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Integrally closed rings which are Prüfer

Başak Ay Saylam

Department of Mathematics, Izmir Institute of Technology, Izmir, Turkey

ABSTRACT

Let R be a commutative ring with zero divisors. It is well known that if R is integrally closed, then R is a Prüfer domain if and only if there is an integer $n>1$ such that, for all $a,b \in R, (a,b)^n = (a^n, b^n)$. We soften this result for commutative rings with zero divisors by proving that this integer n does not have to work for all $a,b \in R$.

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1. Introduction

Let R be a commutative ring with zero divisors. We call an element of R *regular* if it is not a zero-divisor. Let $Reg(R)$ denote the monoid of regular elements of R and $Q(R) = Q$ denote the total ring of fractions R. We note that $Q = (\text{Reg}(R))^{-1}R$. Let $\widetilde{R}$ denote the integral closure of R in $Q(R)$. We say that an ideal I of R is *regular* if I contains a regular element of R. We note that every invertible fractional ideal of R is finitely generated and regular. For a prime ideal P of R, we set $R_{(P)} = (\text{Reg}(R) - P)^{-1}R \subseteq Q$ and $R_{[P]} = \{y \in Q(R) : xy \in R \text{ for some } x \in R - P\}$.

We recall that R is a Prüfer ring if and only if every finitely generated (or two-generated) regular ideal is invertible [6, Theorem 2.13]. In general, we cannot say that the finitely generated regular ideals of a Prüfer ring are generated by two elements. However, in the domain case, we have the following.

Proposition 1.1. [4, Proposition III.1.11] If R is an integrally closed domain and if every finitely generated ideal of R can be generated by two elements, then R is a Prüfer domain.

In fact, a stronger version (Proposition 1.3) of the above proposition can be proven with the help of Lemma 1.2, and it can be shown that an integrally closed domain R is Prüfer if for each finitely generated ideal I of R there is a bound on the number of generators of I. So we do not have to assume that all finitely generated ideals are generated by two elements.

Lemma 1.2. [12, Theorem 6] *Let R be an integrally closed local domain with field of fractions Q. If each $x \in Q$ satisfies $f(x) = 0$, where $f(X)$ is a primitive polynomial in $R[X]$, then either $x \in R$ or $x^{-1} \in R$.*

Proposition 1.3. *If R is an integrally closed domain and for each finitely generated ideal I of R there is a bound on the number of generators of I, then R is a Prüfer domain.*

CONTACT Başak Ay Saylam basakay@iyte.edu.tr Department of Mathematics, Izmir Institute of Technology, Izmir, Turkey.

Proof. We claim that R is a Prüfer domain, that is for any maximal ideal P of R, R_P is a valuation domain. Also, if I is finitely generated ideal of R, then I_P is finitely generated in R_P. So, without loss of generality, suppose that R is local. Let $c \in Q$, so $c = \frac{a}{b}$ for some $a, b, \in R$, both nonzero. Consider $J = Ra + Rb$ and suppose that J^n can be generated by n elements, where n is a positive integer. Observe that $J^n = Ra^n + Ra^{n-1}b + \cdots + Ra^i b^{n-i} + \cdots + Rab^{n-1} + Rb^n$ and that by Nakayama's Lemma it follows that for each i, $0<i<n$, $a^i b^{n-i}$ is in the ideal generated by $a^j b^{n-j}$ for $0 \leqslant j \leqslant n$ such that $j \neq i$. Hence there exists a polynomial expression $f(X, Y) \in R[X, Y]$ of degree n such that $f(a, b) = 0$ with at least one coefficient of f is a unit in R. So, $\frac{f(a,b)}{b^n} = \bar{f}(\frac{a}{b}) = 0$, and hence c satisfies a primitive polynomial. By Lemma 1.2, $c \in R$ or $c^{-1} \in R$ which makes R a valuation domain. □

We aim to show that Proposition 1.3 can be generalized to rings with zero divisors without assuming that all finitely generated regular ideals are generated by two elements, but with the assumption that there is a bound on the number of generators. Furthermore, in Ref. [7] it is shown that for an integrally closed ring R, R is a Prüfer ring if and only if for any elements $a, b \in R$, where a is regular, there is an integer $n > 1$ such that $(a, b)^n = (a^n, b^n)$ [7, Theorem 13]. So, according to this result the same n should work for all $a, b \in R$. Our main purpose is to prove that we do not need the same n to work for all $a, b \in R$, and, in fact, that the number of generators of $(a, b)^n$ to be no more than n for some $n > 0$.

Eakin and Sathaye, in fact, prove Proposition 1.3 via ES-prestable ideals by using a series of lemmas [2]. Although most of these results are proven for commutative rings, a key result [2, Lemma F] is proven only for the domain case. Here, we show that this lemma is also true for commutative rings with zero divisors and hence generalize one of their main results.

For the ideals I and J of R, the colon ideal $[I : J]$ is defined to be $\{q \in Q : qJ \subseteq I\}$. For the ideals I and J of the ring R, with J regular, the natural map from $[I : J]$ to $\mathrm{Hom}_R(J, I)$ is an isomorphism [1, Lemma 1.1]. Thus, the endomorphism ring of a regular ideal I, $\mathrm{End}_R(I) = [I : I]$.

An ideal I of R is called SV-stable if it is projective over its endomorphism ring, $\mathrm{End}_R(I)$. SV-stable ideals are studied thoroughly in [9, 10]. Furthermore, we remark that an SV-stable ideal, over an integral domain, is invertible in $\mathrm{End}_R(I)$. We note that, over a commutative ring, if I is a finitely generated *regular* SV-stable ideal then I is invertible in End(I). R is called an SV-stable (finitely SV-stable, respectively) ring if every (finitely generated, respectively) regular ideal of R is SV-stable. If R is a commutative ring and I is a finitely SV-stable regular ideal, then I is invertible in $\mathrm{End}_R(I)$. An ideal I of R is called ES-stable if $xI = I^2$ for some $x \in I$. R is called an ES-stable (finitely ES-stable) ring if every regular ideal (finitely generated regular ideal) of R is ES-stable. Hence, if I is a regular ES-stable ideal, then $xI = I^2$ for some regular element $x \in I$ since I^2 is regular, also.

We say that an ideal I of a semilocal ring is ES-prestable if some power of I is ES-stable. In a general ring, an ideal I is ES-prestable if IR_P is ES-prestable for any prime ideal P of R. So, the definition of prestability is a local condition. We define a ring R to be finitely ES-prestable if every finitely generated regular ideal in R is ES-prestable.

The facts relating ES-stable and SV-stable integral domains are well-studied (see sections 7.3 and 7.4 in Ref. [3], for example). It is proven in [3, Lemma 7.4.6] that an ES-stable ideal is SV-stable and the converse is also true if the ideal is finitely generated and the ring is local [3, Corollary 7.4.5]. In Section 2, we give a new characterization of Marot valuation rings. In Section 3, we extend some facts relating ES-stable and SV-stable domains in Ref. [3] to commutative rings with zero divisors and generalize some useful facts about ES-prestable ideals which appeared in Ref. [2]. In Section 4, we give some characterizations of finitely ES-prestable rings, and so we prove that R is an integrally closed Prüfer ring if and only if for each finitely generated regular ideal I of R there is a positive integer n such that I^n can be generated by n elements.

2. A new characterization of a Marot valuation ring

We recall that a valuation is a map ν from a ring K onto a totally ordered group G and a symbol ∞, such that for all x and y in K:

1. $\nu(xy) = \nu(x) + \nu(y)$.
2. $\nu(x+y) \geqslant \min\{\nu(x), \nu(y)\}$.
3. $\nu(1) = 0$ and $\nu(0) = \infty$.

The ring $R_\nu = \{x \in Q | \nu(x) \geqslant 0\}$, together with the ideal $P_\nu = \{x \in Q | \nu(x) > 0\}$, denoted (R_ν, P_ν), is called a *Manis valuation pair (of K)*. R_ν is called a Manis valuation ring (of K), and G is called *the value group of* R_ν.

A ring R is called a *Marot ring* if every regular ideal can be generated by a set of regular elements. Moreover, every overring of a Marot ring is Marot. If R is a Marot ring, then $R_{[P]} = R_{(P)}$ for any prime ideal P of R [8, Theorem II.7.6]. Furthermore, in the presence of the Marot property, valuation rings share some properties of valuation domains. For example, it is not true, in general, that given a valuation pair $(R_\nu, P_\nu), P_\nu$ is the unique maximal (regular) ideal of R_ν. But if a Manis valuation ring R_ν is Marot, then we have the following.

Proposition 2.1. [6, Proposition 4.1] Let R be a Marot ring. Assume that $R \neq Q$. Then, the following conditions are equivalent:

a. *R is a valuation ring.*
b. *For each regular element $x \in Q(R)$, either $x \in R$ or $x^{-1} \in R$.*
c. *R has only one maximal regular ideal and R is a Bézout ring; that is each of its finitely generated regular ideals is principal.*
d. *The set of regular ideals of R is totally ordered by inclusion.*

In the Introduction, we mentioned a result of Seidenberg (Lemma 1.2). We prove that his result can be generalized to a Marot ring R.

Lemma 2.2. *Let R be a Marot ring. Then the following are equivalent:*

a. *R is an integrally closed local ring, and each regular element $x \in Q(R)$ satisfies a primitive polynomial in $R[X]$.*
b. *R is a valuation ring.*

Proof. $a \Rightarrow b$: Let x be a regular element of $Q(R)$ such that x satisfies a primitive polynomial $f(X) = a_nX^n + \cdots + a_1X + a_0 \in R[X]$. Since $f(X)$ is primitive, without loss of generality, at least one of the coefficients is a unit in R, say a_n is a unit in R and multiply both sides of $f(x)$ by a_n, so we get $a_nx^n + \cdots + a_n^{n-1}a_1x + a_n^{n-1}a_0 = 0$. Hence, $(a_nx)^n + \cdots + (a_nx)a_n^{n-2}a_1 + a_n^{n-1}a_0 = 0$. Then a_{nx} is integral over R. Since R is integrally closed, $a_nx \in R$. Now, we have two cases. The first case is that if a_{nx} is a unit in R, then $(a_nx)^{-1} = a_n^{-1}x^{-1} \in R$, so $x^{-1} \in R$. The second case is that a_{nx} is not a unit. Then $(a_nx + a_{n-1})x^{n-1} + \cdots + a_1x + a_0 = 0$. Multiplying both sides of this equation by $(a_nx + a_{n-1})^{n-2}$ leads to an equation which shows that $(a_nx + a_{n-1})x$ is integral over R, and so $(a_nx + a_{n-1})x \in R$. If a_{n-1} is a unit in R, then $(a_nx + a_{n-1})$ is a unit in R since R is local. Thus $x \in R$. If a_{n-1} is not a unit in R, then continuing until a unit coefficient we have that $x \in R$ or $x^{-1} \in R$. Then, by Proposition 2.1, R is a valuation ring.

$b \Rightarrow a$: If R is a valuation ring, then for any regular element $x \in Q(R), x \in R$ or $x^{-1} \in R$. If $x \in R$, since R is integrally closed, x satisfies a primitive polynomial in $R[X]$. If $x^{-1} \in R$, since R is integrally closed, then x^{-1} satisfies a primitive polynomial, and so does x. □

3. ES-prestable rings

In Ref. [11], Rush proves that the integral closure of a finitely SV-stable ring is a Prüfer ring. In this section we show that the integral closure of a finitely ES-stable ring is a Prüfer ring, also. Furthermore, we investigate some properties of ES-prestable rings. We begin by proving some facts relating ES-stable and SV-stable rings.

Lemma 3.1. Let R be a commutative ring and I a regular ideal of R. If I is an ES-stable ideal, then I is SV-stable.

Proof. Suppose that I is ES-stable. Then, $xI = I^2$ for some regular element $x \in I$. So, $(x^{-1}I)I = I$, and hence $x^{-1}I \subseteq \mathrm{End}(I)$. Let $y \in \mathrm{End}(I)$. Then, $xyI \subseteq xI = I^2 \subseteq I$. Hence, $y \in x^{-1}I$. Therefore, $x^{-1}I = \mathrm{End}(I)$, so that $x^{-1}\mathrm{End}(I)$ is the inverse of I in $\mathrm{End}(I)$, that is I is SV-stable. □

Corollary 3.2. *Let R be a commutative ring. If R is an ES-stable ring, then R is SV-stable.*

Proof. Follows immediately from Lemma 3.1. □

Lemma 3.3. [11, Proposition 2.1] If R is a finitely SV-stable ring, then $\widetilde{R}$ is a Prüfer ring and every R-submodule of R containing R is a ring.

Lemma 3.4. If R is a finitely ES-stable ring, then $\widetilde{R}$ is a Prüfer ring.

Proof. It immediately follows from Lemmas 3.1 and 3.3. □

The class of finitely ES-prestable rings is larger than the class of finitely ES-stable rings. In the next section we will see that a ring is finitely ES-prestable if and only if its integral closure is Prüfer (Theorem 4.1).

Lemma 3.5. [2, Lemma E] Let $R \subseteq R' \subseteq Q(R)$ be semilocal rings with R' a finite R-module. If I is a finitely generated regular ideal in R, then I is ES-prestable in R if and only if IR' is ES-prestable in R'.

Lemma 3.6. If R is a finitely ES-prestable ring, then so is any overring R' of R, that is $R \subseteq R' \subseteq Q(R)$.

Proof. Let J be a finitely generated ideal of R'. Then $J = R's_1 + \cdots + R's_t$ for some $s_1, s_2, \ldots, s_t \in J$. So, there exists a regular element $c \in R$ such that $cs_i \in R$ for all i. Thus, $I = Rcs_1 + \cdots + Rcs_t$ is an ideal of R, which is isomorphic to J as an R-module, is a finitely generated regular ideal of R. So, J is ES-prestable. □

The following result is the generalization of [2, Lemma F] to commutative rings with zero divisors.

Lemma 3.7. Let I be a finitely generated regular ideal in an integrally closed ring R. Then I is invertible if and only if it is ES-prestable.

Proof. Since I is finitely generated, the definition of ES-prestability and the invertibility of I are both local conditions. Thus, we may suppose that R is local. If I is invertible, then it is principal and hence ES-prestable. Conversely, suppose that I is prestable. Then, for some positive integer n, there is $y \in I^n$ such that $yI^n = (I^n)^2$. Let $r \in I$ be a regular element. Then, $r^{2n} \in I^{2n}$ is regular, also. Since $yI^n = (I^n)^2$, for some $x \in I^n, yx = r^{2n}$. If y were a zero-divisor, then $zy = 0$ for some $z \in R$. So, $zyx = zr^{2n} = 0$ implying that r^{2n} is a zero-divisor, which is not possible. Thus, y must be a regular element. Since I is finitely generated, so is I^n. Hence, by [5, Proposition 24.1], $I^n = Ry$. Therefore, I is invertible. □

4. Bound on the number of generators

In this section, we show that if R is an integrally closed ring, then R is a Prüfer ring if and only if for each finitely generated regular ideal I of R there is a positive integer n such that I^n can be generated by n elements. We begin by proving a key result which is the generalization of a part of [2, Theorem 2].

Theorem 4.1. R is finitely ES-prestable if and only if $\widetilde{R}$ is a Prüfer ring.

Proof. Let I be a finitely generated regular ideal of $\widetilde{R}$. Let us write $I = (x_1, ..., x_n)\widetilde{R}$. So, there exists a regular element $c \in R$ such that $(cx_1, ..., cx_n)$ is a regular ideal of R. Since R is finitely ES-prestable, $(cx_1, ..., cx_n)$ is an ES-prestable ideal. Then, clearly, $(cx_1, ..., cx_n)\widetilde{R} = cI$ is ES-prestable. Thus, by Lemma 3.7, cI is invertible. So, any finitely generated regular ideal of $\widetilde{R}$ is invertible implying that $\widetilde{R}$ is a Prüfer ring.

Suppose that $\widetilde{R}$ is Prüfer. For a finitely generated regular ideal I of R, $I\widetilde{R}$ is invertible and finitely generated. So, there exists an overring R' of R, constructed by adjoining R those finitely many elements that invert $I\widetilde{R}$, which is a finite R-module such that IR' is invertible in R'. So IR' is locally principal, hence IR' is locally ES-prestable. By Lemma 3.5, I is locally ES-prestable, and hence I is ES-prestable. □

Corollary 4.2. *If R is finitely SV-stable, then R is finitely ES-prestable.*

Proof. If R is finitely SV-stable, then by Lemma 3.3, $\widetilde{R}$ is Prüfer, and hence R is finitely ES-prestable by Theorem 4.1. □

Theorem 4.3. [2, Corollary 1] Let I be a finitely generated regular ideal in a local ring R. Then, the following are equivalent.

a. *I is ES-prestable.*
b. *I^n is generated by n elements for some positive integer n,*
c. *There is an integer b(I) such that for each positive integer n, I^n has b(I) generators.*

Corollary 4.4. *Let R be a commutative ring. Then the following are equivalent:*

a. *$\widetilde{R}$ is a Prüfer ring,*
b. *R is finitely ES-prestable,*
c. *for each finitely generated regular ideal I of R, there is a positive integer n such that I^n can be generated by n elements.*

Proof. $a \Longleftrightarrow b$: Follows immediately from Theorem 4.1.
$b \Longleftrightarrow c$: Follows immediately from Theorem 4.3 since ES-prestibility is a local condition. □

In Ref. [7, Theorem 13], a list of characterizations of a Prüfer ring is given. Now we are ready to state a more general version of [7, Theorem 13 (11)].

Theorem 4.5. Let R be a commutative ring. The following are equivalent:

a. *R is a Prüfer ring,*
b. *R is integrally closed and for each $a, b \in R$ with a regular there is a positive integer n such that $(a, b)^n$ can be generated by n elements,*
c. *R is integrally closed and for each $a, b \in R$ with a regular (a, b) is ES-prestable.*

5. Applications

Let R be a commutative ring. The regular height of a regular prime ideal P of R is defined to be the supremum of the length of chains consisting of regular prime ideals contained in P. The regular dimension of R, denoted by $\text{reg}-\dim(R)$, is the supremum of the regular heights of regular prime ideals of R. Thus, if R is an integral domain, then the regular dimension of R is exactly the Krull dimension of R. If there is a non-negative integer n such that $\text{reg}-\dim(V)\leqslant n$ for every valuation overring V of R, with equality for at least one V, then R has valuative dimension n.

Next, we prove a generalization of [2, Corollary 5] to Marot rings.

Corollary 5.1. *Let R be a Marot ring of Krull dimension one. Then $\widetilde{R}$ is a Prüfer ring if and only if the valuative dimension of R is one.*

Proof . If $\widetilde{R}$ is Prufer̈, then, for each maximal ideal P of $\widetilde{R}, \widetilde{R}_{[P]}$ is a Manis valuation ring [7, Theorem 13]. Since R is Marot, so is $\widetilde{R}_{[P]}$, and hence, $\widetilde{R}_{[P]}$ is a valuation ring with a unique maximal regular ideal $P\widetilde{R}_{[P]}$. Since R is of Krull dimension one, so is its valuative dimension. We suppose that there exists a valuation overring of R, V, which has two distinct nonminimal prime regular ideals M_1 and M_2. Since V is a Marot valuation ring, we may assume that $M_1 \subset M_2$. Let us choose $\alpha \in M_1 - M_2$, and let v be the valuation associated to V. We write $\alpha = a/b$, where $a \in R$ and $b \in Reg(R)$. Since $\widetilde{R}$ is a Prüfer ring, by Theorem 4.5, $(a, b)R$ is ES-prestable, and hence, the number of generators of $(a, b)^n$ becomes constant. Then, there exists a form $f(X, Y)$ such that $f(a, b) = 0$ with at least one coefficient of f a unit in R. Let f have degree m. Thus, there is a polynomial expression $g(\alpha) = f(a, b)/b^m = 0$, with coefficients in R, say,

$$r_n(\alpha)^m + r_{n-1}(\alpha)^{m-1} + \cdots + r_1\alpha + r_0 = 0,$$

such that r_i is a unit for the least index j, but then all other terms will have strictly larger value under the valuation v. □

We close this section by providing an example of a Noetherian ring whose integral closure is "close" to being Noetherian, in deed Dedekind.

Corollary 5.2. *Let R be a finitely ES-prestable Noetherian ring. Then $\widetilde{R}$ is a Prüfer ring and every regular ideal of $\widetilde{R}$ is finitely generated.*

Proof. Let R be a finitely ES-prestable Noetherian ring. We can assume that R is local. Thus, $\widetilde{R}$ is a Prüfer ring by Theorem 4.1. By [2, Corollary 6], the Krull dimension of R is at most one, and hence, by [8, Corollary III.11.3], every regular ideal of $\widetilde{R}$ is finitely generated. □

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ORCID

Başak Ay Saylam http://orcid.org/0000-0003-3448-2776

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