimension because we are considering the zero mode. Therefore derivatives w.r.t. x_5 or y is zero. Then the corresponding non-zero Christoffel symbols are;

$$\widetilde{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \frac{1}{2\sqrt{3}}[\delta^{\lambda}_{\mu}\partial_{\nu}\phi + \delta^{\lambda}_{\nu}\partial_{\mu}\phi - g_{\mu\nu}\partial^{\lambda}\phi]$$
$$+\frac{1}{\sqrt{3}}A_{\mu}A_{\nu}\partial^{\lambda}\phi + \frac{1}{2}e^{-\frac{3\phi}{\sqrt{3}}}[A_{\nu}F^{\lambda}_{\mu} + A_{\mu}F^{\lambda}_{\nu}]$$
$$\widetilde{\Gamma}^{\lambda}_{55} = \frac{1}{\sqrt{3}}e^{-\frac{3\phi}{\sqrt{3}}}\partial^{\lambda}\phi$$
$$\widetilde{\Gamma}^{5}_{55} = -\frac{1}{\sqrt{3}}e^{-\frac{3\phi}{\sqrt{3}}}A^{\rho}\partial_{\rho}\phi$$
$$\widetilde{\Gamma}^{5}_{5\lambda} = -\frac{1}{\sqrt{3}}e^{-\frac{3\phi}{\sqrt{3}}}A^{\rho}A_{\lambda}\partial_{\rho}\phi - \frac{1}{2}e^{-\frac{3\phi}{\sqrt{3}}}A^{\rho}F_{\lambda\rho} - \frac{1}{\sqrt{3}}\partial_{\lambda}\phi$$
$$\widetilde{\Gamma}^{\lambda}_{5\mu} = \frac{1}{\sqrt{3}}e^{-\frac{3\phi}{\sqrt{3}}}A_{\mu}\partial^{\lambda}\phi + \frac{1}{2}e^{-\frac{3\phi}{\sqrt{3}}}F^{\lambda}_{\mu}$$
$$\widetilde{\Gamma}^{5}_{\mu\nu} = [\frac{1}{2}(A_{\mu;\nu} + A_{\nu;\mu}) - \frac{1}{2}e^{-\frac{3\phi}{\sqrt{3}}}[A^{\rho}(A_{\nu}F_{\mu\rho} + A_{\mu}F_{\nu\rho}]$$
$$-\frac{3}{2\sqrt{3}}(A_{\nu}\partial_{\mu}\phi + A_{\mu}\partial_{\nu}\phi) + \frac{1}{2\sqrt{3}}(g_{\mu\nu}A^{\rho}\partial_{\rho}\phi)]$$

Here I used the notation $A_{\mu;\nu} = A_{\mu,\nu} - \Gamma^{\rho}_{\mu\nu}A_{\rho}$ where $A_{\mu;\nu}$ denotes the covariant derivative of A_{μ} . Now we are ready to construct the Ricci tensor and scalar respectively by using these symbols. As we wrote before the Ricci tensor can be written in 5D as;

$$\widetilde{R}_{AB} = \widetilde{\Gamma}^{C}_{AB,C} - \widetilde{\Gamma}^{C}_{AC,B} + \widetilde{\Gamma}^{C}_{DC}\widetilde{\Gamma}^{D}_{AB} - \widetilde{\Gamma}^{C}_{DB}\widetilde{\Gamma}^{D}_{AC}$$
$$\widetilde{R}_{\mu\nu} = \widetilde{\Gamma}^{C}_{\mu\nu,C} - \widetilde{\Gamma}^{C}_{\mu C,\nu} + \widetilde{\Gamma}^{C}_{DC}\widetilde{\Gamma}^{D}_{\mu\nu} - \widetilde{\Gamma}^{C}_{D\nu}\widetilde{\Gamma}^{D}_{\mu C}$$
$$\widetilde{R}_{\mu 5} = \widetilde{\Gamma}^{C}_{\mu 5,C} - \widetilde{\Gamma}^{C}_{\mu C,5} + \widetilde{\Gamma}^{C}_{DC}\widetilde{\Gamma}^{D}_{\mu 5} - \widetilde{\Gamma}^{C}_{D5}\widetilde{\Gamma}^{D}_{\mu C}$$
$$\widetilde{R}_{55} = \widetilde{\Gamma}^{C}_{55,C} - \widetilde{\Gamma}^{C}_{5C,5} + \widetilde{\Gamma}^{C}_{DC}\widetilde{\Gamma}^{D}_{55} - \widetilde{\Gamma}^{C}_{D5}\widetilde{\Gamma}^{D}_{5C}$$

One may construct the Ricci scalar in 5D by using the given Christoffels. For the KK zero mode the action become;

$$S = M_*^3\pi L\int d^4x\sqrt{g}[R_4 - \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - \frac{1}{4}e^{-\sqrt{3}\phi}F_{\mu\nu}F^{\mu\nu}] \quad (4.5)$$

As we see in the action in addition to usual 4D Einstein-Hilbert action of gravity we have two terms. The part of action containing field strength tensor is known as 'Maxwell's action' and the other term is an action of a scalar field.

4.2. Large Extra Dimensions (ADD Model)

One way to obtain 4D gravity on a brane is combining the braneworld idea with KK compactification. This was studied by Arkani-Hamed, Dimopoulos and Dvali [42]. In this model Standart Model (SM) particles are located in 4-dimensions and the gravity spreads to all dimensions with compact extra dimensions.

The action for the simplest ADD model may be written as;

$$S_{ADD} = \frac{M_*^{2+N}}{2} \int d^4x \int_0^{2\pi L} d^N y \sqrt{G} R_{4+N} + \int d^4x \sqrt{g}(T + L_{SM}) \qquad (4.6)$$

where $g(x) = G(x, y = 0)$, $M_* \sim (1 - 10)TeV$ and N denotes the number of extra dimensions.

If we integrate the action (4.6) over the extra dimensions we will have the 4D action for zero mode as;

$$\frac{M_*^{2+N}}{2} \int d^4x \int_0^{2\pi L} d^N y \sqrt{G} R_{4+N} = \frac{M_*^{2+N}(2\pi L)^N}{2} \int d^4x \sqrt{g} R \qquad (4.7)$$

here we take the second part of the action (4.6) to be zero. In the above equation the right side is the 4D action with the Plank mass $M_{Pl}^2 = M_*^{2+N}(2\pi L)^N$. From this relation we may find what should be the size of extra dimensions;

$$L = (\frac{M_{Pl}}{M_*})^{2/N} \frac{1}{2\pi M_*} \qquad (4.8)$$

i.e. $L \sim M_*^{-1}(\frac{M_{Pl}}{M_*})^{2/N}$. If the fundamental scale of gravity is taken as $M_* \sim TeV$ then the size of extra dimension becomes;

$$L \sim 10^{-17+30/N} cm \qquad (4.9)$$

Now we may list some of the values of L with the change of number N;

- $for N = 2; L \sim 0.1mm$, $1/L \sim 10^{-3} eV$

- $for N = 3; L \sim 1nm$, $1/L \sim 100eV$
- ...
- $for N = 6; L \sim 10^{-12}cm$, $1/L \sim 10MeV$

When $N = 1$ one obtains $L \sim 10^{13}cm$, and this is excluded within the ADD framework since gravity below this value would have been higher dimensional. The other important value is at $N = 2$ because at this value in which $L \sim 10^{-2}cm$ the modification of the 4D laws of gravity is predicted at sub-millimeter distances.

Now let us suppose two static faraway sources on the brane interact with the following non-relativistic gravitational potential written as;

$$V(r) = -G_N m_1 m_2 \sum_{n=-\infty}^{+\infty} |\Psi_n(y=0)|^2 \frac{e^{-m_n r}}{r} \tag{4.10}$$

where $\Psi_n(y = 0)$ is the wave function of n'th KK mode at a position of a brane and r denotes the distance between masses m_1, m_2. The mass term of KK mode are given as; $m_n = |n|/L$ where n stands for the number of KK modes. In the limit $r \gg L$ the potential given in (4.10) becomes;

$$V(r) = -\frac{G_N m_1 m_2}{r} \tag{4.11}$$

for only $m_n = 0$ contributes. Equation (4.11) is the conventional 4D law of Newtonian dynamics. This limit shows the distances much larger than the size of extra dimensions. But in the opposite limit $r \ll L$ we may get the potential in higher dimensions.

$$V(r) = -\frac{G m_1 m_2}{M_*^{2+N} r^{2+N}} \tag{4.12}$$

As we mentioned before this is the law of (4+N) dimensional gravitational interactions. Therefore we may conclude that the laws of gravity are modified at distances of order L.

4.3. Randall Sundrum Models

Before starting to this model let us first say something about braneworld [15, 16, 43]. The central idea is that the visible, four-dimensional universe is restricted to a brane inside a higher-dimensional space, called the ”bulk”. If the additional dimensions are compact, then the observed universe contains the extra dimensions. Therefore we should obtain the 4D gravity on a brane. In the brane picture 3 fundamental forces are localized on the brane but some fields e.g. the gravity has no such constraint. There are some ways to obtain 4D gravity on a brane. The first one is to combine the braneworld with the KK compactification which was done by Arkani-Hamed, Dimopoulos and Divali (ADD) [42]. Here the extra dimensions are compact. The other possibility is based on the phenomenon of damping or localization of gravity through extra dimensions discovered by Randall and Sundrum (RS) [44, 45]. In this model extra dimensions are strongly curved by a large cosmological constant. Here the extra dimensions are warped. We start with the a so-called RS II model [44] that has two branes one of which is located at infinity. The metric of this form can be written as;

$$ds^2 = e^{-|y|/L}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \tag{4.13}$$

where $\eta_{\mu\nu} = diag(-,+,+,+)$ is the 4D Minkowski metric. The pre-factor $e^{-|y|/L}$, called the warp factor, is written as an exponential for convenience. Its dependence on the extra dimension coordinate y causes this metric to be non-factorizable, which means that, unlike the metrics appearing in the usual Kaluza-Klein scenarios, it cannot be expressed as a product of the 4D Minkowski metric and a manifold of extra dimensions.

This metric is the solution of the equation given below;

$$M_*\sqrt{G}(R_{AB} - \frac{1}{2}G_{AB}R) = -M_*^3\Lambda\sqrt{G}G_{AB} + \sqrt{-g}g_{\mu\nu}T\delta^\mu_A\delta^\nu_B\delta(y) \tag{4.14}$$

where T is the brane tension and Λ is the negative cosmological constant. This equation is found from the action of RS model which is given by;

$$S_{RS} = \frac{M_*^3}{2}\int d^4x \int_{-\infty}^{+\infty} dy\sqrt{G}(R_5 - 2\Lambda) + \int d^4x\sqrt{g}(T + L_{SM}) \tag{4.15}$$

The equation (4.14) is found from the variation of (4.15) w.r.t. 5D metric G^{AB}. For simplicity we take L_{SM} to be zero.

Here the question is that how the gravity is localized. To answer this question let us consider graviton fluctuations. The metric becomes

$$ds^2 = (e^{-|y|/L}\eta_{\mu\nu} + h_{\mu\nu}(x,y))dx^\mu dx^\nu + dy^2 \tag{4.16}$$

where $h_{\mu\nu}(x,y)$ is the perturbation term. To understand the form of this perturbation term we should find the linearized Einstein equations and solve for the perturbation term. I will denote usual metric by $\bar{\eta}_{AB}dx^A dx^B \equiv e^{-|y|/L}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2$ and the terms which contain the perturbation term by δ. Then the linearized Einstein equations are written as;

$$\mathcal{G}_{AB} = R_{AB} - \frac{1}{2}g_{AB}R = (\bar{R}_{AB} + \delta R_{AB}) - \frac{1}{2}(\bar{\eta}_{AB} + h_{AB})(\bar{R} + \delta R) \tag{4.17}$$

where the bar over the quantities refer to the background metric $\bar{\eta}_{AB} \equiv e^{-|y|/L}\eta_{\mu\nu} + dy^2$ while the quantities without bar refer to the pertubed metric. In all calculations we will set higher order terms in $h_{\mu\nu}$ to zero.

Now let us find the terms $\delta R_{AB}, \delta R$ and $\delta\mathcal{G_{AB}}$.

$$\Gamma^\mu_{\rho\nu} = \frac{1}{2}e^{|y|/L}\eta^{\mu\sigma}(h_{\sigma\nu,\rho} + h_{\sigma\rho,\nu} - h_{\nu\rho,\sigma}) \tag{4.18}$$

$$\Gamma^5_{\rho\nu} = -\frac{1}{2}(h_{\rho\nu,5} + \partial_5(e^{-|y|/L})\eta_{\nu\rho}) \tag{4.19}$$

$$\Gamma^\mu_{\rho 5} = \frac{1}{2}e^{|y|/L}\eta^{\mu\sigma}(h_{\rho\sigma,5} + \partial_5(e^{-|y|/L})\eta_{\sigma\rho})) \tag{4.20}$$

where we take the higher order terms in $h_{\mu\nu}$ to be zero and we use $\eta_{\mu\nu}$ for rising and lowering the indices. By using (4.18),(4.19) and (4.20) one may construct the elements of Einstein tensor as;

$$R_{\mu\nu} = -\frac{1}{2}\partial_5\partial_5(e^{-|y|/L})\eta_{\mu\nu} - \frac{1}{2}e^{|y|/L}\eta_{\mu\nu}[\partial_5(e^{-|y|/L})]^2 + \delta R_{\mu\nu}$$
$$R_{\mu\nu} = -\frac{1}{2}e^{|y|/L}\Box h_{\mu\nu} - \frac{1}{2}\partial_5\partial_5 h_{\mu\nu} - \frac{1}{2}\partial_5\partial_5(e^{-|y|/L})\eta_{\mu\nu}$$
$$-\frac{1}{2}e^{|y|/L}\eta_{\mu\nu}[\partial_5(e^{-|y|/L})]^2 - \frac{1}{2}e^{|y|/L}h_{\mu\nu}[\partial_5(e^{-|y|/L})]^2$$
$$R_{55} = -2e^{|y|/L}\partial_5\partial_5(e^{-|y|/L}) - e^{2|y|/L}[\partial_5(e^{-|y|/L})]^2$$

where $\delta R_{\mu\nu}$ contains the $h_{\mu\nu}$ terms and is given as;

$$\delta R_{\mu\nu} = -\frac{1}{2}e^{|y|/L}\Box h_{\mu\nu} - \frac{1}{2}\partial_5\partial_5 h_{\mu\nu} - \frac{1}{2}e^{|y|/L}h_{\mu\nu}[\partial_5(e^{-|y|/L})]^2$$

Here we have used the gauge fixings $\eta^{\mu\nu}h_{\mu\nu} = 0$ and $\partial_\mu h^\mu_\nu = 0$. Also the derivatives w.r.t. extra dimension are given as;

$$\partial_5(e^{-|y|/L}) = -\frac{e^{-|y|/L}}{L}[\Theta(y) - \Theta(-y)]$$
$$\partial_5\partial_5(e^{-|y|/L}) = -\frac{e^{-|y|/L}}{L^2} - \frac{2e^{-|y|/L}}{L}\delta(y)$$
$$[\partial_5(e^{-|y|/L})]^2 = \frac{e^{-2|y|/L}}{L^2}$$

where $\Theta(y)$ is the step function which is related to the first derivative of absolute value function and $\delta(y)$ is Dirac-delta function that is the second derivative. Also we use $[\Theta(y) - \Theta(-y)]^2 = 1$. Hence;

$$\delta\mathcal{G}_{\mu\nu} = +\frac{1}{2}e^{|y|/L}\Box h_{\mu\nu} + \frac{1}{2}\partial_5\partial_5 h_{\mu\nu} + \frac{1}{2L^2}e^{-|y|/L}h_{\mu\nu} + \frac{3}{2L^2}h_{\mu\nu} - \frac{2}{L}h_{\mu\nu}\delta(y)$$

The right side of the Einstein equation is related to the energy-momentum tensor as known. From equation (4.14) one may find the pertubed part of energy-momentum tensor which comes from $h_{\mu\nu}$. In that equation we use for determinant of the metric;

$$\sqrt{G} = \sqrt{\bar{G}} + \delta\sqrt{G}$$

where $\delta\sqrt{G}$ is the determinant containing $h_{\mu\nu}$ terms and is known as;

$$\delta\sqrt{G} = -\frac{1}{2}\sqrt{G}G_{\mu\nu}\delta G^{\mu\nu} = 0$$

where the gauge choise $\eta_{\mu\nu}h_{\mu\nu} = 0$. Also the term $\sqrt{\bar{G}}$ is the determinant of the unperturbed metric. One may write (4.14) for $A, B = \mu, \nu$ and for the perturbed metric as;

$$M_*\sqrt{\bar{G}}\delta G_{\mu\nu} = -M_*^3\Lambda\sqrt{\bar{G}}\delta G^{\mu\nu} + \sqrt{-\bar{g}}\delta G^{\mu\nu}T\delta(y)$$

with $\delta G^{\mu\nu} = h^{\mu\nu}$. This equation may be written in details as ;

$$M_* e^{-2|y|/L}[+\frac{1}{2}e^{|y|/L}\Box h_{\mu\nu} + \frac{1}{2}\partial_5\partial_5 h_{\mu\nu} + \frac{1}{2L^2}e^{-|y|/L}h_{\mu\nu} + \frac{3}{2L^2}h_{\mu\nu} - \frac{2}{L}h_{\mu\nu}\delta(y)]$$
$$= -M_*^3\Lambda e^{-2|y|/L}h_{\mu\nu} + 3\frac{M_*^3}{L}h_{\mu\nu}\delta(y)$$

Also for L and T we use;

$$L \equiv \sqrt{-\frac{3}{2\Lambda}}$$
$$T = \frac{3M_*^3}{L}$$

Then we will try a solution of the form ;

$h_{\mu\nu}(x,y) \equiv u(y)e^{ipx}$

with $p^2 = -m^2$. One gets the equation for $u(y)$;

$[-m^2 e^{|y|/L} - \partial_y^2 - \frac{2}{L}\delta(y) + \frac{3}{2L^2}]u(y) = 0$

When we suppose zero mode m will be taken zero then a simple solution may be found as;

$$u(y) \sim e^{-|y|/L} \tag{4.21}$$

This function of y is important to explain the location of gravitation because it is like a wave function of gravitation. As we see from the form of $u(y)$ as $y \to \infty$, $u(y) \to 0$ which shows us that gravitation is located on the brane in which we live.

In the other Randall-Sundrum model that is RS I [45] model there are again two branes located at the end point of an interval of a certain size. One brane is called 'hidden brane' and the other one is called 'visible brane'. The first one has positive tension while the second one has negative. The action containing the gravity and branes may be written as;

$$S = S_{gravity} + S_{brane1} + S_{brane2} \tag{4.22}$$

If we want to write this total action clearly it becomes;

$$S = \int d^4x \int dy \sqrt{G}(2M_*^3 R_5 + \Lambda) + \int d^4x \sqrt{g_1}(L_1 - T_1) + \int d^4x \sqrt{g_2}(L_2 - T_2) \tag{4.23}$$

where R_5 is five dimensional scalar curvature, M_* is 5D Plank mass, T_1, T_2 are branes tension and L_1, L_2 are matter langrangians. We again take the contributions of matter to be zero. The variation of the action w.r.t. the 5D metric G^{AB} gives the equations of motion.

$$\frac{\delta S}{\delta G^{AB}} = \frac{\delta S_{gravity}}{\delta G^{AB}} + \frac{\delta S_1}{\delta G^{AB}} + \frac{\delta S_2}{\delta G^{AB}} \tag{4.24}$$

where S_1 and S_2 denote the actions of brane one and brane two respectively. The corresponding Einstein equaitons are;

$$M_* \sqrt{G}(R_{AB} - \frac{1}{2}G_{AB}R) - M_*^3 \Lambda \sqrt{G} G_{AB} = T_{hid}\sqrt{g_{hid}} g_{\mu\nu}^{hid} \delta_A^\mu \delta_B^\nu \delta(y) \tag{4.25}$$
$$+ T_{vis}\sqrt{g_{vis}} g_{\mu\nu}^{vis} \delta_A^\mu \delta_B^\nu \delta(y-L)$$

where $g_{\mu\nu}^{hid}(x) = G_{\mu\nu}(x,0)$ and $g_{\mu\nu}^{vis}(x) = G_{\mu\nu}(x, y_0)$. Here the extra dimension y runs in the interval $[-y_0, y_0]$. The solution of the equation of motion is given as [45];

$$ds^2 = e^{-|y|/L} g_{\mu\nu} dx^\mu dx^\nu + dy^2 \tag{4.26}$$

As we said in this section the hidden brane is located on $y = 0$ and visible one is at $y = y_0$. With these values of y we replace $g_{\mu\nu}$ by $\bar{g}_{\mu\nu}$. Then we have the relations;

$$g_{\mu\nu}^{hid}(x) = \bar{g}_{\mu\nu}(x), \quad g_{\mu\nu}^{vis}(x) = \exp^{-|y_0|/L} \bar{g}_{\mu\nu}(x) \tag{4.27}$$

In RSI model it is tought to be that the SM fileds are located on visible brane which has a negative tension. Let us now look for the matter part of the action which is given as;

$$\int d^4x \sqrt{g_{vis}} [g_{vis}^{\mu\nu} (D_\mu H)^\dagger (D_\nu H) - \lambda (H^\dagger H - v_0^2)^2] \tag{4.28}$$

where H is the Higgs field. If we write the form of $g_{\mu\nu}^{vis}(x)$ given in (4.27), we will get the new form for the action given in (4.28);

$$\int d^4x \sqrt{\bar{g}} [\bar{g}^{\mu\nu} (D_\mu H)^\dagger (D_\nu H) - \lambda (H^\dagger H - e^{-|y_0|/L} v_0^2)^2] \tag{4.29}$$

where we rescaled the Higgs field as $H \to e^{-|y_0|/2L} H$ and the new mass term becomes $\lambda e^{-|y_0|/L} v_0^2$. As we see the exponential factor shows the behaviour of gravity. In this model the source of gravity is located at hidden brane and the graviton's probability function is extremely high at the hidden brane. But it drops exponentially as it propagates closer towards the visible brane. Therefore one may see that as it propagates the gravity would be much weaker on the visible brane.

CHAPTER 5

DARK ENERGY IN EXTRA DIMENSIONS

In this chapter we consider the main features and problems of extra dimensional models related to cosmic acceleration. The main aim in this type of models is to account for cosmic acceleration (i.e. dark energy) in a simpler way e.g. through extra dimensional curvature [17, 18, 46]. We will see that the usual 4-dimensional cosmic acceleration tends to dynamic extra dimensions. An attempt to stabilize extra dimensions in general seems to neccesiate introduction of ghost type unordinary fields or fluids into picture. The inclusion of cosmic acceleration into an extra dimensional setting is an attractive idea because of the possibility of treating extra dimensions as a source or a suitable setting for dark energy. Therefore may be seen as a potential to solve many problems of high energy physics.

In fact the tendency of destabilization of extra dimension in the pressence of cosmic acceleration may be seen through the following simple extra dimensional metric, in which we suppose an extra dimensional model with a single static extra dimension.

$$ds^2 = -dt^2 + a^2(t)(dx_1^2 + dx_2^2 + dx_3^2) + dx_5 \qquad (5.1)$$

with $\dot{a}, \ddot{a} > 0$ that is the condition for accelerated expansion as we said before. Here if we denote the pressure of the extra dimensional part by P_5 , then it corresponds to the 55 component of the energy-momentum tensor. We take $8\pi G$ in front of the energy-momentum tensor as 1. This may be written as;

$$T_{55} = R_{55} - \frac{1}{2}g_{55}R$$

where R_{55} is zero since the extra dimensional part is flat. Therefore only the Ricci scalar contributes to the extra dimensional pressure.

$$T_{55} = -\frac{1}{2}g_{55}R = -3(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2})$$

And the equation of state of extra dimensional part is;

$$w_5 = \frac{P_5}{\rho} = \frac{-3(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2})}{3\frac{\dot{a}^2}{a^2}} = -\frac{\frac{\ddot{a}}{a}}{\frac{\dot{a}^2}{a^2}} - 1$$

As we see from here w_5 is less than -1 because as we said before $\dot{a}, \ddot{a} > 0$. This suggests that after integration over extra dimensions one may get a ghost-like fluid, generically, that is not desirable. Let us discuss the situation in more complicated cases.

5.1. An Examplary Model

To see the main problems of extra dimensional models with dark energy we consider the following seven dimensional space. Here we suppose both the ordinary 3-space and the extra dimensions may expand or contract. This model is special case of [19] where the number of extra dimensions are taken to be three. Then the corresponding metric is given by;

$$ds^2 = dt^2 - a^2(t)[\frac{dr_a^2}{1-K_a r_a^2} + r_a^2(d\theta_a^2 + sin^2\theta_a d\phi_a^2)] - b^2(t)[\frac{dr_b^2}{1-K_b r_b^2} + r_b^2(d\theta_b^2 + sin^2\theta_b d\phi_b^2)] \quad (5.2)$$

where both $a(t)$ and $b(t)$ are scale factors , K_a and K_b are related to the curvature of 3-space and extra space, respectively. As we said before we suppose matter content to be a perfect fluid. In order to write down the corresponding Ricci tensors and the Ricci scalar we may write this metric in conformally transformed [47] form as;

$$\begin{aligned} ds^2 = b^2(t)[\frac{1}{b^2(t)}(dt^2 - a^2(t)[\frac{dr_a^2}{1-K_a r_a^2} + r_a^2(d\theta_a^2 + sin^2\theta_a d\phi_a^2)]) \\ - [\frac{dr_b^2}{1-K_b r_b^2} + r_b^2(d\theta_b^2 + sin^2\theta_b d\phi_b^2)]] \end{aligned} \quad (5.3)$$

where $b^2(t) = \Omega_1^2$ and $b^{-2}(t) = \Omega_2^2$. Here we wrote our metric in the form $\widetilde{\widetilde{g}}_{MN} = \Omega_1^2 \widetilde{g}_{MN}$ where $\widetilde{g}_{MN} = \Omega_2^2 g_{MN}$. In the light of conformal transformations [47] one may write the Ricci tensor and the Ricci scalar as;

$$\begin{aligned} &\widetilde{\widetilde{R}}_{MN} = \widetilde{R}_{MN} - (n-2)\widetilde{\nabla}_M\widetilde{\nabla}_N(\ln\Omega_1) - \widetilde{g}_{MN}\widetilde{g}^{AB}\widetilde{\nabla}_A\widetilde{\nabla}_B(\ln\Omega_1) \\ &+(n-2)\widetilde{\nabla}_M(\ln\Omega_1)\widetilde{\nabla}_N(\ln\Omega_1) - (n-2)\widetilde{g}_{MN}\widetilde{g}^{AB}\widetilde{\nabla}_A(\ln\Omega_1)\widetilde{\nabla}_B(\ln\Omega_1) \\ &\widetilde{\widetilde{R}} = \Omega_1^{-2}[\widetilde{R} - 2(n-1)\widetilde{\Box}(\ln\Omega_1) - (n-1)(n-2)\frac{\widetilde{g}^{MN}\widetilde{\nabla}_N(\Omega_1)\widetilde{\nabla}_N(\Omega_1)}{\Omega_1^2}] \end{aligned}$$

This is written for the metric $\widetilde{\widetilde{g}}_{MN}$. And for the transformed metric $\widetilde{g}_{MN}$ the formulations become;

$$\widetilde{R}_{MN} = R_{MN} - (n-2)\nabla_M\nabla_N(\ln\Omega_2) - g_{MN}g^{AB}\nabla_A\nabla_B(\ln\Omega_2)$$
$$+(n-2)\nabla_M(\ln\Omega_2)\nabla_N(\ln\Omega_2) - (n-2)g_{MN}g^{AB}\nabla_A(\ln\Omega_2)\nabla_B(\ln\Omega_2)$$

while the Ricci scalar is written as;

$$\widetilde{R} = \Omega_2^{-2}[R - 2(n-1)\Box(\ln\Omega_1) - (n-1)(n-2)\frac{g^{MN}\nabla_N(\Omega_1)\nabla_N(\Omega_1)}{\Omega_1^2}]$$

where the term $\widetilde{R}$ contains two parts one of which comes from the conformal 4D part and the other one is from the extra dimensional part. Therefore we may write $\widetilde{R}$ as;

$$\widetilde{R} = \widetilde{R}_4 + R_{ex}$$

where R_{ex} is the Ricci scalar of the extra dimensional part of $\tilde{g}_{AB}dx^A dx^B$ and $\widetilde{R}_4$ has the form;

$$\widetilde{R}_4 = \Omega_2^{-2}[R_4 - \frac{6\Box(\Omega_2)}{\Omega_2}]$$

By using the Appendix A we may find R_4 and R_{ex};

$$R_4 = -\frac{6}{a^2}(\ddot{a}a + \dot{a}^2 + K_a)$$
$$R_{ex} = -6K_b$$

where we take $b = 1$ because in the extra dimensional part of the metric $\tilde{g}_{MN}$,where $M, N = 5, 6, 7$, we do not have a conformal factor. Now we are ready to construct the Ricci tensor elements as follow;

$$\widetilde{\widetilde{R}}_{00} = \widetilde{R}_{00} - 5\widetilde{\nabla}_0\widetilde{\nabla}_0(\ln\Omega_1) - \widetilde{g}_{00}(\widetilde{g}^{00}\widetilde{\nabla}_0\widetilde{\nabla}_0(\ln\Omega_1))$$
$$+5\widetilde{\nabla}_0(\ln\Omega_1)\widetilde{\nabla}_0(\ln\Omega_1) - 5\widetilde{g}_{00}\widetilde{(}g^{00}\widetilde{\nabla}_0(\ln\Omega_1)\widetilde{\nabla}_0(\ln\Omega_1))$$
$$\widetilde{\widetilde{R}}_{00} = -3\frac{\ddot{a}}{a} + 6\frac{\dot{a}}{a}\frac{\dot{b}}{b} - 3\frac{\ddot{b}}{b}$$
$$\widetilde{\widetilde{R}}_{11} = \widetilde{R}_{11} - \widetilde{g}_{11}(\widetilde{g}^{00}\widetilde{\nabla}_0\widetilde{\nabla}_0(\ln\Omega_1)) + 5\widetilde{g}_{11}(\widetilde{g}^{00}\widetilde{\nabla}_0(\ln\Omega_1)\widetilde{\nabla}_0(\ln\Omega_1))$$
$$\widetilde{\widetilde{R}}_{11} = \frac{a^2}{1-K_a r_a^2}(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{K_a}{a^2} + 3\frac{\dot{a}}{a}\frac{\dot{b}}{b} + 5\frac{\dot{b}^2}{b^2})$$
$$\widetilde{\widetilde{R}}_{22} = \widetilde{R}_{22} - \widetilde{g}_{22}(\widetilde{g}^{00}\widetilde{\nabla}_0\widetilde{\nabla}_0(\ln\Omega_1)) + 5\widetilde{g}_{22}(\widetilde{g}^{00}\widetilde{\nabla}_0(\ln\Omega_1)\widetilde{\nabla}_0(\ln\Omega_1))$$
$$\widetilde{\widetilde{R}}_{22} = a^2 r_a^2(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{K_a}{a^2} + 3\frac{\dot{a}}{a}\frac{\dot{b}}{b} + 5\frac{\dot{b}^2}{b^2})$$
$$\widetilde{\widetilde{R}}_{33} = \widetilde{R}_{33} - \widetilde{g}_{33}(\widetilde{g}^{00}\widetilde{\nabla}_0\widetilde{\nabla}_0(\ln\Omega_1)) + 5\widetilde{g}_{33}(\widetilde{g}^{00}\widetilde{\nabla}_0(\ln\Omega_1)\widetilde{\nabla}_0(\ln\Omega_1))$$
$$\widetilde{\widetilde{R}}_{33} = a^2 r_a^2 sin^2(\theta_a)(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{K_a}{a^2} + 3\frac{\dot{a}}{a}\frac{\dot{b}}{b} + 5\frac{\dot{b}^2}{b^2})$$
$$\widetilde{\widetilde{R}}_{55} = \widetilde{R}_{55} - \widetilde{g}_{55}(\widetilde{g}^{00}\widetilde{\nabla}_0\widetilde{\nabla}_0(\ln\Omega_1)) + 5\widetilde{g}_{55}(\widetilde{g}^{00}\widetilde{\nabla}_0(\ln\Omega_1)\widetilde{\nabla}_0(\ln\Omega_1))$$
$$\widetilde{\widetilde{R}}_{55} = \frac{b^2}{1-K_b r_b^2}(2\frac{K_b}{b^2} + 3\frac{\dot{a}}{a}\frac{\dot{b}}{b} - 3\frac{\dot{b}^2}{b^2} + \frac{\ddot{b}}{b} + 5\frac{\dot{b}^2}{b^2})$$

Since the conformal factors depend only on the time then we take the derivatives w.r.t. to other dimensions to be zero. Now we may calculate the Ricci scalar with the given formula as;

$$\widetilde{\widetilde{R}} = \widetilde{\widetilde{g}}^{MN}\widetilde{\widetilde{R}}_{MN} = -6(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K_a}{a^2} + \frac{K_b}{b^2} + 3\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2})$$

Now the corresponding elements of energy-momentum tensor are;

$$8\pi\bar{G}\bar{\rho} = 3[(\frac{\dot{a}}{a})^2 + \frac{K_a}{a^2}] + 3[(\frac{\dot{b}}{b})^2 + \frac{K_b}{b^2}] + 15\frac{\dot{a}}{a}\frac{\dot{b}}{b} \tag{5.4}$$

$$-8\pi\bar{G}\bar{P}_a = 2\frac{\ddot{a}}{a} + 3\frac{\ddot{b}}{b} + [(\frac{\dot{a}}{a})^2 + \frac{K_a}{a^2}] + 6\frac{\dot{a}}{a}\frac{\dot{b}}{b} \tag{5.5}$$

$$-8\pi\bar{G}\bar{P}_b = 2\frac{\ddot{b}}{b} + 3\frac{\ddot{a}}{a} + 3[(\frac{\dot{a}}{a})^2 + \frac{K_a}{a^2}] + [(\frac{\dot{b}}{b})^2 + \frac{K_b}{b^2}] + 6\frac{\dot{a}}{a}\frac{\dot{b}}{b} \tag{5.6}$$

where $\bar{G}$ is gravitational constant, $\bar{\rho}$ is energy density in the higher dimensional world , and $\bar{P}_a, \bar{P}_b$ are the pressure of 3-space and the extra space respectively.

Now let us consider some possible cases. In all cases we will take the curvature of the extra dimensions to be zero i.e. the Einstein equations effectively equivalent to 4-dimensional Einstein equations. For a radiation-dominated universe we have $\bar{P}_a = \frac{1}{3}\bar{\rho}$, $\bar{P}_b = 0$ where $w_a = 1/3$ and $w_b = 0$. If we consider static extra dimensions that is with constant b, equaitons (5.4), (5.5), (5.6) read for radiation dominated universe as follow;

$$(\frac{\dot{a}}{a})^2 + \frac{K_a}{a^2} = \frac{8\bar{G}\pi}{3}\bar{\rho} \tag{5.7}$$

$$2\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{K_a}{a^2} = -\frac{8\bar{G}\pi}{3}\bar{\rho} \tag{5.8}$$

$$\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{K_a}{a^2} = 0 \tag{5.9}$$

Since the constant b solution is stable for small perturbations of scale factor, one may conclude that we can reach ordinary evolution of a radiation dominated universe with static extra dimensions. In fact this is expected since the use of (5.9) in (5.8) and (5.9) reduces (5.8) to the 4-dimensional FLRW space. On the other hand when we consider a matter dominated universe (in which the pressures are zero), there is no solution for constant b. In order to have solution for this case, the matter needs to provide a negative pressure in the extra space. This pressure may be calculated by taking b constant and putting equations (5.4) and (5.5) into (5.6) one gets;

$$\bar{P}_b = -\frac{1}{2}\bar{\rho} \tag{5.10}$$

Now let us look for the evolution of extra dimensions which suggest b to be non-static. Let us consider a matter dominated universe with zero pressures and we take the spatial curvatures to be zero. Then equations (5.5) and (5.6) reduce to ;

$$5\frac{\ddot{a}}{a} + 7(\frac{\dot{a}}{a})^2 + 6\frac{\dot{a}}{a}\frac{\dot{b}}{b} + 3(\frac{\dot{b}}{b})^2 = 0 \tag{5.11}$$
$$5\frac{\ddot{b}}{b} - 3(\frac{\dot{a}}{a})^2 + 6\frac{\dot{a}}{a}\frac{\dot{b}}{b} - 2(\frac{\dot{b}}{b})^2 = 0 \tag{5.12}$$

where (5.11) is obtained by multiplying (5.5) by 2 and (5.6) by 3 and substracting the first equation from the second one. And also (5.12) is obtained in a similiar way. These two equations are for the accelerations $\ddot{a}$ and $\ddot{b}$ respectively. Now in the case of accelerated expansion in three-space that is $\frac{\ddot{a}}{a} > 0$, from the first equation we have the condition;

$$\frac{\dot{b}}{b} > [1 + \sqrt{\frac{10}{3}}]\frac{\dot{a}}{a} \equiv J_+\frac{\dot{a}}{a} \quad or$$
$$\frac{\dot{b}}{b} < [1 - \sqrt{\frac{10}{3}}]\frac{\dot{a}}{a} \equiv J_-\frac{\dot{a}}{a}$$

where $J_+ = [1 + \sqrt{\frac{10}{3}}]$ and $J_- = [1 - \sqrt{\frac{10}{3}}]$ are the roots of $\frac{\dot{b}}{b}$. Now we introduce a new parameter which is the ratio of Hubbles' parameter of a to the that of b, $\eta(t) = \frac{H_a}{H_b}$. This quantity will be the key for the acceleration of the three-space. In order to see this we should find the form of $\dot{\eta} = \frac{d\eta}{dt}$. From the form of the η we find its derivative as follows;

$$\frac{d\eta}{dt} = \frac{\dot{H}_a}{H_b} - \frac{H_a}{H_b}\frac{\dot{H}_b}{H_b}$$

After using $\dot{H}_a = \frac{\ddot{a}}{a} - (\frac{\dot{a}}{a})^2$ and $\dot{H}_b = \frac{\ddot{b}}{b} - (\frac{\dot{b}}{b})^2$, equations (5.11) and (5.12) becomes;

$$5\dot{H}_a + 12H_a^2 + 6H_aH_b + 3H_b^2 = 0 \tag{5.13}$$
$$5\dot{H}_b + 3H_b^2 + 6H_aH_b + -3H_a^2 = 0 \tag{5.14}$$

Then in terms of the η and $\dot{\eta}$, equations (5.13) and (5.14) become a single equation after multiplying (5.13) by $\frac{1}{H_b}$ and (5.14) by $\frac{H_a}{H_b^2}$ and the subtracting resulting equations we get;

$$\dot{\eta} - \frac{1}{5}(3H_a - 6H_a\eta + 3H_b - 3H_a\eta^2) = 0 \tag{5.15}$$

In this equation if $\dot{\eta} > 0$ (i.e $3H_a - 6H_a\eta + 3H_b - 3H_a\eta^2 > 0$) then we say that $\eta < -1 + \sqrt{3} \equiv K_{att}$ or $K_{rep} \equiv -1 - \sqrt{3} < \eta$ which come from the equation (5.15). On the other hand $K_{att} < \eta < K_{rep}$ for $\dot{\eta} < 0$. Fast expansion or contraction of extra dimensions may lead to fast variation of some fundamental constants of nature such as Newton's gravitational constant, coupling constants etc. Therefore it is safer to take $H_b \sim 0$, which implies $|\eta| >> 1$. $\dot{\eta} < 0$ and $|\eta| >> 1$ can not be satisfied simultaneously. Hence $\dot{\eta} < 0$ is excluded. If $\dot{\eta} > 0$ and $|\eta| >> 1$ then $\eta < 0$. This implies $H_b < 0$ since $H_a > 0$. In other words the extra dimensions tend to contract under generic conditions for accelerated expansion of the usual three dimensional space. This example shows the difficulty of stabilization of extra dimensions in the context of accelerating cosmic expansion.

5.2. Energy Conditions

In relativistic classical field theories of gravitation, particularly in general relativity, an energy condition is one of various alternative conditions which can be applied to the matter content of the theory. In general relativity, energy conditions [48, 49] are often used (and required) in proofs of various important theorems. As we know in general relativity and allied theories, the distribution of the mass, momentum, and stress due to matter and to any non-gravitational fields is described by the energy-momentum tensor (or matter tensor), $T^{\mu\nu}$. However, the Einstein field equations do not specify what kinds of states of matter or non-gravitational fields are maintained in a space-time model. Because without some further criterion, the Einstein field equations give default solutions with properties most physicists regard as unphysical. The energy conditions represent such criteria. There are some energy conditions namely called 'strong, null, weak and dominant energy conditions'. Now let us write down mathematically these conditions.

- $Null\ energy\ condition\ (NEC)$:

The null energy condition [49] stipulates that for every future-pointing null vector field $\overrightarrow{k}$;

- $T_{\mu\nu}k^{\mu}k^{\nu} \geq 0 \, where \, g_{\mu\nu}k^{\mu}k^{\nu} = 0$

For a perfect fluid NEC becomes;

- $T_{\mu\nu}k^{\mu}k^{\nu} = \rho + P \geq 0$

We can see from here that all cases such as matter, radiation and cosmological constant satisfy NEC.

- $Weak \ energy \ condition : where \ g_{\mu\nu}X^{\mu}X^{\nu} \leq 0$

The weak energy condition [50] stipulates that for every future-pointing timelike vector field $\overrightarrow{X}$, the matter density observed by the corresponding observers is always non-negative:

- $T_{\mu\nu}X^{\mu}X^{\nu} \geq 0$

In the case of perfect fluid we have;

- $T_{\mu\nu}X^{\mu}X^{\nu} \geq 0$,$\rho \geq 0$, $\rho + P \geq 0$

Here again all the sources satisfy WEC.

- $Strong \ energy \ condition : where \ g_{\mu\nu}X^{\mu}X^{\nu} \leq 0$

The strong energy condition stipulates that for every future-pointing time-like vector field $\overrightarrow{X}$, the trace of the tidal tensor measured by the corresponding observers is always non-negative:

- $(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})X^{\mu}X^{\nu} \geq 0$

Again in the case of perfect fluid SEC becomes;

- $(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})X^{\mu}X^{\nu} \geq 0$,$\rho + P \geq 0$,$\rho + 3P \geq 0$

For the SEC one may see from the given conditions that matter and radiation satisfy SEC but cosmological constant does not satisfy the second condition for SEC that is $\rho + 3P \geq 0$.

- $Dominant \ energy \ condition :$

The dominant energy condition stipulates that, in addition to the weak energy condition holding true, for every future-pointing causal vector field (either timelike or null) $\overrightarrow{Y}$, the vector field $T^a_b Y^b$ must be a future-pointing causal vector which means that mass-energy can never be observed to be flowing faster than light. In the case of perfect fluid DEC becomes;

- $\rho \geq |P|$

When we check this for the various sources matter, radiation and cosmological constant satisfy this condition but phantom does not satisfy.

5.3. Constraints on Extra Dimensional Models of Dark Energy From Energy Conditions

Epoch(s) of cosmic acceleration play essential roles in modern cosmological models. As we said in previous chapters observations of type Ia supernovae [3, 4] and the cosmic microwave background [51, 52] indicate that the universe is expanding at an accelerating rate. A complete cosmological model based on more fundamental physics must accommodate or should explain this epoch of acceleration. In this section we consider a broad class of accelerating models with extra spatial dimensions. We see that these higher dimensional models violate either the strong or null energy condition (NEC) respectively. The analysis given here are the review and the discussion of the works in [20, 21]. We have supposed 4 assumptions in our work;

- $GR\ condition$
- $Flatness\ condition$
- $Boundedness\ condition$
- $Metric\ condition$

By General Relativity (GR) condition we mean that we describe both the 4D and higher dimensional theory by General Relativity (GR). Flatness and boundness conditions imply that the 3D is spatially flat and the extra dimensions are bounded, respectively. Finally the metric condition is that the extra dimensional metric is Ricci flat (RF) or conformally Ricci flat (CRF).

The violation of NEC in these models is unavoidable if the universe is de Sitter or nearly de Sitter. Let us consider a class of extra dimensional models for dark energy in the light of energy conditions.

In the subsection before the last subsection in this section we consider some theorems that severely restrict the possibility of realistic models that obey NEC. We will call the theorems as 'no-go theorems'. The no-go theorems depend on the intrinsic curvature of the compactification manifold M. There exist two possibilities for M that are;

- $Curvature\ free$:

In this category all modes are with a single extra dimension such as braneworlds [15, 16, 43]. It also includes compact manifolds with vanishing intrinsic Ricci scalar.

- $Curved$:

This category includes compact manifolds this time with non-vanishing intrinsic Ricci scalar. We mainly consider conformally Ricci flat curved internal manifolds.

In this analysis we consider the shape and size of the compactification space M acts as fields in 4D. Knowing the time evolution of this field gives us chance to work out the time evolution of M. The basic idea is the reverse of the Kaluza Klein philosophy that is instead of starting with a specific matter in higher dimensional model and then reducing to the 4D we go back way the 4D relations and observations are used to put constraints on extra dimensional models. Studying this may be also called 'oxidised cosmic acceleration' [21].

The no-go theorems we consider in this section suggest that there are some thresholds in w . If we want to make w below these thresholds we should violate an energy condition in higher-dimensional theory.

Here in our study we will consider the higher-dimensional action in the Einstein-Hilbert action [53] which may be reduced to;

$$S_{4D} = \frac{1}{2l_4^2} \int R\sqrt{-g} d^4x + otherterms \tag{5.16}$$

where l_4 is the Planck length in 4D and it is constant.

5.3.1. Some Simple Examples

As we said before the NEC is violated if the extra dimensions are flat and static. Here we consider k extra dimensions with the corresponding metric written as;

$$ds^2_{4+k} = ds^2_4 + ds^2_k \tag{5.17}$$

where ds^2_4 describes the 4D part of the metric and ds^2_k shows the metric of extra dimension. As we said it is flat and may be written as;

$$ds^2_k = \delta_{ab} dy^a dy^b \tag{5.18}$$

And also we consider the 4D part to be a flat FLRW universe with the metric may be written as;

$$ds^2_4 = -dt^2 + a^2(t)(dx^2_1 + dx^2_2 + dx^2_3) \tag{5.19}$$

As we know already the 4D universe is accelerating with $\ddot{a}/a > 0$. The Einstein equations in D-dimensions may be written as;

$$G_{MN} = R_{MN} - \frac{1}{2} g_{MN} R = 8\pi G T_{MN} \tag{5.20}$$

Now let us look component by component to this equation. The 00 component is;

$$G_{00} = R_{00} - \frac{1}{2} g_{00} R = 8\pi G T_{00} \tag{5.21}$$

where the 00 component of Ricci tensor is written as;

$$R_{00} = \Gamma^C_{00,C} - \Gamma^C_{0C,0} + \Gamma^C_{DC}\Gamma^D_{00} - \Gamma^C_{D0}\Gamma^D_{0C} \tag{5.22}$$

with ;

$$\Gamma^C_{00} = \frac{1}{2} g^{CD}[g_{0D,0} + g_{D0,0} - g_{00,D}], \quad \Gamma^C_{00} = 0 \tag{5.23}$$

$$\Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j \tag{5.24}$$

Then the 00 component of Ricci tensor is found as;

$$R_{00} = -3\frac{\ddot{a}}{a} + 3\frac{\dot{a}^2}{a^2} \tag{5.25}$$

The other components of Ricci tensor may be formulated as;

$$R_{ij} = \Gamma^C_{ij,C} - \Gamma^C_{iC,j} + \Gamma^C_{DC}\Gamma^D_{ij} - \Gamma^C_{Dj}\Gamma^D_{iC} \tag{5.26}$$

with;

$$\Gamma^C_{ij} = \frac{1}{2} g^{CD}[g_{iD,j} + g_{Dj,i} - g_{ij,D}], \;\; \Gamma^0_{ij} = (a\dot{a})\delta_{ij}, \;\; \Gamma^i_{0j} = (\frac{\dot{a}}{a})\delta^i_j \tag{5.27}$$

using this in R_{ij} we have;

$$R_{ij} = \Gamma^0_{ij,0} + \Gamma^k_{0k}\Gamma^0_{ij} - \Gamma^0_{kj}\Gamma^k_{i0} - \Gamma^k_{0j}\Gamma^0_{ik} = (2\dot{a}^2 + a\ddot{a})\delta_{ij} \tag{5.28}$$

And the other components of the Ricci tensor are zero since extra dimensions are flat and static. Now we are ready to construct the Einstein equations.

$$G_{00} = R_{00} - \frac{1}{2} g_{00}R = -3\frac{\ddot{a}}{a} + 3(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a}) = 3\frac{\dot{a}^2}{a} \tag{5.29}$$

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij}R = -(\dot{a}^2 + 2\ddot{a})\delta_{ij} \tag{5.30}$$

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab}R = -3(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a})\delta_{ab} \tag{5.31}$$

where i, j run over 3 spatial dimensions and a, b run over the extra dimensions. Now let us assume that the 4D cosmology has a power-law scale factor $a(t) \sim t^r$. Now by using the given form of scale factor we may rewrite the Einstein equations as;

$$t^2 G_{00} = 3r^2 \tag{5.32}$$

$$t^2 G_{ij} = r(2-3r)\delta_{ij} \tag{5.33}$$

$$t^2 G_{ab} = 3r(1-2r)\delta_{ab} \tag{5.34}$$

where $a(t) \propto t^r$. Here the pressure along the extra dimensions is negative therefore the corresponding stress energy violates the NEC. We may check this by using the definition of NEC given by;

$$T_{MN}k^M k^N = T_{00}k^0 k^0 + T_{ij}k^i k^j + T_{ab}k^a k^b \geq 0 \tag{5.35}$$

$$t^2 T_{MN}k^M k^N = 3r(1-r) \tag{5.36}$$

Here as we said before since $r > 1$ or $r < 0$ for an accelerated 4D universe (5.36) is negative, indicating NEC violation.

In an another model we may suppose a universe in which the extra dimensions evolve as power laws in time as in three spatial dimensions. This is also a possibility and may be explored by using a metric called 'Kasner metric' [54]. It may be written as;

$$ds^2_{Kasner} = -dt^2 + \sum_{j=1}^{3+k} t^{2r_j} dx_j^2 \tag{5.37}$$

where k denotes the number of extra dimensions. As we see both the three spatial dimensions and the extra dimensions have scale factors which depends on time. Here we suppose that the volume of extra dimensions behaves like t^q with $q = \sum_{j=4}^{3+k} r_k$. We are ready to calculate the corresponding Ricci tensors and the Ricci scalar to construct the Einstein equations.

$$\Gamma^0_{ij} = (rt^{2r-1})\delta_{ij}, \quad \Gamma^0_{ab} = (r_a t^{2r_a-1})\delta_{ab} \tag{5.38}$$

$$\Gamma^i_{ok} = (r_k t^{2r_k-2r_i-1})\delta^i_k, \quad \Gamma^a_{0b} = (r_b t^{2r_b-2r_a-1})\delta^a_b \tag{5.39}$$

The corresponding elements of Ricci tensor are;

$$R_{00} = \Gamma^{C}_{00,C} - \Gamma^{C}_{0C,0} + \Gamma^{C}_{DC}\Gamma^{D}_{00} - \Gamma^{C}_{D0}\Gamma^{D}_{0C} \tag{5.40}$$

$$R_{00} = \frac{3r}{t^2} + \frac{\sum r_a}{t^2} - \frac{3r^2}{t^2} - \frac{\sum r_a^2}{t^2} \tag{5.41}$$

$$R_{00} = \frac{3r}{t^2}(1-r) + \frac{\sum r_a}{t^2}(1-r_a) \tag{5.42}$$

$$R_{ij} = \Gamma^{0}_{ij,0} + \Gamma^{k}_{0k}\Gamma^{0}_{ij} - \Gamma^{0}_{kj}\Gamma^{k}_{i0} - \Gamma^{k}_{0j}\Gamma^{0}_{ik} \tag{5.43}$$

$$R_{ij} = t^{2r-2}[r(r-1) + 3r^2 + r\sum r_a - 2r^2]\delta_{ij} \tag{5.44}$$

$$R_{ij} = t^{2r-2}[3r^2 + r(\sum r_a - 1)]\delta_{ij} \tag{5.45}$$

$$R_{ab} = \Gamma^{C}_{ab,C} - \Gamma^{C}_{aC,b} + \Gamma^{C}_{DC}\Gamma^{D}_{ab} - \Gamma^{C}_{Da}\Gamma^{D}_{bC} \tag{5.46}$$

$$R_{ab} = t^{2r_a-2}[2r_a^2 - r_a + 3rr_a + r_a\sum r_a - 2r_ar_b]\delta_{ab} \tag{5.47}$$

where we have written $r_1 = r_2 = r_3 = r$ for spatial dimensions. We may calculate the Ricci scalar R.

$$R = g^{00}R_{00} + g^{ij}R_{ij} + g^{ab}R_{ab}$$
$$R = t^{-2}[6r^2 - 6r - \sum r_a^2 + 6r\sum r_a + (\sum r_a)^2]$$

We consider vacuum Einstein solutions. Therefore all components of Einstein tensor G_{MN} must be equal to zero respectively.

$$G_{00} = R_{00} - \frac{1}{2}g_{00}R = t^{-2}[\sum r_a - 3/2\sum r_a^2 + 3r\sum r_a + (\sum r_a)^2] = 0$$
$$G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R = t^{2r-2}[2r - 2r\sum r_a + \frac{\sum r_a^2}{2} - \frac{(\sum r_a)^2}{2}]\delta_{ij} = 0$$
$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$$
$$G_{ab} = t^{2r-2}[2r_a^2 - r_a + 3rr_a + r_a\sum r_a - 2r_ar_b - 3r^2 + 3r$$
$$+\frac{\sum r_a^2}{2} - 3r\sum r_a - \frac{(\sum r_a)^2}{2}]\delta_{ab} = 0$$

When we solve these three equations we will get two conditions for r and q.

- $\sum_{j=1}^{3+k} r_j = 1$
- $\sum_{j=1}^{3+k} r_j^2 = 3r^2 + \sum_{j=4}^{3+k} r_j^2 = 1$

The first condition implies $q < 0$ if $r > 1/3$. So $r > 1$ is excluded which means the three non-compact dimensions are expanding and the other directions tend to contract. (On the other hand the other case that leads to accelerated expansion in the non-compact dimensions i.e. the case $r < 0$ corresponds to contracting universe and is inagreement with cosmological observations.) To intercept this a negative pressure is required as in the case of cosmological constant. When the three noncompact dimensions are expanding at an accelerated rate then NEC is violated.

In order to evade this problem let us try a new metric in which the extra dimensional part evolve with time. The corresponding metric is written as;

$$ds^2_{4+k} = A^2(\eta)(-d\eta^2 + dx_1^2 + dx_2^2 + dx_3^2) + \exp[\frac{2c}{k}\psi(\eta)]ds_k^2 \tag{5.48}$$

where $A(\eta)$ is the scale factor of the 4D universe measured in (4+k) dimensional Einstein frame. Here η is the conformal time. Also the term ψ canonically normalised scalar field in the 4D Einstein frame. Therefore it is called 'universal Kaluza-Klein breathing mode modulus'. The constant c is given by;

$$c = \sqrt{\frac{2k}{k+2}} \tag{5.49}$$

The 4-D Einstein frame scale factor $a(\eta)$ may be found as;

$$a(\eta) = e^{c\psi/2}A(\eta) \tag{5.50}$$

which comes from the determinant of the metric tensor of (4+k) dimensions as we reduce it to 4D. Now we may write the equations of motion of Friedmann universe in terms of the derivative of ψ as;

$$\rho + P = \rho(1+w) = 3(1+w)H^2 = (\frac{d\psi}{d\eta})^2 \tag{5.51}$$

where $\rho = P/w = 3H^2$ from Friedmann equations and $H = \frac{\dot{a}}{a}$ where dot denotes the derivative with respect to time, t. If we put $H = \frac{\dot{a}}{a}$ into equation and make a change of variable from t to η one gets a solution for ψ;

$$\psi(\eta) = \pm\frac{2\sqrt{3(1+w)}}{1+3w}\ln\eta + \psi_0 \tag{5.52}$$

We will use this solution later in the elements of energy-momentum tensor. Now from the Einstein equations we may calculate the energy-momentum tensor of (5.48). The corresponding Christoffels are;

$$\Gamma^0_{\mu\nu} = \frac{\dot{A}}{A}\delta_{\mu\nu}, \Gamma^0_{ab} = \frac{c}{kA}e^{\frac{2c}{k}\psi}(\frac{d\psi}{d\eta})$$
$$\Gamma^i_{0k} = \frac{\dot{A}}{A}\delta^i_k, \Gamma^a_{0b} = \frac{c}{k}(\frac{d\psi}{d\eta})\delta^a_b$$

From now on I will use dot for the derivatives w.r.t. η. Then using these Christoffels one may find Ricci tensors as;

$$R_{00} = -2\frac{\ddot{A}}{A} + 2(\frac{\dot{A}}{A})^2 + c\frac{\dot{A}}{A}\dot{\psi} - \frac{c^2}{k}\dot{\psi}^2 - c\ddot{\psi}$$
$$R_{ij} = (\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} + c\frac{\dot{A}}{A}\dot{\psi})\delta_{ij}$$
$$R_{ab} = e^{\frac{2c}{k}\psi}[\frac{2c^2}{k^2A}\ddot{\psi} + (\frac{4}{A} - \frac{2c}{k})\frac{c}{kA}\dot{A}\dot{\psi} + \frac{c^2}{kA}\dot{\psi}^2]\delta_{ab}$$

with the Ricci scalar;

$$R = 5\frac{\ddot{A}}{A^3} + \frac{\dot{A}^2}{A^4} + \frac{2c}{A^2}\frac{\dot{A}}{A}\dot{\psi} + \frac{c}{A}(\frac{1}{A} + \frac{2c}{k})\ddot{\psi} + \frac{c^2}{A}(\frac{1}{A} + 1)\dot{\psi}^2 + 2c\frac{\dot{A}}{A}\dot{\psi}(\frac{2}{A} - \frac{c}{k})$$

One may put all these into Einstein equations and find energy-momentum tensor components. We may write these components in terms of w and η by using the equations (5.50) and (5.52). I will call the whole functions $F(\eta, w)$. Finally the components of energy-momentum tensor become;

$$T_{00} = F(1-w), T_{ij} = F(1-w)\delta_{ij}, \qquad (5.53)$$
$$T_{ab} = -F(1-w)[2 \mp \sqrt{\frac{3(1+w)}{2}\frac{(k+2)}{k}}]\delta_{ab}$$

From the NEC with the null vector $n^M = (1, \overrightarrow{0}, \hat{u})$ one gets;

$$-F(1-w)[\pm\sqrt{\frac{3(1+w)}{2}\frac{(k+2)}{k}} - 1] \geq 0 \Rightarrow \pm\sqrt{\frac{3(1+w)}{2}\frac{(k+2)}{k}} \geq 1 \qquad (5.54)$$

which is only possible for one of the branches. In other words NEC is violated by one branch at least. When the inside of square root is equal to 1 then w becomes;

$$w_k = -\frac{k+6}{3(k+2)} \qquad (5.55)$$

Here for only one extra dimension $w = -7/9$ which is its most negative value closest to -1. We see that we can not approach de Sitter expansion without violating NEC.

In these models we have shown that flat extra dimensions with only breathing mode dynamics could let accelerating universes without violating NEC but these models can not be close to de Sitter because of the value of w. There are many different ideas that may be supposed such as the extra dimensional space could be static then some other fields could cause acceleration. Also there is a possibility with warp factors but none of these make a difference. As the compactification manifold M is curvature free then the higher dimensional theory violate the NEC up to a value of w called critical value. This relation of these values means that there is a gap between pure de Sitter case and NEC satisfying condition.

There is only one possibility in which $w \simeq -1$ and NEC is not violated is to have a curvature for internal manifold M. To see this consider R, R_{ab} whose values for conformally flat extra dimensions are;

$$\mathring{R} = \Omega_1^{-2}[2(n-1)\widetilde{\Box}(\ln\Omega_1) - (n-1)(n-2)\frac{\widetilde{g}^{MN}\widetilde{\nabla}_N(\Omega_1)\widetilde{\nabla}_N(\Omega_1)}{\Omega_1^2}]$$
$$\mathring{R}_{ab} = (n-2)\widetilde{\nabla}_M\widetilde{\nabla}_N(\ln\Omega_1) - \widetilde{g}_{MN}\widetilde{g}^{AB}\widetilde{\nabla}_A\widetilde{\nabla}_B(\ln\Omega_1)$$

We will not calculate these terms because we may interpret the results in this general form. When we write the elements of Einstein tensor G_{MN} these additional terms will give;

$$\mathring{G}^0_0 = -\frac{1}{2}\mathring{R}, \quad \mathring{G}^n_m = -\frac{1}{2}\mathring{R}\delta^n_m, \quad \mathring{G}^b_a = \mathring{R}^b_a - \frac{1}{2}\mathring{R}\delta^b_a$$

where m, n run over three spatial dimensions, a, b run over the extra dimensions and $\mathring{R}$, $\mathring{R}^b_a$ are the extra dimensional contributions for (5.48). Now if we look for the NEC condition with a null vector of the form $n^M = (1, \hat{u}, \overrightarrow{0})$, we will see that $\mathring{R}$ does not appear. When we consider the another form of a null vector with $n^M = (1, \overrightarrow{0}, \hat{u})$ where $\hat{u}$ is k-dimensional unit vector. With this null vector the NEC becomes;

$$\mathring{T}_{MN} n^M n^N = \mathring{R}$$

Here if it is possible to adjust the additional term $\mathring{R}$, then the NEC may be satisfied without any other contributions. This is in fact a kind of fine-tuning. In 5.3.3 we will see that even with such a fine tuning it is impossible to attain $\omega \simeq -1$ for a sufficiently long time for conformally Ricci flat extra dimensional spaces. However the situation in the case of general curved extra dimensional spaces remain open.

5.3.2. General Analysis

In this section we will consider how the accelerated expansion orders strong constraints on many extra dimensional models. As we said before the accelerated expansion could be due to inflation or dark energy. We are interested in dark energy in our thesis. In fact most of the conclusions obtained for dark energy are true for inflation as well while inflation may impose stronger constraints since it needs many e-folds.

In the compactified theories the expansion of the non-compact directions has affinity to cause the extra dimensions to contract. But this contraction is physically problematic because it may let the physical constants to vary with time. What about the constraints? The models containing dark energy are described by Einstein gravity either in 4D or efective theory and in the higher dimensional theory. But the problem is that the 4D effective theory may require some extra constraints when lifted into the higher dimensional Einstein gravity.

First we will consider the metric of the form;

$$ds^2 = e^{2\Omega}(-dt^2 + \tilde{a}^2 dx_i dx^i) + g_{ab} dy^a dy^b$$

where g_{ab} and Ω depens on time and extra the dimensions and a, b run over the extra dimensions , $\tilde{a}$ is the usual FRW scale factor and the scalar curvature for g_{ab} is zero. If $\mathring{R}$ corresponding to g_{ab} is zero then we say that extra dimensional space is Ricci flat while it is conformally Ricci flat if $g_{ab} = e^{2\Omega_2}\bar{g}_{ab}$ where $\bar{g}_{ab}$ is Ricci flat. For the calculations we will use Maurer-Cartan formalism. Then any metric may be written in vielbeins e^A as;

$$g_{MN} dX^M \wedge dX^N = \eta_{AB} e^A \wedge e^B$$

where η_{AB} flat Minkowski metric of all space. This form is used to introduc tensors in a non-coordinate basis which are defined by vielbeins. Now the time derivative of a vielbein e^a may be taken as;

$$\frac{de^a}{dt} \equiv \xi^a_b e^b \tag{5.56}$$

where ξ_{ab} is defined as the velocity and may be written in terms of a symmetric and an antisymmetric part as;

$$\xi_{ab} = w_{ab} + \frac{\delta_{ab}}{k}\xi + \sigma_{ab} \tag{5.57}$$

where k is the number of extra dimensions, w_{ab} is antisymmetric part, $\xi = \delta^{ab}\xi_{ab}$ is the trace and σ_{ab} is symmetric but traceless. Now in the light of these properties one may write the time derivative of metric $g_{\alpha\beta}$ by using (5.56) and (5.57) as;

$\frac{1}{2}\frac{dg_{ab}}{dt} = \frac{1}{k}\xi g_{ab} + \sigma_{ab}$

where the terms ξ, σ are functions of time and the extra dimensions. The pressures along the 3 spatial dimensions and the extra dimensions are defined as;

$$P_3 = \frac{1}{3}\delta_{ij}T^{ij} \tag{5.58}$$

$$P_k = \frac{1}{k}\delta_{ab}T^{ab} \tag{5.59}$$

where k is the number of compact extra dimensions and T^{ij}, T^{ab} are energy momentum tensor of 3-space and the extra space respectively. The violation of NEC requires;

$T_{MN}n^M n^N < 0$

where M, N run over all the dimensions. One may show that when $\rho + P_3$ or $\rho + P_k$ is less than zero NEC is necessarily violated. And also when $\langle \rho + P_3 \rangle_A < 0$ or $\langle \rho + P_k \rangle_A < 0$ NEC is again violated. Here $\langle Q \rangle$ denotes the average of a quantity. In general this average is defined as;

$$\langle Q \rangle_A = (\int Qe^{A\Omega}\sqrt{g}d^k y)/(\int e^{A\Omega}\sqrt{g}d^k y) \tag{5.60}$$

where A is a constant and the term $e^{A\Omega}$ is called 'weight factor'. We take the average of the weight factor over A to be positive. This term is the average of Q in the warped metric on M. The averaging process defines a projection operator which is acting on M. We may divide Q into to parts a constant part and a perpendicular part, $Q(t, y^a) = Q_0 + Q_\perp$ where Q_0 is the constant part with $\langle Q \rangle = Q_0$. The average of the perpendicular part is given to be zero and the constant part is equal to the average of total quantity Q. Differantiating (5.60) w.r.t. time one may find;

$$\langle \dot{Q}_\perp \rangle = -\langle 2\dot{\Omega}_\perp Q_\perp + \xi_\perp Q_\perp \rangle \tag{5.61}$$

Since $\langle Q_\perp \rangle = 0$ one may see from (5.61) that ;

$$2\dot{\Omega} + \xi_\perp = 0 \tag{5.62}$$

We will use these informations while constructing the elements of Ricci tensor and Einstein equations.

We should convert the extra dimension scale factor $\tilde{a}$ into 4D scale factor a. Now we will introduce CRF metric to express the $\tilde{a}$ dependent terms in terms of 4D effective scale factor $a(t)$. To do this we use the relation $a(t) \equiv e^{\phi/2}\tilde{a}$ with;

$$e^{\phi/2} = \ell_{4+k}^{-k} \int e^{2\Omega}\sqrt{g}d^k y \tag{5.63}$$

where ℓ_{4+k} is (4+k) dimensional Planck length. Now we may find the elements of Ricci tensor by using the given Christoffels in terms of $\tilde{a}$ below;

$$\Gamma^C_{MN} = \frac{1}{2}g^{CF}(g_{FM,N} + g_{FN,M} - g_{MN,F}) \tag{5.64}$$
$$\Gamma^0_{00} = \dot{\Omega} \tag{5.65}$$
$$\Gamma^0_{ij} = (\dot{\Omega}\tilde{a}^2 + \dot{\tilde{a}}\tilde{a})\delta_{ij} \tag{5.66}$$
$$\Gamma^0_{ab} = \frac{1}{2}e^{-2\Omega}\dot{g}_{ab} \tag{5.67}$$
$$\Gamma^\mu_{0\nu} = \frac{d\Omega}{dy^a}\delta^\mu_\nu \tag{5.68}$$
$$\Gamma^i_{0k} = (\dot{\Omega} + \frac{\dot{\tilde{a}}}{\tilde{a}})\delta^i_k \tag{5.69}$$
$$\Gamma^a_{00} = e^{2\Omega}g^{ab}\frac{d\Omega}{dy^b} \tag{5.70}$$
$$\Gamma^a_{ij} = -g^{ab}e^{2\Omega}\tilde{a}^2\delta_{ij} \tag{5.71}$$
$$\Gamma^a_{0b} = \frac{1}{2}g^{ab}\dot{g}^{ab} \tag{5.72}$$
$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(g_{db,c} + g_{dc,b} - g_{bc,d}) \tag{5.73}$$

where a, b, c, d run over the extra dimensions while μ, ν run over the 4 dimensions. The elements of Ricci tensor is given in [20, 21] as;

$$R_{00} = e^{-2\Omega+\phi}[-3\frac{\ddot{a}}{a} - \frac{k+2}{2k}\xi_0^2 - \sigma^2 - \frac{1}{k}\xi\perp^2 + \dot{\Omega}\xi\perp + \frac{1}{2a^3}\frac{d}{dt}(a^3\xi_0)$$

$$-3\ddot{\Omega} - 3\frac{\ddot{a}}{a}\dot{\Omega} + \dot{\Omega}\xi_0 - \dot{\xi}_\perp - \frac{k+2}{2k}\xi_0\xi\perp] + 4(\delta\Omega)^2 + \mathring{\triangle}\Omega$$

$$R_{ij} = e^{-2\Omega+\phi}[\frac{\ddot{a}}{a} + 2(\frac{\dot{a}}{a})^2 + \dot{\Omega}\xi_\perp - \frac{1}{2a^3}\frac{d}{dt}(a^3\xi_0) + 2\dot{\Omega}^2 + \ddot{\Omega}$$

$$+5\frac{\dot{a}}{a}\dot{\Omega} - \dot{\Omega}\xi_0 + \frac{\dot{a}}{a}\xi_\perp - \frac{1}{2}\xi_0\xi\perp]\delta_{ij} - [4(\delta\Omega)^2 + \mathring{\triangle}\Omega]\delta_{ij}$$

$$R_{ab} = e^{-2\Omega+\phi}[\frac{1}{ka^3}\frac{d}{dt}(a^3\xi_0) + \frac{1}{k}\xi\perp^2 + \frac{2}{k}\xi\perp\dot{\Omega} + \frac{1}{k}\xi\perp\xi_0 + \frac{2}{k}\xi_0\dot{\Omega}$$

$$+\frac{1}{ka^3}\frac{d}{dt}(a^3\xi_\perp)]\delta_{ab} + e^{-2\Omega+\phi}[\frac{1}{a^3}\frac{d}{dt}(a^3\sigma_{ab}) + \xi\perp\sigma_{ab} + 2\dot{\Omega}\sigma_{ab}]$$

And the Ricci scalar is found in [20, 21] as;

$$R = \mathring{R} - 8\mathring{\triangle}\Omega - 20(\partial\Omega)^2 + e^{-2\Omega+\phi}[6\frac{\ddot{a}}{a} + 6(\frac{\dot{a}}{a})^2 - \frac{1}{a^3}\frac{d}{dt}(a^3\xi_0) + \frac{k+2}{2k}\xi_0^2 \tag{5.74}$$

$$+ \sigma^2 + \frac{k+2}{k}\xi_\perp^2 + 4\xi_\perp\dot{\Omega} + 6(\dot{\Omega})^2 + 2\frac{1}{a^3}\frac{d}{dt}(a^3\xi_\perp) + 6\frac{1}{a^3}\frac{d}{dt}(a^3\dot{\Omega}) - 2\dot{\Omega}\xi_0 + \frac{2}{k}\xi_0\xi_\perp]$$

By using these Ricci tensors and the Ricci scalar 4D Einstein equaitons are found in [20, 21] as;

$$3(\frac{\dot{a}}{a})^2 = \rho_T \tag{5.75}$$

$$-2\frac{\ddot{a}}{a} - (\frac{\dot{a}}{a})^2 = P_T \tag{5.76}$$

where ρ_T and P_T are total efective energy density and efective pressure in 4D. As we mentioned before to satisfy NEC, $\rho + P_3$ and $\rho + P_k$ must be grater than zero. Now let us look for these conditions by using given Einstein equations. The elements of Einstein tensor are given in [20, 21] as;

$$G_{00} = \frac{1}{2}\mathring{R} - 3\mathring{\triangle}\Omega - 6(\partial\Omega)^2 + e^{-2\Omega+\phi}[3(\frac{\dot{a}}{a})^2 - \frac{k+2}{4k}(\xi_0 + \xi_\perp)^2 - \frac{1}{2}\sigma^2] \tag{5.77}$$

$$G_{ij} = [-\frac{1}{2}\mathring{R} + 3\mathring{\triangle}\Omega + 6(\partial\Omega)^2]\delta_{ij} \tag{5.78}$$

$$+ e^{-2\Omega+\phi}[-2\frac{\ddot{a}}{a} - (\frac{\dot{a}}{a})^2 - \frac{k+2}{4k}(\xi_0 + \xi_\perp)^2 - \frac{1}{2}\sigma^2]\delta_{ij}$$

$$G_{ab} = \mathring{R}_{ab} - \frac{1}{2}\mathring{R}\delta_{ab} - 4\mathring{\nabla}_a\mathring{\nabla}_b\Omega + 4\mathring{\triangle}\Omega\delta_{ab} - 4\partial_a\Omega\partial_b\Omega \tag{5.79}$$

$$+ 10(\partial\Omega)^2\delta_{ab} + e^{-2\Omega+\phi}[-3\frac{\ddot{a}}{a} - 3(\frac{\dot{a}}{a})^2 - \frac{k+2}{4k}(\xi_0 + \xi_\perp)^2 - \frac{1}{2}\sigma^2]\delta_{ab}$$

$$+ e^{-2\Omega+\phi}[\frac{k+2}{2ka^3}\frac{d}{dt}(a^3(\xi_0 + \xi_\perp))\delta_{ab} + \frac{1}{a^3}\frac{d}{dt}(a^3\sigma_{ab})]$$

where $\mathring{R}$ is the intrinsic curvature of M in unwarped metric. In these given tensor elements by ignoring the intrinsic curvature and warp terms one may obtain the standart Friedmann equations for the scale factor $a(t)$ with the terms containing $\xi_\perp, \xi_0, \sigma$. Now let us calculate the pressure along the extra dimensions which is given in (5.59). We have;

$$\begin{aligned} P_k = (\frac{1}{k} - \frac{1}{2})\mathring{R} + 4(1 - \frac{1}{k})\mathring{\triangle}\Omega + (10 - \frac{4}{k})(\partial\Omega)^2 \\ + e^{-2\Omega+\phi}[-3\frac{\ddot{a}}{a} - 3(\frac{\dot{a}}{a})^2 - \frac{k+2}{4k}(\xi_0 + \xi_\perp)^2 - \frac{1}{2}\sigma^2] \\ + e^{-2\Omega+\phi}[\frac{k+2}{2ka^3}\frac{d}{dt}(a^3(\xi_0 + \xi_\perp))] \end{aligned} \tag{5.80}$$

where $\mathring{\triangle} = \delta^{ab}\mathring{\nabla}_a\mathring{\nabla}_b$ and as defined before $\delta^{ab}\sigma_{ab} = 0$. The total pressure is equal to the zero-zero component of the Einstein tensor. It may be written as;

$$\rho = \frac{1}{2}\mathring{R} - 3\mathring{\triangle}\Omega - 6(\partial\Omega)^2 + e^{-2\Omega+\phi}[3(\frac{\dot{a}}{a})^2 - \frac{k+2}{4k}(\xi_0 + \xi_\perp)^2 - \frac{1}{2}\sigma^2] \tag{5.81}$$

Finally for the pressure along the 3 spatial dimensions which is defined in (5.58) may be calculated as;

$$P_3 = -\frac{1}{2}\mathring{R} + 3\mathring{\triangle}\Omega + 6(\partial\Omega)^2 + e^{-2\Omega+\phi}[-2\frac{\ddot{a}}{a} - (\frac{\dot{a}}{a})^2 - \frac{k+2}{4k}(\xi_0 + \xi_\perp)^2 - \frac{1}{2}\sigma^2]] \tag{5.82}$$

As it is seen we calculated three elements of NEC and now let us construct the conditions and see if they satisfy the NEC or not. One of the condition is ;

$$\rho + P_3 = e^{-2\Omega+\phi}[-2\frac{\ddot{a}}{a} + 2(\frac{\dot{a}}{a})^2 - \frac{k+2}{2k}(\xi_0 + \xi_\perp)^2 - \sigma^2] \tag{5.83}$$

As one may see in (5.83) we have summation of derivatives of scale factor. This term is exactly equal to the summation of (5.75) and (5.76). Then we may rewrite the equation (5.83) as;

$$\rho + P_3 = e^{-2\Omega+\phi}[(\rho_T + P_T) - \frac{k+2}{2k}(\xi_0 + \xi_\perp)^2 - \sigma^2] \tag{5.84}$$

where ρ_T and P_T are total energy density and total pressure respectively. It must be said that these density and pressure are different from ρ and P_3 because the warp term is non-trivial. By the same way we may construct the other element of NEC as;

$$\rho + P_k = \frac{1}{k}\mathring{R} + 4(1-\frac{4}{k})\mathring{\triangle}\Omega + 4(1-\frac{1}{k})(\partial\Omega)^2 \quad (5.85)$$
$$+ e^{-2\Omega+\phi}[-3\frac{\ddot{a}}{a} - \frac{k+2}{2k}(\xi_0+\xi_\perp)^2 - \sigma^2] + e^{-2\Omega+\phi}[\frac{k+2}{2k}\frac{1}{a^3}\frac{d}{dt}(a^3(\xi_0+\xi_\perp))]$$

To detect the NEC conditions let us find averages of these two elements given in (5.84) and (5.85) respectively. They are found as;

$$e^{-\phi}\langle e^{2\Omega}(\rho+P_3)\rangle_A = (\rho_T+P_T) - \frac{k+2}{2k}\langle\xi\rangle_A^2 - \frac{k+2}{2k}\langle(\xi-\langle\xi\rangle_A)^2\rangle_A - \langle\sigma^2\rangle_A \quad (5.86)$$

$$e^{-\phi}\langle e^{2\Omega}(\rho+P_k)\rangle_A = \frac{1}{2}(\rho_T+3P_T) + 2(\frac{A}{4}-1)\frac{k+2}{2k}\langle(\xi-\langle\xi\rangle_A)^2\rangle_A - \quad (5.87)$$
$$\frac{k+2}{2k}\langle\xi\rangle_A^2 - \langle\sigma^2\rangle_A + [k-5+\frac{10}{k}+A(\frac{6}{k}-3)]\langle e^{2\Omega}(\partial\Omega)^2\rangle_A + \frac{k+2}{2k}\frac{1}{a^3}\frac{d}{dt}(a^3\langle\xi\rangle_A)$$

Now let us interpret these two conditions whether they violate the NEC or not and if they violate, the question is under which conditions?

As we see in equation (5.86) the first term on the right hand side is positive as known from Friedmann equaitons since $w > -1$. And the other terms are positive also because in all terms we have square of them. But since they all have minus sign in front they are non-positive. Therefore in order to satisfy NEC, summation of last three terms must be taken to be very close to zero to have a positive term on the right hand side. What about the equation (5.87). When we look at the term on the right side coming from the 4D Friedmann equaitons is not positive because of the accelerated expansion. And sign of the other terms depend on the average number A. We find a range for A which must include the case $A = 2$. In order to include this value we impose the following conditions;

$$\frac{A}{4} - 1 \leq 0 \Rightarrow A \leq 4 \quad (5.88)$$

$$A \geq \frac{k^2-5k-10}{3k-6} \quad (5.89)$$

$$4 \geq A \geq \frac{k^2-5k-10}{3k-6}$$

In terms of the equation of state we may rewrite (5.86) and (5.87) as;

$$e^{-\phi}\langle e^{2\Omega}(\rho+P_3)\rangle_A = \rho_T(1+w) - \frac{k+2}{2k}\langle\xi\rangle_A^2 + (non-positive-terms-for-all-A) \tag{5.90}$$

$$e^{-\phi}\langle e^{2\Omega}(\rho+P_k)\rangle_A = \frac{1}{2}\rho_T(1+3w) + \frac{k+2}{2k}\frac{1}{a^3}\frac{d}{dt}(a^3\langle\xi\rangle_A) + (non-positive-terms-for-some-A) \tag{5.91}$$

As we see from these equations there are some terms which are non-positive. We should have a value for A by which these terms become positive if there exist that value.

In equation (5.90) one may see that when $w=-1$ the first term on the right side is zero. Then we have other terms which are non-positive. Therefore in order to satisfy NEC the only possibility is that these non-positive terms must be zero. But situation is different in (5.91). When we look at this equation the first term on the right side is negative as said before and last terms are non-positive. Therefore there exists only one possibility in which the term containing the derivative of ξ must be non-zero to satisfy NEC. We will see in the next subsection that it is very difficult (if not impossible) to satisfy NEC for both of (5.90) and (5.91).

5.3.3. Steinhardt-Wesley No-Go Theorems

Now let us focus on the theorems that satisfy NEC or violate NEC for dark energy cases. We have two types of such theorems those are theorems that satisfy NEC and violate NEC. Firstly we focus on the theorems that satisfy NEC.

- $Dark\ Energy\ No\text{-}go\ Theorem\ IA$:

ΛCDM (the current concordance model in cosmology) is incompatible with compactified models satisfying the NEC.

- $Proof$:

In the ΛCDM model in which the universe is a mixture of matter and a positive cosmological constant or the equaiton of state between $-1/3$ and -1 because of the presence of matter existency. (At the value when $w=-1$ that is pure de Sitter case, the first term on the right hand side of (5.90) is zero and we have some non-positive terms also. Therefore in order to satisfy NEC in (5.90), we should have the last two terms to be zero. However when we look at the equaiton (5.91), the first term on the right hand side is negative if $w = -1$. Since we have non-positive terms also, we should have the middle term to be positive in order to satisfy NEC; but this requires ξ and its time derivative to be non-zero. But when $\xi = 0$,NEC is satisfied in (5.90) while it is violated in (5.91). Additionally when time derivative of ξ is positive then (5.90) is violated while (5.91) is satisfied. Therefore as we see, NEC can not be satisfied in both equaitons when $w = -1$).

In the ΛCDM model the current energy density of the universe is a mixture of matter and cosmological constant. The density of matter is proportional to $\frac{1}{a^3}$ while that of cosmological constant is constant. Hence $w \to -1$ as $t \to \infty$. Therefore ΛCDM model is incompatible with compactified models satisfying NEC at least in future.

- $Dark\ Energy\ No\text{-}go\ Theorem\ IB$:

Dark energy models with constant w_{DE} less than $w_{transient}$ or time-varying w_{DE} whose value remain less than $w_{transient}$ for a continuous period lasting more than a few Hubble times are incompatible with compactified models satisfying the NEC. Here w_{DE} denotes the equation of state for dark energy.

- $Proof$:

In the case where $w < -1$, 4D does not satisfy NEC. Since this does not go with our asumption then this case is forbidden also. As we mentioned, in ΛCDM universe, w is in the interval between $-1/3$ to -1. Therefore we should focus on this range. Depending on the number of extra dimensions, k, there exists an w called '$w_{transient}$' which changes between $-1/3$ and -1. When w is less than the transient one, then NEC is violated in (5.91) if ξ and its time derivative are small or negative; or NEC is violated in (5.90) if ξ is large and positive. the only possibility for NEC satisfying is to have a ξ to be nearly zero and its time derivative to be large and positive enough. This is only compatible for a short period. When we look at (5.91), in order to satisfy NEC, we should have the following condition on the right hand side;

$$\frac{k+2}{2k}\frac{1}{a^3}\frac{d}{dt}(a^3\langle\xi\rangle_A) > -\frac{1}{2}\rho_T(1+3w) \tag{5.92}$$

The right side of inequality is positive if $w < -1/3$. Then we see from (5.92), on the left hand side from derivative of a we have the Hubble's parameter, and also the derivative of ξ. If this $\langle\xi\rangle_A$ is small in the beginning of its evolution then we say that (5.90) satisfies

NEC. Then (5.92) suggest that;

$$H^{-2}\frac{d\langle\xi\rangle_A}{dt} = O(1) \tag{5.93}$$

if $\langle\xi\rangle_A \ll H$ and $1+3w \simeq 0$, where $O(1)$ shows the magnitude. Then integrating this over time till $t \sim \dfrac{1}{H}$ one may have a condition for ξ itself ;

$$\langle\xi\rangle_A/H = O(1) \tag{5.94}$$

One may conclude from here that in order to satisfy NEC in (5.90) that is having small $\langle\xi\rangle_A$, is only possible for a few Hubble times.

- $Dark\ Energy\ No\text{-}go\ Theorem\ IC$:

All dark energy models are incompatible with compactified models satisfying the NEC if the moduli fields are frozen.

- $Proof$:

As we mentioned before, any form of dark energy requires w to reach a value less than $-1/3$. When we look at equation (5.90), NEC is satisfied for $w < -1/3$ if the other two terms are nearly zero. But in equation (5.91) the first term as we said is negative and also last terms are non-positive. Then we say that moduli fields must vary with time in order to satisfy NEC.

Now let us focus on the theorems that violate NEC.

- $Dark\ Energy\ No\text{-}go\ Theorem\ IIA$:

Dark energy is incompatible with compactified models (with fixed moduli) if the NEC is satisfied in the compact dimensions (i.e. $\rho + P_k \geq 0$ for all t and y^m) whether or not NEC is violated in the non-compact directions.

- $Proof$:

In this theorem as we mentioned, the moduli fields must be frozen that means time derivative of them is zero or very small. Therefore the middle term in equation (5.91) is zero. And at best we may take the last term zero, that may be achieved by choosing the averaging parameter A to be $\dfrac{k^2-5k-10}{3k-6}$. In the light of these conditions we have only the first term on the right hand side of (5.91) which is proportional to $(1 + 3w)$. This term has turning point for its sign at the value of $w = -1/3$. When w is less than this value then the term is negative and it is positive when w is grater. But the acceleration of universe imposes the w to be less than $-1/3$. Therefore one may conclude from here that whenever universe is accelerating, NEC is violated in the compact dimensions.

- $Dark\ Energy\ No\text{-}go\ Theorem\ IIB$:

Dark energy is incompatible with compactified models (with fixed moduli) for which the net NEC violation along the compact directions is time-independent

- $Proof$:

As we mentioned before, at $A = \dfrac{k^2 - 5k - 10}{3k - 6}$ the equation (5.91) is proportional to $(1 + 3w)$. However cosmic evolution needs time dependent w since it includes cosmic acceleration (i.e. inflationary era and present era) and cosmic deceleration (i.e. matter dominated eras). Therefore the NEC violation in the compact directions needs to be time-dependent.

- $Dark\ Energy\ No\text{-}go\ Theorem\ IIC$:

Dark energy is incompatible with compactified models with fixed moduli if the warp factor $\Omega(t, y)$ is non-trivial and has continuous first derivative and if any of the following quantities is homogeneous in y:

- $\rho + P_3$:
- $x\rho + P_k\ for\ RF\ metric\ for\ x\ to\ be\ (1/2)(1 - 3w) > x > 4(k - 1)/3k$:
- $\rho\ for\ CRF\ metric\ for\ k > 4$:
- $2\rho + P_k\ for\ CRF\ metric\ for\ k > 3\ and\ w > -1$:
- $Proof$:

The first quantity that is found in Appendix D, is inhomogeneous because of the factor $e^{-2\Omega+\phi}$. This exponential factor has the term Ω which is the function of extra dimensions. In the definition of exponential function we have the all powers of Ω. Therefore we say that $\rho + P_3$ is inhomogeneous in y.

The second quantity may be found by using equations (B.1) and (B.2). Then it becomes;

$$x\rho + P_k = (4 - \frac{4}{k} - 3x)\triangle\Omega + (10 - \frac{4}{k} - 6x)(\partial\Omega)^2 + e^{-2\Omega+\phi}\rho_T(x + \frac{3w}{2} - \frac{1}{2}) \quad (5.95)$$

As we mentioned before the last term here is inhomogeneous. The other terms are also inhomogeneous for the given range of x that insures NEC violation.

For the third one we have from the appendix B, in the equation (B.6) the first term on the right hand side is positive for the given condition k. But this term changes its sign with the extremum points of Ω. Then ρ is inhomogeneous in y.

Finally the for the last quantity we have the equaiton;

$$\rho + P_k = (k - 1 - \frac{6}{k})\triangle\Omega + (\frac{k^2}{2} - \frac{k}{2} - \frac{2}{k} - 4)(\partial\Omega)^2 + e^{-2\Omega+\phi}\rho_T(\frac{3}{2} + \frac{3w}{2}) \quad (5.96)$$

As it may be seen from this equation, for the given range of k, the first term is positive and also the second one on the right hand side while the last term is also positive for the values of $w > -1$. Because of the same reason for $\triangle\Omega$, this quantity is also inhomogeneous in y.

- $Dark\ Energy\ No\text{-}go\ Theorem\ IID$:

Dark energy is incompatible with compactified models with fixed moduli if the warp factor $\Omega(t, y)$ is non-trivial if $\rho + P_k$ is homogeneous.

- $Proof$:

For $w \leq -1/3$ the last term for both RF and CRF case in $\rho + P_k \leq 0$. And the other term $\triangle\Omega$ is non-zero since $\Omega(t, y)$ is non-trivial. This term changes its sign at the maximum and minimum of $\Omega(t, y)$. Then $\rho + P_k$ is inhomogeneous at least for some y.

- $Dark\ Energy\ No\text{-}go\ Theorem\ IIE$:

Dark energy is incompatible with compactified models with fixed moduli if $w_k(A) > -1$ for $\langle\rho\rangle_A > 0$ or if $w_k(A) < -1$ for $\langle\rho\rangle_A < 0$ both at $A = \dfrac{k^2 - 5k - 10}{3k - 6}$.

- $Proof$:

Let us first construct this w_k, the ratio of the average of pressure along extra dimensions to average of energy density, it may be found for RF metric by dividing (B.2) to (B.1) as follows;

$$w_k(A) = \frac{[(10 - 4A) + \frac{4A}{k} - \frac{4}{k}](\partial\Omega)^2 + (\frac{3w}{2} - \frac{1}{2})e^{\phi}\langle e^{-2\Omega}\rho_T\rangle_A}{(3A - 6)(\partial\Omega)^2 + e^{\phi}\langle e^{-2\Omega}\rho_T\rangle_A} \quad (5.97)$$

Here we consider for dark energy case $\langle e^{-2\Omega}\rho_T\rangle > 0$. At the critical value of A that is $\frac{k^2-5k-10}{3k-6}$, the ratios of the multipliers of $(\partial\Omega)^2$ is -1. Also for $w < -1/3$, as in the case of dark energy, then $\frac{3w}{2} - \frac{1}{2} < -1$. Then we can see from (5.97) that at this value of A w_k is less than -1 which is in contradiction with theorem. On the other hand for $\langle e^{-2\Omega}\rho_T\rangle < 0$ we see that $w_k > -1$ again in contradiction with theorem. This proves the theorem.

In the case of CRF we have the value for w_k which can be found by dividing (B.8) to (B.6);

$$w_k(A) = \frac{[-(7-\frac{6}{k}-k)+(6-\frac{2}{k}+\frac{5k}{2}-\frac{k^2}{2})](\partial\Omega)^2+(\frac{3w}{2}-\frac{1}{2})e^{\phi}\langle e^{-2\Omega}\rho_T\rangle_A}{[-(k-4)A+\frac{1}{2}(k^2-3k-10)](\partial\Omega)^2+e^{\phi}\langle e^{-2\Omega}\rho_T\rangle_A} \tag{5.98}$$

Here again by the same way that is taking $\langle e^{-2\Omega}\rho_T\rangle > 0$ and putting $A = \frac{k^2-5k-10}{3k-6}$ it may be seen that $w_k > -1$. Therefore we say that this is also incompatible with compactified models.

5.3.4. Critical Analysis of No-Go Theorems

Two crucial ingredients of the no-go theorems of Steinhardt and Wesley [20, 21] are;

- A - $averaging\ as\ the\ averaging\ tool\ to\ higher\ dimensional\ results\ to\ 4D$
- $the\ assumption\ of\ the\ necessecity\ of\ the\ applicability\ of\ the\ higher\ dimensional\ null\ energy\ condition$

We give a critical discussion of these assumptions before considering each of the theorems in [20, 21]. First consider A-averaging whose definiton is given in (5.60). At distance much larger than the size of extra dimension(s) we see the extra dimensions integrated. In analogy this is similiar to what we see when look at a hose $R^1 \times S^1$. We may see smaller smaller patch on the side of the hose as we examine it close and closer while at very large distances we can not see the details , we see the circle S^1 integrated out and hence we see the hose just as a line, R^1. Therefore in reduction to 4-dimensions the extra dimensions must be integrated out. To do this one may consider extra dimensional classical solution (metric) in the action and integrate out. For example for Einstein-Hilbert action we may take;

$$M_*^2 \int \widetilde{R}\sqrt{\widetilde{g}}d^{4+k}x = M_{pl}^2 \int R\sqrt{g}d^4x \tag{5.99}$$

Of course the intermadiate steps depend on the form of the metric. For the Ricci flat metric considered in [21] i.e. for ;

$ds^2 = e^{2\Omega}h_{\mu\nu}dx^\mu dx^\nu + g_{\alpha\beta}dx^\alpha dx^\beta$

where $\mu, \nu = 0, 1, 2, 3$ and $\alpha, \beta = 1, 2, , , k$. With this form of metric one may write (5.99) as;

$$M_*^2 \int \widetilde{R}\sqrt{\widetilde{g}}d^{4+k}x = M_{pl}^2 \int e^{2\Omega}\sqrt{g}[R_4 + f(\Omega, g_{\mu\nu}, g_{\alpha\beta})]d^4x \tag{5.100}$$

with $\widetilde{g} = (-1)^S det(g_{AB})$ where S is the number of spatial dimensions. For flat Robertson-Walker metric (5.100) becomes ;

$$M_{pl}^2 \int a^3[\bar{R}_4 + f(\Omega, g_{\mu\nu}, g_{\alpha\beta})]d^4x \tag{5.101}$$

where;

$$a^3 = \bar{a}^3 \int \sqrt{\bar{g}}e^{2\Omega}d^ky \tag{5.102}$$

Therefore it is naturel to define $a(t) = e^{\phi/3}\bar{a}(t)$ where ;

$$e^\phi = \ell^{-k} \int \sqrt{\bar{g}}e^{2\Omega}d^ky \tag{5.103}$$

where ℓ^{-k} is the higher dimensional Planck length rather than the definition given in [20, 21] ;

$$a(t) = e^{\phi/2}\bar{a}(t) \tag{5.104}$$

where the definiton of $e^{\phi/2}$ is given in (5.63). And the corresponding averaging is also given in (5.60).

The second questionable point in our opinion is the imposing the higher dimensional energy condition, $T_{AB}n^A n^B$, as a strict physical condition (that insures absence of ghost,instabilities etc.) In fact this has a ligitimate basis since the extra dimensional components of energy-momentum tensor seem like energy (e.g. masses) when viewed from 4-dimensions. However a concrete analysis is needed to arrive clear out, definite conclusions. In principle it seems possible that an equaiton of state smaller than -1 violation of NEC in extra dimensions may be due to an unconventional form of the extra dimensional piece of the Lagrangian (extra dimensional metric part) rather than a wrong sign in kinetic energy (i.e. ghost). It is possible that such a case may lead to a case where extra dimensional energy conditions are violated while there is no ghost. In our opinion the correct procedure to get the 4-dimensional energy conditions is not the averaging of the extra dimensional $T_{AB}n^A n^B$ done in [20, 21]. The unambigous way to derive the 4-dimensional energy conditions is to integrate the action over the extra dimensions and than obtain Einstein equations and construct $T_{\mu\nu}n^\mu n^\nu$ to check validity of NEC. In fact all these points should be considered in a seperate study to see how the conclusions of [20, 21] survive.

Another point to mention is that the no-go theorems discussed in previous subsection employ the assumption of the applicability of general relativity (i.e. Einstein-Hilbert action), three dimensional flatness, boundness of extra dimensions and extra dimensions being Ricci flat or conformally Ricci flat. The cosmological observations confirm the assumptions of general relativity and three dimensional flatness (at least up to a very high degree approximation) for present time hence these are wholly valid assumptions for dark energy. However [20, 21] uses these assumptions for the time of inflation where their applicability is questionable. It is possible that gravitational action is in a form other than Einstein-Hilbert action and this is pronounced at inflationary era while it approaches usual Einstein-Hilbert form at late times. Therefore in our opinion the applicability of constraints obtained in [20, 21] are not so restrictive as given in [20, 21]. Still another point is that they consider only extra dimensionally Ricci flat and conformally Ricci flat extra dimensional metrics. In fact the conformally Ricci flat metrics considered in [20, 21] are not the possible most general ones where the conformal factors multiplying the four dimensional and the extra dimensional pieces of the metric being independent. Now let us also make some analysis on some no-go theorems of the previous subsection.

- $Dark\ Energy\ No\text{-}go\ Theorem\ IA$:

The pure de Sitter universe (i.e. $w = -1$), since in that case the right hand side of (5.91) is negative, hence violates NEC. The ΛCDM (i.e. the standart model of cosmology) approach $w = -1$ as time goes to infinity. Therefore the current standart model of cosmology (i.e. ΛCDM) is in contradiction this type of extra dimensional models. However there are other viable models of dark energy whose equaiton of state does not go to -1 at infinite future, such as thawing quintessence, tachyon or phantom dark energy models [55, 56]. In other words this theorem alone does not rule out extra dimensional cosmological models.

- $Dark\ Energy\ No\text{-}go\ Theorem\ IB$:

This theorem states that w cannot be less than $w_{transient}$ with $-1 < w_{transient} < -\frac{1}{3}$ for more than a few Hubble times. However a constant or an almost constant w is compatible with observations and is in fact more compatible then the case when w varies large amount. In fact the result of this theorem introduces a problem for inflationary models [57, 58] formulated in the framework of the assumptions of [20, 21] since inflation needs at least 40 e-fold expansion [59]. However as we mentioned before, it is possible use an extension of general relativity which effectively reduces to general relativity at late times or one may adopt extra dimensional models more general than those given in [20, 21] (i.e. those that are not conformally Ricci flat in the extra dimensions).

- $Dark\ Energy\ No\text{-}go\ Theorem\ IC$:

This theorem states that the models in which the moduli ξ are fixed (and specifically G_N is constant) are incompatible with NEC.

The dynamical nature of ξ is a direct consequence of NEC and the equations (5.90) and (5.91). This conclusion is the framework of the assumptions of [20, 21] is inescable. However variation of G_N is not an inevitable consequence of this result. If gravitaion and matter fields live in the same extra dimensional space then their extra dimensional volume varies at the some rate , so both sides of Einstein equaiton are multiplied by the same extra dimensional volume factor hence G_N remains constant. The remaining theorems (i.e. those for the models with NEC violation) are specific technical theorems.

CHAPTER 6

CONCLUSION

In this thesis first we have reviewed the basic concepts of cosmic expansion, dark energy, and extra dimensions. Then we have reviewed and reexamined the constraints derived from energy conditions on extra dimensional cosmological models [19–21].

In the second chapter we have seen the basic formulation of cosmic expansion and the observational evidence for the expansion of the universe. We have derived Einstein equations for the Friedmann-Lemaitre-Robertson-Walker metric that describes universe at cosmological scales. And by using these results in the following chapter we construct the dark energy models. In Chapter 3 we have reviewed dark energy and the corresponding models of dark energy. We have seen that there are some scalar fields that are viable, standart candidates for dark energy.

In Chapter 4 we have gone through extra dimensions. We have reviewed the basic models of extra dimensions; Kaluza-Klein model, ADD model and Randall-Sundrum models. In Kaluza-Klein model we got the form of Einstein-Hilbert action in 4D by using 5D metric. In this model we aimed to combine Einsten gravity with Maxwell's theory and got the corresponding action. In other models we have used extra dimensions to explain the weakness and the localization of gravity.

In Chapter 5 we have reviwed the dark energy models with extra dimensions. In this chapter we have given some examples for the models of dark energy in extra dimensions. We have seen that there are constraints on these models. In order to understand these constraints, we have given some information about energy conditions. In all models we have checked the possibility of null energy condition (NEC). Therefore by obtaning the corresponding Einstein equations for each model we have constructed the null energy condition (NEC) on the models and checked whether it is satisfied or not. We have seen that it is not easy to accommodate accelerated expansion of the universe in extra dimensional models. It seems that the only possibility of accommodating ΛCDM (i.e the Standard model of cosmology) in the context of extra dimensional models in the context of Einstein-Hilbert action is to have an intrinsic curvature for the extra dimensions. In the next subsections we have reviewed the study of [20, 21].

In the subsection 5.3.2 we have considered extra dimensional models satisfying general relativity, flatness condition, boundedness condition, and Ricci flat and conformally Ricci flat metrics. We have found the NEC elements that are the sum of energy density and pressure along both the three space and extra space. Next we have considered some no-go theorems for dark energy in extra dimensions [20]. We have seen that these theorems suggest the difficulty of constructing extra dimensional models in the context of the accelerated expansion of the universe.

In the last subsection we have given our critical analysis on these theorems. We have argued that one may define a physically more relevant averaging than the one given [20, 21]. In future one should check consequences of such a change in the identification of averaging process. The second thought provoking point is the assumption of the applicability of the higher dimensional NEC. We have seen that imposing higher higher dimensional NEC is not well founded. Although the implications of NEC in the usual four dimensions is well known, its implications for higher dimensions is not studied well, and if the higher dimensional NEC lead to four dimensional conclusions (even after averaging) is not evident. All these points must be studied carefully and in detail in further studies.

APPENDIX A

EINSTEIN TENSOR FOR FRIEDMANN-LEMAITRE-ROBERTSON-WALKER SPACE

After using equation (2.17) we obtain;

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}[g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}]$$

$$\Gamma^{0}_{00} = \frac{1}{2}g^{0\sigma}[g_{0\sigma,0} + g_{\sigma 0,0} - g_{00,\sigma}]$$

$$\Gamma^{0}_{00} = 0$$

$$\Gamma^{0}_{ij} = \frac{1}{2}g^{0\sigma}[g_{i\sigma,j} + g_{j\sigma,i} - g_{ij,\sigma}]$$

$$\Gamma^{0}_{ij} = \frac{1}{2}g^{00}[g_{i0,j} + g_{0j,i} - g_{ij,0}]$$

$$\Gamma^{0}_{ij} = \frac{1}{2}\partial_0[g_{ij}]$$

$$\Gamma^{0}_{11} = \frac{1}{2}\partial_0[g_{11}]$$

$$\Gamma^{0}_{11} = \frac{a\dot{a}}{1 - Kr^2}$$

$$\Gamma^{0}_{22} = \frac{1}{2}\partial_0[g_{22}]$$

$$\Gamma^{0}_{22} = a\dot{a}r^2$$

$$\Gamma^{0}_{33} = \frac{1}{2}\partial_0[g_{33}]$$

$$\Gamma^0_{33} = a\dot{a}r^2 sin^2\theta$$

$$\Gamma^i_{jk} = \frac{1}{2}g^{i\sigma}[g_{j\sigma,k} + g_{\sigma k,j} - g_{jk,\sigma}]$$

$$\Gamma^1_{11} = \frac{1}{2}g^{11}[g_{11,1} + g_{11,1} - g_{11,1}]$$

$$\Gamma^1_{11} = \frac{2K}{1 - Kr^2}$$

$$\Gamma^1_{22} = \frac{1}{2}g^{11}[g_{12,2} + g_{21,2} - g_{22,1}]$$

$$\Gamma^1_{22} = \frac{1}{2}g^{11}[-\partial_1(g_{22})]$$

$$\Gamma^1_{22} = -r(1 - Kr^2)$$

$$\Gamma^1_{33} = \frac{1}{2}g^{11}[-\partial_1(g_{33})]$$

$$\Gamma^1_{33} = -r(1 - Kr^2)sin^2\theta$$

$$\Gamma^2_{21} = \frac{1}{2}g^{22}[g_{22,1} + g_{12,2} - g_{21,2}]$$
$$\Gamma^2_{21} = \frac{1}{2}g^{22}\partial_1(g_{22})$$
$$\Gamma^2_{21} = \frac{1}{r}$$
$$\Gamma^3_{31} = \frac{1}{2}g^{33}\partial_1(g_{33})$$
$$\Gamma^3_{31} = \frac{1}{r}$$
$$\Gamma^2_{33} = \frac{1}{2}g^{22}[g_{23,3} + g_{32,3} - g_{33,2}]$$
$$\Gamma^2_{33} = \frac{1}{2}g^{22}[-\partial_2(g_{33})]$$
$$\Gamma^2_{33} = -sin\theta cos\theta$$
$$\Gamma^3_{23} = \frac{1}{2}g^{33}[g_{23,3} + g_{33,2} - g_{23,3}]$$
$$\Gamma^3_{23} = \frac{1}{2}g^{33}\partial_2(g_{33})$$
$$\Gamma^3_{23} = cot\theta$$
$$\Gamma^i_{j0} = \frac{1}{2}g^{i\sigma}[g_{j\sigma,0} + g_{\sigma 0,j} - g_{j0,\sigma}]$$
$$\Gamma^i_{j0} = \frac{1}{2}g^{ik}\partial_0(g_{jk})$$
$$\Gamma^i_{j0} = \delta^i_j\frac{\dot{a}}{a}$$

After calculating these elements we can also calculate $R_{\mu\nu}$ and also R which is equal to $R = g^{\mu\nu}R_{\mu\nu}$

$$R_{\mu\nu} = \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\mu\rho,\nu} + \Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu}$$
$$R_{00} = \Gamma^\rho_{00,\rho} - \Gamma^\rho_{0\rho,0} + \Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{00} - \Gamma^\rho_{0\sigma}\Gamma^\sigma_{\rho 0}$$
$$R_{00} = -3\frac{\ddot{a}}{a}$$
$$R_{ij} = \Gamma^\rho_{ij,\rho} - \Gamma^\rho_{i\rho,j} + \Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{ij} - \Gamma^\rho_{i\sigma}\Gamma^\sigma_{\rho j}$$
$$R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2K}{1 - Kr^2}$$
$$R_{22} = r^2(a\ddot{a} + 2\dot{a}^2 + 2K)$$
$$R_{33} = r^2(a\ddot{a} + 2\dot{a}^2 + 2K)sin^2(\theta)$$

Now we are ready to calculate R which we have written as;

$$
\begin{aligned}
R &= g^{\mu\nu} R_{\mu\nu} \\
R &= g^{00} R_{00} + g^{11} R_{11} + g^{22} R_{22} + g^{33} R_{33} \\
R &= \frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + K)
\end{aligned}
$$

After finding Ricci elements we can calculate Einstein tensor element $G_{\mu\nu}$ from the Einstein equation.We have written it as;

$$
\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\
G_{00} &= R_{00} - \frac{1}{2} g_{00} R \\
G_{00} &= -3\frac{\ddot{a}}{a} + \frac{1}{2}\frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + K) \\
G_{00} &= \frac{3}{a^2}(\dot{a}^2 + K) \\
G_{11} &= R_{11} - \frac{1}{2} g_{11} R \\
G_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2K}{1 - Kr^2} - \frac{1}{2}\frac{a^2}{(1 - Kr^2)}\frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + K) \\
G_{11} &= (Kr^2 - 1)(2\ddot{a}a + \dot{a}^2 + K) \\
G_{22} &= R_{22} - \frac{1}{2} g_{22} R \\
G_{22} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2K) - \frac{1}{2}a^2 r^2 \frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + K) \\
G_{22} &= -r^2(2a\ddot{a} + \dot{a}^2 + K) \\
G_{33} &= R_{33} - \frac{1}{2} g_{33} R \\
G_{33} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2K) sin^2(\theta) - \frac{1}{2}a^2 r^2 sin^2(\theta)\frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + K) \\
G_{33} &= -r^2 sin^2(\theta)(2a\ddot{a} + \dot{a}^2 + K)
\end{aligned}
$$

APPENDIX B

COSMOLOGICAL CONSTANT IN EINSTEIN STATIC UNIVERSE

Although Einstein static universe is not a viable model it is instructive to see how a cosmological constant arises and to see its effect in a simple way. As we found in Chapter 2, when the energy momentum tensor is taken to be $T_{\mu\nu} = Diag(\rho, P, P, P)$, that is of a perfect fluid, then we have from (2.24) for a static universe i.e for $\dot{a} = 0$;

$$K = \frac{8\pi G\rho a^2}{3} \tag{B.1}$$

Here if $\rho > 0$ then we see K is positive and this means that the universe is positively curved to make a^2 positive. From equation (2.25) taking $\ddot{a}$ to be zero and the value of ρ we get;

$$K = -8\pi G P a^2 \tag{B.2}$$

We see that to have a positive K we should have negative pressure, but as we know in all forms of energy pressure is not negative. To describe the static universe, we should add this new term to the Einstein equations. Now the Einstein equations fort his fluid and dust become;

$$K = \frac{8\pi G\rho a^2}{3} + \frac{8\pi G\rho_b a^2}{3} \tag{B.3}$$

$$K = -8\pi G P a^2 + \frac{8\pi G\rho_b a^2}{3} \tag{B.4}$$

Here $P_b = 0$ and ρ_b is the energy density of matter. Cosmological constant contributes positively to the background energy density and negatively to the pressure. This corresponds to a new form of energy where $\rho = -P$. This is called 'cosmological constant'. Cosmological constant can be considered as a perfect fluid with; $\rho_\Lambda = \frac{\Lambda}{8\pi G} = -P_\Lambda$ which shows us that equation of state is -1.

APPENDIX C

KALUZA-KLEIN TOWER

Now let us see how ’cylinder condition’ arises naturally in detail with an example of a real scalar field in 5D space-time. Let us write the Lagrangian density for this scalar field in 5D;

$$L = -\frac{1}{2}\partial_A \Phi \partial^A \Phi, A = 0, 1, 2, 3, 5 \tag{C.1}$$

Here the field $\Phi(t, \overrightarrow{x}, y) \equiv \Phi(x_\mu, y)$ with $\mu = 0, 1, 2, 3$. Here x_μ denotes the 4-dimensional space-time and y is the fifth dimension that is assumed to be compactified on a circle S with radius L. As we said before the extra dimension should be periodic with $y \to y + 2\pi L$. Now we can expand the field in the harmonics on a circle of radius L.

$$\Phi(x, y) = \sum_{n=-\infty}^{+\infty} \phi_n(x) e^{\frac{iny}{L}} \tag{C.2}$$

here I denote x_μ by x. Then (4.1) reduces to ;

$$L = -\frac{1}{2} \sum_{n,m=-\infty}^{+\infty} (\partial_\mu \phi_n \partial^\mu \phi_m - \frac{nm}{L^2} \phi_n \phi_m) e^{\frac{i(n+m)y}{L}} \tag{C.3}$$

Taking $\Phi(x, y)$ real or assuming ϕ_n even under $y \to -y$ implies $\phi_{-n} = \phi_n^*$. If we use this in (4.3)

$$S = \int d^4x \int_0^{2\pi L} L dy = -\pi L \int d^4x \sum_{n=-\infty}^{+\infty} (\partial_\mu \phi_n \partial^\mu \phi_n^* + \frac{n^2}{L^2} \phi_n \phi_n^*) \tag{C.4}$$

Here we performed the integration with respect to extra dimension y. This resulting expression is the action for an infinite number of 4-dimensional fields $\phi_n(x)$. Now let us study some properties of these fields. We introduce the notation $\varphi_n \equiv \sqrt{2\pi L}\phi_n$. Then writing the action ;

$$S=\int d^4x[-\frac{1}{2}\partial_\mu\varphi_0\partial^\mu\varphi_0]-\int d^4x\sum_{k=-\infty}^{+\infty}(\partial_\mu\phi_k\partial^\mu\phi_k^*+\frac{k^2}{L^2}\phi_k\phi_k^*) \tag{C.5}$$

Now let us interpret this picture. The action above consist of :

- $A\,real\,single\,massless\,scalar\,field\,also\,called\,'zero\,mode'$
- $infinite\,number\,of\,scalar\,fields\,with\,masses\,\frac{k^2}{L^2}$

These massive states are called the Kaluza-Klein modes. They are relevant at high energies. And the zero mode is relevant at low energies. These massive states are not observed because they are too heavy to be produced in current accelerators since L is very small. For example if we take $L=10^{-16}cm$ then for the first KK mode i.e $k=1$ $mc^2=\frac{\hbar c}{L}\simeq 200GeV$. We do not observe such particles in current high energy physics experiments. Therefore L must be smaller than 10^{-16} cm. Therefore we may say that since KK modes are not observed yet, extra dimensions are not observed yet.

As a next step let us consider a (4+1) dimensional example of Abelian gauge fields. For this let us consider the Lagrangian density given as;

$$L=-\frac{1}{4g_5^2}F_{AB}F^{AB} \tag{C.6}$$

where g_5 is coupling term with the dimension of $[mass]^{-1}$ and $F_{AB}F^{AB}=F_{\mu\nu}F^{\mu\nu}+2(\partial_\mu A_5-\partial_5 A_\mu)^2$ and F_{AB} is called 'Field strength tensor' in 5-D and the 4-D part of this tensor is $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. Here again expanding the fields A_μ, A_5 in the harmonics on a circle of lenth L as;

$$A_\mu(x,y)=\sum_{n=-\infty}^{+\infty}A_\mu^{(n)}(x)e^{iny/L}, A_5(x,y)=\sum_{n=-\infty}^{+\infty}A_5^{(n)}(x)e^{iny/L} \tag{C.7}$$

The 5-D action can be reduced to 4-D by integrating Lagrangian density over the extra dimensions we have;

$$S=\int d^4x\int_0^{2\pi L}L\equiv\int d^4xL_4 \tag{C.8}$$

The derivatives of the Abelian gauge fields A_μ, A_5 give ;

$$\partial_\mu A_\nu = \sum_{n=-\infty}^{+\infty} \partial_\mu A_\nu^n(x) e^{iny/L}, \partial_\mu A_5 = \sum_{n=-\infty}^{+\infty} \partial_\mu A_5^n(x) e^{iny/L} \tag{C.9}$$

$$\partial_5 A_\mu = \sum_{n=-\infty}^{+\infty} A_\nu^n(x)(\frac{in}{L}) e^{iny/L}, \partial_5 A_5 = \sum_{n=-\infty}^{+\infty} A_5^n(x)(\frac{in}{L}) e^{iny/L} \tag{C.10}$$

Using (4.9) and (4.10) in $F_{AB}F^{AB}$ we have;

$$\begin{aligned} F_{AB}F^{AB} = [(&\sum_{n=-\infty}^{+\infty} (\partial_\mu A_\nu^{(n)}(x) - \partial_\nu A_\mu^{(n)}(x)) e^{iny/L} \\ &\sum_{m=-\infty}^{+\infty} (\partial^\mu A^{(\nu m)}(x) - \partial^\nu A^{(\mu m)}(x)) e^{imy/L}] \\ &+ 2[(\sum_{n=-\infty}^{+\infty} \partial_\mu A_5^n(x)) e^{iny/L} - (\sum_{n=-\infty}^{+\infty} A_5^n(x)(\frac{in}{L})) e^{iny/L}]^2 \end{aligned}$$

Here we use the condition $A_\mu^{(-n)} = (A_\mu^{(-n)})^*$ then some terms will cancel and some integrals over the extra dimension will be zero. Finally we get;

$$L_4 = -\frac{1}{4g_4^2}[F_{\mu\nu}^{(0)} F^{\mu\nu(0)} + 2\sum_{k=1}^{+\infty} [F_{\mu\nu}^{(k)} F^{\mu\nu*(k)} + \frac{2k^2}{L^2} A_\mu^{(k)} A^{\mu*(k)}] + 2(\partial_\mu A_5^{(0)})] \tag{C.11}$$

where we see the following physical states;

- $A\ zero\ mode\ massless\ gauge\ field\ A_\mu^{(0)}$
- $Massive\ KK\ gauge\ bosons$
- $Massless\ scalar\ field\ A_5^{(0)}$

As we see from these results all the KK modes are massive except for the zero mode. This can be interpreted as an effect of Higgs mechanism. In a similiar way one may do Kaluza-Klein reduction for $g_{\mu\nu}$ and R. For the metric tensor $\tilde{g}_{\mu\nu}(x,y)$, we may decompose it as;

$$\tilde{g}_{MN}(x,y) = \sum_n g_{MN}^{(n)}(x) e^{iny/L} \tag{C.12}$$

APPENDIX D

PRESSURE AND ENERGY DENSITY FOR FIXED MODULI

For (RF) models, we have the following useful relations in the case of fixed ξ (breathing mode) and metric g_{mn} :

$$G_{00} = -3\triangle\Omega - 6(\partial\Omega)^2 + e^{-2\Omega+\phi}\rho_T \tag{D.1}$$

$$P_k = +4(1-\frac{1}{k})\triangle\Omega + (10-\frac{4}{k})(\partial\Omega)^2 + e^{-2\Omega+\phi}\frac{1}{2}\rho_T(-1+3w) \tag{D.2}$$

$$P_3 = +3\triangle\Omega + 6(\partial\Omega)^2 + e^{-2\Omega+\phi}P_T \tag{D.3}$$

Now we can calculate the elements of NEC by just summing energy density with pressures respectively. We fnd;

$$\rho + P_3 = e^{-2\Omega+\phi}(\rho_T + P_T) \tag{D.4}$$

$$\rho + P_k = (1-\frac{4}{k})\triangle\Omega + 4(1-\frac{1}{k})(\partial\Omega)^2 + e^{-2\Omega+\phi}\frac{1}{2}\rho_T(1+3w) \tag{D.5}$$

For (CRF) models we have ;

$$G_{00} = (k-4)\triangle\Omega + \frac{1}{2}(k^2-3k-10)(\partial\Omega)^2 + e^{-2\Omega+\phi}\rho_T \tag{D.6}$$

$$P_3 = -(k-4)\triangle\Omega - \frac{1}{2}(k^2-3k-10)(\partial\Omega)^2 + e^{-2\Omega+\phi}\frac{1}{2}\rho_T(1+3w) \tag{D.7}$$

$$P_k = (7-k-\frac{6}{k}\triangle\Omega + (6\frac{2}{k} + \frac{5k}{2} - \frac{k^2}{2}(\partial\Omega)^2 + e^{-2\Omega+\phi}\frac{1}{2}\rho_T(-1+3w) \tag{D.8}$$

Ant the corresponding NEC elements are ;

$$\rho + P_3 = e^{-2\Omega+\phi}(\rho_T + P_T) \tag{D.9}$$

$$\rho + P_k = (3-\frac{6}{k})\triangle\Omega + (k+1-\frac{2}{k})(\partial\Omega)^2 + e^{-2\Omega+\phi}\frac{1}{2}\rho_T(1+3w) \tag{D.10}$$

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