GENERALIZATION OF COFINITELY SUPPLEMENTED MODULES TO LATTICES

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Yasin ÇETİNDİL

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İZMİR

We approve the thesis of **Yasin ÇETİNDİL**

Date of Signature

. . . . **1 November 2005**
Prof. Dr. Rafail ALİZADE
Supervisor
Department of Mathematics
İzmir Institute of Technology

. . . . **1 November 2005**
Asst. Prof. Dr. Murat ATMACA
Department of Mathematics
Muğla University

. . . . **1 November 2005**
Asst. Prof. Dr. Orhan COŞKUN
Department of Electrical and Electronics Engineering
Izmir Institute of Technology

. . . . **1 November 2005**
Prof. Dr. Oğuz YILMAZ
Head of Department
İzmir Institute of Technology

. . . .
Assoc. Prof. Dr. Semahat ÖZDEMİR
Head of the Graduate School

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ABSTRACT

In this thesis we study how to extend the notion of cofinitely supplemented module to lattice theory. A submodule N of a module M is called cofinite if the factor module M/N is finitely generated and we say that M is a cofinitely supplemented module if every cofinite submodule of M has a supplement. We analogously define the notions of cofinite element and cofinitely supplemented lattice for lattices. Inspired by the similarities between the properties of modules and modular lattices, we obtain results for cofinitely supplemented modular lattices, analogous to results for cofinitely supplemented modules.

ÖZET

Bu tezde dual sonlu tümleyen modül kavramının kafes teorisinde tanımlanması ve genelleştirilmesi araştırılmaktadır. Bir M modülü ve N altmodülü için M/N bölüm modulü sonlu üretilmiş ise N altmodülüne dual sonlu ve eğer M modülünün her dual sonlu altmodülünün tümleyeni varsa M modülüne dual sonlu tümleyen modül denir. Benzer şekilde dual sonlu eleman ve dual sonlu tümleyen kafes kavramlarını tanımladık. Modüller ve modüler kafesler arasındaki benzerliklerden esinlenerek, dual sonlu tümleyen modüller için geçerli olan bazı sonuçları modüler kafeslere genelleştirdik.

TABLE OF CONTENTS

CHAPTER 1 . INTRODUCTION 1

CHAPTER 2 . MODULES 2

2.1 Modules and Submodules 2

2.2 Isomorphism Theorems 3

2.3 Direct Sum 6

CHAPTER 3 . COFINITELY SUPPLEMENTED MODULES 7

3.1 Superfluous (Small) Submodules 7

3.2 Complements and Supplements of a Submodule 9

3.3 Cofinitely Supplemented Modules 10

CHAPTER 4 . LATTICES 11

4.1 Sublattices 12

4.2 Modular Lattices 12

4.3 Lattice of Submodules 13

CHAPTER 5 . COFINITELY SUPPLEMENTED LATTICES 14

5.1 Superfluous (Small) Elements 14

5.2 Complements and Supplements 14

5.3 Cofinitely Supplemented Lattices 15

REFERENCES 26

CHAPTER 1

INTRODUCTION

R will be an associative ring with identity and we will consider left unital R-modules. Let M be an R-module. A module M is supplemented, if every submodule K of M has a supplement, i.e., a submodule L minimal with respect to $K + L = M$. It is well known that a submodule L of M is a supplement of a submodule K if $K + L = M$ and $K \cap L \ll L$. If every cofinite submodule K of M (that is $K \leq M$ with ${}^{M}/_{K}$ finitely generated) has a supplement in M is called a cofinitely supplemented module.

A lattice L is called supplemented if every element b of L has a supplement in L, i.e., an element c which is minimal with respect to $b \vee c = 1$. It is well known that if L is a bounded modular lattice, then c is a supplement of b in L if and only if $b \vee c = 1$ and $b \wedge c \ll {}^{c}/_{0}$. An element a of a lattice L is called cofinite in L if the quotient sublattice ${}^{1}/_{a}$ is compact (that is, the element 1 is compact). A lattice L is called cofinitely supplemented if every cofinite element of L has a supplement in L.

There are many similarities between the properties of cofinitely supplemented modules and cofinitely supplemented lattices. The properties of the former are extensively studied in (Alizade, Bilhan, Smith, 2001). We study generalizations of these properties to lattice theory.

CHAPTER 2

MODULES

2.1 Modules and Submodules

Definition 2.1.1. Let R be a ring with identity 1 and M be an abelian group. Suppose there is a function $f : R \times M \longrightarrow M$ (we will denote $f(r,m)$ by rm) where $r \in R$ and $m \in M$. Then M is called a **left *R*-module** (or briefly a **module**) if the followings are satisfied. (1) For every $r \in R$ and $m, n \in M$,

$$r(m+n) = rm + rn.$$

(2) For every $r, s \in R$ and $m \in M$,

$$(r+s)m = rm + sm.$$

(3) For every $r, s \in R$ and $m \in M$,

$$(rs)m = r(sm).$$

(4) For every $m \in M$,

$$1 \cdot m = m.$$

Definition 2.1.2. A subset N of an R-module M is called a **submodule** if N itself is a module with respect to the same operations. Notation: $N \leq M$.

Definition 2.1.3. Let M be module and let N be a submodule of M. The set of cosets

$$M/_N = \{x + N \mid x \in M\}$$

is a module with respect to the addition and scalar multiplication defined by

$$(x+N) + (y+N) = (x+y) + N \ , \quad r(x+N) = rx + N.$$

This module $M/_N$ is called a **factor module** of M by N.

Lemma 2.1.4. ***(Modular Law)*** *Let N, K, L be submodules of a module M and $K \leq N$, then*

$$N \cap (K + L) = K + (N \cap L).$$

Proof. Any x from $N \cap (K + L)$ can be represented as

$$x = n = k + l$$

for some $n \in N$, $k \in K$, and $l \in L$. Since $K \leq N$, $k \in N$. Therefore

$$l = n - k \in N \cap L.$$

Thus

$$x = k + l \in K + (N \cap L).$$

Converse is obvious. □

2.2 Isomorphism Theorems

Definition 2.2.1. If M and N are two modules then a function $f : M \longrightarrow N$ is a **homomorphism** in case for all $r, s \in R$ and $m, a \in M$,

$$f(rm + sa) = rf(m) + sf(a).$$

Definition 2.2.2. A homomorphism $f : M \longrightarrow N$ is called an **epimorphism** in case it is onto. It is called a **monomorphism** in case it is one to one.

Definition 2.2.3. **Kernel of f**: $\ker f = \{m \in M \mid f(m) = 0\} \leq M$.
Image of f: $\operatorname{Im} f = \{f(m) \mid m \in M\} \leq N$.

So f is an epimorphism if and only if $\operatorname{Im} f = N$, and it can be easily verified that f is a monomorphism if and only if $\ker f = 0$.

Definition 2.2.4. A homomorphism f is called an **isomorphism** if it is both an epimorphism and a monomorphism (i.e. it is a bijection).

Theorem 2.2.5. ***(Fundamental Homomorphism Theorem)*** *Let M and N be left modules and $f : M \longrightarrow N$ be a homomorphism, then*

$$M/_{\ker f} \cong \mathrm{Im} f \ .$$

In particular if f is an epimorphism then

$$M/_{\ker f} \cong N \ .$$

Proof. Define $\overline{f} : M/_K \to N$ where $K = \ker f$ by

$$\overline{f}(m + K) = f(m).$$

$m + K = n + K$ implies $m - n \in K$, so $f(m - n) = 0$, then $f(m) = f(n)$. Thus $\overline{f}$ is well-defined. Also

$$\overline{f}((m+K)+(n+K)) = \overline{f}((m+n)+K) = f(m+n) = f(m)+f(n) = \overline{f}(m+K)+\overline{f}(n+K).$$

Hence

$$\overline{f}(r(m + K)) = \overline{f}(rm + K) = f(rm) = rf(m) = r\overline{f}(m + K).$$

So $\overline{f}$ is a homomorphism. If

$$\overline{f}(m + K) = \overline{f}(n + K)$$

then

$$f(m) = f(n) \Rightarrow f(m - n) = 0 \Rightarrow m - n \in K \Rightarrow m + K = n + K$$

which gives us $\overline{f}$ is one-to-one.

At last, since for every $n \in N$ we have

$$n = f(m) = \overline{f}(m + K),$$

$\overline{f}$ is onto. So $\overline{f}$ is an isomorphism. □

Theorem 2.2.6. ***(Second Isomorphism Theorem)*** *If N, K are submodules of M, then*

$$(N + K)/_K \cong N/_{(N \cap K)} \ .$$

Proof. Define $f : N \to {}^{(N+K)}/_{K}$ by

$$f(n) = n + K\ .$$

Since

$$(n+k) + K = n + K = f(n),$$

f is an epimorphism.

$$\ker f = \{n \in N \mid\ n \in K\} = N \cap K\ .$$

So by Fundamental Homomorphism Theorem

$$ {}^{(N+K)}/_{K} \cong {}^{N}/_{(N \cap K)}\ .$$

$\square$

Theorem 2.2.7. ***(Third Isomorphism Theorem)*** *If $K \leq N \leq M$, then*

$$\left({}^{M}/_{K}\right)\Big/\left({}^{N}/_{K}\right) \cong {}^{M}/_{N}\ .$$

Proof. Define $f : {}^{M}/_{K} \to {}^{M}/_{N}$ by

$$f(m+K) = m + N\ .$$

Suppose that $m_1 + K = m_2 + K$. Then

$$m_1 - m_2 \in K \leq N \Rightarrow m_1 - m_2 \in N \Rightarrow m_1 + N = m_2 + N.$$

Hence f is well-defined. Also

$$f(r(m_1{+}K){+}s(m_2{+}K)) = f((rm_1{+}sm_2){+}K) = (rm_1{+}sm_2){+}N = r(m_1{+}N){+}s(m_2{+}N)$$

which gives us

$$f(r(m_1 + K) + s(m_2 + K)) = rf(m_1 + K) + sf(m_2 + K),$$

i.e. f is a homomorphism. Since for all $m + N \in {}^{M}/_{N}$ we have $f(m+K) = m+N$, f is an epimorphism.

$$\ker f = \{m + K \mid m \in N\} = {}^{N}/_{K}\ .$$

So by Fundamental Homomorphism Theorem

$$\left({}^{M}/_{K}\right)\Big/\left({}^{N}/_{K}\right) \cong {}^{M}/_{N}$$

$\square$

2.3 Direct Sum

Definition 2.3.1. Let $\{N_i\}_{i\in I}$ be a family of submodules of a module M. M is the **internal direct sum** of submodules N_i if every element $m \in M$ can be uniquely represented as

$$m = \sum_{i\in I} n_i$$

where $n_i \in N_i$ and $n_i = 0$ for almost all $i \in I$.

Proposition 2.3.2. $M = \bigoplus_{i\in I} N_i$ *if and only if*

$$M = \sum_{i\in I} N_i \text{ and } N_i \cap (\sum_{j\neq i} N_j) = 0$$

for every $i \in I$.

Proof. ($\Rightarrow$) 1) For every $m \in M$ we have

$$m = \sum_{i\in I} n_i \in \sum_{i\in I} N_i \Rightarrow M \subseteq \sum_{i\in I} N_i$$

which gives us

$$M = \sum_{i\in I} N_i.$$

2) Let

$$x = n_i = \sum_{i\neq j} n_j \in N_i \cap \sum_{j\neq i} N_j$$

then by uniqueness of representation $x = n_i = 0$.

($\Leftarrow$) $\forall m \in M = \sum_{i\in I} N_i$,

$$m = \sum_{i\in I} n_i.$$

To prove uniqueness let $m = \sum_{i\in I} n_i = \sum_{i\in I} n_i^{'}$. For every $i \in I$ we have

$$n_i - n_i^{'} = \sum_{i\neq j}(n_j - n_j^{'}) \in N_i \cap (\sum_{i\neq j} N_j) = 0$$

Therefore $n_i = n_i^{'}$. ∎

Definition 2.3.3. If $M = N \oplus K$ then N, K are called **direct summands** of M.

CHAPTER 3

COFINITELY SUPPLEMENTED MODULES

3.1 Superfluous (Small) Submodules

The most important notion in the study of supplements is the small submodule.

Definition 3.1.1. A submodule N of a module M is called **superfluous** or **small** if there is no proper submodule K of M such that $N + K = M$.

Equivalently $N + K = M$ implies that $K = M$. It is denoted by $N \ll M$.

Proposition 3.1.2. *Let M be a module*

1. *If $K \leq N \leq M$ and K is small in N then K is small in M.*
2. *Let N be a small submodule of a module M, then any submodule of N is also small in M.*
3. *If K is a small submodule of a module M and K is contained in a direct summand N of M then K is small in N.*
4. *$K \ll M$ and $N \ll M$ if and only if $K + N \ll M$.*
5. *If $K \leq N \leq M$, then $N \ll M$ if and only if*
$$K \ll M \text{ and } {}^{N}\!/_{K} \ll {}^{M}\!/_{K} .$$
6. *Finite sum of small submodules N_i of M is a small submodule of M.*
7. *Let $f : M \longrightarrow N$ be a homomorphism of modules M and N, let K be a submodule of M. If K is a small submodule of M, then $f(K)$ is a small submodule of N.*

Proof. 1. Let $K + L = M$ for a submodule L of M.

$$N = N \cap M = N \cap (K + L) = K + (N \cap L).$$

Since K is small in N, $N = N \cap L$ so $N \leq L$. $K \leq N$ and $N \leq L$ so $K \leq L$. Therefore $M = K + L = L$. Thus $K \ll M$.

2. Let K be a submodule of N and $K+L=M$ for a submodule L of M. Since $K \leq N$, $N+L=M$ and also since $N \ll M$, $L=M$. So $K \ll M$.

3. $K \leq N \leq M$,$K \ll M$ and $M=N \oplus L$ for a submodule L of M. Let $K+U=N$ for a submodule U of N.

$$M=N+L=K+U+L$$

since $K \ll M$, $M=U+L$ and $U \cap L \leq N \cap L=0$ implies $U \cap L=0$ so $M=U \oplus L$.

$$N=N \cap M=N \cap (U \oplus L)=U \oplus (N \cap L)=U.$$

So $K \ll N$.

4. $(\Rightarrow)$ Let $(K+N)+L=M$ for some $L \leq M$. Since

$$K+(N+L)=(K+N)+L=M \text{ and } K \ll M,$$

we have $N+L=M$. Since $N \ll M$, $L=M$.

$(\Leftarrow)$ $K \leq K+N \ll M$ by 2) $K \ll M$. Similarly $N \leq K+N \ll M$ by 2) $N \ll M$.

5. $(\Rightarrow)$ Since $N \ll M$ by (2) $K \ll M$. Suppose that

$$N/K + X/K = M/K$$

where X/K is a submodule of M/K, then $N+X=M$. By assumption $X=M$ i.e. $X/K = M/K$.

$(\Leftarrow)$ Let $N+X=M$ then

$$(N+X)/K = M/K$$

i.e.

$$N/K + (X+K)/K = M/K \text{ or } N+X+K=M\ .$$

Since $N \ll M$ we have $X+K=M$. Now $K \ll M$ implies $X=M$.

6. Let

$$N=\sum_{i=1}^{n} N_i \text{ and } N_1+\cdots+N_n+X=M$$

for some $X \leq M$. Since $N_i \ll M$,

$$N_1+(N_2+\cdots+N_n+X)=M$$

then

$$N_2+\cdots+N_n+X=M.$$

Continuing this way, we obtain $N_n+X=M$ and hence $N_n \ll M$, $X=M$.

7. Suppose that
$$f(K) + L = f(M)$$
for some $L \leq f(M)$. Then
$$f^{-1}(f(K) + L) = f^{-1}(f(K)) + f^{-1}(L) = f^{-1}(f(M)) = M$$
and therefore
$$M = K + \ker f + f^{-1}(L) = K + f^{-1}(L).$$
Since $K \ll M$, $f^{-1}(L) = M$, hence
$$f(f^{-1}(L)) = f(M)$$
implies that
$$L \cap f(M) = f(M).$$
So $L = f(M)$.

□

3.2 Complements and Supplements of a Submodule

Definition 3.2.1. Let M be a module. A submodule N of module M is said to be a **complement** of a submodule L of M if $N \cap L = 0$ and N is maximal with respect to this property.

Definition 3.2.2. Let M be a module. A submodule N of module M is called a **supplement** of a submodule L of M if $N + L = M$ and N is minimal with respect to this property.

Proposition 3.2.3. *N is a supplement of L in M if and only if*
$$N + L = M \text{ and } N \cap L \ll N.$$

Proof. ($\Rightarrow$) Let N be a supplement of L in M. Then we know that $M = N + L$ and N is minimal with respect to this property. For $K \leq N$ let $N = K + (N \cap L)$. By modular law
$$N = K + (N \cap L) = N \cap (K + L)$$

that is $N \leq L + K$.

$$M = N + L = L + K.$$

By minimality of N we have $K = N$.

($\Leftarrow$) Let $M = L + K$ for some submodule K of N.

$$N = N \cap M = N \cap (K + L) = K + (N \cap L).$$

Since $N \cap L \ll N$, $K = N$. So N is minimal with respect to $N + L = M$. $\square$

Unlike complements, supplements need not exist always.

Definition 3.2.4. A module M is called **supplemented** if every submodule of M has a supplement.

3.3 Cofinitely Supplemented Modules

Definition 3.3.1. Let M be an R-module. For $K \leq M$ if ${}^{M}\!/_{K}$ is finitely generated then K is called a **cofinite submodule** of M.

Definition 3.3.2. If every cofinite submodule of M has a supplement in M then M is called a **cofinitely supplemented** module.

CHAPTER 4

LATTICES

Definition 4.0.3. A partially ordered set (or **poset**) is a set taken together with a partial order (reflexive, antisymmetric and transitive relation) on it.

Definition 4.0.4. The **infimum** is the greatest lower bound of a set S, defined as a quantity m such that no member of the set is less than m. When it exists (which is not required by this definition, e.g., $\inf R$ does not exist), the infimum is denoted $\inf S$ or $\inf_{x \in S} x$.

Definition 4.0.5. The **supremum** is the least upper bound of a set S, defined as a quantity M such that no member of the set exceeds M. When it exists (which is not required by this definition, e.g., $\sup R$ does not exist), the supremum is denoted $\sup S$ or $\sup_{x \in S} x$.

Definition 4.0.6. A **lattice** is any non-empty poset L in which any two elements x and y have a supremum $x \vee y$ and an infimum $x \wedge y$.

Another equivalent definition is that a triple $\langle L; \wedge, \vee \rangle$ is called a **lattice** if L is a nonempty set, $\wedge$ (meet) and $\vee$ (join) are binary operations on L, both $\wedge$ and $\vee$ are idempotent, commutative and associative and they satisfy the absorption law

$$a \wedge (a \vee b) = a \vee (a \wedge b) = a \ .$$

The partial order relation can be recovered from meet and join by defining

$$x \leq y \quad \Leftrightarrow \quad x \wedge y = x \ \text{ and } \ x \vee y = y.$$

The study of lattices is called lattice theory.

Definition 4.0.7. A lattice is said to be **bounded** if it has a greatest element often denoted by 1 and a least element often denoted by 0.

Definition 4.0.8. A lattice is said to be **complete** if every nonempty subset of it has a supremum and infimum.

4.1 Sublattices

Definition 4.1.1. A subset B of a lattice L is called a **sublattice** if for each $b, b' \in B$,

$$\inf_L \{b, b'\} \in B \text{ and } \sup_L \{b, b'\} \in B.$$

Clearly in this case, B is also a lattice and

$$\inf_B \{b, b'\} = \inf_L \{b, b'\} \quad \text{and} \quad \sup_B \{b, b'\} = \sup_L \{b, b'\}.$$

Definition 4.1.2. In a complete lattice L, a subset B is called a **complete sublattice** if for each subset $X \subseteq B$,

$$\inf_L X \in B \text{ and } \sup_L X \in B.$$

Definition 4.1.3. A **quotient sublattice** b/a for $a \leq b$ represents the sublattice

$$\{x \in L \mid a \leq x \leq b\}.$$

4.2 Modular Lattices

Definition 4.2.1. A lattice L which satisfies the identity

$$x \vee (y \wedge z) = (x \vee y) \wedge z$$

for all $x, y, z \in L$ such that $x \leq z$ is said to be **modular**.

4.3 Lattice of Submodules

The set of submodules of a module ordered by inclusion forms a lattice. The supremum is given by the sum of submodules and the infimum by the intersection of them.

Proposition 4.3.1. *The lattice of submodules is modular. Namely, if K, H, L are submodules of M and $K \subset H$ then*

$$H \cap (K + L) = K + (H \cap L) .$$

Proof. First observe

$$K + (H \cap L) = (H \cap K) + (H \cap L) \subset H \cap (K + L) .$$

If

$$h = k + l \in H \cap (K + L)$$

with $h \in H, k \in K, l \in L$, then

$$k \in K \subset H \text{ and } l = h - k \in H \cap L.$$

Therefore

$$H \cap (K + L) \subset K + (H \cap L) .$$

So the lattice of submodules of a module is a modular lattice. □

CHAPTER 5

COFINITELY SUPPLEMENTED LATTICES

5.1 Superfluous (Small) Elements

Definition 5.1.1. In a lattice with 1, an element a is called **superfluous** if $a \vee b \neq 1$ holds for every $b \neq 1$.

Lemma 5.1.2. *Let L be a lattice and let $a \leq b$ and $b_i \in L$ $(1 \leq i \leq n)$ for some positive integer n,*

1. *$b \ll L$ if and only if $a \ll L$ and $b \ll {}^{1}\!/_{a}$.*
2. *$(b_1 \vee b_2 \vee \ldots \vee b_n) \ll L$ if and only if $b_i \ll L$.*

Proof. 1. $(\Rightarrow)$ If $b \ll L$, it is clear that $a \ll L$ and $b \ll {}^{1}\!/_{a}$.
$(\Leftarrow)$ If $b \vee c = 1$, then $b \vee (a \vee c) = 1$. Since $b \ll {}^{1}\!/_{a}$, $a \vee c = 1$ and since $a \ll L$, $c = 1$. Therefore $b \ll L$.

2. It is enough to show the property for $n = 2$.
$(\Rightarrow)$ By previous alternative, $b_1 \leq b_1 \vee b_2 \ll L$ implies $b_1 \ll L$. Similarly $b_2 \ll L$.
$(\Leftarrow)$ Let $(b_1 \vee b_2) \vee b = 1$. Since $b_1 \ll L$,

$$1 = (b_1 \vee b_2) \vee b = b_1 \vee (b_2 \vee b)$$

implies $b_2 \vee b = 1$. Similarly, since $b_2 \ll L$, $b_2 \vee b = 1$ requires $b = 1$. Therefore $b_1 \vee b_2 \ll L$.

□

5.2 Complements and Supplements

Definition 5.2.1. Let L be a lattice with 0 and 1 and $a \in L$. An element $a' \in L$ is called a **complement** of a if

$$a \wedge a' = 0 \text{ and } a \vee a' = 1.$$

Definition 5.2.2. If $a' \in L$ is a complement of $a \in L$, we use the notation

$$a \oplus a' = 1$$

and we call this a **direct sum** and a and a' **direct summands**.

Definition 5.2.3. In a lattice with 1, an element c is called a **supplement** of b in L if it is minimal relative to the property $b \vee c = 1$.

Lemma 5.2.4. *If L is a bounded modular lattice, then c is a supplement of b in L if and only if*

$$b \vee c = 1 \ \ \textit{and} \ \ b \wedge c \ll {}^{c}/_{0}.$$

Proof. ($\Rightarrow$) Let c be a supplement of b in L. Then by definition, c is minimal relative to the property $b \vee c = 1$. Now we have to show that $b \wedge c \ll {}^{c}/_{0}$. Suppose that $b \wedge c \not\ll {}^{c}/_{0}$. This means that there is an element d in ${}^{c}/_{0}$ such that $(b \wedge c) \vee d = c$. In that case

$$1 = b \vee c = b \vee (b \wedge c) \vee d = b \vee d$$

contradicts the fact that c is minimal with the property $b \vee c = 1$. Hence $b \wedge c \ll {}^{c}/_{0}$.
($\Leftarrow$) Let

$$b \vee c = 1 \ \ \text{and} \ \ b \wedge c \ll {}^{c}/_{0} \, .$$

Now we have to show that c is a supplement of b. Suppose $b \vee c' = 1$ for some $c' \leq c$. In that case, by modular law

$$c = c \wedge (b \vee c') = (b \wedge c) \vee c'$$

Since $b \wedge c \ll c$ we have $c' = c$. Therefore c is a supplement of b in L. $\square$

5.3 Cofinitely Supplemented Lattices

Definition 5.3.1. A lattice L is called **supplemented** if each element of L has a supplement in L.

Definition 5.3.2. An element c of a complete lattice L is called **compact** if for every subset X of L and $c \leq \vee X$ there is a finite subset $F \subseteq X$ such that $c \leq \vee F$.

Definition 5.3.3. A lattice with greatest element 1 is called **compact** if the element 1 is compact.

Definition 5.3.4. A complete lattice L is called **compactly generated** if each element of L is a join of compact elements.

Definition 5.3.5. An element a of a lattice L is called **cofinite** in L if the quotient sublattice ${}^{1}\!/_{a}$ is compact.

Definition 5.3.6. A lattice L is called **cofinitely supplemented** if every cofinite element of L has a supplement in L.

Lemma 5.3.7. *Let L be a cofinitely supplemented lattice. Then ${}^{1}\!/_{a}$ is cofinitely supplemented for any $a \in L$.*

Proof. Suppose that L is cofinitely supplemented and let $a \in L$. Since L is cofinitely supplemented, any cofinite element b of ${}^{1}\!/_{a}$ has a supplement $c \in L$. By Lemma 5.2.4,

$$1 = b \vee c \ \text{ and } \ b \wedge c \ll {}^{c}\!/_{0}.$$

Hence

$$(b \vee c) \vee a = 1 \vee a = 1 = b \vee (c \vee a)\,.$$

By Modular law,

$$b \wedge (c \vee a) = (b \wedge c) \vee a \ll {}^{(c \vee a)}\!/_{0}$$

by Lemma 5.1.2(2). By Lemma 5.2.4, $c \vee a$ is a supplement of b in ${}^{1}\!/_{a}$. It follows that ${}^{1}\!/_{a}$ is cofinitely supplemented. □

Lemma 5.3.8. *Let $a < b$ be elements in a modular lattice L. If a is superfluous in ${}^{b}\!/_{0}$ then a is also superfluous in L. If b is a direct summand in L, the converse also holds.*

Proof. Suppose $a \vee u = 1$. Then by modularity

$$b = b \wedge 1 = b \wedge (a \vee u) = a \vee (b \wedge u)$$

and since a is superfluous in ${}^{b}\!/_{0}$,

$$b \wedge u = b \text{ or } b \leq u.$$

Hence

$$a < b \leq u \text{ and } u = a \vee u = 1.$$

Therefore a is small in L.

Conversely, let $v \in {}^{b}\!/_{0}$ be such that

$$a \vee v = b \text{ and } b \oplus c = 1.$$

Then $a \vee v \vee c = 1$ and so $v \vee c = 1$, a being superfluous in L. Hence

$$v = v \vee 0 = v \vee (b \wedge c) = b \wedge (v \vee c) = b$$

by modularity. □

Definition 5.3.9. A subset I of a lattice L is called an **ideal** in L if
(i) $x \vee y \in I$ for every $x, y \in I$.
(ii) $x \wedge y \in I$ for every $x \in I$ and $y \in L$.

Lemma 5.3.10. *The superfluous elements of a lattice form an ideal.*

Proof. Let us denote the set of superfluous elements by I.
(i) $x, y \in I$ means $x \ll L$ and $y \ll L$. By Lemma 5.1.2(2), $x \vee y \ll L$.
(ii) Since $x \wedge y \leq x$ and $x \ll L$, by Lemma 5.1.2(1) $x \wedge y \ll L$ and hence $x \wedge y \in I$. □

Lemma 5.3.11. *In a modular lattice L, let c' be superfluous in ${}^{c}\!/_{0}$ and d' be superfluous in ${}^{d}\!/_{0}$. Then*

$$c' \vee d' \ll {}^{(c \vee d)}\!/_{0}.$$

Proof. By taking $L = {}^{(c \vee d)}\!/_{0}$ in Lemma 5.3.8, we get c' and d' are superfluous and by Lemma 5.3.10, $c' \vee d'$ is superfluous in ${}^{(c \vee d)}\!/_{0}$. □

Lemma 5.3.12. *In any modular lattice*

$$(c \vee d) \wedge b \leq [c \wedge (b \vee d)] \vee [d \wedge (b \vee c)]$$

holds for every $b, c, d \in L$.

Proof. Since

$$c \wedge (b \vee d) \leq b \vee c$$

by modularity

$$[c \wedge (b \vee d)] \vee [d \wedge (b \vee c)] = [[c \wedge (b \vee d)] \vee d] \wedge (b \vee c)$$

and again by modularity, since $d \leq b \vee d$,

$$[c \wedge (b \vee d)] \vee d = (d \vee c) \wedge (b \vee d)\,.$$

These equalities gives us

$$[c \wedge (b \vee d)] \vee [d \wedge (b \vee c)] = [[c \wedge (b \vee d)] \vee d] \wedge (b \vee c) = [(d \vee c) \wedge (b \vee d)] \wedge (b \vee c)\,.$$

It is clear that

$$b \leq (b \vee d) \wedge (b \vee c)\,.$$

And hence

$$(c \vee d) \wedge b \leq [c \wedge (b \vee d)] \vee [d \wedge (b \vee c)]\,.$$

$\square$

Theorem 5.3.13. *If a and b are elements in a modular lattice L then the quotient sublattices*

$$^{(a \vee b)}\!/_{b} \text{ and } ^{a}\!/_{(a \wedge b)}$$

are isomorphic.

Proof. It is easily verified that the maps

$$f : {}^{a \vee b}\!/_{b} \to {}^{a}\!/_{a \wedge b}\,, \quad f(x) = x \wedge a$$

for all $x \in {}^{a \vee b}\!/_{b}$ and

$$g : {}^{a}\!/_{a \wedge b} \to {}^{a \vee b}\!/_{b}\,, \quad g(y) = y \vee b$$

for all $y \in {}^{a}\!/_{a \wedge b}$ are (in a modular lattice L) mutually inverse lattice morphisms. $\square$

Proposition 5.3.14. *Let c be a supplement of b in a lattice L. If $a \leq b$ and $a \vee c = 1$, then c is a supplement of a in L.*

Proof. Let $c' \leq c$ with $a \vee c' = 1$. $a \leq b$ implies $b \vee c' = 1$. Since c is a supplement of b, $c' = c$. $\square$

Lemma 5.3.15. *Let a and b be elements of a lattice L such that b is cofinite, $a/0$ is cofinitely supplemented and $a \vee b$ has a supplement in L. Then b has a supplement in L.*

Proof. Let c be a supplement of $a \vee b$ in L and d a supplement of $(c \vee b) \wedge a$ in $a/0$. These mean that

$$c \vee (a \vee b) = 1, \ c \wedge (a \vee b) \ll c/0$$

and

$$d \vee [(c \vee b) \wedge a] = a, \ d \wedge [(c \vee b) \wedge a] \ll d/0$$

First observe that d is a supplement of $c \vee b$ in L. Indeed

$$1 = c \vee (a \vee b) = c \vee b \vee [[(c \vee b) \wedge a] \vee d] = (c \vee b) \vee d$$

and

$$(c \vee b) \wedge d = (c \vee b) \wedge d \wedge a = [(c \vee b) \wedge a] \wedge d \ll d/0.$$

Using Proposition 5.3.14(1), as $b \vee d \leq a \vee b$,

$$(a \vee b) \vee c = (b \vee d) \vee c = 1$$

and c is a supplement of $a \vee b$, we obtain that c is also a supplement of $b \vee d$. Hence

$$(b \vee d) \wedge c \ll c/0 \ .$$

Finally, we prove that $c \vee d$ is a supplement of b in L. We already know that

$$b \vee (c \vee d) = 1$$

so that we only need to show that

$$b \wedge (c \vee d) \ll (c \vee d)/0 \ .$$

By Lemma 5.3.11,

$$[c \wedge (b \vee d)] \vee [d \wedge (b \vee c)] \ll (c \vee d)/0$$

and using Lemma 5.3.12,

$$(c \vee d) \wedge b \ll (c \vee d)/0 \ .$$

$\square$

Lemma 5.3.16. *Let* $a_i/0$ $(i \in I)$ *be any collection of cofinitely supplemented sublattices of a lattice* L*. Then*

$$\bigvee_{i\in I} a_i \Big/ 0$$

is a cofinitely supplemented sublattice of L*.*

Proof. Let

$$A = \bigvee_{i\in I} a_i \Big/ 0$$

and b be a cofinite element of A. Because $\bigvee_{i\in I} a_i \Big/ b$ is compact, there exists a finite set $F \subseteq I$ such that

$$\bigvee_{i\in I} a_i = \bigvee_{i\in F} a_i \ .$$

Since

$$\left(\bigvee_{i\in F} a_i\right) \vee b = \bigvee_{i\in F} a_i \ ,$$

we have

$$A = b \vee \left(\bigvee_{i\in F} a_i\right) \Big/ 0 \ .$$

Hence

$$\bigvee_{i\in I} a_i = \bigvee_{i\in F} a_i = b \vee \left(\bigvee_{i\in F} a_i\right) = \left[b \vee \left(\bigvee_{i\in F-j_1} a_i\right)\right] \vee a_{j_1}.$$

It is clear that $\bigvee_{i\in I} a_i$ has 0 as its supplement and we know that $a_{j_1}/0$ is cofinitely supplemented. By Lemma 5.3.15,

$$b \vee \left(\bigvee_{i\in F-j_1} a_i\right)$$

has a supplement in A. By repeated use of Lemma 5.3.15, we deduce that b has a supplement in A. It follows that

$$\bigvee_{i\in I} a_i \Big/ 0$$

is cofinitely supplemented. □

Corollary 5.3.17. *Any direct sum of cofinitely supplemented sublattices of a lattice* L *is cofinitely supplemented.*

Proof. This immediately follows from Lemma 5.3.16. □

Definition 5.3.18. An element $m \in L$ is called **maximal** in L if there is no element greater than m.

Definition 5.3.19. In a complete lattice L the meet of all the maximal elements different from 1 in L is called the **radical** of L, denoted by $r(L)$.

Proposition 5.3.20. *Each finite join of compact elements is a compact element.*

Proof. By induction, it suffices to verify the assertion for only two compact elements, say $a, b \in L$. Since a and b are compact, for

$$a = \bigvee_{i \in I_1} a_i \text{ and } b = \bigvee_{i \in I_2} b_i$$

we can find finite sets $F_1 \subseteq I_1$ and $F_2 \subseteq I_2$ such that

$$a = \bigvee_{i \in F_1} a_i \text{ and } b = \bigvee_{i \in F_2} b_i.$$

Clearly

$$a \vee b = \left(\bigvee_{i \in F_1} a_i\right) \vee \left(\bigvee_{i \in F_2} b_i\right)$$

is a finite join. Therefore $a \vee b$ is compact. $\square$

Lemma 5.3.21. *In a compactly generated lattice L, an element $k \in {}^{b}\!/\!_{a}$ is compact in a quotient sublattice ${}^{b}\!/\!_{a}$ if and only if there is a compact element c in L such that*

$$k = a \vee c \text{ and } a \vee c \leq b\,.$$

Proof. ($\Rightarrow$) Let k be a compact element in ${}^{b}\!/\!_{a}$. The lattice being compactly generated, there is a family of compact elements $\{c_i\}_{i \in I}$ such that $k = \bigvee_{i \in I} c_i$. Clearly

$$k = a \vee k = a \vee \left(\bigvee_{i \in I} c_i\right) = \bigvee_{i \in I} (a \vee c_i)$$

with $a \vee c_i \in {}^{b}\!/\!_{a}$ (because $c_i \leq k \leq b$). Thus there exists a finite subset $F \subseteq I$ such that

$$k = \bigvee_{i \in F} (a \vee c_i) = a \vee \left(\bigvee_{i \in F} c_i\right) = a \vee c \leq b$$

where $c = \bigvee_{i \in F} c_i$ is compact in L as finite join of compact elements by Proposition 5.3.20.
($\Leftarrow$) Suppose c is compact in L, $X \subseteq {}^{b}\!/\!_{a}$ and $a \vee c \leq \vee X$. Then $c \leq \vee X$ and so there is a finite subset $F \subseteq X$ such that $c \leq \vee F$. But $F \subseteq {}^{b}\!/\!_{a}$ implies $a \vee c \leq \vee F$ and hence $a \vee c$ is compact in ${}^{b}\!/\!_{a}$. $\square$

Proposition 5.3.22. *Let a be a superfluous element in a compactly generated lattice L. L is compact if and only if ${}^{1}\!/_{a}$ is a compact sublattice.*

Proof. ($\Rightarrow$) If L is compact and $1 = \bigvee_{i \in I} c_i$ is a join of compact elements, then

$$1 = 1 \vee a = \bigvee_{i \in I} (c_i \vee a)$$

is a join of compact elements in ${}^{1}\!/_{a}$.
($\Leftarrow$) If ${}^{1}\!/_{a}$ is a compact sublattice, then

$$1 = \bigvee_{i \in I} k_i = \bigvee_{i \in I} (a \vee c_i) = a \vee \left(\bigvee_{i \in I} c_i \right)$$

by Lemma 5.3.21. Then since a is superfluous in L,

$$1 = \bigvee_{i \in I} c_i \ .$$

$\square$

Lemma 5.3.23. *If a is superfluous in L, then $a \leq r(L)$.*

Proof. If $m \neq 1$ is a maximal element in L, then since a is superfluous $a \vee m \neq 1$. Hence $a \vee m = m$, namely $a \leq m$. $\square$

Lemma 5.3.24. *If a is compact in L and $a \leq r(L)$, then a is superfluous in L.*

Proof. Suppose that a is not superfluous. Then there is an element $b \neq 1$ such that $a \vee b = 1$. Clearly $a \nleq b$ and if we consider the set

$$D = \{x \in L \mid \ a \nleq x, x \neq 1, a \vee x = 1\}$$

D is nonempty. If C is a chain in D, a being compact, $a \nleq \vee C$ and $\vee C \in D$. Denote again by b a maximal element in D. The element b is also maximal in L (indeed, if $b < c$, by the maximality of b in D, $a \leq c$ and hence, $1 = a \vee b \leq c$, namely $c = 1$) and so

$$a \leq r(L) \leq b \ ,$$

the required contradiction. $\square$

Theorem 5.3.25. *If L is compactly generated, then $r(L)$ is the join of all the superfluous elements from L.*

Proof. Let u is the join of all the superfluous elements from L. Since the superfluous elements of a lattice forms an ideal, u is also superfluous. Hence by Lemma 5.3.23, $u \leq r(L)$.
Conversely, if $u < r(L)$, the lattice being compactly generated, there is a compact element a such that $a \leq r(L)$, $a \nleq u$. But Lemma 5.3.24 implies a superfluous and we contradict $a \nleq u$. Therefore $u = r(L)$. □

Lemma 5.3.26. *Let L be a cofinitely supplemented lattice. Then every cofinite element of the quotient sublattice ${}^{1}/_{r(L)}$ is a direct summand.*

Proof. Any cofinite element a of ${}^{1}/_{r(L)}$ is also a cofinite element of L. Since L is cofinitely supplemented, by Lemma 5.2.4 there exists an element b of L such that

$$1 = a \vee b \text{ and } a \wedge b \ll {}^{b}/_{0}.$$

Now $a \wedge b \ll L$ by Lemma 5.3.8 and hence by Lemma 5.3.23, $a \wedge b \leq r(L)$. Therefore

$$a \wedge (b \vee r(L)) \leq r(L).$$

Thus

$${}^{1}/_{r(L)} = {}^{a}/_{r(L)} \oplus {}^{[b \vee r(L)]}/_{r(L)}$$

as required. □

Definition 5.3.27. An element a in a lattice L is called an **atom** if there is no element $b \in L$ such that $0 < b < a$.

Definition 5.3.28. The join of all atoms of L, denoted by $s(L)$, is called the **socle** of the lattice L.

Lemma 5.3.29. *The following statements are equivalent for a lattice L:*

1. *Every cofinite element of L is a direct summand of 1.*

2. *Every maximal element of L is a direct summand of 1.*

3. ${}^{1}/_{s(L)}$ *does not contain a maximal element.*

Proof. (1)⇒(2) It is clear that a maximal element, say m, is cofinite. Really ${}^{1}/_{m}$ is compact. Because if we represent 1 as a join of some indexed elements, these elements can only be either m or 1. Therefore 1 can be represented as a finite join.
(2)⇒(3) Let ${}^{1}/_{s(L)}$ contains a maximal m. Then for any atom $m' \leq s(L)$, $m \vee m' = m$ since $m \vee s(L) = m$. Therefore $m \vee m' \neq 1$ for every $m' \in L$. Hence no maximal elements can be a direct summand of 1.
(3)⇒(1) Let a be any cofinite element of L. Then $a \vee s(L)$ is cofinite and hence $1 = a \vee s(L)$ by (3). It follows that $1 = a \vee a'$ for any element a' such that

$$s(L) = [a \wedge s(L)] \oplus a'$$

This proves (1). □

Definition 5.3.30. Let a and b be elements of a lattice L such that $b > a$. If there is no element $a \in L$ such that $a < c < b$ we say that the quotient sublattice

$${}^{b}/_{a} = \{a, b\}$$

is **simple**.

Definition 5.3.31. A lattice L is called **local** if it has a largest element $\neq 1$.

Proposition 5.3.32. *A lattice L is local if and only if $r(L)$ is superfluous and maximal.*

Theorem 5.3.33. *Let $Loc(L)$ is defined as*

$$Loc(L) = \bigvee_{i \in I} \{a_i \mid \ {}^{a_i}/_{0} \text{ is local } \forall i \in I\}$$

and $Cof(L)$ is defined as

$$Cof(L) = \bigvee_{i \in I} \{a_i \mid \ {}^{a_i}/_{0} \text{ is cofinitely supplemented } \forall i \in I\}.$$

Then the following statements are equivalent for a lattice L:

1. *L is cofinitely supplemented.*

2. *Every maximal element of L has a supplement in L.*

3. *The quotient sublattice $1/_{Loc\,(L)}$ doesn't contain a maximal element.*

4. *The quotient sublattice $1/_{Cof\,(L)}$ doesn't contain a maximal element.*

Proof. (1)⇒(2) Clear because maximal elements are cofinite.

(2)⇒(3) Let m be a maximal element of L. There exists an element l of L such that

$$1 = m \vee l \text{ and } m \wedge l \ll l$$

by Lemma 5.2.4. Note that

$$l/_{(m \wedge l)} \cong (l \vee m)/_{m} = 1/_{m}$$

and since $1/_{m}$ is simple $m \wedge l$ is a maximal element of $l/_{0}$. Therefore l is a local element of L. It follows that $Loc\,(L)$ is not an element of $m/_{0}$. Hence $1/_{Loc\,(L)}$ does not contain a maximal element.

(3)⇒(4) Local lattices are cofinitely supplemented. Because it has a largest element which is not equal to 1. Therefore since any element is superfluous, 1 is the supplement of each proper element. Hence $Loc\,(L) \leq Cof\,(L)$. This gives (4).

(4)⇒(1) Let c be a cofinite element of L. Then $c \vee Cof\,(L)$ is a cofinite element of L and hence (4) gives that

$$1 = c \vee Cof\,(L)\,.$$

Since $1/_{c}$ is compact it follows that

$$1 = c \vee k_1 \vee k_2 \vee \ldots \vee k_n$$

for some c and cofinitely supplemented sublattices $k_i/_{0}$ $(1 \leq i \leq n)$. By repeated use of Lemma 5.3.15, c has a supplement in L. It follows that L is cofinitely supplemented. □

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