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Quantum Group Symmetry for Kaleidoscope of Hydrodynamic Images and Quantum States

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Abstract. The hydrodynamic flow in several bounded domains can be formulated by the image theorems, like the two circle, the wedge and the strip theorems, describing flow by q-periodic functions. Depending on geometry of the domain, parameter q has different geometrical meanings and values. In the special case of the wedge domain, with q as a primitive root of unity, the set of images appears as a regular polygon kaleidoscope. By interpreting the wave function in the Fock-Barman representation as complex potential of a flow, we find $mod\, n$ projection operators in the space of quantum coherent states, related with operator q-numbers. They determine the units of quantum information as kaleidoscope of quantum states with quantum group symmetry of the q-oscillator. Expansion of Glauber coherent states to these units and corresponding entropy are discussed.

1. Introduction

The relation between hydrodynamics and quantum theory has long history starting probably from Madelung representation of the Schrödinger equation in 1926. Here we propose another type of relation based on Fock-Bargman representation of a quantum state by analytic function. This function $f(z) = \varphi(x, y) + i\chi(x, y)$ can be interpreted as complex potential of incompressible and irrotational hydrodynamic flow with complex velocity $\bar{V}(z) = df(z)/dz$. The problem is, for given boundary curve C to find analytic function (complex potential) $F(z)$ with boundary condition $\Im F|_C = \chi|_C = 0$, where χ is the stream function. This condition show that the boundary curve is the stream curve and normal velocity across the boundary vanishes: $v_n|_C = 0$. For simple geometry of boundary curves, several theorems solving this problem exist. The Miln-Thomson circle theorem for one circle added to the planar flow, two circles theorem for the flow in annular domain [1] and the strip theorem for the flow in a strip [2]. For two circle theorem the flow is determined by q-periodic function, where parameter q is given by ratio of two circle radiuses $q = r_2^2/r_1^2$. The strip theorem corresponds to the limit $q \to 1$, and the circle theorem to $q \to \infty$. In the case, when q is the root of unity, we have the wedge theorem [3].

2. Wedge theorem in hydrodynamics

For a given in plane flow $f(z)$, introduction of boundary wedge with angle $\alpha = \frac{2\pi}{N} = \frac{\pi}{n}$, where $N = 2n$ - positive even number, produces the flow

$$F_q(z) = \sum_{k=0}^{n-1} f(q^{2k} z) + \sum_{k=0}^{n-1} \bar{f}(q^{2k} z), \tag{1}$$

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where $q = e^{i\frac{2\pi}{N}} = e^{i\frac{\pi}{n}}$ is primitive root of unity $q^N = q^{2n} = 1$, [2]. This flow is q^2-periodic (rotation invariant), $F_q(q^2 z) = F_q(z)$, and corresponding complex velocity is rotation self-similar $\bar{V}(q^2 z) = \bar{q}^2 \bar{V}_q(z)$, where $q^2 z = e^{i\frac{2\pi}{n}} z$ is rotation to angle $\frac{2\pi}{n}$.

2.1. Vortex kaleidoscope

By this theorem for point vortex $f(z) = \frac{i\Gamma}{2\pi}\ln(z - z_0)$ we have kaleidoscope of $2n$ vortices:

$$F_q(z) = \frac{i\Gamma}{2\pi}\sum_{k=0}^{n-1}\ln\frac{z - z_0 q^{2k}}{z - \bar{z}_0 q^{2k}} = \frac{i\Gamma}{2\pi}\ln\prod_{k=0}^{n-1}\frac{z - z_0 q^{2k}}{z - \bar{z}_0 q^{2k}} = \frac{i\Gamma}{2\pi}\ln\frac{z^n - z_0^n}{z^n - \bar{z}_0^n},$$

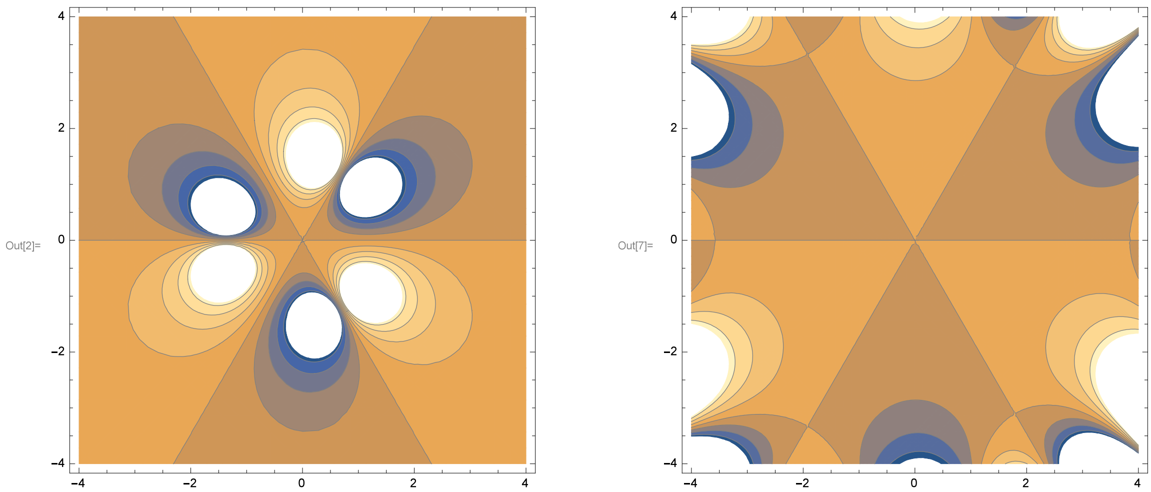
with positive and negative strength Γ, located at $z_0, z_0 q^2, ..., z_0 q^{2(n-1)}$, and $\bar{z}_0, \bar{z}_0 q^2, ..., \bar{z}_0 q^{2(n-1)}$, correspondingly.

2.1.1. Trinity vortex flow

For point vortex in wedge domain $\alpha = \frac{\pi}{3}$ complex potential

$$F_0(z) = \frac{i\Gamma}{2\pi}\ln(z^3 - z_0^3) - \frac{i\Gamma}{2\pi}\ln(z^3 - \bar{z}_0^3),$$

describes the trinity flow shown in Figure 1a.



$a)$ Classical trinity vortex flow $b)$ Quantum qutrit flow

Figure 1. Classical and quantum flow with $q^6 = 1$ symmetry

2.2. q-periodicity

The wedge theorem (1) determines kaleidoscope of images, which can be rewritten as

$$F_q(z) = (1 + q^{2z\frac{d}{dz}} + ...q^{2(n-1)z\frac{d}{dz}})(f(z) + \bar{f}(z)) = [n]_{q^{2z\frac{d}{dz}}}(f(z) + \bar{f}(z)). \quad (2)$$

This form implies that operator $P_0 = \frac{1}{n}[n]_{q^{2z\frac{d}{dz}}}$ is the projection operator: $P_0^2 = P_0$. Applied to the half plane flow $f(z) + \bar{f}(z)$ this operator gives flow in the wedge domain.

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3. Dilatation and rotation operators in Fock-Bargman representation

The projection operator P_0 is related with rotation and dilatation of coherent states in quantum mechanics. The number operator $\widehat{N} = a^+a$, acting on m - particle states $\widehat{N}|m\rangle = m|m\rangle$, $m = 0, 1, ...$ implies following transformation of coherent state $|z\rangle$:

$$q^{2\widehat{N}}|m\rangle = q^{2m}|m\rangle \quad \rightarrow \quad q^{2\widehat{N}}|z\rangle = |q^2 z\rangle$$

(up to normalization factor) as multiplication of z with complex number q^2: $z \rightarrow q^2 z$. For real q^2 this is the dilatation transformation, and for $q^2 = e^{i\theta}$ it is rotation on angle θ. For an arbitrary complex number $q^2 = |q^2|e^{i\arg q^2}$ it is a combination of these two transformations.

In Fock-Bargman representation the number operator and corresponding eigenvalue problem are $\widehat{N} = z\frac{d}{dz}$, $\widehat{N}z^n = nz^n$, so that for any analytic function we have

$$q^{2z\frac{d}{dz}} z^n = q^{2n} z^n \quad \rightarrow \quad q^{2z\frac{d}{dz}} f(z) = f(q^2 z).$$

This allows us to rewrite projection operator (2) in Fock space as operator valued q^2-number.

4. mod 3 states

4.1. Projection mod 3 operators in Fock space

The orthogonal and Hermitian operators

$$P_0 = \frac{1}{3}(I + q^{2N} + q^{4N}) = \frac{1}{3}[3]_{q^{2N}} = \sum_{k=0}^{\infty} |3k\rangle\langle 3k|, \tag{3}$$

$$P_1 = \frac{1}{3}(I + q^{2N-2} + q^{4N-4}) = \frac{1}{3}[3]_{q^{2(N-1)}} = \sum_{k=0}^{\infty} |3k+1\rangle\langle 3k+1|, \tag{4}$$

$$P_2 = \frac{1}{3}(I + q^{2N-4} + q^{4N-2}) = \frac{1}{3}[3]_{q^{2(N-2)}} = \sum_{k=0}^{\infty} |3k+2\rangle\langle 3k+2|, \tag{5}$$

satisfy algebra of projection operators: $P_i P_j = \delta_{ij} P_i$, $P_i^\dagger = P_i$, $i, j = 0, 1, 2$. Due to completeness relation $P_0 + P_1 + P_2 = I$, the Fock space can be decomposed to orthogonal subspaces $\mathcal{H}_F = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2$, where for any $|\psi_i\rangle \in \mathcal{H}_i$, $i = 0, 1, 2$,

$$|\psi_i\rangle = P_i|\psi\rangle = \sum_{k=0}^{\infty} c_{3k+i}|3k+i\rangle = \sum_{n=i\,mod\,3}^{\infty} c_n|n\rangle,$$

and $|\psi\rangle = |\psi_0\rangle + |\psi_1\rangle + |\psi_2\rangle$.

4.2. q-number operators and quantum Fourier transform

Combining these state vectors as column matrix we get

$$\begin{pmatrix} |\psi_0\rangle \\ |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} = \begin{pmatrix} P_0 \\ P_1 \\ P_2 \end{pmatrix} |\psi\rangle = \frac{1}{3} \begin{pmatrix} I & q^{2N} & q^{4N} \\ I & q^{2N-2} & q^{4N-4} \\ I & q^{2N-4} & q^{4N-2} \end{pmatrix} \begin{pmatrix} |\psi\rangle \\ |\psi\rangle \\ |\psi\rangle \end{pmatrix}$$

or

$$\begin{pmatrix} |\psi_0\rangle \\ |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \bar{q}^2 & \bar{q}^4 \\ 1 & \bar{q}^4 & \bar{q}^2 \end{pmatrix} \begin{pmatrix} |\psi\rangle \\ q^{2N}|\psi\rangle \\ q^{4N}|\psi\rangle \end{pmatrix}. \tag{6}$$

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4.3. Glauber trinity States

Let $|\alpha\rangle$ is the Glauber coherent state, then $q^{2N}|\alpha\rangle = |q^2\alpha\rangle$, $q^{4N}|\alpha\rangle = |q^4\alpha\rangle$ and due to (6)

$$\begin{pmatrix} |\alpha_0\rangle \\ |\alpha_1\rangle \\ |\alpha_2\rangle \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \bar{q}^2 & \bar{q}^4 \\ 1 & \bar{q}^4 & \bar{q}^2 \end{pmatrix}\begin{pmatrix} |\alpha\rangle \\ |q^2\alpha\rangle \\ |q^4\alpha\rangle \end{pmatrix}.$$

This gives three orthonormal states, $s = 0, 1, 2$, as a basis in $\mathcal{H}$:

$$|s\rangle_\alpha = \frac{|\alpha_s\rangle}{\sqrt{\langle\alpha_s|\alpha_s\rangle}} = \frac{P_s|\alpha\rangle}{\sqrt{\langle\alpha|P_s|\alpha\rangle}},$$

where

$$|\alpha_s\rangle = P_s|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{k=0}^{\infty}\frac{\alpha^{3k+s}}{\sqrt{(3k+s)!}}|3k+s\rangle.$$

The states $|s\rangle_\alpha$ we called as kaleidoscope of $mod\,3$ coherent states [6].

4.4. Fock-Bargman decomposition

For arbitrary state $|\psi\rangle$ in Fock-Bargman representation we have decomposition $\langle\alpha|\psi\rangle = \langle\alpha_0|\psi_0\rangle + \langle\alpha_1|\psi_1\rangle + \langle\alpha_3|\psi_3\rangle$, implying $\psi(\bar{\alpha}) = \psi_0(\bar{\alpha}) + \psi_1(\bar{\alpha}) + \psi_2(\bar{\alpha})$ for the wave function $\psi(\bar{\alpha}) = e^{|\alpha|^2/2}\langle\alpha|\psi\rangle$. This is just expansion of analytic function $\psi(z) = \psi_0(z) + \psi_1(z) + \psi_2(z)$ to the sum of three analytic functions

$$\begin{pmatrix} \psi_0(z) \\ \psi_1(z) \\ \psi_2(z) \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \bar{q}^2 & \bar{q}^4 \\ 1 & \bar{q}^4 & \bar{q}^2 \end{pmatrix}\begin{pmatrix} \psi(z) \\ \psi(q^2 z) \\ \psi(q^4 z) \end{pmatrix},$$

with $mod\,3$ symmetry: $\psi_0(q^2z) = \psi_0(z)$, $\psi_1(q^2z) = q^2\psi_1(z)$, $\psi_2(q^2z) = q^4\psi_2(z)$. These functions result from application of projection operators (3)-(5) in space of analytic functions

$$\begin{aligned} P_0 &= \frac{1}{3}(I + q^{2z\frac{d}{dz}} + q^{4z\frac{d}{dz}}) = \frac{1}{3}[3]_{q^{2z\frac{d}{dz}}}, \\ P_1 &= \frac{1}{3}(I + q^{2z\frac{d}{dz}-2} + q^{4z\frac{d}{dz}-4}) = \frac{1}{3}[3]_{q^{2(z\frac{d}{dz}-1)}}, \\ P_2 &= \frac{1}{3}(I + q^{2z\frac{d}{dz}-4} + q^{4z\frac{d}{dz}-2}) = \frac{1}{3}[3]_{q^{2(z\frac{d}{dz}-2)}}, \end{aligned}$$

with properties $q^{2z\frac{d}{dz}}P_s = q^{2s}P_s, s = 0, 1, 2$ and explicitly we have

$$\psi_s(z) = \sum_{k=0}^{\infty} c_{3k+s}\frac{z^{3k+s}}{\sqrt{(3k+s)!}}.$$

5. Wedge theorem and q^2 periodic states

5.1. q^2 periodic quantum state

By analogy with the wedge theorem (1) we can construct q^2-periodic quantum state as superposition of coherent states

$$|0\rangle_\alpha \equiv |\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle + ... + |q^{2(n-1)}\alpha\rangle = (I + q^{2\widehat{N}} + q^{4\widehat{N}} + ... + q^{2(n-1)\widehat{N}})|\alpha\rangle = [n]_{q^{2\widehat{N}}}|\alpha\rangle,$$

where due to $q^{2n\widehat{N}} = I$ we have $q^{2\widehat{N}}|0\rangle_\alpha = |0\rangle_\alpha$.

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5.2. Self-similar quantum states

In addition to q^2-periodic quantum state $|0\rangle_\alpha$, exists the set of q^2- self-similar quantum states as superpositions of coherent states, determined by q^2-operator numbers

$$|1\rangle_\alpha \equiv [n]_{q^{2\hat{N}+2}}|\alpha\rangle, \quad |2\rangle_\alpha \equiv [n]_{q^{2\hat{N}+4}}|\alpha\rangle, \quad ... \ , \ |n-1\rangle_\alpha \equiv [n]_{q^{2\hat{N}+2(n-1)}}|\alpha\rangle.$$

These states are orthogonal quantum states, satisfying the following self-similarity conditions:

$$q^{2\hat{N}}|1\rangle_\alpha = q^2|1\rangle_\alpha, \quad q^{2\hat{N}}|2\rangle_\alpha = q^4|2\rangle_\alpha, \quad ... \ , \ q^{2\hat{N}}|n-1\rangle_\alpha = q^{2(n-1)}|n-1\rangle_\alpha.$$

5.3. Wedge theorem for quantum states

Now we can formulate quantum analog of the wedge theorem (1). For arbitrary state

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n\,|n\rangle, \qquad |\bar{\psi}\rangle = \sum_{n=0}^{\infty} \bar{c}_n\,|n\rangle, \tag{7}$$

where $\langle\psi|\psi\rangle = \sum_{n=0}^{\infty}|c_n|^2 = 1$, by replacing $\bar{\alpha} \to z$ in the wave functions $\langle\alpha|\psi\rangle = e^{-\frac{|\alpha|^2}{2}}\psi(\bar{\alpha})$, $\langle\alpha|\bar{\psi}\rangle = e^{-\frac{|\alpha|^2}{2}}\bar{\psi}(\bar{\alpha})$, the kaleidoscope expansion is definite by $mod\,n$ analytic function, $\psi_0(z) \equiv {}_0\psi(z)$, where

$$\psi_0(z) = \frac{1}{n}(\psi(z) + \psi(q^2 z) + \psi(q^4 z) + ... + \psi(q^{2(n-1)}z),$$
$$\bar{\psi}_0(z) = \frac{1}{n}(\bar{\psi}(z) + \bar{\psi}(q^2 z) + \bar{\psi}(q^4 z) + ... + \bar{\psi}(q^{2(n-1)}z).$$

Then for the state

$$|\Psi\rangle = |\psi\rangle + |\bar{\psi}\rangle = 2\sum_{n=0}^{\infty}\Re(c_n)\,|n\rangle$$

expansion coefficients are real. Corresponding wave function $\langle\alpha|\Psi\rangle = e^{-\frac{|\alpha|^2}{2}}(\psi(\bar{\alpha}) + \bar{\psi}(\bar{\alpha}))$ determines the kaleidoscope of q^2 - periodic function

$$\Psi_0(z) = \psi_0(z) + \bar{\psi}_0(z) =$$
$$= \frac{1}{n}(\psi(z) + \psi(q^2 z) + \psi(q^4 z) + ... + \psi(q^{2(n-1)}z) + \bar{\psi}(z) + \bar{\psi}(q^2 z) + \bar{\psi}(q^4 z) + ... + \bar{\psi}(q^{2(n-1)}z),$$

which is just the wedge theorem (1). This way we have identification of Fock-Bargman state $\psi(z)$ with complex potential $f(z)$ in plane. In the wedge domain with $q^{2n} = 1$, complex potential $F(z)$ corresponds to Fock-Bargman state $\Psi_0(z)$. Both, the wave function and the complex potential are real on the wedge boundaries. For the wave function it means that the state is from real vector space, like for the rebit state[4], and for the complex potential, that the stream function vanishes on real line and the flow is along this line.

6. Coherent state decompositions

6.1. Even and odd mod 2 decomposition

For arbitrary vector in Fock space (7) the wave function is

$$\langle\pm\alpha|\psi\rangle = e^{-\frac{|\alpha|^2}{2}}(\sqrt{\cosh|\alpha|^2}\,{}_\alpha\langle 0|\psi\rangle \pm \sqrt{\sinh|\alpha|^2}\,{}_\alpha\langle 1|\psi\rangle),$$

and $\langle\pm\alpha|\psi\rangle \equiv e^{-\frac{|\alpha|^2}{2}}\psi(\pm\bar{\alpha})$, where Fock-Bargman cat states are:

$${}_\alpha\langle 0|\psi\rangle = \frac{\psi_0(\bar{\alpha})}{\sqrt{\cosh|\alpha|^2}} = \frac{\psi(\bar{\alpha}) + \psi(-\bar{\alpha})}{2\sqrt{\cosh|\alpha|^2}}, \quad {}_\alpha\langle 1|\psi\rangle = \frac{\psi_1(\bar{\alpha})}{\sqrt{\sinh|\alpha|^2}} = \frac{\psi(\bar{\alpha}) - \psi(-\bar{\alpha})}{2\sqrt{\sinh|\alpha|^2}}.$$

Then $\psi(\alpha) = \psi_0(\alpha) + \psi_1(\alpha)$, which means that every analytic function is superposition of even and odd functions.

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6.2. Kaleidoscope mod n decomposition
The Glauber coherent state

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{s=0}^{n-1} \sqrt{{}_s e^{|\alpha|^2}}\, |s\rangle_\alpha$$

and rotated states can be decomposed to $mod\, n$ states as Quantum Fourier Transform,

$$\begin{pmatrix} |\alpha\rangle \\ |q^2\alpha\rangle \\ ... \\ |q^{2(n-1)}\alpha\rangle \end{pmatrix} = \frac{1}{n} \begin{pmatrix} 1 & 1 & ... & 1 \\ 1 & q^2 & ... & q^{2(n-1)} \\ . & . & ... & . \\ 1 & q^{2(n-1)} & ... & q^{2(n-1)^2} \end{pmatrix} N \begin{pmatrix} |0\rangle_\alpha \\ |1\rangle_\alpha \\ ... \\ |n-1\rangle_\alpha \end{pmatrix}, \tag{8}$$

with normalization matrix $N = diag[\sqrt{{}_0 e^{|\alpha|^2}}, \sqrt{{}_1 e^{|\alpha|^2}}, ..., \sqrt{{}_{n-1} e^{|\alpha|^2}}]$. This determines the set of entire complex functions

$$_\alpha\langle s|\psi\rangle = \frac{1}{\sqrt{{}_s e^{|\alpha|^2}}} \psi_s(\bar{\alpha}),$$

$s = 0, 1, ..., n-1$, and $\langle\alpha|\psi\rangle = e^{-\frac{|\alpha|^2}{2}} \psi(\bar{\alpha})$. By changing argument notation $\bar{\alpha} \to z$ it can be seen as Discrete Fourier Transform coming from Quantum Fourier Transform (8),

$$\begin{pmatrix} \psi_0(z) \\ \psi_1(z) \\ ... \\ \psi_{n-1}(z) \end{pmatrix} = \frac{1}{n} \begin{pmatrix} 1 & 1 & ... & 1 \\ 1 & \bar{q}^2 & ... & \bar{q}^{2(n-1)} \\ . & . & ... & . \\ 1 & \bar{q}^{2(n-1)} & ... & \bar{q}^{2(n-1)^2} \end{pmatrix} \begin{pmatrix} \psi(z) \\ \psi(q^2 z) \\ ... \\ \psi(q^{2(n-1)} z) \end{pmatrix}.$$

Then, an arbitrary analytic function is superposition of kaleidoscope functions

$$\psi(z) = \psi_0(z) + \psi_2(z) + ... + \psi_{n-1}(z)$$

with self-similar (rotation) symmetry

$$\psi_0(q^2 z) = \psi_0(z),\ \psi_1(q^2 z) = q^2 \psi_1(z),\ ...\ ,\ \psi_{n-1}(q^2 z) = q^{2(n-1)} \psi_{n-1}(z).$$

7. Quantum group structure

7.1. Clock and shift matrices
The dilatation (rotation) operator

$$q^{2\hat{N}} = I \otimes \Sigma_3 = I \otimes \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & q^2 & ... & 0 \\ . & . & ... & . \\ 0 & 0 & ... & q^{2(n-1)} \end{pmatrix}$$

is related with Sylvester clock and shift matrices $\Sigma_3 = H \Sigma_1^\dagger H^\dagger$, where

$$\Sigma_1 = \begin{pmatrix} 0 & 0 & ... & 0 & 1 \\ 1 & 0 & ... & 0 & 0 \\ . & . & ... & . & \\ 0 & 0 & ... & 1 & 0 \end{pmatrix}, \quad H = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & ... & 1 \\ 1 & \bar{q}^2 & ... & \bar{q}^{2(n-1)} \\ . & . & ... & . \\ 1 & \bar{q}^{2(n-1)} & ... & \bar{q}^{2(n-1)^2} \end{pmatrix}.$$

The matrices are q^2-commutative $\Sigma_1 \Sigma_3 = q^2 \Sigma_3 \Sigma_1$ and satisfy $\Sigma_1^n = I,\ \ \Sigma_3^n = I$. From dilatation operator $q^{2\hat{N}}$ we have q^2-number operator

$$[\hat{N}]_{\tilde{q}^2} = \frac{q^{2\hat{N}} - q^{-2\hat{N}}}{q^2 - q^{-2}} = I \otimes diag([0]_{\tilde{q}^2}, [1]_{\tilde{q}^2}, ..., [n-1]_{\tilde{q}^2})$$

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for the symmetric calculus, as matrix with diagonal elements given by q-numbers: $[n]_{\tilde{q}^2} = \frac{q^{2n}-q^{-2n}}{q^2-q^{-2}}$. The operator can be factorized as $[\hat{N}]_{\tilde{q}^2} = \hat{B}^+\hat{B}$, $[\hat{N}+1]_{\tilde{q}^2} = \hat{B}\hat{B}^+$, where $\hat{B}^n = 0$, $(\hat{B}^+)^n = 0$, and

$$\hat{B} = I \otimes \hat{a}\sqrt{\frac{[N]_{\tilde{q}^2}}{N}}.$$

Explicitly in matrix form it is

$$\hat{B} = I \otimes \begin{pmatrix} 0 & \sqrt{[1]} & 0 & ... & 0 \\ 0 & 0 & \sqrt{[2]} & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}, \quad \hat{B}^+ = I \otimes \begin{pmatrix} 0 & 0 & ... & 0 \\ \sqrt{[1]} & 0 & ... & 0 \\ 0 & \sqrt{[2]} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 0 \end{pmatrix}.$$

7.2. Quantum q^2 oscillator

The operators satisfy quantum algebra

$$\hat{B}\hat{B}^+ - q^2\hat{B}^+\hat{B} = q^{-2\hat{N}}, \quad \hat{B}\hat{B}^+ - q^{-2}\hat{B}^+\hat{B} = q^{2\hat{N}}$$

and determine quantum q^2-oscillator with Hamiltonian

$$\hat{H} = \frac{\hbar\omega}{2}\left([\hat{N}]_{\tilde{q}^2} + [\hat{N}+I]_{\tilde{q}^2}\right).$$

The kaleidoscope states for arbitrary state (7) are the eigenstates of operators:

$$q^{2\hat{N}}|\psi_s\rangle = q^{2s}|\psi_s\rangle, \quad [\hat{N}]_{\tilde{q}^2}|\psi_s\rangle = [n]_{\tilde{q}^2}|\psi_s\rangle,$$

and eigenstates of Hamiltonian

$$\hat{H}|\psi_s\rangle = E_s|\psi_s\rangle = \frac{\hbar\omega}{2}\left([n]_{\tilde{q}^2} + [n+1]_{\tilde{q}^2}\right)|\psi_s\rangle,$$

with finite spectrum, $s = 0, 1, ..., n-1$,

$$E_s = \frac{\hbar\omega}{2}\frac{\sin\frac{2\pi}{n}(s+\frac{1}{2})}{\sin\frac{\pi}{n}}. \tag{9}$$

The same spectrum was obtained in [5] for description of physical system of two anyons.

In Fock-Bargman representation, the operators $B =\longrightarrow D_z$, $B^+ \to z$ and $[\hat{N}]_{\tilde{q}^2} = B^+B \to zD_z$ are acting on analytic functions $\psi(z)$, so that for $mod\, n$ functions $\psi_s(z)$ we have eigenvalue problem

$$zD_z\psi_s(z) = \frac{q^{2z\frac{d}{dz}} - q^{-2z\frac{d}{dz}}}{q^2-q^{-2}}\psi_s(z) = \frac{\psi_s(q^2z) - \psi_s(q^{-2}z)}{q^2-q^{-2}} = \frac{q^{2s}-q^{-2s}}{q^2-q^{-2}}\psi_s(z) = [n]_{\tilde{q}^2}\psi_s(z).$$

For every kaleidoscope function in this representation the spectrum of Hamiltonian is (9):

$$H\psi_s(z) = \frac{\hbar\omega}{2}(zD_z + D_z z)\psi_s(z) = E_s\psi_s(z).$$

In particular case of the Glauber coherent states, the wave function $\psi(z) = e^{\alpha z}$ can be decomposed on the set of kaleidoscope wave functions, given by $mod\, n$ exponential functions $\psi_s(z) =_s e^{\alpha z}$, as eigenfunctions of this Hamiltonian.

7.2.1. Qutrit flow As an example, in $mod\,3$ case with $q^6 = 1$, the wave function

$$\psi_0(z) =_0 e^{\alpha z} = \frac{1}{3}(e^{\alpha z} + 2e^{-\frac{1}{2}\alpha z}\cos(\frac{\sqrt{3}}{2}\,\alpha z)),$$

implies the flow in the wedge domain $\alpha = \frac{\pi}{3}$ according to the complex potential

$$\Psi_0(z) = \psi_0(z) + \bar{\psi}_0(z),$$

satisfying the wedge theorem in this domain. The stream function of this flow for $\alpha = 1 + i$ is shown in Figure 1b.

8. Entropy of kaleidoscope expansion

8.1. Cat states and entropy in Fock space

The even and odd states (the cat states) can be introduced for any vector in Fock space (7), by projection operators to even and odd states, $P_0 = \sum_{k=0}^{\infty} |2k\rangle\langle 2k|$, $P_1 = \sum_{k=0}^{\infty} |2k+1\rangle\langle 2k+1|$, $P_0 + P_1 = I$. The expansion is

$$|\psi\rangle = |\psi_0\rangle + |\psi_1\rangle = \sqrt{\langle\psi|P_0|\psi\rangle}\,|0\rangle_\psi + \sqrt{\langle\psi|P_1|\psi\rangle}\,|1\rangle_\psi, \tag{10}$$

where even and odd states are:

$$|\psi_0\rangle = P_0|\psi\rangle = \sum_{k=0}^{\infty} c_{2k}\,|2k\rangle,\ \ |\psi_1\rangle = P_1|\psi\rangle = \sum_{k=0}^{\infty} c_{2k+1}\,|2k+1\rangle.$$

After normalization, the cat states become:

$$|0\rangle_\psi = \frac{|\psi_0\rangle}{\sqrt{\langle\psi|P_0|\psi\rangle}}, \quad |1\rangle_\psi = \frac{|\psi_1\rangle}{\sqrt{\langle\psi|P_1|\psi\rangle}}.$$

The expansion (10) represents an arbitrary state as a qubit, with probabilities to measure even and odd outcome states: $p_0 = \langle\psi|P_0|\psi\rangle$ and $p_1 = \langle\psi|P_1|\psi\rangle$, $p_0 + p_1 = 1$. As a random variable, this state can be characterized by the level of randomness. The Shannon entropy

$$S = -p_0 \log_2 p_0 - p_1 \log_2 p_1 = -\langle\psi|P_0|\psi\rangle \log_2\langle\psi|P_0|\psi\rangle - \langle\psi|P_1|\psi\rangle, \log_2\langle\psi|P_1|\psi\rangle$$

is measuring the level of randomness for state $|\psi\rangle$ to be in even or odd part of Fock space, and we call is as $mod\,2$ entropy.

8.1.1. Glauber coherent states as qubits As an example we consider $mod\,2$ qubit expansion of Glauber coherent state:

$$|\alpha\rangle = |\alpha_0\rangle + |\alpha_1\rangle = e^{-\frac{|\alpha|^2}{2}}\sqrt{\cosh|\alpha|^2}\,|0\rangle_\alpha + e^{-\frac{|\alpha|^2}{2}}\sqrt{\sinh|\alpha|^2}\,|1\rangle_\alpha,$$

to the pair of cat states

$$|0\rangle_\alpha = \frac{P_0|\alpha\rangle}{\sqrt{\langle\psi|P_0|\psi\rangle}} = \frac{e^{|\alpha|^2/2}}{\sqrt{\cosh|\alpha|^2}}\sum_{k=0}^{\infty}\frac{\alpha^{2k}}{\sqrt{(2k)!}}|2k\rangle,$$

$$|1\rangle_\alpha = \frac{P_1|\alpha\rangle}{\sqrt{\langle\psi|P_1|\psi\rangle}} = \frac{e^{|\alpha|^2/2}}{\sqrt{\sinh|\alpha|^2}}\sum_{k=0}^{\infty}\frac{\alpha^{2k+1}}{\sqrt{(2k+1)!}}|2k+1\rangle.$$

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The ratio $p_1/p_0 = \tanh|\alpha|^2$, of probabilities $p_0 = e^{-|\alpha|^2}\cosh|\alpha|^2$, $p_0 = e^{-|\alpha|^2}\sinh|\alpha|^2$, related with level of randomness, is constant on circles $|\alpha|^2 = r^2$, giving number of photons in state $|\alpha\rangle$. The Shannon entropy of the state is

$$S = \frac{|\alpha|^2}{\ln 2} - \frac{e^{-|\alpha|^2}}{\ln 2}(\cosh|\alpha|^2 \ln\cosh|\alpha|^2 + \sinh|\alpha|^2 \ln\sinh|\alpha|^2)$$

and it is shown in Figure 2. The minimal value of entropy corresponds to the limit $\alpha \to 0$, $|\alpha\rangle \to |0\rangle_\alpha$, $S \to 0$. The maximal value is at the Hadamard state $\alpha \to \infty$, $|\alpha\rangle \to \frac{|0\rangle_\alpha + |1\rangle_\alpha}{\sqrt{2}}$, $S \to 1$.

8.2. Entropy of qutrit in Fock Space

The qutrit or *mod* 3 expansion of arbitrary state is

$$|\psi\rangle = \sqrt{\langle\psi|P_0|\psi\rangle}\,|0\rangle_\psi + \sqrt{\langle\psi|P_1|\psi\rangle}\,|1\rangle_\psi + \sqrt{\langle\psi|P_2|\psi\rangle}\,|2\rangle_\psi,$$

where projection operators to trinity states are given in (3)-(5) and three orthonormal states are

$$|0\rangle_\psi = \frac{P_0|\psi\rangle}{\sqrt{\langle\psi|P_0|\psi\rangle}},\quad |1\rangle_\psi = \frac{P_1|\psi\rangle}{\sqrt{\langle\psi|P_1|\psi\rangle}},\quad |2\rangle_\psi = \frac{P_2|\psi\rangle}{\sqrt{\langle\psi|P_2|\psi\rangle}}.$$

The corresponding *mod* 3 Shannon entropy

$$S = -\langle\psi|P_0|\psi\rangle \log_2\langle\psi|P_0|\psi\rangle - \langle\psi|P_1|\psi\rangle \log_2\langle\psi|P_1|\psi\rangle - \langle\psi|P_2|\psi\rangle \log_2\langle\psi|P_2|\psi\rangle$$

is measure of randomness for state $|\psi\rangle$ to be in 0,1 or 2 (*mod* 3) parts of Fock space.

8.2.1. Glauber coherent state as qutrit

For Glauber coherent state the expansion is

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\left(\sqrt{{}_0e^{|\alpha|^2}}\,|0\rangle_\alpha + \sqrt{{}_1e^{|\alpha|^2}}\,|1\rangle_\alpha + \sqrt{{}_2e^{|\alpha|^2}}\,|2\rangle_\alpha\right),$$

with probabilities $p_s = e^{-|\alpha|^2}\,{}_se^{|\alpha|^2}$, $s = 0, 1, 2$.

The Shannon entropy (rotational invariant) is

$$S = \frac{|\alpha|^2}{\ln 2} - \frac{e^{-|\alpha|^2}}{\ln 2}\left({}_0e^{|\alpha|^2} \ln {}_0e^{|\alpha|^2} +_1 e^{|\alpha|^2} \ln {}_1e^{|\alpha|^2} +_2 e^{|\alpha|^2} \ln {}_2e^{|\alpha|^2}\right)$$

and it is shown in Figure 2. For $\alpha \to 0$, $|\alpha\rangle \to |0\rangle_\alpha$, $S \to 0$ and maximal value is $\alpha \to \infty$, $|\alpha\rangle \to \frac{|0\rangle_\alpha + |1\rangle_\alpha + |2\rangle_\alpha}{\sqrt{3}}$, $S \to \frac{\ln 3}{\ln 2}$.

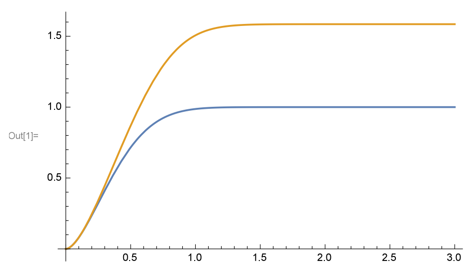

Figure 2. Entropy of qubit (blue) and qutrit (yellow) expansions

8.3. Entropy of qudit in Fock Space

The above expansion can be extended to arbitrary kaleidoscope states

$$|\psi\rangle = \sum_{s=0}^{n-1} |\psi_s\rangle = \sum_{s=0}^{n-1} \sqrt{\langle\psi|P_s|\psi\rangle}\, |s\rangle_\psi,$$

where projection operators to $mod\, n$ states are $P_s = \sum_{k=0}^{\infty} |nk+s\rangle\langle nk+s|$, and qudit basis states, s = 0,1,...,n-1, are

$$|s\rangle_\psi = \frac{P_s|\psi\rangle}{\sqrt{\langle\psi|P_s|\psi\rangle}}.$$

The $mod\, n$ entropy

$$S = -\sum_{s=0}^{n-1} p_s \log_2 p_s = -\sum_{s=0}^{n-1} \langle\psi|P_s|\psi\rangle \log_2 \langle\psi|P_s|\psi\rangle$$

is measure of randomness for state $|\psi\rangle$ to be in 0,1,...,n-1 $mod\, n$ kaleidoscope in Fock space.

8.3.1. Glauber coherent state as qudit For Glauber state the kaleidoscope $mod\, n$ expansion is

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{s=0}^{n-1} \sqrt{{}_s e^{|\alpha|^2}}\, |s\rangle_\alpha,$$

with probabilities $p_s = e^{-|\alpha|^2}\, {}_s e^{|\alpha|^2}$, $s = 0, 1, ..., n-1$. The Shannon entropy

$$S = \frac{|\alpha|^2}{\ln 2} - \frac{e^{-|\alpha|^2}}{\ln 2} \sum_{s=0}^{n-1} {}_s e^{|\alpha|^2} \ln\, {}_s e^{|\alpha|^2}$$

is minimal for $\alpha \to 0$, $|\alpha\rangle \to |0\rangle_\alpha$, $S \to 0$ and maximal for $\alpha \to \infty$, $|\alpha\rangle \to \frac{|0\rangle_\alpha + |1\rangle_\alpha + ... + |n-1\rangle_\alpha}{\sqrt{n}}$, $S \to \frac{\ln n}{\ln 2}$.

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