IOP Conf. Series: Journal of Physics: Conf. Series **1194** (2019) 012059 doi:10.1088/1742-6596/1194/1/012059

Special functions with mod n symmetry and kaleidoscope of quantum coherent states

Aygül Koçak[1] **and Oktay K. Pashaev**[2]

Department of Mathematics , Izmir Institute of Technology, Izmir, 35430, Turkey

E-mail: [1]aygulkocak@iyte.edu.tr, [2]oktaypashaev@iyte.edu.tr

Abstract. The set of mod n functions associated with primitive roots of unity and discrete Fourier transform is introduced. These functions naturally appear in description of superposition of coherent states related with regular polygon, which we call kaleidoscope of quantum coherent states. Displacement operators for kaleidoscope states are obtained by mod n exponential functions with operator argument and non-commutative addition formulas. Normalization constants, average number of photons, Heinsenberg uncertainty relations and coordinate representation of wave functions with mod n symmetry are expressed in a compact form by these functions.

1. Introduction

In description of superposition of Glauber coherent states associated with regular n polygon and roots of unity[1], main characteristics of these states as normalization factors, coordinate representations, uncertainty relations,etc. appear as an infinite sum of exponential functions with mod n symmetry [2, 3, 4]. For n=2, mod 2 exponential functions coincide with hyperbolic functions, this is why, for arbitrary n these functions were called as generalized hyperbolic functions[5]. Here, we introduce generic special functions with mod n symmetry and called them mod n functions so that generalized hyperbolic functions becomes particular case of these functions. Kaleidoscope states represent of coherent states as Gaussian wave functions , generating function for these states is described by mod n Gaussian exponential function and $n-$paricle states by mod n Hermite polynomials. In the present paper, we describe main properties of mod n functions and their applications for description of kaleidoscope of quantum coherent states.

1.1. Scale and Phase Invariance

The set of mod n functions satisfy self-similarity property under rotations. A function is said to be scale-invariant if it satisfies following property;

$$f(\lambda z) = \lambda^d f(z)\,,$$

for some choice of exponent $d \in \mathbb{R}$ and fixed scale factor $\lambda > 0$, which can be taken as a length or size of re-scaling. If $\lambda = e^{i\varphi}$ and as follows $|\lambda| = 1$, then this formula gives

$$f(e^{i\varphi} z) = e^{i\varphi d} f(z)$$

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.
Published under licence by IOP Publishing Ltd

IOP Conf. Series: Journal of Physics: Conf. Series **1194** (2019) 012059 doi:10.1088/1742-6596/1194/1/012059

In this case, rotation of argument z to angle φ implies rotation of function f to angle φd, and scale invariance becomes rotational or phase(gauge) invariance. If $\lambda = q^2$ is the primitive root of unity

$$q^{2n} = 1$$

so that $q^2 = e^{i\frac{2\pi}{n}}$, then

$$f(e^{i\frac{2\pi}{n}} z) = e^{i\frac{2\pi}{n}d} f(z) .$$

This means that rotation of argument z to angle $\frac{2\pi}{n}$ of $n-$sided polygon, leads to rotation of f on $d-$times of this angle. We call this as discrete phase gauge invariant function, with order d. Simplest example of phase invariant functions is given by even and odd functions with $q^4 = 1$;

$$f_{even}(q^2 x) = f_{even}(x) \ , \ f_{odd}(q^2 x) = q^2 f_{odd}(x)$$

where $\lambda = q^2 = -1$ and $d = 0$, $d = 1$ respectively.

2. Mod n functions

For calculation of normalization constants and average number of photons in kaleidoscope of quantum coherent states, we introduce mod n functions. For $q^{2n} = 1$ primitive root of unity, we consider n values of argument $x, q^2 x, q^4 x, ..., q^{2(n-1)} x$ as rotated by angle $\frac{2\pi}{n}$ and associated with vertices of regular polygon. Mod n functions are defined by relation

$$\begin{bmatrix} f_0(x) \\ f_1(x) \\ f_2(x) \\ \vdots \\ f_{n-1}(x) \end{bmatrix} = \frac{1}{n} \begin{bmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & \bar{q}^{\,2} & \bar{q}^{\,4} & ... & \bar{q}^{\,2(n-1)} \\ 1 & \bar{q}^{\,4} & \bar{q}^{\,8} & ... & \bar{q}^{\,4(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \bar{q}^{\,2(n-1)} & \bar{q}^{\,4(n-1)} & ... & \bar{q}^{\,2(n-1)^2} \end{bmatrix} \begin{bmatrix} f(x) \\ f(q^2 x) \\ f(q^4 x) \\ \vdots \\ f(q^{2(n-1)} x) \end{bmatrix} , \tag{1}$$

where transformation matrix is discrete Fourier transformation, related with unitary generalized Hadamard gate matrix $\hat{H}$. Every mod n function is a superposition of functions with these arguments, such that addition of coefficients for $f(q^2 x), f(q^4 x), ..., f(q^{2(n-1)} x)$ are equal to zero, due to

$$1 + \bar{q}^{\,2k} + \bar{q}^{\,4k} + ... + \bar{q}^{\,2(n-1)k} = 0 \, , 1 \le k \le n-1 \, .$$

By inverting transformation (1), It is evident that arbitrary function $f(x)$ can be written as a superposition of mod n functions

$$f_k(x) = \frac{1}{n} \sum_{s=0}^{n-1} \bar{q}^{2sk} f(q^{2s} x) \tag{2}$$

in a unique way, $f(x) = \sum_{k=0}^{n-1} f_k(x)$.

IOP Conf. Series: Journal of Physics: Conf. Series **1194** (2019) 012059 doi:10.1088/1742-6596/1194/1/012059

2.1. Mod n exponential functions

Most important example of this expansion is given by exponential function;

$$e^x = \sum_{m=0}^{\infty}\frac{x^m}{m!} = \sum_{k=0}^{\infty}\frac{x^{nk}}{nk!} + \sum_{k=0}^{\infty}\frac{x^{nk+1}}{(nk+1)!} + \sum_{k=0}^{\infty}\frac{x^{nk+2}}{(nk+2)!} + ... + \sum_{k=0}^{\infty}\frac{x^{nk+(n-1)}}{(nk+(n-1))!}.$$

Every sum here represent mod n exponential function $f_s(x) \equiv {}_s e^x(mod\, n)$,$0 \leq s \leq n-1$,

$${}_s e^x(mod\, n) \equiv \sum_{k=0}^{\infty}\frac{x^{nk+s}}{(nk+s)!}.$$

Due to (2) for $f(x) = e^x$, they can be expressed as superposition of standard exponentials by discrete Fourier transformation,

$${}_s e^x(mod\, n) = \frac{1}{n}\sum_{k=0}^{n-1}\bar{q}^{2sk}e^{q^{2k}x} \ .$$

For n=2, the mod 2 exponential functions coincide with hyperbolic functions:

$${}_0 e^x = \cosh x \quad , \quad {}_1 e^x = \sinh x$$

This is why, it is natural to call mod n exponential functions for arbitrary n, as generalized hyperbolic functions[5].
The derivative operator is acting on mod n exponential functions in following way;

$$\frac{d}{dx}\, {}_0 e^x = {}_{n-1}e^x \quad , \quad \frac{d}{dx}\, {}_s e^x = {}_{s-1}e^x \quad , 1 \leq s \leq n-1.$$

Applying this derivative n times, we find that function $f_s(x) = {}_s e^x$ is a solution of ordinary differential equation of degree n,

$$f_s^{(n)} = f_s\,, \quad \text{where } 0 \leq s \leq n-1,$$

with initial values: $f_s^{(s)}(0) = 1, f_s(0) = f_s'(0) = ... = f_s^{(s-1)}(0) = f_s^{(s+1)}(0) = ... = f_s^{(n-1)}(0) = 0$.
This differential equation is the eigenvalue problem $\hat{a}^n f = f$ for annihilation operator $\hat{a} = \frac{d}{dz}$ in the Fock-Bargmann representation, acting on analytic function $f = f(z)$.

3. Displacement operators for kaleidoscope states

As a first application of mod n exponential functions, we consider displacement operators for kaleidoscope of coherent states. Application of this displacement operator to vacuum state requires factorization of mod n exponential functions with operator argument. In next section, we describe in details this factorization for mod 2 exponential functions.

3.1. Factorization of mod 2 exponential functions with operator argument

As well known exponential function with operator argument can be factorized in the form

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}\,, \tag{3}$$

IOP Conf. Series: Journal of Physics: Conf. Series **1194** (2019) 012059 doi:10.1088/1742-6596/1194/1/012059

where $\hat{A}$ and $\hat{B}$ are c-commutative: $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$. Here, we derive factorization formulas for mod 2 exponential functions:
Let $\hat{A}$ and $\hat{B}$ are two c-commutative operators, then

$$ {}_0e^{\hat{A}+\hat{B}} = \left({}_0e^{\hat{A}}\,{}_0e^{\hat{B}} + {}_1e^{\hat{A}}\,{}_1e^{\hat{B}}\right) e^{-\frac{1}{2}[\hat{A},\hat{B}]}\,, \tag{4} $$

$$ {}_1e^{\hat{A}+\hat{B}} = \left({}_0e^{\hat{A}}\,{}_1e^{\hat{B}} + {}_1e^{\hat{A}}\,{}_0e^{\hat{B}}\right) e^{-\frac{1}{2}[\hat{A},\hat{B}]}. \tag{5} $$

Factorization formula (3) gives $q-$commutative relation between operators $e^{\hat{A}}$ and $e^{\hat{B}}$,

$$ e^{\hat{A}}e^{\hat{B}} = e^{[\hat{A},\hat{B}]}e^{\hat{B}}e^{\hat{A}} = qe^{\hat{B}}e^{\hat{A}}\,. $$

We have analogue of this formula for mod 2 exponential functions.
For operators $\hat{A}$ and $\hat{B}$ such that $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$ following identities hold

$$ \begin{aligned} {}_0e^{\hat{A}}\,{}_0e^{\hat{B}} &= {}_0e^{\hat{B}}\,{}_0e^{\hat{A}}\,{}_0e^{[\hat{A},\hat{B}]} + {}_1e^{\hat{B}}\,{}_1e^{\hat{A}}\,{}_1e^{[\hat{A},\hat{B}]},\\ {}_1e^{\hat{A}}\,{}_1e^{\hat{B}} &= {}_1e^{\hat{B}}\,{}_1e^{\hat{A}}\,{}_0e^{[\hat{A},\hat{B}]} + {}_0e^{\hat{B}}\,{}_0e^{\hat{A}}\,{}_1e^{[\hat{A},\hat{B}]},\\ {}_0e^{\hat{A}}\,{}_1e^{\hat{B}} &= {}_1e^{\hat{B}}\,{}_0e^{\hat{A}}\,{}_0e^{[\hat{A},\hat{B}]} + {}_0e^{\hat{B}}\,{}_1e^{\hat{A}}\,{}_1e^{[\hat{A},\hat{B}]},\\ {}_1e^{\hat{A}}\,{}_0e^{\hat{B}} &= {}_0e^{\hat{B}}\,{}_1e^{\hat{A}}\,{}_0e^{[\hat{A},\hat{B}]} + {}_1e^{\hat{B}}\,{}_0e^{\hat{A}}\,{}_1e^{[\hat{A},\hat{B}]}. \end{aligned} $$

These identities can be rewritten in terms of hyperbolic functions of operator argument:

$$ \begin{aligned} \cosh\hat{A}\cosh\hat{B} &= \cosh\hat{B}\cosh\hat{A}\cosh[\hat{A},\hat{B}] + \sinh\hat{B}\sinh\hat{A}\sinh[\hat{A},\hat{B}],\\ \sinh\hat{A}\sinh\hat{B} &= \sinh\hat{B}\sinh\hat{A}\cosh[\hat{A},\hat{B}] + \cosh\hat{B}\cosh\hat{A}\sinh[\hat{A},\hat{B}],\\ \cosh\hat{A}\sinh\hat{B} &= \sinh\hat{B}\cosh\hat{A}\cosh[\hat{A},\hat{B}] + \cosh\hat{B}\sinh\hat{A}\sinh[\hat{A},\hat{B}],\\ \sinh\hat{A}\cosh\hat{B} &= \cosh\hat{B}\sinh\hat{A}\cosh[\hat{A},\hat{B}] + \sinh\hat{B}\cosh\hat{A}\sinh[\hat{A},\hat{B}]. \end{aligned} $$

Formulas (4) and (5), imply also addition formulas for hyperbolic functions of operator argument:

$$ \cosh\left(\hat{A}+\hat{B}\right) = \left(\cosh\hat{A}\cosh\hat{B} + \sinh\hat{A}\sinh\hat{B}\right) e^{-\frac{1}{2}[\hat{A},\hat{B}]}, \tag{6} $$

$$ \cosh\left(\hat{A}-\hat{B}\right) = \left(\cosh\hat{A}\cosh\hat{B} - \sinh\hat{A}\sinh\hat{B}\right) e^{\frac{1}{2}[\hat{A},\hat{B}]}, \tag{7} $$

$$ \sinh\left(\hat{A}+\hat{B}\right) = \left(\sinh\hat{A}\cosh\hat{B} + \sinh\hat{B}\cosh\hat{A}\right) e^{-\frac{1}{2}[\hat{A},\hat{B}]}, \tag{8} $$

$$ \sinh\left(\hat{A}-\hat{B}\right) = \left(\sinh\hat{A}\cosh\hat{B} - \sinh\hat{B}\cosh\hat{A}\right) e^{\frac{1}{2}[\hat{A},\hat{B}]}. \tag{9} $$

For special case, when $[\hat{A},\hat{B}] = 0$, these addition formulas reduce to usual formulas for hyperbolic functions. In the next section, we apply these formulas for factorization of displacement operators for Schrödinger's cat states.

3.2. Mod n displacement operator

3.2.1. Mod 2 case The displacement operators $D(\mp\alpha)$ as exponential function of operator argument

$$ D(\mp\alpha) = e^{\mp\alpha\hat{a}^\dagger \pm\bar{\alpha}\hat{a}} = e^{-\frac{1}{2}|\alpha|^2}e^{\mp\alpha\hat{a}^\dagger}e^{\pm\bar{\alpha}\hat{a}}\,, $$

generate coherent states $|\mp\alpha\rangle$;

$$ |\mp\alpha\rangle = D(\mp\alpha)|0\rangle\,. $$

IOP Conf. Series: Journal of Physics: Conf. Series **1194** (2019) 012059 doi:10.1088/1742-6596/1194/1/012059

Superpositions of these states as the cat states can be created by mod 2 displacement operators ${}_0D(\alpha)$ and ${}_1D(\alpha)$:

$$
\begin{aligned}
|\tilde{0}\rangle_\alpha &= \frac{|\alpha\rangle + |-\alpha\rangle}{2} = \left(\frac{D(\alpha)+D(-\alpha)}{2}\right)|0\rangle = {}_0D(\alpha)|0\rangle\,,\\
|\tilde{1}\rangle_\alpha &= \frac{|\alpha\rangle - |-\alpha\rangle}{2} = \left(\frac{D(\alpha)-D(-\alpha)}{2}\right)|0\rangle = {}_1D(\alpha)|0\rangle\,.
\end{aligned}
$$

Due to identities $(6)-(9)$, these operators can be written as

$$
\begin{aligned}
{}_0D(\alpha) &= e^{-\frac{1}{2}|\alpha|^2}(\cosh\alpha\hat{a}^\dagger\cosh\alpha\hat{a} - \sinh\alpha\hat{a}^\dagger\sinh\alpha\hat{a})\,,\\
{}_1D(\alpha) &= e^{-\frac{1}{2}|\alpha|^2}(\sinh\alpha\hat{a}^\dagger\cosh\alpha\hat{a} + \cosh\alpha\hat{a}^\dagger\sinh\alpha\hat{a})\,,
\end{aligned}
$$

and the cat states become

$$|\tilde{0}\rangle_\alpha = {}_0D(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2}\cosh\alpha\hat{a}^\dagger|0\rangle\,, \tag{10}$$

$$|\tilde{1}\rangle_\alpha = {}_1D(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2}\sinh\alpha\hat{a}^\dagger|0\rangle\,. \tag{11}$$

Normalization of these states is given in $(14), (15)$.

3.2.2. Mod 3 case Displacement operators defined by mod 3 operator valued exponential functions, determine the kaleidoscope of three states(trinity states) as;

$$
\begin{aligned}
{}_0D(\alpha) = \frac{D(\alpha)+D(q^2\alpha)+D(q^4\alpha)}{3} \quad &\Rightarrow \quad |\tilde{0}\rangle_\alpha = {}_0D(\alpha)|0\rangle\,,\\
{}_1D(\alpha) = \frac{D(\alpha)+\bar{q}^2D(q^2\alpha)+\bar{q}^4D(q^4\alpha)}{3} \quad &\Rightarrow \quad |\tilde{1}\rangle_\alpha = {}_1D(\alpha)|0\rangle\,,\\
{}_2D(\alpha) = \frac{D(\alpha)+\bar{q}^4D(q^2\alpha)+\bar{q}^2D(q^4\alpha)}{3} \quad &\Rightarrow \quad |\tilde{2}\rangle_\alpha = {}_2D(\alpha)|0\rangle\,.
\end{aligned}
$$

3.2.3. Mod n kaleidoscope states The above construction can be generalized to arbitrary mod n case ,$q^{2n}=1$, described by displacement operators

$${}_kD(\alpha) = \frac{1}{n}\sum_{j=0}^{n-1}\bar{q}^{2jk}D(q^{2j}\alpha)\ , 0\le k\le n-1\,.$$

Acting to vacuum state, they produce kaleidoscope of coherent states

$$|\tilde{k}\rangle_\alpha = {}_kD(\alpha)|0\rangle\,.$$

Operators ${}_kD(\alpha)$ are not unitary, this is why, the states $|\tilde{k}\rangle_\alpha$ are not normalized. In following sections, we show that normalization of these states can be written in a compact form by using mod n exponential functions with argument $|\alpha|^2$.

4. Generating function for mod n Hermite polynomials

4.1. Mod 2 Hermite polynomials

Coordinate representation of cat states $(18), (19)$ is related with generating functions for Hermite polynomials of even $H_{2k}(x)$ and odd $H_{2k+1}(x)$ order as;

$$
\begin{aligned}
\sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!}H_{2k}(x) &= e^{-z^2}\cosh(2zx) = {}_0e^{-z^2+2zx}\,,\\
\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!}H_{2k+1}(x) &= e^{-z^2}\sinh(2zx) = {}_1e^{-z^2+2zx}\,.
\end{aligned}
$$

4.2. Mod 3 Hermite polynomials

In a similar way, coordinate representation of trinity states is related with mod 3 exponential functions, which are generating functions for $H_{3k}(x), H_{3k+1}(x)$ and $H_{3k+2}(x)$ Hermite polynomials;

$$\sum_{k=0}^{\infty} \frac{z^{3k}}{(3k)!} H_{3k}(x) = {}_0e^{-z^2+2zx} = \frac{1}{3}\left(e^{-z^2+2zx} + 2e^{\frac{z^2}{2}-zx} \cos\left(\frac{\sqrt{3}}{2}(z^2+2zx)\right)\right),$$

$$\sum_{k=0}^{\infty} \frac{z^{3k+1}}{(3k+1)!} H_{3k+1}(x) = {}_1e^{-z^2+2zx} = \frac{1}{3}\left(e^{-z^2+2zx} + 2e^{\frac{z^2}{2}-zx} \cos\left(\frac{\sqrt{3}}{2}(z^2+2zx) - \frac{2\pi}{3}\right)\right),$$

$$\sum_{k=0}^{\infty} \frac{z^{3k+2}}{(3k+2)!} H_{3k+2}(x) = {}_2e^{-z^2+2zx} = \frac{1}{3}\left(e^{-z^2+2zx} + 2e^{\frac{z^2}{2}-zx} \cos\left(\frac{\sqrt{3}}{2}(z^2+2zx) + \frac{2\pi}{3}\right)\right).$$

4.3. Mod n Hermite polynomials

To describe wave functions for kaleidoscope states in coordinate representation for arbitrary n, we introduce mod n Hermite polynomials. Generating function for these polynomials

$$\sum_{s=0}^{\infty} \frac{z^{ns+k}}{(ns+k)!} H_{ns+k}(x) = {}_ke^{-z^2+2zx}, \tag{12}$$

is defined by mod n composite exponential functions;

$${}_ke^{-z^2+2zx} \equiv \frac{1}{n} \sum_{s=0}^{n-1} \bar{q}^{2ks} e^{-(q^{2s}z)^2+2(q^{2s}z)x}, \tag{13}$$

where in (2), we have used $f(z) = e^{-z^2+2zx}$, for $0 \leq k \leq n-1$.

5. Schrödinger's mod 2 cat states

The Schrödinger cat states as an even and odd superposition of $|\alpha\rangle$ and $|-\alpha\rangle$ states, represent mod 2 kaleidoscope states. Here by mod 2 exponential functions, we calculate several characteristics of these states as normalization constants, average number of photons, the uncertainty relations and coordinate representation, explicitly. The normalization of Schrödinger cat states (10), (11) is represented in terms of mod 2 exponential functions as;

$$|0\rangle_\alpha = \frac{{}_0e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_0e^{|\alpha|^2}}}|0\rangle = \frac{\cosh \alpha\hat{a}^\dagger}{\sqrt{\cosh |\alpha|^2}}|0\rangle \quad (mod\, 2), \tag{14}$$

$$|1\rangle_\alpha = \frac{{}_1e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_1e^{|\alpha|^2}}}|0\rangle = \frac{\sinh \alpha\hat{a}^\dagger}{\sqrt{\sinh |\alpha|^2}}|0\rangle \quad (mod\, 2) \tag{15}$$

Average number of photons in these cat states is propotional to ratio of normalization constants;

$${}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = |\alpha|^2 \frac{{}_1e^{|\alpha|^2}}{{}_0e^{|\alpha|^2}} = |\alpha|^2 \tanh |\alpha|^2, \tag{16}$$

$${}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha = |\alpha|^2 \frac{{}_0e^{|\alpha|^2}}{{}_1e^{|\alpha|^2}} = |\alpha|^2 \coth |\alpha|^2. \tag{17}$$

As easy to evaluate, asymptotically these numbers are approaching the usual coherent states number $|\alpha|^2$:

$$\lim_{|\alpha|\to\infty} {}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = \lim_{|\alpha|\to\infty} {}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha \approx |\alpha|^2 = \langle \pm\alpha|\widehat{N}| \pm \alpha\rangle.$$

IOP Conf. Series: Journal of Physics: Conf. Series **1194** (2019) 012059 doi:10.1088/1742-6596/1194/1/012059

In the limit $|\alpha| \to 0$, we get number of photons in the so called Schrödinger's kitten states:

$$\lim_{|\alpha|\to 0} {}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = 0, \quad \lim_{|\alpha|\to 0} {}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha = 1.$$

It is known that in contrast to coherent states, Schrödinger's cat states are not satisfying minimum uncertainty relation, but instead

$$\begin{aligned}(\Delta\hat{q})_{|0\rangle_\alpha}(\Delta\hat{p})_{|0\rangle_\alpha} &= \frac{\hbar}{2}\sqrt{\left(1+2\,{}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha\right) - \left(\alpha^2+\overline{\alpha}^2\right)^2},\\ (\Delta\hat{q})_{|1\rangle_\alpha}(\Delta\hat{p})_{|1\rangle_\alpha} &= \frac{\hbar}{2}\sqrt{\left(1+2\,{}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha\right) - \left(\alpha^2+\overline{\alpha}^2\right)^2},\end{aligned}$$

where average number of photons are given by (16), (17).
Coordinate representation of cat states is given in terms of mod 2 Hermite polynomials which can be described by mod 2 generating functions;

$$\langle x|0\rangle_\alpha = \frac{e^{-\frac{x^2}{2}}}{\pi^{1/4}\sqrt{\cosh|\alpha|^2}}\sum_{n=0}^{\infty}\frac{H_{2k}(x)}{(2k)!}\left(\frac{\alpha}{\sqrt{2}}\right)^{2k} = \frac{e^{-\frac{x^2}{2}}}{\pi^{1/4}\sqrt{{}_0e^{|\alpha|^2}}}\ {}_0e^{-\frac{\alpha^2}{2}+\sqrt{2}\alpha x}, \tag{18}$$

$$\langle x|1\rangle_\alpha = \frac{e^{-\frac{x^2}{2}}}{\pi^{1/4}\sqrt{\sinh|\alpha|^2}}\sum_{n=0}^{\infty}\frac{H_{2k+1}(x)}{(2k+1)!}\left(\frac{\alpha}{\sqrt{2}}\right)^{2k+1} = \frac{e^{-\frac{x^2}{2}}}{\pi^{1/4}\sqrt{{}_1e^{|\alpha|^2}}}\ {}_1e^{-\frac{\alpha^2}{2}+\sqrt{2}\alpha x}. \tag{19}$$

6. Trinity states

As a first generalization of Schrödinger cat states, we introduce the trinity states. If the cat states are associated with $q^4 = 1$, the trinity states are related with $q^6 = 1$, so that $q^2 = e^{i\frac{2\pi}{3}}$. We start this generalization from the set of coherent states, rotated by angle $\frac{2\pi}{3}$ and associated with vertices of equilateral triangle. Then, the set of three orthonormal states $|0\rangle_\alpha, |1\rangle_\alpha$ and $|2\rangle_\alpha$,the trinity states, is

$$\begin{aligned}|0\rangle_\alpha &= e^{\frac{|\alpha|^2}{2}}\frac{|\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle}{\sqrt{3}\sqrt{e^{|\alpha|^2}+e^{q^2|\alpha|^2}+e^{q^4|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}}\frac{|\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle}{3\sqrt{{}_0e^{|\alpha|^2}(mod\,3)}},\\ |1\rangle_\alpha &= e^{\frac{|\alpha|^2}{2}}\frac{|\alpha\rangle + \overline{q}^2|q^2\alpha\rangle + \overline{q}^4|q^4\alpha\rangle}{\sqrt{3}\sqrt{e^{|\alpha|^2}+\overline{q}^2e^{q^2|\alpha|^2}+\overline{q}^4e^{q^4|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}}\frac{|\alpha\rangle + \overline{q}^2|q^2\alpha\rangle + \overline{q}^4|q^4\alpha\rangle}{3\sqrt{{}_1e^{|\alpha|^2}(mod\,3)}},\\ |2\rangle_\alpha &= e^{\frac{|\alpha|^2}{2}}\frac{|\alpha\rangle + \overline{q}^4|q^2\alpha\rangle + \overline{q}^2|q^4\alpha\rangle}{\sqrt{3}\sqrt{e^{|\alpha|^2}+\overline{q}^4e^{q^2|\alpha|^2}+\overline{q}^2e^{q^4|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}}\frac{|\alpha\rangle + \overline{q}^4|q^2\alpha\rangle + \overline{q}^2|q^4\alpha\rangle}{3\sqrt{{}_2e^{|\alpha|^2}(mod\,3)}}.\end{aligned}$$

By using definition of mod 3 exponential functions, we can obtain trinity states in a compact form;

$$|0\rangle_\alpha = \frac{{}_0e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_0e^{|\alpha|^2}}}|0\rangle, \quad |1\rangle_\alpha = \frac{{}_1e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_1e^{|\alpha|^2}}}|0\rangle, \quad |2\rangle_\alpha = \frac{{}_2e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_2e^{|\alpha|^2}}}|0\rangle \qquad (mod\ 3).$$

6.1. Number of photons in trinity states

For calculating number of photons in trinity states, it is convenient to apply annihilation operator $\hat{a}$ to the states $|0\rangle_\alpha, |1\rangle_\alpha$ and $|2\rangle_\alpha$. The operator $\hat{a}$ acts on these states as cyclic permutation and number of photons is determined by ratio of two consecutive mod 3 exponential functions,

$$
\begin{aligned}
{}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha &= |\alpha|^2 \left(\frac{{}_2e^{|\alpha|^2}}{{}_0e^{|\alpha|^2}} \right), &(20)\\
{}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha &= |\alpha|^2 \left(\frac{{}_0e^{|\alpha|^2}}{{}_1e^{|\alpha|^2}} \right), &(21)\\
{}_\alpha\langle 2|\widehat{N}|2\rangle_\alpha &= |\alpha|^2 \left(\frac{{}_1e^{|\alpha|^2}}{{}_2e^{|\alpha|^2}} \right). &(22)
\end{aligned}
$$

For small number of photons, we get in the limit $|\alpha|^2 \to 0$;

$$\lim_{|\alpha|^2\to 0} {}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = 0\,, \quad \lim_{|\alpha|^2\to 0} {}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha = 1\,, \quad \lim_{|\alpha|^2\to 0} {}_\alpha\langle 2|\widehat{N}|2\rangle_\alpha = 2\,.$$

6.2. Heinsenberg uncertainty relation for trinity states

The uncertainty relations can be obtained by direct calculations as,

$$(\Delta\hat{q})_{|k\rangle_\alpha}\,(\Delta\hat{p})_{|k\rangle_\alpha} = \frac{\hbar}{2}\left(1 + 2\,{}_\alpha\langle k|\widehat{N}|k\rangle_\alpha\right),$$

where ${}_\alpha\langle k|\widehat{N}|k\rangle_\alpha\,, 0 \le k \le 2$ are given by (20)-(22). In the limiting case $|\alpha|^2 \to 0$, uncertainty is growing with states number k.

7. Kaleidoscope of quantum coherent states

As a generalization of previous results, here we consider superposition of n coherent states, which are belonging to vertices of regular n-polygon and are rotated by angle $\frac{2\pi}{n}$, related with primitive roots of unity $q^{2n} = 1$.

7.1. Mod n form of kaleidoscope states

The compact expression for the kaleidoscope of quantum coherent states has the form

$$|k\rangle_\alpha = \frac{{}_ke^{\alpha\hat{a}^\dagger}}{\sqrt{{}_ke^{|\alpha|^2}}}|0\rangle \quad (mod\, n), \;\; 0 \le k \le n-1.$$

7.2. Number of photons in kaleidoscope states

This allows us to calculate number of photons by simple application of annihilation operator, then kaleidoscope of quantum states is generated by annihilation operator $\hat{a}$ as cyclic permutation of these states;

$$\hat{a}|0\rangle_\alpha = \alpha\sqrt{\frac{{}_{n-1}e^{|\alpha|^2}}{{}_0e^{|\alpha|^2}}}|n-1\rangle_\alpha\,, \qquad \hat{a}|k\rangle_\alpha = \alpha\sqrt{\frac{{}_{k-1}e^{|\alpha|^2}}{{}_ke^{|\alpha|^2}}}|k-1\rangle_\alpha\,. \tag{23}$$

By taking norm of these states, we get average number of photons as,$1 \le k \le n-1$,

$${}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = |\alpha|^2\left(\frac{{}_{n-1}e^{|\alpha|^2}}{{}_0e^{|\alpha|^2}}\right), \quad {}_\alpha\langle k|\widehat{N}|k\rangle_\alpha = |\alpha|^2\left(\frac{{}_{k-1}e^{|\alpha|^2}}{{}_ke^{|\alpha|^2}}\right).$$

Asymptotically, for small occupation numbers they approach the integer values

$$\lim_{|\alpha|\to 0} {}_\alpha\langle k|\widehat{N}|k\rangle_\alpha = k\,, 0 \le k \le n-1\,.$$

IOP Conf. Series: Journal of Physics: Conf. Series **1194** (2019) 012059 doi:10.1088/1742-6596/1194/1/012059

7.3. *Heinsenberg uncertainty relation for kaleidoscope states*

In the construction of uncertainty relation for kaleidoscope states, it is noticed that the form of variance is different for the cat states ($n = 2$), since the cat states are eigenstates of operator $\hat{a}^2$. The following uncertainty relations are valid for $n \geq 3$;

$$(\Delta \hat{q})_{|k\rangle_\alpha} (\Delta \hat{p})_{|k\rangle_\alpha} = \frac{\hbar}{2}\left(1 + 2|\alpha|^2 \frac{{}_{k-1}e^{|\alpha|^2}}{{}_{k}e^{|\alpha|^2}}\right),$$

where

$$(\Delta \hat{q})_{|k\rangle_\alpha} \equiv (\Delta \hat{p})_{|k\rangle_\alpha} = \sqrt{\frac{\hbar}{2}\left(1 + 2|\alpha|^2 \frac{{}_{k-1}e^{|\alpha|^2}}{{}_{k}e^{|\alpha|^2}}\right)}.$$

This uncertainty relation for kaleidoscope states $|k\rangle_\alpha$ have the following limit;

$$\lim_{|\alpha|^2 \to 0} (\Delta \hat{q})_{|k\rangle_\alpha} (\Delta \hat{p})_{|k\rangle_\alpha} = \frac{\hbar}{2}(2k+1) \quad , 0 \leq k \leq n-1\,.$$

It is noticed that above uncertainty relation include mod n exponential funtions, and in the limit $|\alpha|^2 \to 0$ coincide with spectrum of harmonic oscillator with finite number of energy levels

$$E_k = \hbar\left(k + \frac{1}{2}\right) \quad , 0 \leq k \leq n-1\,.$$

7.4. *Coordinate representation of kaleidoscope states*

The wave function for kaleidoscope of quantum coherent states $|k\rangle_\alpha$, $0 \leq k \leq n-1$,in coordinate representation is given by

$$\langle x|k\rangle_\alpha = \frac{e^{-\frac{x^2}{2}}}{\pi^{1/4}\sqrt{{}_{k}e^{|\alpha|^2}}} \sum_{s=0}^{\infty} \frac{H_{ns+k}(x)}{(ns+k)!}\left(\frac{\alpha}{\sqrt{2}}\right)^{ns+k},$$

and with (12), (13), it appears as superposition of Gaussian wave functions

$$\langle x|k\rangle_\alpha = \frac{e^{-\frac{x^2}{2}}}{\pi^{1/4}\sqrt{{}_{k}e^{|\alpha|^2}}}\ {}_{k}e^{-\frac{\alpha^2}{2}+\sqrt{2}\alpha x},$$

where mod n composite exponential functions are defined by (2) with $f(\alpha) = {}_{k}e^{-\frac{\alpha^2}{2}+\sqrt{2}\alpha x}$,

$${}_{k}e^{-\frac{\alpha^2}{2}+\sqrt{2}\alpha x} \equiv \frac{1}{n}\sum_{s=0}^{n-1} \bar{q}^{2ks} e^{-\frac{1}{2}(q^{2s}\alpha)^2 + \sqrt{2}(q^{2s}\alpha)x}$$

In particular case, the wave functions for mod 4 quartet states with $q^8 = 1$ are following;

$$\langle x|0\rangle_\alpha = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2}\pi^{1/4}} \frac{e^{-\frac{\alpha^2}{2}}\cosh(\sqrt{2}\alpha x) + e^{\frac{\alpha^2}{2}}\cos(\sqrt{2}\alpha x)}{\sqrt{\cosh|\alpha|^2 + \cos|\alpha|^2}},$$

$$\langle x|1\rangle_\alpha = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2}\pi^{1/4}} \frac{e^{-\frac{\alpha^2}{2}}\sinh(\sqrt{2}\alpha x) + e^{\frac{\alpha^2}{2}}\sin(\sqrt{2}\alpha x)}{\sqrt{\sinh|\alpha|^2 + \sin|\alpha|^2}},$$

$$\langle x|2\rangle_\alpha = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2}\pi^{1/4}} \frac{e^{-\frac{\alpha^2}{2}}\cosh(\sqrt{2}\alpha x) - e^{\frac{\alpha^2}{2}}\cos(\sqrt{2}\alpha x)}{\sqrt{\cosh|\alpha|^2 - \cos|\alpha|^2}},$$

$$\langle x|3\rangle_\alpha = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2}\pi^{1/4}} \frac{e^{-\frac{\alpha^2}{2}}\sinh(\sqrt{2}\alpha x) - e^{\frac{\alpha^2}{2}}\sin(\sqrt{2}\alpha x)}{\sqrt{\sinh|\alpha|^2 - \sin|\alpha|^2}}.$$

Corresponding probabilities with mod 4 symmetries for $\alpha = 1 + i$ are shown in Figures 1-4.

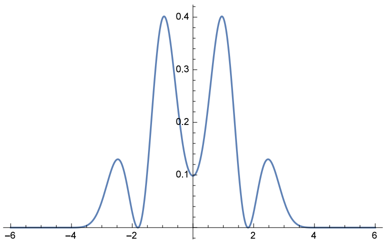

Figure 1. Probability for mod 4 $|0\rangle$ state

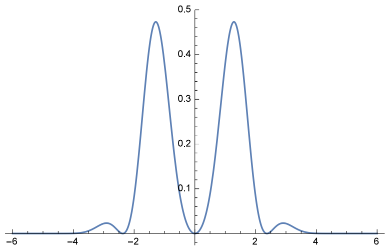

Figure 2. Probability for mod 4 $|1\rangle$ state

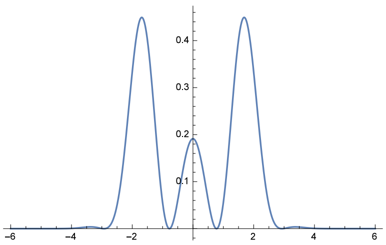

Figure 3. Probability for mod 4 $|2\rangle$ state

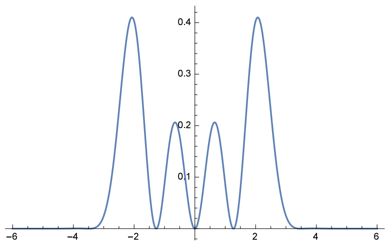

Figure 4. Probability for mod 4 $|3\rangle$ state

As we can see for even states $|0\rangle$ and $|2\rangle$ probabilites at origin are not zero, while for odd states $|1\rangle$ and $|3\rangle$, these probabilities vanishes at origin. More detail analysis of probability distribution for different mod n kaleidoscope states is under investigation.

8. Conclusions

The cat states as orthogonal coherent states were proposed for description of qubit unit of quantum information in quantum optics. Generalization of these states to trinity states, quartet states and generic mod n kaleidoscope states can provide units of quantum information as qutrit and ququad. For quantum information processing ternary, quaternary could be more efficient with base 3 and 4 as basis for position notation. Then, mod n kaleidoscope states provides basis for generic qudit unit of quantum information. It is clear that in the description of physical characteristics of such states as entaglenmet, entropy and randomness mod n generalized hyperbolic function will provide essential roles.

9. Acknowledgements

This work is supported by TUBITAK grant 116F206.

10. References

[1] Pashaev O K and Kocak A 2018 *Symmetries, Differential Equations and Applications* (*Springer Proc. Math. Stat.* vol 266) ed V G Kac and P J Olver *et al* (Zurich: Springer) pp 179-199
[2] Bialynicka-Birula Z 1968 *Phys. Rev.* **173** 1207
[3] Stoler D 1971 *Phys. Rev. D* **4** 2309
[4] Spiridonov V V 1995 *Phys. Rev. A* **52** 1909
[5] Ungar A 1982 *Amer. Math. Monthly* **89** 688-691