Kaleidoscope of Classical Vortex Images and Quantum Coherent States

Oktay K. Pashaev and Aygül Koçak

Abstract The Schrödinger cat states, constructed from Glauber coherent states and applied for description of qubits are generalized to the kaleidoscope of coherent states, related with regular n-polygon symmetry and the roots of unity. This quantum kaleidoscope is motivated by our method of classical hydrodynamics images in a wedge domain, described by q-calculus of analytic functions with q as a primitive root of unity. First we treat in detail the trinity states and the quartet states as descriptive for qutrit and ququat units of quantum information. Normalization formula for these states requires introduction of specific combinations of exponential functions with mod 3 and mod 4 symmetry, which are known also as generalized hyperbolic functions. We show that these states can be generated for an arbitrary n by the Quantum Fourier transform and can provide in general, qudit unit of quantum information. Relations of our states with quantum groups and quantum calculus are discussed.

Keywords Coherent states · Quantum information · Qubit · Qutrit
Qudit · Quantum Fourier transform

1 Introduction

1.1 Classical Vortex Kaleidoscope

The classical problem of point vortices in a domain bounded by two infinite circular cylinders with arbitrary radiuses and positions in the plane, can be formulated as the Apollonius circles problem, reducible by Möbius transformation to the one in annular domain between two concentric circles [1]. Recently we have formulated the two circles theorem, allowing one to construct an arbitrary flow in such annular domain

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by the complex potential $F(z)$ as q-periodic analytic function, $F(qz) = F(z)$, where $q = R^2/r^2$ is determined by ratio of two circle radiuses. Depending on the number and the position of vortices, sources or sinks, one can fix singularities of this function in terms of q-elementary functions [2]. Similar theorem [3] formulated for the flow in the wedge domain with angle $\frac{\pi}{n}$, requires construction of complex potential $F(z)$ as q^2-periodic function, $F(q^2 z) = F(z)$ with q as a root of unity $q^{2n} = 1$. It determines complex velocity $\overline{V}(z) = dF(z)/dz$ as q^2- self-similar analytic function

$$\overline{V}(q^2 z) = q^{-2}\, \overline{V}(z).$$

The wedge theorem describes the fluid flow as superposition of complex analytic functions

$$F(z) = \sum_{k=0}^{n-1} f(q^{2k} z) + \sum_{k=0}^{n-1} \bar{f}(q^{2k} z)\,, \tag{1}$$

representing the kaleidoscope of images associated with the regular $2n$ - polygon. For the point vortex located at z_0, the theorem gives q^2-periodic complex potential

$$F(z) = \frac{i\Gamma}{2\pi} \ln \frac{z^n - z_0^n}{z^n - \bar{z}_0^n} = F(q^2 z), \tag{2}$$

which due to the Kummer expansion

$$z^n - z_0^n = (z - z_0)(z - q^2 z_0)(z - q^4 z_0)\ldots(z - q^{2(n-1)} z_0)\,,$$

appears as the set of vortices with even images at points $z_0,\ q^2 z_0,\ q^4 z_0, \ldots,\ q^{2(n-1)} z_0$ and with odd images at $\bar{z}_0,\ q^2\bar{z}_0,\ q^4\bar{z}_0, \ldots,\ q^{2(n-1)}\bar{z}_0$. This kaleidoscope of vortex images we called the Kummer kaleidoscope.

1.2 Quantum Kaleidoscope and Coherent States

Since analytical functions are related intrinsically with quantum coherent states and the Fock–Bargman representation, here we extend our ideas to the Hilbert space for the coherent states. The problem is to construct q-periodic quantum states and q-self-similar quantum states. Similar problem, relating self-similarity properties of fractals, the theory of entire analytical functions and the q-deformed algebra with coherent states was discussed recently in [4]. In the present paper we consider the case, when q is the primitive root of unity $q^{2n} = 1$ and show that it leads to the kaleidoscope of coherent states $|\alpha\rangle$, $|q^2\alpha\rangle$, ..., $|q^{2(n-1)}\alpha\rangle$, located at vertices of the regular polygon. By acting with dilatation operator on analytic function

$$f(q^2 z) = q^{2z\frac{d}{dz}} f(z) \tag{3}$$

we can rewrite the wedge theorem (1) in a compact form

$$F(z) = \sum_{k=0}^{n-1} \left(q^{2z\frac{d}{dz}}\right)^k [f(z) + \bar{f}(z)] = [n]_{q^{2z\frac{d}{dz}}} [f(z) + \bar{f}(z)], \tag{4}$$

where we have used non-symmetric $\hat{Q}$-number

$$[n]_{\hat{Q}} = 1 + \hat{Q} + \hat{Q}^2 + \cdots \hat{Q}^{n-1} = \frac{\hat{Q}^n - 1}{\hat{Q} - 1},$$

with the operator base

$$\hat{Q} \equiv q^{2z\frac{d}{dz}}. \tag{5}$$

From this representation, q^2-periodicity of function $F(z)$ follows easily. Due to the identity $\hat{Q}^n = q^{2nz\frac{d}{dz}} = 1$ we have

$$\hat{Q}[n]_{\hat{Q}} = [n]_{\hat{Q}}$$

and as follows

$$F(q^2 z) = \hat{Q} F(z) = \hat{Q}[n]_{\hat{Q}}[f(z) + \bar{f}(z)] = [n]_{\hat{Q}}[f(z) + \bar{f}(z)] = F(z).$$

It is noticed that the differential operator (5) is the Fock–Bargman representation for the dilatation operator $\hat{Q} = q^{2\widehat{N}}$, acting on coherent states as

$$q^{2\widehat{N}} |\alpha\rangle = |q^2 \alpha\rangle, \tag{6}$$

where $\widehat{N} = \hat{a}^\dagger \hat{a}$ is the number operator. Then, by analogy with the wedge theorem (1) and (4), we can construct q^2-periodic quantum state as superposition of coherent states

$$\begin{aligned} |0\rangle_\alpha &\equiv |\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle + \cdots + |q^{2(n-1)}\alpha\rangle \\ &= (I + q^{2\widehat{N}} + q^{4\widehat{N}} + \cdots + q^{2(n-1)\widehat{N}})|\alpha\rangle = [n]_{q^{2\widehat{N}}} |\alpha\rangle. \end{aligned} \tag{7}$$

The q^2-periodicity for this quantum state

$$q^{2\widehat{N}} |0\rangle_\alpha = |0\rangle_\alpha$$

follows easily from the relation $\hat{Q}^n = q^{2n\widehat{N}} = I$. This suggests also that to find q^2-self-similar quantum states, one can take the following superpositions of coherent states

$$|1\rangle_\alpha \equiv [n]_{q^{2\widehat{N}+2}} |\alpha\rangle, \quad |2\rangle_\alpha \equiv [n]_{q^{2\widehat{N}+4}} |\alpha\rangle, \quad \ldots \quad , \quad |n-1\rangle_\alpha \equiv [n]_{q^{2\widehat{N}+2(n-1)}} |\alpha\rangle,$$

satisfying self-similarity conditions

$$q^{2\widehat{N}}|1\rangle_\alpha = q^2|1\rangle_\alpha, \;\; q^{2\widehat{N}}|2\rangle_\alpha = q^4|2\rangle_\alpha, \ldots, \;\; q^{2\widehat{N}}|n-1\rangle_\alpha = q^{2(n-1)}|n-1\rangle_\alpha.$$

It turns out that this construction provides the set of orthogonal quantum states. The similar superpositions of coherent states were discussed in different context by several authors, as the generalized coherent states [5, 6], as factorization problem for the Schrödinger equation with self-similar potential [7] and as the Schrödinger cat states [8]. The Schrödinger cat states [9] as superposition of Glauber's optical coherent states with opposite phases, become important tool for construction of qubits, as a units of quantum information [10] in quantum optics [11]. They correspond to even and odd quantum states with $q^2 = -1$. Here we generalize this construction to the kaleidoscope of coherent states, related with regular n-polygon symmetry and the roots of unity. Superposition of coherent states with such symmetry plays the role of the quantum Fourier transform and provides the set of orthonormal quantum states, as a description of qutrits, ququats and qudits. Such quantum states, considered as a units of quantum information processing and corresponding to an arbitrary base number n, could have advantage in secure quantum communication.

1.3 Glauber Coherent States

We consider the Heisenberg–Weyl algebra, written in terms of creation and annihilation operators, satisfying bosonic commutation relation

$$\left[\hat{a}, \hat{a}^\dagger\right] = \widehat{\mathrm{I}}.$$

The annihilation operator determines the vacuum state $\hat{a}|0\rangle = 0$ from the Hilbert space $|0\rangle \in H$ and the creation operator $\hat{a}^\dagger$ repeatedly applied to this state, gives orthonormal set of states $|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$. Coherent states are defined as eigenstates of annihilation operator [12]:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle\,,$$

where $\alpha \in C$. This gives us a relation between complex plane and the Hilbert space, such that $\alpha \in C \leftrightarrow |\alpha\rangle \in H$. Another equivalent definition is given by the displacement operator,

$$D(\alpha) = e^{\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}} = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^\dagger} e^{-\bar{\alpha}\hat{a}} \tag{8}$$

so that,

$$|\alpha\rangle = D(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle\,. \tag{9}$$

From this we get the following representation of coherent states:

$$|\alpha\rangle = \frac{e^{\alpha \hat{a}^\dagger}}{\sqrt{e^{|\alpha|^2}}}|0\rangle,$$

which is instructive for our generalizations. The inner product of coherent states,

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2+\bar{\alpha}\beta}$$

is never zero, $|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2} \neq 0$. This is why coherent states are not orthogonal. The aim of the present paper is to construct an orthogonal set of states as superposition of coherent states with discrete regular polygon symmetry.

2 Schrödinger's Cat States

In description of the Schrödinger cat states one introduces two orthogonal states as superpositions of $|\alpha\rangle$ and $|-\alpha\rangle$ states, which are called even and odd cat states [8],

$$|\text{Cat}_{even}\rangle \sim |\alpha\rangle + |-\alpha\rangle \quad , \quad |\text{Cat}_{odd}\rangle \sim |\alpha\rangle - |-\alpha\rangle.$$

The states in this superpositions are related by rotation to angle π, which corresponds to primitive root of unity $q^2 = \bar{q}^{\,2} = -1$, so that $q^4 = 1$. The normalization constants for these states

$$|0\rangle_\alpha = \frac{N_0}{\sqrt{2}}(\,|\alpha\rangle + |q^2\alpha\rangle\,), \quad |1\rangle_\alpha = \frac{N_1}{\sqrt{2}}(\,|\alpha\rangle + \bar{q}^{\,2}|q^2\alpha\rangle\,), \tag{10}$$

are calculated as:

$$N_0 = \frac{e^{\frac{|\alpha|^2}{2}}}{\sqrt{2}\sqrt{\cosh|\alpha|^2}}, \quad N_1 = \frac{e^{\frac{|\alpha|^2}{2}}}{\sqrt{2}\sqrt{\sinh|\alpha|^2}}. \tag{11}$$

Transformation to these states can be described in the matrix form as an action by the Hadamard gate,

$$\begin{bmatrix} |0\rangle_\alpha \\ |1\rangle_\alpha \end{bmatrix} = \mathbf{N} \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & \bar{q}^{\,2} \end{bmatrix}}_{Hadamard\ gate} \begin{bmatrix} |\alpha\rangle \\ |q^2\alpha\rangle \end{bmatrix}, \tag{12}$$

where the normalization matrix

$$\mathbf{N} = \frac{e^{\frac{|\alpha|^2}{2}}}{\sqrt{2}} \operatorname{diag}\left({}_0e^{|\alpha|^2},\ {}_1e^{|\alpha|^2}\right)^{-1/2} (mod\, 2) \equiv \operatorname{diag}(N_0, N_1) \tag{13}$$

is defined by the even (0 mod 2) and the odd (1 mod 2) exponential functions, coinciding with hyperbolic functions,

$$(mod\, 2) \quad {}_0e^{|\alpha|^2} \equiv \sum_{k=0}^{\infty} \frac{\left(|\alpha|^2\right)^{2k}}{(2k)!} = \frac{e^{|\alpha|^2} + e^{q^2|\alpha|^2}}{2} = \cosh|\alpha|^2,$$

$$(mod\, 2) \quad {}_1e^{|\alpha|^2} \equiv \sum_{k=0}^{\infty} \frac{\left(|\alpha|^2\right)^{2k+1}}{(2k+1)!} = \frac{e^{|\alpha|^2} + \bar{q}^2 e^{q^2|\alpha|^2}}{2} = \sinh|\alpha|^2.$$

2.1 Mod 2 Representation of Cat States

In terms of these exponential functions we can rewrite the Schrödinger cat states in a compact form:

$$|0\rangle_\alpha = \frac{{}_0e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_0e^{|\alpha|^2}}}|0\rangle \quad (mod\, 2) = \frac{\cosh\alpha\hat{a}^\dagger}{\sqrt{\cosh|\alpha|^2}}|0\rangle\,,$$

$$|1\rangle_\alpha = \frac{{}_1e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_1e^{|\alpha|^2}}}|0\rangle \quad (mod\, 2) = \frac{\sinh\alpha\hat{a}^\dagger}{\sqrt{\sinh|\alpha|^2}}|0\rangle.$$

2.2 Eigenvalue Problem for Cat States

Since $|\alpha\rangle$ is an eigenstate of annihilation operator $\hat{a}$, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, it is also the eigenstate of operator $\hat{a}^2$:

$$\hat{a}^2|\alpha\rangle = \alpha^2|\alpha\rangle.$$

However, the last equation admits one more eigenstate $|-\alpha\rangle$ with the same eigenvalue α^2, so that

$$\hat{a}^2|\mp\alpha\rangle = \alpha^2|\mp\alpha\rangle.$$

Hence, any superposition of states $\{|+\alpha\rangle, |-\alpha\rangle\}$ is also an eigenstate of operator $\hat{a}^2$, with the same eigenvalue. This implies that Schrödinger cat states are eigenstates of this operator,

$$\hat{a}^2|0\rangle_\alpha = \alpha^2|0\rangle_\alpha, \quad \hat{a}^2|1\rangle_\alpha = \alpha^2|1\rangle_\alpha,$$

constituting orthonormal basis $\{|0\rangle_\alpha, |1\rangle_\alpha\}$. It can be used to define the qubit coherent state:

$$|\psi\rangle_\alpha = c_0|0\rangle_\alpha + c_1|1\rangle_\alpha,$$

where $|c_0|^2 + |c_1|^2 = 1$, representing a unit of quantum information in quantum optics. This qubit state is an eigenstate of operator $\hat{a}^2$ as well:

$$\hat{a}^2|\psi\rangle_\alpha = \alpha^2|\psi\rangle_\alpha.$$

2.3 Number of Photons in Cat States

The cat states are not eigenstates of the annihilation operator $\hat{a}$. On the contrary, action of this operator gives flipping between cat states $|0\rangle_\alpha$ and $|1\rangle_\alpha$:

$$\hat{a}|0\rangle_\alpha = \alpha\frac{N_0}{N_1}|1\rangle_\alpha\,, \quad \hat{a}|1\rangle_\alpha = \alpha\frac{N_1}{N_0}|0\rangle_\alpha\,.$$

By using these equations we find number of photons in Schrödinger's cat states as :

$$\begin{aligned}
{}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha &= |\alpha|^2\frac{N_0^2}{N_1^2} = |\alpha|^2\frac{{}_1e^{|\alpha|^2}}{{}_0e^{|\alpha|^2}} = |\alpha|^2\tanh|\alpha|^2,\\
{}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha &= |\alpha|^2\frac{N_1^2}{N_0^2} = |\alpha|^2\frac{{}_0e^{|\alpha|^2}}{{}_1e^{|\alpha|^2}} = |\alpha|^2\coth|\alpha|^2.
\end{aligned}$$

It shows deviation from number of photons in coherent states

$$\langle\alpha|\widehat{N}|\alpha\rangle = |\alpha|^2$$

shown in Fig. 1. In the limiting case $|\alpha| \to \infty$ both distributions asymptotically goes to this value

$$\lim_{|\alpha|\to\infty} {}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = \lim_{|\alpha|\to\infty} {}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha \approx |\alpha|^2.$$

The cat states for $|\alpha|^2 \ll 1$ are reduced to the so called Schrödinger's kitten states with number of photons 0 and 1:

$$\lim_{|\alpha|\to 0} {}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = 0, \quad \lim_{|\alpha|\to 0} {}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha = 1.$$

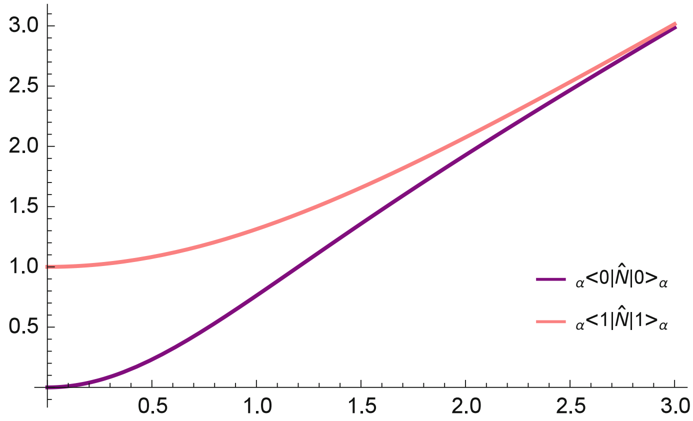

Fig. 1 Photon numbers in Schrödinger's cat states

2.4 Fermionic Representation of Cat States

The dilatation operator $q^{2\widehat{N}} = e^{i\pi\widehat{N}} = (-1)^{\widehat{N}}$ is the parity operator for cat states, so that $|0\rangle_\alpha$ and $|1\rangle_\alpha$ states are eigenstates of this operator. The first state is the q^2-periodic state and the second one is q^2-self-similar state,

$$q^{2\widehat{N}}|0\rangle_\alpha = |0\rangle_\alpha\,, \quad q^{2\widehat{N}}|1\rangle_\alpha = q^2|1\rangle_\alpha. \tag{14}$$

These states represent kaleidoscope of two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$, rotated by angle π, and can be rewritten in terms of parity operator

$$\begin{aligned} |0\rangle_\alpha &= N_0\,[2]_{q^{2\widehat{N}}}|\alpha\rangle = N_0(I + q^{2\widehat{N}})|\alpha\rangle, \\ |1\rangle_\alpha &= N_1\,[2]_{q^{2\widehat{N}+2}}|\alpha\rangle = N_1(I + q^2 q^{2\widehat{N}})|\alpha\rangle, \end{aligned} \tag{15}$$

or

$$\begin{aligned} |0\rangle_\alpha &= N_0(I + (-1)^{\widehat{N}})|\alpha\rangle\,, \\ |1\rangle_\alpha &= N_1(I - (-1)^{\widehat{N}})|\alpha\rangle\,. \end{aligned} \tag{16}$$

It is noticed that the cat states are eigenstates also of q^2- non-symmetric number operator

$$[\widehat{N}]_{q^2} = \frac{q^{2\widehat{N}} - 1}{q^2 - 1},$$

where $q^2 = -1$,

$$[\widehat{N}]_{q^2}|0\rangle_\alpha = [0]_{q^2}|0\rangle_\alpha, \qquad [\widehat{N}]_{q^2}|1\rangle_\alpha = [1]_{q^2}|1\rangle_\alpha,$$

with eigenvalues $[0]_{q^2} = 0$ and $[1]_{q^2} = 1$. In the Fock basis $|n\rangle$, $n = 0, 1, 2...$, these number operator is diagonal, with eigenvalues 0 for even numbers $n = 2k$, and 1 for odd numbers $n = 2k + 1$. This number operator in the cat basis is matrix of the fermion number operator

$$[\widehat{N}]_{q^2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \widehat{N}_F$$

factorized by fermionic creation and annihilation operators $\widehat{N}_F = \hat{b}^\dagger \hat{b}$, with algebra

$$\hat{b}\hat{b}^\dagger + \hat{b}^\dagger \hat{b} = I, \qquad \hat{b}^2 = 0, \qquad (\hat{b}^\dagger)^2 = 0,$$

and matrix representation

$$\hat{b} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \hat{b}^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The cat states in this basis then are just computational basis qubit states:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

3 Trinity States

The Schrödinger cat states can be generalized to the kaleidoscope of coherent states. We start this generalization from the set of three coherent states, rotated by angle $\frac{2\pi}{3}$ and located at vertices of equilateral triangle, which corresponds to roots of unity $q^6 = 1$. First we define superposition

$$|0\rangle_\alpha = \frac{N_0}{\sqrt{3}} \left(|\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle\right).$$

Due to identity

$$q^{6n} - 1 = (q^{2n} - 1)(1 + q^{2n} + q^{4n}) = 0 \;\Rightarrow$$

$$1 + q^{2n} + q^{4n} = 3\, \delta_{n,0(mod\, 3)},$$

with

$$\delta_{k,0\,(mod\, 3)} = \begin{cases} 1, \; k = 0\,(mod\, 3); \\ 0, \; k \neq 0\,(mod\, 3), \end{cases} \tag{17}$$

the normalization constant is $N_0 = e^{\frac{|\alpha|^2}{2}}(3\,{}_0e^{|\alpha|^2})^{-1/2}$, where we have introduced (mod 3) exponential function

$$ {}_0e^{|\alpha|^2}(mod\,3) \equiv \sum_{k=0}^{\infty} \frac{\left(|\alpha|^2\right)^{3k}}{(3k)!} = \frac{1}{3}\left(e^{|\alpha|^2} + e^{q^2|\alpha|^2} + e^{q^4|\alpha|^2}\right). $$

In a similar way we obtain the set of orthonormal states $|0\rangle_\alpha$, $|1\rangle_\alpha$ and $|2\rangle_\alpha$:

$$ |0\rangle_\alpha = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + |q^{\,2}\alpha\rangle + |q^{\,4}\alpha\rangle}{\sqrt{3}\sqrt{e^{|\alpha|^2} + e^{q^2|\alpha|^2} + e^{q^4|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + |q^{\,2}\alpha\rangle + |q^{\,4}\alpha\rangle}{3\sqrt{{}_0e^{|\alpha|^2}(mod\,3)}}, $$

$$ |1\rangle_\alpha = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \overline{q}^2|q^{\,2}\alpha\rangle + \overline{q}^4|q^{\,4}\alpha\rangle}{\sqrt{3}\sqrt{e^{|\alpha|^2} + \overline{q}^2 e^{q^2|\alpha|^2} + \overline{q}^4 e^{q^4|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \overline{q}^2|q^{\,2}\alpha\rangle + \overline{q}^4|q^{\,4}\alpha\rangle}{3\sqrt{{}_1e^{|\alpha|^2}(mod\,3)}}, $$

$$ |2\rangle_\alpha = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \overline{q}^4|q^{\,2}\alpha\rangle + \overline{q}^2|q^{\,4}\alpha\rangle}{\sqrt{3}\sqrt{e^{|\alpha|^2} + \overline{q}^4 e^{q^2|\alpha|^2} + \overline{q}^2 e^{q^4|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \overline{q}^4|q^{\,2}\alpha\rangle + \overline{q}^2|q^{\,4}\alpha\rangle}{3\sqrt{{}_2e^{|\alpha|^2}(mod\,3)}}. $$

3.1 Matrix Form of Trinity States

These states appear by action of the trinity gate, playing the role of three dimensional analogue of Hadamard gate

$$ \begin{bmatrix} |0\rangle_\alpha \\ |1\rangle_\alpha \\ |2\rangle_\alpha \end{bmatrix} = \mathbf{N} \underbrace{\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \overline{q}^2 & \left(\overline{q}^2\right)^2 \\ 1 & \overline{q}^4 & \left(\overline{q}^4\right)^2 \end{bmatrix}}_{Trinity\ gate} \begin{bmatrix} |\alpha\rangle \\ |q^2\alpha\rangle \\ |q^4\alpha\rangle \end{bmatrix}, \qquad (18) $$

with normalization constants

$$ \mathbf{N} = \frac{e^{\frac{|\alpha|^2}{2}}}{\sqrt{3}} \,\mathrm{diag}\left({}_0e^{|\alpha|^2}, {}_1e^{|\alpha|^2}, {}_2e^{|\alpha|^2}\right)^{-1/2} (mod\ 3) \equiv \mathrm{diag}\,(N_0, N_1, N_2) \qquad (19) $$

and identity

$$ 1 + \overline{q}^{2(n-k)} + \overline{q}^{4(n-k)} = 3\,\delta_{n,k(mod\ 3)}, 0 \le k \le 2, $$

where

$$ \delta_{n,k\,(mod\,3)} = \begin{cases} 1, & n = k\,(mod\,3); \\ 0, & n \ne k\,(mod\,3). \end{cases} \qquad (20) $$

Trinity states as superposition of coherent states have the following explicit phase shift :

$$
\begin{aligned}
|0\rangle_\alpha &= N_0(|\alpha\rangle + |e^{i\frac{2\pi}{3}}\alpha\rangle + |e^{-i\frac{2\pi}{3}}\alpha\rangle),\\
|1\rangle_\alpha &= N_1(|\alpha\rangle + e^{-i\frac{2\pi}{3}}|e^{i\frac{2\pi}{3}}\alpha\rangle + e^{i\frac{2\pi}{3}}|e^{-i\frac{2\pi}{3}}\alpha\rangle),\\
|2\rangle_\alpha &= N_2(|\alpha\rangle + e^{i\frac{2\pi}{3}}|e^{i\frac{2\pi}{3}}\alpha\rangle + e^{-i\frac{2\pi}{3}}|e^{-i\frac{2\pi}{3}}\alpha\rangle).
\end{aligned}
$$

By using three different (mod 3) exponential functions, we can rewrite these states in a compact form:

$$
|0\rangle_\alpha = \frac{{}_0e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_0e^{|\alpha|^2}}}|0\rangle\,, \quad |1\rangle_\alpha = \frac{{}_1e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_1e^{|\alpha|^2}}}|0\rangle\,, \quad |2\rangle_\alpha = \frac{{}_2e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_2e^{|\alpha|^2}}}|0\rangle \qquad (mod\ 3).
$$

3.2 Eigenvalue Problem for Trinity States

Coherent states $\{|\alpha\rangle, |q^2\alpha\rangle, |q^4\alpha\rangle\}$ are eigenstates of operator $\hat{a}$ with different eigenvalues $\alpha, q^2\alpha, q^4\alpha$, and the eigenstates of operator $\hat{a}^3$ with the same eigenvalue α^3. Due to this, our trinity states $\{|0\rangle_\alpha, |1\rangle_\alpha, |2\rangle_\alpha\}$ are also eigenstates of operator $\hat{a}^3$:

$$
\hat{a}^3|q^{2k}\alpha\rangle = \alpha^3|q^{2k}\alpha\rangle \quad \Rightarrow \quad \hat{a}^3|k\rangle_\alpha = \alpha^3|k\rangle_\alpha, \quad k = 0, 1, 2.
$$

From trinity states we can construct the qutrit coherent state

$$
|\psi\rangle_\alpha = c_0|0\rangle_\alpha + c_1|1\rangle_\alpha + c_2|2\rangle_\alpha,
$$

where $|c_0|^2 + |c_1|^2 + |c_2|^2 = 1$, as a unit of quantum information with base 3. It turns out that this state is an eigenstate of operator $\hat{a}^3$:

$$
\hat{a}^3|\psi\rangle_\alpha = \alpha^3|\psi\rangle_\alpha.
$$

3.3 Number of Photons in Trinity States

The annihilation operator $\hat{a}$ acts on states $|0\rangle_\alpha$, $|1\rangle_\alpha$ and $|2\rangle_\alpha$ as cyclic permutation:

$$
\hat{a}|0\rangle_\alpha = \alpha\frac{N_0}{N_2}|2\rangle_\alpha\,,\ \ \hat{a}|1\rangle_\alpha = \alpha\frac{N_1}{N_0}|0\rangle_\alpha\,,\ \ \hat{a}|2\rangle_\alpha = \alpha\frac{N_2}{N_1}|1\rangle_\alpha\,. \tag{21}
$$

This equation allows us to calculate number of photons in trinity states (see Fig. 2):

$$
{}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = |\alpha|^2\left[\frac{{}_2e^{|\alpha|^2}}{{}_0e^{|\alpha|^2}}\right] = |\alpha|^2\left[\frac{1 + 2e^{\frac{-3|\alpha|^2}{2}}\cos\left(\frac{\sqrt{3}}{2}|\alpha|^2 + \frac{2\pi}{3}\right)}{1 + 2e^{\frac{-3|\alpha|^2}{2}}\cos\left(\frac{\sqrt{3}}{2}|\alpha|^2\right)}\right],
$$

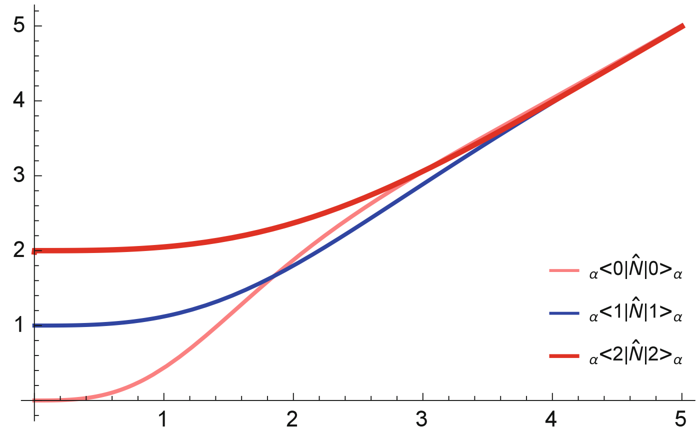

Fig. 2 Photon numbers in trinity states

$$
{}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha = |\alpha|^2 \left[\frac{{}_0e^{|\alpha|^2}}{{}_1e^{|\alpha|^2}}\right] = |\alpha|^2 \left[\frac{1+2e^{\frac{-3|\alpha|^2}{2}}\cos\left(\frac{\sqrt{3}}{2}|\alpha|^2\right)}{1+2e^{\frac{-3|\alpha|^2}{2}}\cos\left(\frac{\sqrt{3}}{2}|\alpha|^2-\frac{2\pi}{3}\right)}\right],
$$

$$
{}_\alpha\langle 2|\widehat{N}|2\rangle_\alpha = |\alpha|^2 \left[\frac{{}_1e^{|\alpha|^2}}{{}_2e^{|\alpha|^2}}\right] = |\alpha|^2 \left[\frac{1+2e^{\frac{-3|\alpha|^2}{2}}\cos\left(\frac{\sqrt{3}}{2}|\alpha|^2-\frac{2\pi}{3}\right)}{1+2e^{\frac{-3|\alpha|^2}{2}}\cos\left(\frac{\sqrt{3}}{2}|\alpha|^2+\frac{2\pi}{3}\right)}\right].
$$

3.3.1 Matrix Representation

Due to $\widehat{N}|n\rangle = n|n\rangle$, $n \geq 0$ from the eigenvalue problem

$$
q^{2\widehat{N}}|0\rangle_\alpha = |0\rangle_\alpha, \quad q^{2\widehat{N}}|1\rangle_\alpha = q^2|1\rangle_\alpha, \quad q^{2\widehat{N}}|2\rangle_\alpha = q^4|2\rangle_\alpha\,,
$$

we find the matrix representation of operators in our kaleidoscope basis as the clock and the shift matrix

$$
q^{2\widehat{N}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & q^4 \end{pmatrix}, \quad \hat{a} = \alpha \begin{pmatrix} 0 & \frac{N_1}{N_0} & 0 \\ 0 & 0 & \frac{N_2}{N_1} \\ \frac{N_0}{N_2} & 0 & 0 \end{pmatrix} = \alpha \begin{pmatrix} \frac{N_1}{N_0} & 0 & 0 \\ 0 & \frac{N_2}{N_1} & 0 \\ 0 & 0 & \frac{N_0}{N_2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (22)
$$

This gives for the q^2-number operator $[\widehat{N}]_{q^2} = \frac{q^{2\widehat{N}}-1}{q^2-1}$, the diagonal form with matrix elements

$$_{\alpha}\langle 0|[\widehat{N}]_{q^2}|0\rangle_{\alpha} = [0]_{q^2}, \quad _{\alpha}\langle 1|[\widehat{N}]_{q^2}|1\rangle_{\alpha} = [1]_{q^2}, \quad _{\alpha}\langle 2|[\widehat{N}]_{q^2}|2\rangle_{\alpha} = [2]_{q^2},$$

as q^2 numbers: $[0]_{q^2} = 0$, $[1]_{q^2} = 1$, $[2]_{q^2} = \frac{1+i\sqrt{3}}{2}$.

4 Quartet States

We define four states, rotated by angle $\frac{\pi}{2}$ and determined by primitive roots of unity: $q^8 = 1$. Superposition of these states with proper coefficients give us quartet of orthonormal basis states:

$$\begin{bmatrix} |0\rangle_{\alpha} \\ |1\rangle_{\alpha} \\ |2\rangle_{\alpha} \\ |3\rangle_{\alpha} \end{bmatrix} = \mathbf{N} \underbrace{\frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \overline{q}^2 & (\overline{q}^2)^2 & (\overline{q}^2)^3 \\ 1 & \overline{q}^4 & (\overline{q}^4)^2 & (\overline{q}^4)^3 \\ 1 & \overline{q}^6 & (\overline{q}^6)^2 & (\overline{q}^6)^3 \end{bmatrix}}_{Quartet\ gate} \begin{bmatrix} |\alpha\rangle \\ |q^2\alpha\rangle \\ |q^4\alpha\rangle \\ |q^6\alpha\rangle \end{bmatrix}, \tag{23}$$

where normalization constants are defined as

$$\mathbf{N} = \frac{e^{\frac{|\alpha|^2}{2}}}{\sqrt{4}} \operatorname{diag}\left({}_0e^{|\alpha|^2}, {}_1e^{|\alpha|^2}, {}_2e^{|\alpha|^2}, {}_3e^{|\alpha|^2}\right)^{-1/2} (mod\, 4) \equiv \operatorname{diag}(N_0, N_1, N_2, N_3)$$

and the identity is

$$1 + \overline{q}^{2(n-k)} + \overline{q}^{4(n-k)} + \overline{q}^{6(n-k)} = 4\,\delta_{n,k(mod\,4)} \quad , 0 \le k \le 3$$

with

$$\delta_{n,k\,(mod\,4)} = \begin{cases} 1, & n = k\,(mod\,4); \\ 0, & n \neq k\,(mod\,4). \end{cases} \tag{24}$$

The quartet states are superpositions of cat states with explicit form of phase shift as

$$\begin{aligned} |0\rangle_{\alpha} &= N_0\,[(|\alpha\rangle + |-\alpha\rangle) + (|i\alpha\rangle + |-i\alpha\rangle)], \\ |1\rangle_{\alpha} &= N_1\,[(|\alpha\rangle - |-\alpha\rangle) - i(|i\alpha\rangle - |-i\alpha\rangle)], \\ |2\rangle_{\alpha} &= N_2\,[(|\alpha\rangle + |-\alpha\rangle) - (|i\alpha\rangle + |-i\alpha\rangle)], \\ |3\rangle_{\alpha} &= N_3\,[(|\alpha\rangle - |-\alpha\rangle) + i(|i\alpha\rangle - |-i\alpha\rangle)]. \end{aligned}$$

By using $(mod\, 4)$ exponential functions we get representation of these states in a compact form:

$$|0\rangle_\alpha = \frac{{}_0e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_0e^{|\alpha|^2}}}|0\rangle, \quad |1\rangle_\alpha = \frac{{}_1e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_1e^{|\alpha|^2}}}|0\rangle, \quad |2\rangle_\alpha = \frac{{}_2e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_2e^{|\alpha|^2}}}|0\rangle, \quad |3\rangle_\alpha = \frac{{}_3e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_3e^{|\alpha|^2}}}|0\rangle\,.$$

4.1 Eigenvalue Problem for Quartet States

As easy to see, the quartet states are eigenstates of operator $\hat{a}^4$ with eigenvalue α^4:

$$\hat{a}^4|q^{2k}\alpha\rangle = \alpha^4|q^{2k}\alpha\rangle \quad \Rightarrow \quad \hat{a}^4|k\rangle_\alpha = \alpha^4|k\rangle_\alpha \quad k = 0, 1, 2, 3.$$

As a result, the ququat state, defined as

$$|\psi\rangle_\alpha = c_0|0\rangle_\alpha + c_1|1\rangle_\alpha + c_2|2\rangle_\alpha + c_3|3\rangle_\alpha,$$

where $|c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 = 1$, describes a unit of quantum information with base 4, and is an eigenstate of operator $\hat{a}^4$:

$$\hat{a}^4|\psi\rangle_\alpha = \alpha^4|\psi\rangle_\alpha.$$

4.2 Number of Photons in Quartet States

The annihilation operator $\hat{a}$ implements cyclic permutation of states $|k\rangle_\alpha$, $k = 0, 1, 2, 3$:

$$\hat{a}|0\rangle_\alpha = \alpha\frac{N_0}{N_3}|3\rangle_\alpha\,,\ \hat{a}|1\rangle_\alpha = \alpha\frac{N_1}{N_0}|0\rangle_\alpha\,,\ \hat{a}|2\rangle_\alpha = \alpha\frac{N_2}{N_1}|1\rangle_\alpha\,,\ \hat{a}|3\rangle_\alpha = \alpha\frac{N_3}{N_2}|2\rangle_\alpha\ ,$$

allowing us to calculate number of photons in quartet states (See Fig. 3):

$$\begin{aligned}
{}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha &= |\alpha|^2\left[\frac{{}_3e^{|\alpha|^2}}{{}_0e^{|\alpha|^2}}\right] = |\alpha|^2\left[\frac{\sinh|\alpha|^2 - \sin|\alpha|^2}{\cosh|\alpha|^2 + \cos|\alpha|^2}\right],\\
{}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha &= |\alpha|^2\left[\frac{{}_0e^{|\alpha|^2}}{{}_1e^{|\alpha|^2}}\right] = |\alpha|^2\left[\frac{\cosh|\alpha|^2 + \cos|\alpha|^2}{\sinh|\alpha|^2 + \sin|\alpha|^2}\right],\\
{}_\alpha\langle 2|\widehat{N}|2\rangle_\alpha &= |\alpha|^2\left[\frac{{}_1e^{|\alpha|^2}}{{}_2e^{|\alpha|^2}}\right] = |\alpha|^2\left[\frac{\sinh|\alpha|^2 + \sin|\alpha|^2}{\cosh|\alpha|^2 - \cos|\alpha|^2}\right],\\
{}_\alpha\langle 3|\widehat{N}|3\rangle_\alpha &= |\alpha|^2\left[\frac{{}_2e^{|\alpha|^2}}{{}_3e^{|\alpha|^2}}\right] = |\alpha|^2\left[\frac{\cosh|\alpha|^2 - \cos|\alpha|^2}{\sinh|\alpha|^2 - \sin|\alpha|^2}\right].
\end{aligned}$$

The quartet states are also eigenstates of q^2-number operator $[\widehat{N}]_{q^2}$:

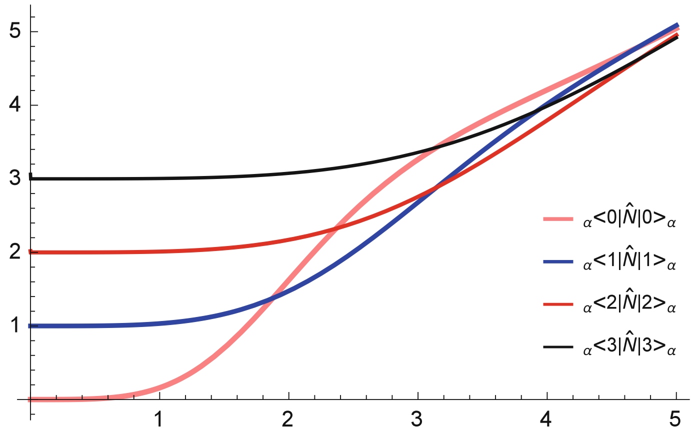

Fig. 3 Photon numbers in quartet states

$$q^{2\widehat{N}}|k\rangle_\alpha = q^{2k}|k\rangle_\alpha \quad \Rightarrow \quad [\widehat{N}]_{q^2}|k\rangle_\alpha = [k]_{q^2}|k\rangle_\alpha \text{ , where } k = 0, 1, 2, 3. \tag{25}$$

5 Kaleidoscope of Quantum Coherent States

As a generalization of previous results, here we consider superposition of n coherent states, which are belonging to vertices of regular n-polygon, rotated by angle $\frac{\pi}{n}$ (Fig. 4). It is related with primitive roots of unity: $q^{2n} = 1$. For the inner product of q^{2k} rotated coherent states we have

$$\langle q^{2k}\alpha|q^{2k}\alpha\rangle = 1,$$

$$\langle q^{2k}\alpha|q^{2l}\alpha\rangle = e^{|\alpha|^2(q^{2(l-k)}-1)},\ 0 \le k, l \le n-1.$$

To calculate orthogonality and normalization conditions we apply the following lemma; For $q^{2n} = 1$, $0 \le s \le n-1$,

- $1 + q^{2m} + q^{4m} + \cdots + q^{2m(n-1)} = n\delta_{m,0(mod\, n)}$
- $1 + q^{2(m-s)} + q^{4(m-s)} + \cdots + q^{2(m-s)(n-1)} = n\delta_{m,s(mod\, n)}$

where

$$\delta_{m,s\,(mod\, n)} = \begin{cases} 1, & m = s\ (mod\, n); \\ 0, & m \neq s\ (mod\, n). \end{cases} \tag{26}$$

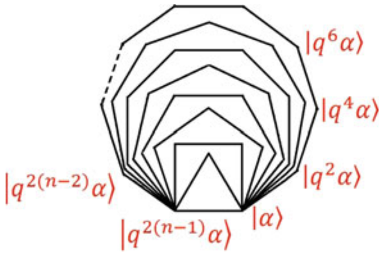

Fig. 4 The regular n-polygon

$$\begin{bmatrix} |0\rangle_\alpha \\ |1\rangle_\alpha \\ |2\rangle_\alpha \\ |3\rangle_\alpha \\ \vdots \\ |n-1\rangle_\alpha \end{bmatrix} = \mathbf{NQ} \begin{bmatrix} |\alpha\rangle \\ |q^2\alpha\rangle \\ |q^4\alpha\rangle \\ |q^6\alpha\rangle \\ \vdots \\ |q^{2(n-1)}\alpha\rangle \end{bmatrix} \tag{27}$$

Fig. 5 General structure of kaleidoscope states

5.1 Quantum Fourier Transformation

Our construction (Fig. 5) shows that orthogonal kaleidoscope of coherent states can by described by the Quantum Fourier transform

$$\begin{bmatrix} |\widetilde{0}\rangle_\alpha \\ |\widetilde{1}\rangle_\alpha \\ |\widetilde{2}\rangle_\alpha \\ |\widetilde{3}\rangle_\alpha \\ \vdots \\ |\widetilde{n-1}\rangle_\alpha \end{bmatrix} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w & w^2 & \dots & w^{n-1} \\ 1 & w^2 & w^4 & \dots & w^{2(n-1)} \\ 1 & w^3 & w^6 & \dots & w^{3(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{(n-1)} & w^{2(n-1)} & \dots & w^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} |\alpha\rangle \\ |q^2\alpha\rangle \\ |q^4\alpha\rangle \\ |q^6\alpha\rangle \\ \vdots \\ |q^{2(n-1)}\alpha\rangle \end{bmatrix}, \tag{28}$$

where $w = e^{\frac{-2\pi i}{n}} = \bar{q}^2$ is the nth root of unity, so that corresponding transformation matrix, the Vandermonde matrix as generalized Hadamard gate,

$$|\widetilde{k}\rangle_\alpha = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} w^{jk} |q^{2j}\alpha\rangle \quad 0 \le k \le n-1, \tag{29}$$

is the unitary gate $QQ^\dagger = Q^\dagger Q = I$. For orthonormal states we define normalization matrix,

$$\mathbf{N} = \frac{e^{\frac{|\alpha|^2}{2}}}{\sqrt{n}} \,\mathrm{diag}\left({}_0e^{|\alpha|^2}, {}_1 e^{|\alpha|^2}, {}_2 e^{|\alpha|^2}, \ldots, {}_{n-1} e^{|\alpha|^2}\right)^{-1/2} (mod\, n)$$

in terms of $(mod\ n)$ exponential functions:

$$f_s(|\alpha|^2) = {}_s e^{|\alpha|^2}(mod\, n) \equiv \sum_{k=0}^{\infty} \frac{(|\alpha|^2)^{nk+s}}{(nk+s)!}\ , \quad 0 \le s \le n-1\,. \tag{30}$$

These functions represent superposition of standard exponentials

$$ {}_s e^{|\alpha|^2}(mod\, n) = \frac{1}{n}\sum_{k=0}^{n-1} \overline{q}^{2sk} e^{q^{2k}|\alpha|^2}\ , \quad 0 \le s \le n-1\,, \tag{31}$$

related to each other by derivatives

$$\frac{\partial}{\partial|\alpha|^2}\left[{}_s e^{|\alpha|^2}\right] = {}_{s-1}e^{|\alpha|^2}\ , \quad \frac{\partial}{\partial|\alpha|^2}\left[{}_0 e^{|\alpha|^2}\right] = {}_{n-1} e^{|\alpha|^2}\,.$$

According to this, function f_s defined in (30) is a solution of ordinary differential equation of degree n

$$f_s^{(n)} = f_s\,, \quad \text{where } 0 \le s \le n-1, \tag{32}$$

with proper initial values: $f_s^{(s)}(0) = 1$ and

$$f_s(0) = f_s'(0) = \cdots = f_s^{(s-1)}(0) = f_s^{(s+1)}(0) = \cdots = f_s^{(n-1)}(0) = 0.$$

As we have learned recently, these functions as the generalized hyperbolic functions were introduced also in [13]. By using these functions one can derive compact expression for the kaleidoscope states as

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^\dagger}|0\rangle \ \Rightarrow\ |s\rangle_\alpha = \frac{{}_s e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_s e^{|\alpha|^2}}}|0\rangle \quad (mod\, n), \ \ 0 \le s \le n-1. \tag{33}$$

5.2 *Number of Photons in Kaleidoscope of Quantum Coherent States*

Cyclic permutation of kaleidoscope states, generated by annihilation operator $\hat{a}$, allows us to calculate average number of photons in these states

$$\hat{a}|s\rangle_\alpha = \alpha \frac{N_s}{N_{s-1}} |s-1\rangle_\alpha \quad \Rightarrow \tag{34}$$

$$_\alpha\langle s|\widehat{N}|s\rangle_\alpha = |\alpha|^2 \left[\frac{_{s-1}e^{|\alpha|^2}}{_{s}e^{|\alpha|^2}} \right] , 1 < s \leq n-1 \,, \tag{35}$$

$$\hat{a}|0\rangle_\alpha = \alpha \frac{N_0}{N_{n-1}} |n-1\rangle_\alpha \quad \Rightarrow \tag{36}$$

$$_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = |\alpha|^2 \left[\frac{_{n-1}e^{|\alpha|^2}}{_{0}e^{|\alpha|^2}} \right] . \tag{37}$$

Asymptotically they approach the coherent states average number value

$$\lim_{|\alpha|\to\infty} {_\alpha\langle s|\widehat{N}|s\rangle_\alpha} \approx |\alpha|^2 = \langle\alpha|\widehat{N}|\alpha\rangle$$

while for small occupation numbers give integers

$$\lim_{|\alpha|\to 0} {_\alpha\langle s|\widehat{N}|s\rangle_\alpha} = s \,.$$

6 Quantum Algebra

Our kaleidoscope coherent states (33) are eigenstates of operator $q^{2\hat{N}}$:

$$q^{2\hat{N}}|k\rangle_\alpha = q^{2k}|k\rangle_\alpha, \quad k = 0, 1 \ldots, n-1.$$

In the Fock space this operator is an infinite matrix of the form

$$\Sigma_3 \equiv q^{2\hat{N}} = I \otimes \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & q^2 & 0 & \ldots & 0 \\ 0 & 0 & q^4 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & q^{2(n-1)} \end{pmatrix}, \quad \Sigma_1 = I \otimes \begin{pmatrix} 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} . \tag{38}$$

Here the $n \times n$ matrices are called the Sylvester clock and shift matrices correspondingly. They are q-commutative

$$\Sigma_1 \Sigma_3 = q^2 \Sigma_3 \Sigma_1,$$

satisfy relations

$$\Sigma_1^n = I, \quad \Sigma_3^n = I$$

and are connected by the unitary transformation:

$$\Sigma_1 = (I \otimes Q) q^{2\hat{N}} (I \otimes Q^+).$$

Hermann Weyl in book [14] proposed them for description of quantum mechanics of finite dimensional systems. By dilatation operator $q^{2\hat{N}}$ we define q^2-number operator

$$[\hat{N}]_{q^2} = \frac{q^{2\hat{N}} - 1}{q^2 - 1}$$

for non-symmetrical q-calculus, and

$$[\hat{N}]_{\tilde{q}^2} = \frac{q^{2\hat{N}} - q^{-2\hat{N}}}{q^2 - q^{-2}}$$

for the symmetrical one. In our kaleidoscope basis, these number operators are diagonal and given by q-numbers:

$$[\hat{N}]_{q^2} = diag([0]_{q^2}, [1]_{q^2}, \ldots, [n-1]_{q^2}),$$

with $[n]_{q^2} = \frac{q^{2n}-1}{q^2-1}$ for non-symmetric case, and

$$[\hat{N}]_{\tilde{q}^2} = diag([0]_{\tilde{q}^2}, [1]_{\tilde{q}^2}, \ldots, [n-1]_{\tilde{q}^2}),$$

with $[n]_{\tilde{q}^2} = \frac{q^{2n}-q^{-2n}}{q^2-q^{-2}}$ for the symmetrical one.

For symmetric case the q-number operator is Hermitian and can be factorized as

$$[\hat{N}] = \hat{B}^+ \hat{B}, \quad [\hat{N}+1] = \hat{B}\hat{B}^+,$$

where

$$\hat{B} = \hat{a}\sqrt{\frac{[N]_{\tilde{q}^2}}{N}}.$$

Explicitly in matrix form it is

$$\hat{B} = I \otimes \begin{pmatrix} 0 & \sqrt{[1]} & 0 & \ldots & 0 \\ 0 & 0 & \sqrt{[2]} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad \hat{B}^+ = I \otimes \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ \sqrt{[1]} & 0 & 0 & \ldots & 0 \\ 0 & \sqrt{[2]} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \tag{39}$$

and $\hat{B}^n = 0$, $(\hat{B}^+)^n = 0$. In non-symmetric case the number operator is not Hermitian.

6.1 Symmetric Case

For symmetric case we have the quantum algebra

$$\hat{B}\hat{B}^{+} - q^{2}\hat{B}^{+}\hat{B} = q^{-2\hat{N}}, \tag{40}$$

$$\hat{B}\hat{B}^{+} - q^{-2}\hat{B}^{+}\hat{B} = q^{2\hat{N}}, \tag{41}$$

and quantum q^2-oscillator with Hamiltonian

$$\hat{H} = \frac{\hbar\omega}{2}\left([\hat{N}]_{\tilde{q}^2} + [\hat{N} + I]_{\tilde{q}^2}\right).$$

In the kaleidoscope states as the eigenstates, the spectrum of this Hamiltonian is

$$E_k = \frac{\hbar\omega}{2}\frac{\sin\frac{2\pi}{n}(k+\frac{1}{2})}{\sin\frac{\pi}{n}}. \tag{42}$$

The same spectrum was obtained in [15] for description of physical system of two anyons. Appearance of quantum algebraic structure in two different physical systems, as optical coherent states and the anyons problem is instructive.

6.2 Non-symmetric Case

In this case the quantum algebra of operators is q^2-deformed

$$\hat{B}\hat{B}^{+} - q^{2}\hat{B}^{+}\hat{B} = I, \tag{43}$$

$$\hat{B}\hat{B}^{+} - \hat{B}^{+}\hat{B} = q^{2\hat{N}}, \tag{44}$$

with periodic (mod n) ($[k+n]_{q^2} = [k]_{q^2}$) q^2-numbers

$$[k]_{q^2} = e^{i\frac{\pi}{n}(k-1)}\frac{\sin\frac{\pi}{n}k}{\sin\frac{\pi}{n}}. \tag{45}$$

7 Conclusions

Kaleidoscope of coherent states considered in present paper can be realized by proper phase superposition of coherent states of light (the Gaussian states) and it can provide a unit of quantum information corresponding not only to diadic, but also to an arbitrary number base n. These states furnish the representation of quantum symmetry related with quantum q-oscillator.

As a generalization of the Schrödinger cat states, from our kaleidoscope states one can construct multi qudits entangled quantum states. This work is in progress.

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References

1. Pashaev, O.K.: J. Phys. Conf. Ser. **482**, 012033 (2014)
2. Pashaev, O.K.: J. Phys. Conf. Ser. **766**, 012015 (2016)
3. Pashaev, O.K.: Phys. Scr. **90**, 074010 (2015)
4. Vitiello, G.: Phys. Lett. A **376**, 2527–2532
5. Bialynicka-Birula, Z.: Phys. Rev. **173**, 172 (1968)
6. Stoler, D.: Phys. Rev. D **4**, 2309 (1971)
7. Spiridonov, V.: Phys. Rev. A **52**, 1909 (1995)
8. Dodonov, V.V., Malkin, I.A., Man'ko, V.I.: Physica **72**, 597 (1974)
9. Schrödinger, E.: Naturwissenschaften **14**, 664 (1926)
10. Wolfgang, P.S.: Quantum Optics in Phase Space. Wiley, London (2001)
11. Ralph, T.C., Gilchrist, A., Milburn ,G.J., Munro ,W.J., Glancy, S.: Phys. Rev. A **68**(4) (2003)
12. Peremolov, A.M.: Generalized Coherent States and Their Applications. Text and Monographs in Physics. Springer, Berlin (1986)
13. Ungar, A.: Am. Math. Mon. **98**, 688 (1982)
14. Weyl, H.: The Theory of Groups and Quantum Mechanics. Dover, New York (1931)
15. Floratos, E.G., Tomaras, T.N.: Phys. Lett. B **251**, 163–166 (1990)