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Apollonius Representation and Complex Geometry of Entangled Qubit States

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Abstract. A representation of one qubit state by points in complex plane is proposed, such that the computational basis corresponds to two fixed points at a finite distance in the plane. These points represent common symmetric states for the set of quantum states on Apollonius circles. It is shown that, the Shannon entropy of one qubit state depends on ratio of probabilities and is a constant along Apollonius circles. For two qubit state and for three qubit state in Apollonius representation, the concurrence for entanglement and the Cayley hyperdeterminant for tritanglement correspondingly, are constant on the circles as well. Similar results are obtained also for n- tangle hyperdeterminant with even number of qubit states. It turns out that, for arbitrary multiple qubit state in Apollonius representation, fidelity between symmetric qubit states is also constant on Apollonius circles. According to these, the Apollonius circles are interpreted as integral curves for entanglement characteristics. The bipolar and the Cassini representations for qubit state are introduced, and their relations with qubit coherent states are established. We proposed the differential geometry for qubit states in Apollonius representation, defined by the metric on a surface in conformal coordinates, as square of the concurrence. The surfaces of the concurrence, as surfaces of revolution in Euclidean and Minkowski spaces are constructed. It is shown that, curves on these surfaces with constant Gaussian curvature becomes Cassini curves.

1. Introduction

The multiple qubit states belong to multidimensional Hilbert space $\mathbb{C}^n$ and have entanglement property, playing fundamental role in processing of quantum information. To develop quantification of entanglement in terms of simple geometrical structures, in addition to the traditional Bloch sphere, one can look for alternative representations of qubit. The stereographic projection $z = e^{i\varphi}\tan\theta/2$ from the south pole of unit sphere to complex plane $\mathbb{C}$ gives the coherent qubit state representation

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\varphi}|1\rangle \quad \Rightarrow \quad |z\rangle = \frac{|0\rangle + z|1\rangle}{\sqrt{1+|z|^2}}. \tag{1}$$

The ratio of probabilities p_0 and p_1, in this state, $\sqrt{\frac{p_1}{p_0}} = |z| \equiv r$, $p_0 + p_1 = 1$, is constant along concentric circles with radius r and it is related with the level of states randomness. Disadvantage of this representation is that one of the computational basis states is at infinity. This makes difficult to construct simple geometrical characteristics, related with distance between states in the plane. Here we propose new representation of qubit by complex numbers, such that computational states are common symmetric states, placed at two finite points in complex plane. Then the Möbius transformation of concentric circles determines the set of Apollonius qubit states, with constant randomness along Apollonius circles.

2. Classification of two qubit states

The generic two qubit state in $\mathbb{C}^2 \times \mathbb{C}^2$

$$|\psi\rangle = \sum_{i,j=0,1} c_{ij}|i\rangle \otimes |j\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle, \quad (2)$$

where $|c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2 = 1$, admits representation

$$|\psi\rangle = |0\rangle \otimes |c_0\rangle + |1\rangle \otimes |c_1\rangle = |0\rangle \otimes (c_{00}|0\rangle + c_{01}|1\rangle) + |1\rangle \otimes (c_{10}|0\rangle + c_{11}|1\rangle). \quad (3)$$

2.0.1. Separability, linear dependence and determinant The state $|\psi\rangle$ is separable iff states $|c_0\rangle$ and $|c_1\rangle$ are linearly dependent $|c_0\rangle = \lambda|c_1\rangle$. This implies that it is separable if and only if the determinant of the coefficients vanishes $D \equiv c_{00}\, c_{11} - c_{01}\, c_{10} = 0$.

2.0.2. Determinant and parallelogram area For real vectors $\vec{c}_0 = (c_{00}, c_{01})$ and $\vec{c}_1 = (c_{10}, c_{11})$ this determinant describes area of the corresponding parallelogram

$$A = |\vec{c}_0 \times \vec{c}_1| = \begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix} = |\vec{c}_0||\vec{c}_1| \sin\theta.$$

If $A = 0$ - the state is separable, and if $A \neq 0$ - the state is entangled. Solving simple optimization problem, to find possible values for area of this parallelogram with fixed sum $(\vec{c}_0)^2 + (\vec{c}_1)^2 = 1$, we obtain that, $0 \leq A \leq \frac{1}{2}$, and the double area $C = 2A$ is bounded as $0 \leq C \leq 1$.

2.0.3. Two qubit characteristics In generic complex case for two qubit state (3), $|\psi\rangle = |0\rangle|c_0\rangle + |1\rangle|c_1\rangle$, this becomes definition of the concurrence

$$C = \left| 2 \begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix} \right|, \qquad 0 \leq C \leq 1. \quad (4)$$

If $C = 0$ - the state is separable, and if $C = 1$ - it is maximally entangled state.

2.0.4. Concurrence and fidelity The concurrence can be represented as fidelity $C = F = |\langle\tilde{\psi}|\psi\rangle|$ between two states, $|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle = |0\rangle \otimes |c_0\rangle + |1\rangle \otimes |c_1\rangle$ and $|\tilde{\psi}\rangle = \tilde{c}_{00}|00\rangle + \tilde{c}_{01}|01\rangle + \tilde{c}_{10}|10\rangle + \tilde{c}_{11}|11\rangle = |0\rangle \otimes |\tilde{c}_0\rangle + |1\rangle \otimes |\tilde{c}_1\rangle$, if the symmetric state is defined as

$$|\tilde{\psi}\rangle = -\bar{c}_{11}|00\rangle + \bar{c}_{01}|01\rangle + \bar{c}_{10}|10\rangle - \bar{c}_{00}|11\rangle. \quad (5)$$

This symmetric state $|\tilde{\psi}\rangle$ results from application of Y gate and anti - unitary gate K [1],

$$|\tilde{\psi}\rangle = Y \otimes Y\, |\bar{\psi}\rangle = (Y \otimes Y)\, K|\psi\rangle.$$

2.0.5. Concurrence and inner product metric Decomposition (3) determines complex Hermitian inner product metric

$$(G)_{ij} = g_{ij} = \langle c_i | c_j \rangle, \quad \bar{g}_{ij} = g_{ji}, \quad i, j = 0, 1,$$

with elements $g_{00} = |c_{00}|^2 + |c_{01}|^2$, $g_{11} = |c_{10}|^2 + |c_{11}|^2$, $g_{01} = \bar{c}_{00}\, c_{10} + \bar{c}_{01}\, c_{11}$, and $g_{10} = \bar{g}_{01} = c_{00}\, \bar{c}_{10} + c_{01}$. Then, the concurrence becomes expressed by area determined by this metric

$$A \equiv \begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix}, \quad G = (AA\dagger)^{\intercal} \Longrightarrow \det G = |\det A|^2,$$

so that $C = 2|\det A| = 2\sqrt{\det G}$.

2.0.6. Concurrence and Reduced Density Matrix The above geometrical interpretation of concurrence can be completed by physical characteristics. The pure state has density matrix $\rho = |\psi\rangle\langle\psi|$, $tr\rho = 1$, and the reduced density matrix is $\rho_A = tr_B \rho$. For pure state, $tr\,(\rho_A^2) = 1$, and for mixed state, $tr\,(\rho_A^2) < 1$. Expansion (3) gives the reduced density matrix

$$\rho_A = |c_0\rangle\langle c_0| + |c_1\rangle\langle c_1| = \begin{pmatrix} |c_{00}|^2 + |c_{10}|^2 & c_{00}\bar{c}_{01} + c_{10}\bar{c}_{11} \\ c_{01}\bar{c}_{00} + c_{11}\bar{c}_{10} & |c_{01}|^2 + |c_{11}|^2 \end{pmatrix},$$

as the inner product metric $\rho_A = G^T$ and the concurrence becomes determined by reduced density matrix $C = 2|\det A| = 2\sqrt{\det G} = 2\sqrt{\det \rho_A}$. The concurrence C and reduced density matrix ρ_A satisfy the following Pythagoras theorem

$$\rho_A^2 + \frac{C^2}{2} = 1,$$

and as follows, condition of separability becomes related with property of reduced state to be pure or mixed. For separable state $C = 0 \Rightarrow tr\,\rho_A^2 = 1$ - the reduced state is pure state and for entangled state $C \neq 0 \Rightarrow tr\,\rho_A^2 = 1 - \frac{C^2}{2} < 1$ - it is a mixed state. This leads to definition of entanglement in terms of ρ_A [1]: the entanglement E for a pure two qubit state $|\psi\rangle$ is defined as von Neumann entropy $E(\psi) = -tr(\rho_A \log_2 \rho_A)$. In terms of concurrence, this entanglement takes form of the Shannon entropy

$$E(C) = -\frac{1+\sqrt{1-C^2}}{2}\log_2\left(\frac{1+\sqrt{1-C^2}}{2}\right) - \frac{1-\sqrt{1-C^2}}{2}\log_2\left(\frac{1-\sqrt{1-C^2}}{2}\right).$$

As an example, we have two qubit state and corresponding concurrence

$$|z\rangle = \frac{|00\rangle + z|11\rangle}{\sqrt{1+|z|^2}}, \qquad C = \frac{2|z|}{1+|z|^2},$$

so that the concurrence is constant for states along concentric circles $|z| = r$. The states on unit circle $|z| = 1$ are maximally entangled.

3. Apollonius qubit states

3.1. Apollonius one qubit states

Applying Hadamard gate $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ to state $|z\rangle$ in (1) gives symmetric Apollonius state

$$H|z\rangle = \frac{(z-1)|0\rangle + (z+1)|1\rangle}{\sqrt{|z-1|^2 + |z+1|^2}}.$$

Computational basis states $|0\rangle$ and $|1\rangle$ are located now in complex plane at points $z = -1$ and $z = 1$, correspondingly. These points can be transformed to arbitrary points in plane. To have more close analogy with classical bits as integer numbers 0 and 1, we can use another representation. We place $|0\rangle$ and $|1\rangle$ states at values of $z = 0$ and $z = 1$, by scaling and shifting $z \to 2z - 1$, then we get non-symmetric Apollonius qubit

$$|a\rangle = \frac{(z-1)|0\rangle + z|1\rangle}{\sqrt{|z-1|^2 + |z|^2}}.$$

The ratio of probabilities to measure states $|0\rangle$ and $|1\rangle$ as level of randomness, is equal to ratio of distances in plane, coinciding with Apollonius definition of circles

$$\frac{p_1}{p_0} = \frac{|z|^2}{|z-1|^2} \equiv r^2,$$

so that the states are common symmetric states for these circles. The Shannon entropy for Apollonius qubit state is completely determined in terms of r only and is constant for states along the circles: $S(r^2) = \log_2(1 + r^2) - \frac{r^2}{1+r^2} \log_2 r^2$.

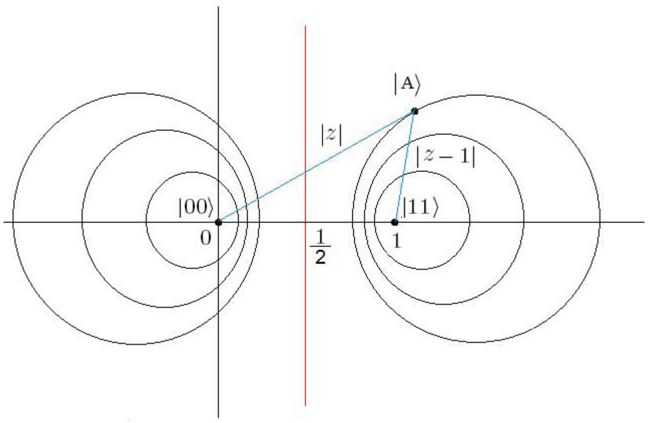

Figure 1. Apollonius qubit states

3.2. Apollonius two qubit states

1) The non-symmetric Apollonius two qubit state, Figure 1, is generated by circuit

$$|a\rangle \otimes |0\rangle \text{——} \boxed{\text{CNOT}} \text{——} |A\rangle$$

and is written below with corresponding concurrence

$$|A\rangle = \frac{(z-1)|00\rangle + z|11\rangle}{\sqrt{|z-1|^2 + |z|^2}}, \quad C = \frac{2|z||z-1|}{|z-1|^2 + |z|^2}. \tag{6}$$

2) The symmetric Apollonius state and corresponding concurrence are

$$|Z\rangle = \frac{(z-1)|00\rangle + (z+1)|11\rangle}{\sqrt{|z-1|^2 + |z+1|^2}}, \quad C = \frac{2|z^2-1|}{|z-1|^2 + |z+1|^2}.$$

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3.3. Concurrence distribution for Apollonius two qubit states

The entanglement for state (6) is completely determined by ratio of distances, as shown in Figure 2 and Figure 3 and it is a constant for states along Apollonius circles:

$$E(r^2) = \log_2(1+r^2) - \frac{r^2}{1+r^2}\log_2 r^2, \quad \left|\frac{z}{z-1}\right| = r.$$

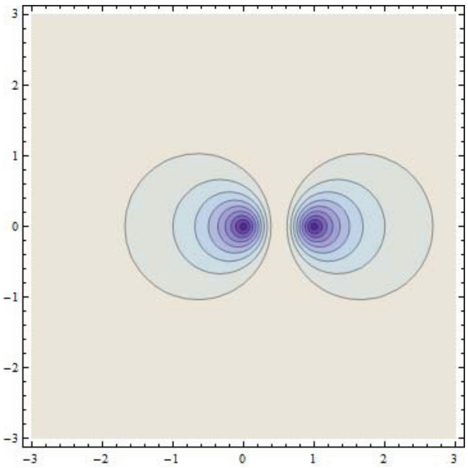

Figure 2. Entanglement for Apollonius two qubit state - Contour plot

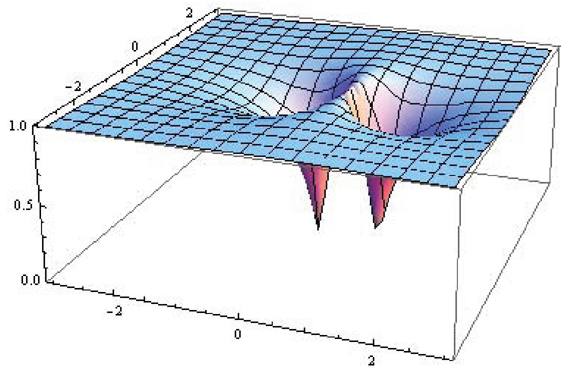

Figure 3. Entanglement for Apollonius two qubit state - 3D plot

3.4. Multiple qubit Apollonius states

The multiple qubit Apollonius states are generated by circuit

$$|a\rangle \otimes |0\rangle ... |0\rangle \otimes |0\rangle \text{———} \boxed{\text{CNOT}} \otimes ... I \otimes I \text{———} ... \text{———} I \otimes I ... \otimes \boxed{\text{CNOT}} \text{———} |A\rangle$$

The state and its symmetric one,

$$|A\rangle = \frac{(z-1)|00...0\rangle + z|11...1\rangle}{\sqrt{|z-1|^2+|z|^2}}, \quad |\tilde{A}\rangle = \frac{-\bar{z}|00...0\rangle + (1-\bar{z})|11...1\rangle}{\sqrt{|z-1|^2+|z|^2}},$$

are giving fidelity

$$F = |\langle\tilde{A}|A\rangle| = \frac{2|z||z-1|}{|z-1|^2+|z|^2} = \frac{2r}{1+r^2},$$

which is a constant on Apollonius circles.

4. Entanglement for multiqubit states

4.1. Concurrence determinant and Levi Civita symbols

For two qubit state (2) the concurrence determinant (4) can be rewritten in terms of Levi-Civita tensors

$$C = 2\,|\vec{c}_0 \times \vec{c}_1| = 2|c_{00}c_{11} - c_{01}c_{10}| = 2\,|\,\epsilon_{ij}\,(\vec{c}_0)_i\,(\vec{c}_1)_j\,| = 2\,\left|\frac{1}{2}\,\epsilon_{ij}\,\epsilon_{kl}\,c_{ik}\,c_{jl}\right|.$$

4.2. Cayley hyperdeterminant and 3 - tangle
For three qubit state $|\psi\rangle = \sum_{i,j,k} c_{ijk}|ijk\rangle$, the analog of determinant is the hyperdeterminant (A.Cayley, 1889)

$$\det\psi = -\frac{1}{2}\,\epsilon_{i_1i_2}\,\epsilon_{j_1j_2}\,\epsilon_{i_3i_4}\,\epsilon_{j_3j_4}\,\epsilon_{k_1k_3}\,\epsilon_{k_2k_4}\,c_{i_1j_1k_1}\,c_{i_2j_2k_2}\,c_{i_3j_3k_3}\,c_{i_4j_4k_4}.$$

It determines the 3-tangle formula for three qubit state $\tau = 4\,|\det\psi|$.

4.3. 3 - tangle for Apollonius three qubit state
For Apollonius 3-qubit state

$$|A\rangle = \frac{(z-1)|000\rangle + z|111\rangle}{\sqrt{|z-1|^2+|z|^2}},$$

it gives tritangle

$$\tau = 4|c^2_{000}\,c^2_{111}| = 4\frac{|z-1|^2|z|^2}{(|z-1|^2+|z|^2)^2} = C^2,$$

which is a constant along Apollonius circles.

4.4. n - tangle of n - qubit states
The 3- tangle determinant formula can be generalized to even number of multiple qubit states [2]. For even $n = 2k$ - qubit state

$$|\psi\rangle = \sum_{i_1i_2...i_n} c_{i_1i_2...i_n}|i_1i_2...i_n\rangle$$

the n-tangle is defined by contraction of $4n$-rank tensor $c_{i_1i_2...i_n}$, and $2n$ - Levi Civita tensors

$$\begin{aligned}\tau_{12...n} &= 2\Bigg|\sum_{0,1} c_{\alpha_1\alpha_2...\alpha_n}\,c_{\beta_1\beta_2...\beta_n}\,c_{\gamma_1\gamma_2...\gamma_n}\,c_{\delta_1\delta_2...\delta_n}\,\epsilon_{\alpha_1\beta_1}\,\epsilon_{\alpha_2\beta_2}\cdots \\ &\qquad \epsilon_{\alpha_n\beta_n}\,\epsilon_{\gamma_1\delta_1}\,\epsilon_{\gamma_2\delta_2}\ldots\epsilon_{\gamma_{n-1}\delta_{n-1}}\,\epsilon_{\alpha_n\gamma_n}\,\epsilon_{\beta_n\delta_n}\Bigg|.\end{aligned}$$

4.5. n - tangle for Apollonius $n = 2k$ qubit state
For $n = 2k$ multiqubit Apollonius state it gives n-tangle as square of concurrence

$$|z\rangle = \frac{(z-1)\,|00...0\rangle + (z+1)\,|11...1\rangle}{\sqrt{|z-1|^2+|z+1|^2}}, \quad \tau_{12...n} = \frac{4\,|z^2-1|^2}{(|z-1|^2+|z+1|^2)^2} = C^2,$$

which is a constant along Apollonius circles as is shown in Figure 4 and Figure 5.

5. Cassini Qubit States

5.0.1. Conformal mapping from Cassini curves to Apollonius circles The Cassini curve is defined as a constant product of distances from fixed complex points $-c$ and c,

$$|z-c||z+c| = a^2.$$

For $w = z^2$ it is a circle: $|w-c^2| = a^2 \implies |w|^2 - c^2(w+\bar{w}) + c^4 = a^4$. Translating the origin $\xi = w - c^2$, the equation becomes $|w-c^2| = |\xi| = a^2$. In ξ plane, 0 and ∞ are

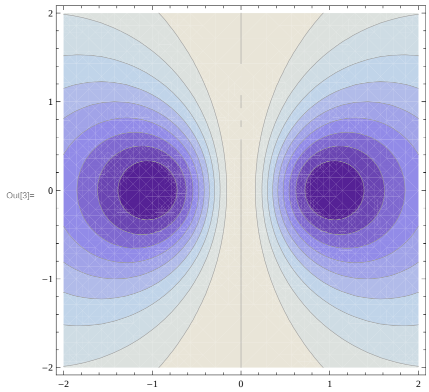

Figure 4. n-tangle of Apllonius n=2k states - contour plot

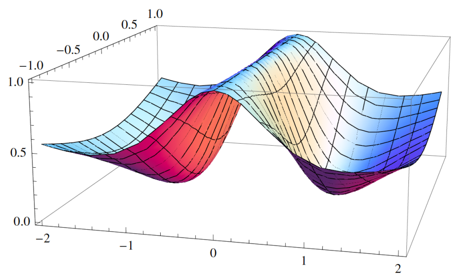

Figure 5. n-tangle of Apllonius n=2k states - 3D plot

symmetric points with respect to concentric circles. Corresponding Möbius transformation is $\eta = -c\,\frac{\xi+c^2}{\xi-c^2} \Longrightarrow \xi = c^2\frac{\eta-c}{\eta+c}$, and for circle $|\xi| = a^2$, it gives the Apollonius circle

$$\frac{|\eta - c|}{|\eta + c|} = \frac{a^2}{c^2}.$$

Combining all transformations $w = z^2,\ \xi = w - c^2,\ \eta = -c\frac{\xi+c^2}{\xi-c^2}$ we get relation between Cassini curves and Apollonius circles:

$$\eta = -c\frac{z^2}{z^2 - 2c^2}. \tag{7}$$

6. Cassini Qubit State

Due to transformation (7), the Apollonius qubit state $|\eta\rangle = \frac{(\eta-c)|0\rangle+(\eta+c)|1\rangle}{\sqrt{|\eta-c|^2+|\eta+c|^2}}$ can be represented as the Cassini qubit state

$$|z\rangle = \frac{(z^2 - c^2)|0\rangle + c^2|1\rangle}{\sqrt{|z^2 - c^2|^2 + c^4}} = \frac{1}{\sqrt{|z^2 - c^2|^2 + c^4}}\begin{pmatrix} z^2 - c^2 \\ c^2 \end{pmatrix}.$$

Probabilities $p_0 = \frac{|z^2-c^2|^2}{|z^2-c^2|^2+c^4}$ and $p_1 = \frac{c^4}{|z^2-c^2|^2+c^4}$, with ratio

$$\frac{p_1}{p_0} = \frac{c^4}{|z^2 - c^2|^2} = \frac{c^4}{a^4} = r^2$$

give the Shannon entropy $S(r^2) = \log_2(1 + r^2) - \dfrac{r^2}{1 + r^2}\log_2 r^2$. This entropy is constant along Cassini curves. For $r = 1$, the states are maximally random states, and the curve becomes the Bernoulli lemniscate. The similar observations are valid for the concurrence, the 3-tangle and the n- tangle for multiple qubit Cassini states. The maximally n-tangled Cassini qubit states along the Bernoulli lemniscate are shown in Figure 6.

7. Bipolar Representation

The Apollonius circles, combined together with the set of orthogonal circles, give the bipolar coordinates $-\infty < \tau < \infty, -\pi < \sigma < \pi$ in complex plane $z = x + iy$,

$$z = \frac{e^\tau}{e^\tau - e^{i\sigma}}.$$

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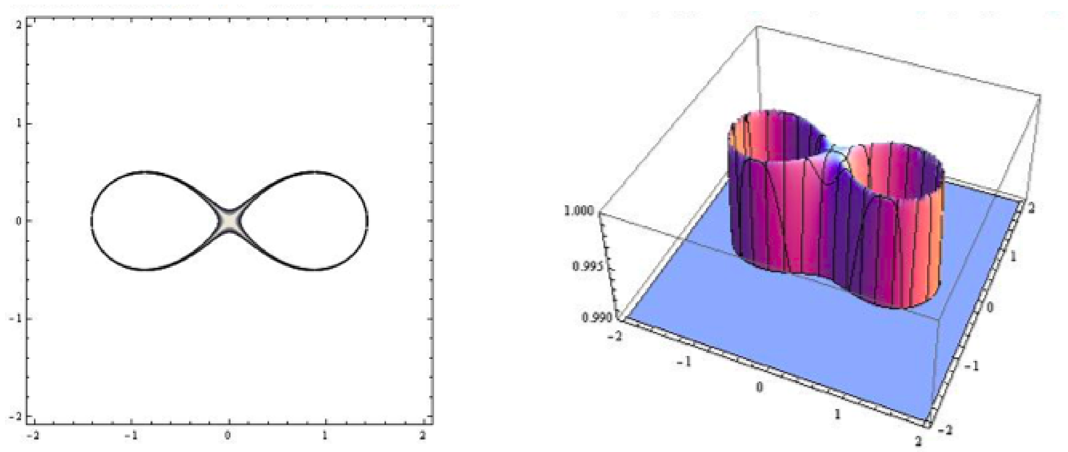

Figure 6. Bernoulli lemniscate for maximally n-tangled Cassini states

The one qubit state in bipolar representation is defined as

$$|\tau, \sigma\rangle = \frac{e^{i\sigma}|0\rangle + e^{\tau}|1\rangle}{\sqrt{1+e^{2\tau}}}.$$

The corresponding entropy depends only of τ,

$$S(\tau) = \log_2(1+e^{2\tau}) - \frac{e^{2\tau}}{1+e^{2\tau}} \log_2 e^{2\tau} = 1 + \frac{\ln\cosh\tau - \tau\tanh\tau}{\ln 2},$$

so that for $\tau = 0 \Longrightarrow S(0) = 1$ and for $\tau = \pm\infty \Longrightarrow S(\pm\infty) = 0$. The two qubit bipolar state and corresponding concurrence are given as follows

$$|\tau, \sigma\rangle = \frac{e^{i\sigma}|00\rangle + e^{\tau}|11\rangle}{\sqrt{1+e^{2\tau}}}, \quad C = \frac{1}{\cosh\tau} = sech\,\tau\,.$$

The fidelity or concurrence written in complex form

$$\mathcal{C} = \mathcal{F} = \langle\tilde{A}|A\rangle = e^{-i\sigma} sech^2\tau$$

is one soliton solution of the Nonlinear Schrödinger equation

$$i\mathcal{C}_\sigma = \mathcal{C}_{\tau\tau} + 2|\mathcal{C}|^2\mathcal{C}.$$

8. Concurrence as conformal metric

The Cassini curves and Apollonius circles, as curves of constant entanglement can be interpreted as integral curves of the concurrence flow, with stream function depending on concurrence. This suggests also to consider these curves as the level curves of some two dimensional surface, which we call the concurrence surface. The distance formula on this surface can be taken proportional to n-tangle $\tau = C^2$. This is why we consider conformal metric on a surface as $C^2(x, y)$:

$$dl^2 = g(z, \bar{z})\; dzd\bar{z} = C^2(z, \bar{z})\; dzd\bar{z}.$$

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For complex analytical changes $z = z(w)$, the metric remains conformal: $dl^2 = g(z(w), \overline{z(w)}) \left|\frac{dz}{dw}\right|^2 dw d\bar{w}$, and the Gaussian curvature of the surface in conformal coordinates acquires simple form

$$K = -\frac{1}{2g(x,y)} \Delta \ln g(x,y). \tag{8}$$

8.1. Apollonius concurrence metric

For Apollonius two qubit state

$$|A\rangle = \frac{(z-1)|00\rangle + z|11\rangle}{\sqrt{|z-1|^2 + |z|^2}}, \tag{9}$$

the metric is

$$dl^2 = \frac{4|z|^2|z-1|^2}{(|z-1|^2 + |z|^2)^2} dz d\bar{z}.$$

In bipolar representation (8) it becomes

$$g(\tau, \sigma) = \frac{1}{4\cosh^2\tau(\cosh\tau - \cos\sigma)^2}. \tag{10}$$

Corresponding Gaussian curvature in bipolar coordinates

$$K = -\frac{1}{2g(\tau,\sigma)} \left(\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\sigma^2}\right) \ln(g(\tau,\sigma)) \tag{11}$$

takes the form

$$K = 4(\cosh\tau - \cos\sigma)^2 = \frac{1}{|z|^2|z-1|^2}.$$

It is a constant along Cassini curves with fixed points 0 and 1: $|z||z-1| = \frac{1}{\sqrt{K}}$. For maximally entangled states with $\tau = 0$, the Gaussian curvature is positive number

$$K = 16\sin^4\frac{\sigma}{2}.$$

8.2. Concurrence surface as surface of revolution

The qubit state (1) determines conformal metric with Gaussian curvature

$$dl^2 = \frac{4|z|^2}{(1+|z|^2)^2} dz\, d\bar{z}, \quad K = \frac{1}{|z|^2}.$$

It can be considered as surface of revolution, generated by rotation of curve $z = \phi(\sqrt{x^2+y^2}) = \phi(r)$. For $z = u + iv$ the metric is

$$dl^2 = g(u,v)(du^2 + dv^2) = (1 \pm (\phi'(r))^2)\, dr^2 + r^2 dv^2, \tag{12}$$

where sign + corresponds to Euclidean space, and sign − to Minkowski space. The surface can be recovered partially in Euclidean space, Figure 7, for $1 \le r \le 2$ by revolution curve

$$z = \phi(x) = z(1) \pm \left(\ln\frac{1+\sqrt{(3-x)(x-1)}}{2-x} - \sqrt{(3-x)(x-1)}\right)$$

and in Minkowski space, Figure 8, for $0 \le r \le 1$ with curve

$$z = \phi(x) = z(1) \mp (\sqrt{(3-x)(1-x)} - \arctan\sqrt{(3-x)(1-x)}).$$

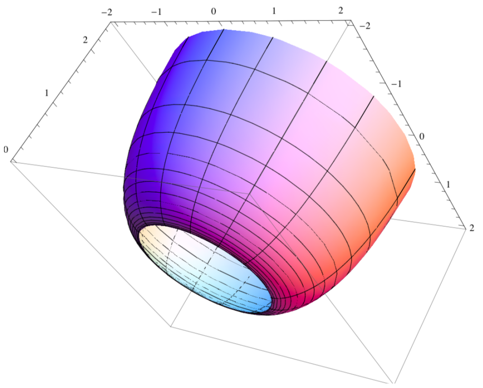

Figure 7. Concurrence surface in Euclidean space

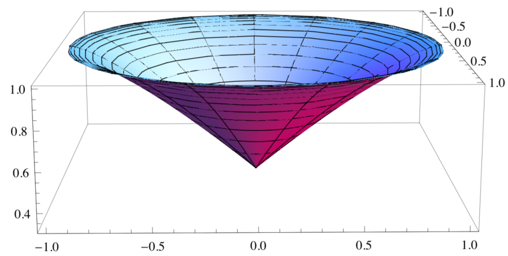

Figure 8. Concurrence surface in Minkowski space

9. Conformal transformation of qubit coherent state

Instead of Möbius transformation to get Apollonius states we can consider more general qubit states, determined by arbitrary analytic function $\mu(z)$. For two qubits we have:

$$|\mu(z)\rangle = \frac{|00\rangle + \mu(z)\,|11\rangle}{\sqrt{1+|\mu(z)|^2}}.$$

The concurrence for this state gives the Riemannian metric

$$g(z,\bar{z}) = C^2 = \frac{4\mu(z)\overline{\mu(z)}}{(1+\mu(z)\overline{\mu(z)})^2},$$

representing the general solution of the Liouville equation with variable Gaussian curvature K:

$$\Delta \ln C^2 = -2K(z,\bar{z})C^2, \quad K(z,\bar{z}) = \left|\frac{\mu_z(z)}{\mu(z)}\right|^2.$$

In Apollonius case, $\mu(z) = \frac{z+1}{z-1}$, the Gaussian curvature and the Liouville equation become

$$K(z,\bar{z}) = \frac{4}{|z^2-1|^2}, \qquad \Delta\psi = -\frac{8}{|z^2-1|^2}e^{\psi},$$

where $\psi = \ln C^2$. Curves on the surface with constant Gaussian curvature are Cassini curves, $|z^2-1|^2 = \frac{4}{K} \equiv a^2$, and the Liouville equation is in the canonical form $\Delta\psi = -\frac{8}{a^2}e^{\psi}$.

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11. References

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