DE GRUYTER Georgian Math. J. 2018; aop

Research Article

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A quantitative Balian–Low theorem for higher dimensions

https://doi.org/10.1515/gmj-2018-0046
Received June 27, 2016; revised February 22, 2017; accepted March 16, 2017

Abstract: We extend the quantitative Balian–Low theorem of Nitzan and Olsen to higher dimensions. We use Zak transform methods and dimension reduction. The characterization of the Gabor–Riesz bases by the Zak transform allows us to reduce the problem to the quasiperiodicity and the boundedness from below of the Zak transforms of the Gabor–Riesz basis generators, two properties for which dimension reduction is possible.

Keywords: Uncertainty principle, Balian–Low theorem

MSC 2010: Primary 42C15; secondary 42A38

1 Introduction

Uncertainty principles are statements that limit the simultaneous concentration of functions and their Fourier transforms. In the last two decades significant attention has been paid to quantifying the maximum concentration that can be achieved. In the same vein, Nazarov proved, in his seminal work [9], that for a function $g \in L^2(\mathbb{R})$ and two sets of finite measure $\mathcal{R}$, $\mathcal{L}$, we have

$$\int_{\mathbb{R}\setminus\mathcal{R}} |g(x)|^2\,dx + \int_{\mathbb{R}\setminus\mathcal{L}} |\widehat{g}(\xi)|^2\,d\xi \geq e^{-C|\mathcal{R}|\,|\mathcal{L}|}\|g\|^2_{L^2(\mathbb{R})}$$

for an absolute constant $C > 0$. This result quantifies the Heisenberg uncertainty principle. Similarly, it is possible to quantify the Balian–Low theorem [1, 3, 8], which states that if the Gabor system

$$G(g) := \{e^{2\pi inx}g(x-m)\}_{(m,n)\in\mathbb{Z}^2},$$

generated by the function g, is a Riesz basis, then we must have

$$\int_{\mathbb{R}} |g(x)|^2x^2\,dx = \infty \quad \text{or} \quad \int_{\mathbb{R}} |\widehat{g}(\xi)|^2\xi^2\,d\xi = \infty.$$

Nitzan and Olsen [10] quantified this theorem by proving for g as above and two real numbers R, L, with $R, L \geq 1$, that

$$\int_{|x|\geq R} |g(x)|^2\,dx + \int_{|\xi|\geq L} |\widehat{g}(\xi)|^2\,d\xi \geq \frac{C}{RL},$$

where C depends only on the Riesz basis bounds for the function g.

As is seen, all these results are one-dimensional in nature. Although analogous results for higher dimensions are conjectured, due to possibly much more complicated geometry of an arbitrary set in higher dimensions, progress has been limited. The most basic results for higher dimensions are the extensions of

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the Balian–Low theorem in its qualitative form, and these were carried out in [6]. One quantitative result in this direction is that of Jaming [7] stating that for $g \in L^2(\mathbb{R}^d)$ and two sets of finite measure $\mathcal{R}$, $\mathcal{L}$, we have

$$\int_{\mathbb{R}^d \setminus \mathcal{R}} |g(x)|^2 \, dx + \int_{\mathbb{R}^d \setminus \mathcal{L}} |\widehat{g}(\xi)|^2 \, d\xi \geq e^{-CD} \|g\|^2_{L^2(\mathbb{R}^d)},$$

where $D = \min\{|\mathcal{R}|\,|\mathcal{L}|, \alpha(\mathcal{R})|\mathcal{L}|^{1/d}, \alpha(\mathcal{L})|\mathcal{R}|^{1/d}\}$, with α denoting the mean width of a set S given by

$$\alpha(S) := \int_{\mathrm{SO}(d)} P_\rho(S) \, d\nu_d(\rho),$$

with $d\nu_d$ being the normalized Haar measure on the group of rotations $\mathrm{SO}(d)$, and $P_\rho(S)$ being the measure of the projection of S on the line obtained by applying the rotation ρ to the line spanned by the vector $(1, 0, \dots, 0)$. It is conjectured that D can be replaced by $|\mathcal{R}|^{1/d}|\mathcal{L}|^{1/d}$. Thus, this result is essentially optimal if one of the sets $\mathcal{R}$, $\mathcal{L}$ is round, but it is far from optimal even when both sets are simple rectangles. Our aim in this work is to extend the work of Nitzan and Olsen [10] to higher dimensions and to investigate localization on rectangles.

Theorem 1.1. *Let $g \in L^2(\mathbb{R}^d)$ be such that the Gabor system generated by g*

$$G(g) := \{e^{2\pi i n x} g(x - m)\}_{(m,n) \in \mathbb{Z}^{2d}}$$

is a Riesz basis. Let $R_i, L_i \geq 1$ be real numbers for each $1 \leq i \leq d$. Let $\mathcal{R}$ and $\mathcal{L}$ be the d-dimensional rectangles $\mathcal{R} := (-R_1, R_1) \times \cdots \times (-R_n, R_n)$ and $\mathcal{L} := (-L_1, L_1) \times \cdots \times (-L_n, L_n)$. Then for a constant C depending only on the Riesz basis bounds of g, we have

$$\int_{\mathbb{R}^n \setminus \mathcal{R}} |g(x)|^2 \, dx + \int_{\mathbb{R}^n \setminus \mathcal{L}} |\widehat{g}(\xi)|^2 \, d\xi \geq \frac{C}{R_i L_i}$$

for any $1 \leq i \leq d$. The theorem is sharp in the sense that the term $C/R_i L_i$ cannot be replaced by $C \log R_i L_i / R_i L_i$.

We observe that the theorem allows us to choose the index i that makes the right-hand side the largest. Since we must have $R_i L_i \leq |\mathcal{R}|^{1/d}|\mathcal{L}|^{1/d}$ at least for some values of i, the term $C/R_i L_i$ can be replaced by $C/|\mathcal{R}|^{1/d}|\mathcal{L}|^{1/d}$ in the theorem.

The rest of the paper is organized as follows. In Section 2, we introduce some standard definitions and results needed for our discussion. Then, in Section 3, we give certain properties of quasiperiodic functions that Nitzan and Olsen uncovered in their work. In Section 4, we use these properties to prove our estimate, and then discuss certain extensions of it. Also, using the function introduced in [2], we construct a function to show that our estimate is sharp.

2 Preliminaries

In this section we will introduce the concepts that will be used throughout the rest of the paper. Further information on these concepts can be found in [5]. We start with Riesz bases. For a separable Hilbert space H, a system $\{v_n\}$ in H is a Riesz basis if it is complete in H and

$$A \sum |a_n|^2 \leq \Big\| \sum a_n v_n \Big\|^2 \leq B \sum |a_n|^2$$

for any sequence $\{a_n\} \in \ell^2$ and two positive constants A and B. The largest such A and smallest such B are called the Riesz basis bounds. An equivalent definition of a Riesz basis is that it is the image of an orthonormal basis under a bounded and invertible linear operator.

We now introduce the Zak transform, which is an extremely useful tool in the study of Gabor systems. Let $g \in L^1(\mathbb{R}^d)$. The Zak transform of g is defined for $(x, y) \in \mathbb{R}^{2d}$ as

$$Zg(x, y) = \sum_{k \in \mathbb{Z}^d} g(x - k) e^{2\pi i k \cdot y}.$$

It immediately follows from this definition and the Plancherel theorem that the Zak transform induces a unitary operator from $L^2(\mathbb{R}^d)$ to $L^2([0,1]^{2d})$. Thus, for $g \in L^2(\mathbb{R}^d)$, the Zak transform Zg takes complex values for almost all $(x,y) \in \mathbb{R}^{2d}$. We let $e_1, e_2, \dots, e_d$ be the canonical basis of $\mathbb{R}^d$. For $g \in L^2(\mathbb{R}^d)$ and $1 \le i \le d$, the function Zg satisfies

$$Zg(x, y + e_i) = Zg(x,y) \quad \text{and} \quad Zg(x + e_i, y) = e^{2\pi i y_i} Zg(x,y). \tag{2.1}$$

We call this property quasiperiodicity. The Zak transform relates to the Fourier transform as follows:

$$Z\hat{g}(x,y) = e^{2\pi i x\cdot y} Zg(-y, x). \tag{2.2}$$

Furthermore, for any Schwarz class function ϕ, we have

$$Z(g * \phi) = Zg *_1 \phi, \tag{2.3}$$

where for $Zg = Zg(x,y)$, the notation $*_1$ means convolution in the first variable x. With the Zak transform we can easily characterize the Gabor systems that are Riesz bases. A Gabor system $G(g)$ is a Riesz basis if and only if

$$A \le |Zg(x,y)|^2 \le B, \tag{2.4}$$

where A, B are Riesz basis bounds. This fact makes the Zak transform a fundamental tool in the study of the Gabor systems. For the proofs and an extensive discussion of formulas (2.1)–(2.4), see [5, Sections 8.2–8.3].

3 Properties of quasiperiodic functions

Nitzan and Olsen deduced their result by quantifying the discontinuous behavior of arguments of quasiperiodic functions. It is well known that a branch of the argument of a quasiperiodic function on $\mathbb{R}^2$ cannot be continuous. Nitzan and Olsen went further and quantified this fact with the following lemma. For the sake of completeness, we provide a proof.

Lemma 3.1. *Let G be a complex valued quasiperiodic function on $\mathbb{R}^2$, and let H be a branch of its argument, that is,*

$$G(x,y) = |G(x,y)| e^{2\pi i H(x,y)}.$$

Let $k, n \ge 8$ be two integers, and let $(x,y) \in [0,1/k) \times [0,1/n)$. Then there exist two integers $1 \le i < k$, $1 \le j < n$ such that at least one of the following is true for every $m \in \mathbb{Z}$:

$$\big|H(x + (i+1)/k, y + j/n) - H(x + i/k, y + j/n) - m\big| > 1/8,$$
$$\big|H(x + i/k, y + (j+1)/n) - H(x + i/k, y + j/n) - m\big| > 1/8.$$

Proof. We assume to the contrary that there is a branch of the argument H for which the claim does not hold for a point $(x,y) \in [0,1/k) \times [0,1/n)$, with $k, n \ge 8$. We let for i, j the integers $h_{i,j}$ denote $H(x + i/k, y + j/n)$. We observe that if this H presents a counterexample to the lemma, then so do infinitely many others, because by adding integers to H at points $(x + i/k, y + j/n)$, we obtain other counterexamples. Since H can be chosen from an infinite collection of counterexamples, we can, to some extent, dictate the values $h_{i,j}$. Below we will do this to obtain a contradiction with quasiperiodicity.

We fix $h_{0,0}$ and choose $h_{i,0}$, $1 \le i \le k$, so as to satisfy $|h_{i,0} - h_{i-1,0}| \le 1/8$. Thus, given $h_{0,0}$ fixed, we choose $h_{i,0}$, $1 < i < k$, one by one, starting with $h_{1,0}$, so that their distance from the choice before is not more than $1/8$. Now we have $h_{i,0}$, $0 \le i \le k$, all fixed. Using $h_{i,0}$, $0 \le i < k$, we choose $h_{i,j}$, $1 \le j \le n$, so as to satisfy $|h_{i,j} - h_{i,j-1}| \le 1/8$. Finally, we choose $h_{k,j}$ for $1 < j \le n$. By quasiperiodicity we must have $h_{k,0} = h_{0,0} + y + l$ for some integer l. We choose $h_{k,j} = h_{0,j} + y + j/n + l$ for $1 < j \le n$. Thus, we have $|h_{k,j} - h_{k,j-1}| \le 1/4$ for $1 \le j \le n$.

We claim that with these choices we also have $|h_{i,n} - h_{i-1,n}| \le 1/8$ for $1 \le i \le k$. This we will prove through iteration. We observe that since H is assumed to be a counterexample to the lemma, $|h_{i,1} - h_{i-1,1} - m_{i,1}| \le 1/8$

for an integer $m_{i,1}$ for each $1 \le i \le k$. But it is also clear, from the construction of H and the triangle inequality, that we have $|h_{i,1} - h_{i-1,1}| \le 1/2$, since for $i < k$,

$$|h_{i,1} - h_{i-1,1}| \le |h_{i,1} - h_{i,0}| + |h_{i,0} - h_{i-1,0}| + |h_{i-1,1} - h_{i-1,0}| \le 1/8 + 1/8 + 1/8,$$

whereas for $i = k$,

$$|h_{k,1} - h_{k-1,1}| \le |h_{k,1} - h_{k,0}| + |h_{k,0} - h_{k-1,0}| + |h_{k-1,1} - h_{k-1,0}| \le 1/4 + 1/8 + 1/8.$$

Therefore, $m_{i,1} = 0$ for each value of i. If we apply the same reasoning, we can obtain that for each $0 \le j < n$, we have $|h_{i,j} - h_{i-1,j}| \le 1/8$. By quasiperiodicity, $|h_{i,n} - h_{i-1,n} - m_{i,n}| \le 1/8$, for some integer $m_{i,n}$ for each $1 \le i \le k$. But we have just discovered that $|h_{i,n-1} - h_{i-1,n-1}| \le 1/8$ for each $1 \le i \le k$. This, together with the triangle inequality, and the construction of H establishes the claim.

We now obtain the contradiction promised by calculating the two sides of the obvious equality

$$(h_{k,n} - h_{k,0}) - (h_{0,n} - h_{0,0}) = (h_{k,n} - h_{0,n}) - (h_{k,0} - h_{0,0})$$

in two different ways. By the quasiperiodicity of H, for any $0 \le i \le k$, the difference $h_{i,n} - h_{i,0}$ must be an integer. But since we know that $|h_{i,n} - h_{i-1,n}| \le 1/8$ and $|h_{i,0} - h_{i-1,0}| \le 1/8$ for each $0 < i \le k$, the integers $h_{i,n} - h_{i,0}$ and $h_{i-1,n} - h_{i-1,0}$ must be the same. Thus, $h_{k,n} - h_{k,0}$ and $h_{0,n} - h_{0,0}$ must be the same, hence $(h_{k,n} - h_{k,0}) - (h_{0,n} - h_{0,0})$ must be zero. On the other hand, from our construction of H, we have $(h_{k,j} - h_{0,j}) - (h_{k,j-1} - h_{0,j-1}) = 1/n$ for each $1 \le j \le n$. Thus, it follows that $(h_{k,n} - h_{0,n}) - (h_{k,0} - h_{0,0}) = 1$, a contradiction. □

The lemma we have just proved suggests that the set of points for which a branch of the argument of a quasiperiodic function changes very quickly must have a measure at least $k^{-1} \cdot n^{-1}$. The next lemma makes this rigorous.

Lemma 3.2. *Let $A > 0$ be a constant, and let G be a complex valued quasiperiodic function on $\mathbb{R}^2$ with $|G| \ge A$. Then for any two integers $k, n \ge 8$, we have a set $S \subseteq [0,1]^2$ of measure at least $k^{-1} \cdot n^{-1}$ such that for all $(x, y) \in S$, we have*

$$|G(x + k^{-1}, y) - G(x, y)| \ge A/3 \quad \text{or} \quad |G(x, y + n^{-1}) - G(x, y)| \ge A/3.$$

Proof. Let H be a measurable branch of the argument of G. We will apply Lemma 3.1 to this H. Let $1 \le i < k$, $1 \le j < n$, and let m be an integer. Let $S'_{i,j,m,1}$ be the set of all $(x, y) \in [0, k^{-1}) \times [0, n^{-1})$ for which the first inequality of the previous lemma holds. We similarly define $S'_{i,j,m,2}$. Clearly, these sets are measurable. From these sets, we define

$$S'_{i,j,1} := \bigcap_{m \in \mathbb{Z}} S_{i,j,m,1}, \quad S'_{i,j,2} := \bigcap_{m \in \mathbb{Z}} S_{i,j,m,2},$$

and we let $S'_{i,j} := S'_{i,j,1} \cup S'_{i,j,2}$. Then from the previous lemma, we have the equality

$$[0, k^{-1}) \times [0, n^{-1}) = \bigcup_{i,j} S'_{i,j}.$$

Thus, the sum of measures of the sets on the right-hand side is at least $k^{-1} \cdot n^{-1}$. For each i, j, we define $S_{i,j}$ to be the translate of $S'_{i,j}$ by (ik^{-1}, jn^{-1}). Thus, the sets $S_{i,j}$ are disjoint, and for fixed i, j the set $S_{i,j}$ has the same measure as $S'_{i,j}$. If we define S to be the union of all $S_{i,j}$, its measure is at least $k^{-1} \cdot n^{-1}$, and for an element $(x, y) \in S$, one of the following is true for all integers m:

$$|H(x + k^{-1}, y) - H(x, y) - m| > 1/8, \quad |H(x, y + n^{-1}) - H(x, y) - m| > 1/8. \tag{3.1}$$

Now suppose the first inequality is true for all m. We know that $|G(x, y)|, |G(x + k^{-1}, y)| \ge A$, but we do not know their exact relation to each other, and this prevents us from immediately concluding the proof. To circumvent this we consider two cases. If $||G(x + k^{-1}, y)| - |G(x, y)|| \ge A/3$, then we have the crude estimate

$$|G(x + k^{-1}, y) - G(x, y)| \ge ||G(x + k^{-1}, y)| - |G(x, y)|| \ge A/3.$$

If, on the other hand, $||G(x+k^{-1},y)|-|G(x,y)|| < A/3$, then by adding and subtracting the same term, we can write $|G(x+k^{-1},y)-G(x,y)|$ as

$$\big|[|G(x+k^{-1},y)|-|G(x,y)|]e^{2\pi i H(x+k^{-1},y)}+|G(x,y)|[e^{2\pi i H(x+k^{-1},y)}-e^{2\pi i H(x,y)}]\big|.$$

This, by the triangle inequality, cannot be less than

$$|G(x,y)||e^{2\pi i H(x+k^{-1},y)}-e^{2\pi i H(x,y)}|-||G(x+k^{-1},y)|-|G(x,y)||.$$

We observe that

$$|e^{2\pi i H(x+k^{-1},y)}-e^{2\pi i H(x,y)}|=\big|e^{2\pi i H(x,y)}[e^{2\pi i[H(x+k^{-1},y)-H(x,y)]}-1]\big|=|e^{2\pi i[H(x+k^{-1},y)-H(x,y)]}-1|.$$

We know that the distance of $H(x+k^{-1},y)-H(x,y)$ to any integer is more than $1/8$, which means that the last term is, as can be seen by considering the location of these numbers on the unit circle, more than $2/3$. Thus returning with this information back to our estimate we have

$$|G(x,y)||e^{2\pi i H(x+k^{-1},y)}-e^{2\pi i H(x,y)}|-||G(x+k^{-1},y)|-|G(x,y)|| \ge A/3.$$

Thus, in any case, if the first inequality in (3.1) holds for all integers, we have $|G(x+k^{-1},y)-G(x,y)| \ge A/3$. Similarly, if the second inequality in (3.1) holds for all integers, we have $|G(x,y+n^{-1})-G(x,y)| \ge A/3$, and this concludes the proof. □

4 Proof of the main result

We start with a lemma that will be the fundamental tool in proving our theorem. The last lemma tells us that given a quasiperiodic function, there is a set of certain size near which the function changes rapidly. Therefore, on this set the function must also differ from its average over balls of large enough size. The next lemma makes rigorous this idea, using convolutions with Schwartz class functions instead of averages over balls.

Lemma 4.1. *Let $A, B > 0$ and $1 \le i \le d$. Given two Schwartz functions ϕ, ψ on $\mathbb{R}^d$ and any $g \in L^2(\mathbb{R}^d)$, with $A \le |Zg| \le B$ a.e., there exists a set $S_i \subseteq [0,1]^{2d}$ of measure at least $A^2/4000B^2(1+\|\phi_i\|_1)(1+\|\psi_i\|_1)$, with ϕ_i, ψ_i denoting the ith partial derivatives of ϕ, ψ, such that for all $(x,y) \in S_i$, we have*

$$|Zg(x,y)-Z(g*\phi)(x,y)| \ge A/12 \quad \text{or} \quad |Z\hat{g}(x,y)-Z(\hat{g}*\psi)(x,y)| \ge A/12.$$

Proof. For any $k_i > 0$, and any $1 \le i \le d$ from property (2.3) of the Zak transform, we can write

$$I=\big|Z(g*\phi)(x+k_i^{-1}\cdot e_i,y)-Z(g*\phi)(x,y)\big|=\big|Zg *_1 \phi(x+k_i^{-1}\cdot e_i,y)-Zg *_1 \phi(x,y)\big|.$$

Then we have

$$I \le \int_{\mathbb{R}^d} |Zg(u,y)|\big|\phi(x-u+k_i^{-1}\cdot e_i)-\phi(x-u)\big|\,du \le B\cdot \int_{\mathbb{R}^d} \big|\phi(x-u+k_i^{-1}\cdot e_i)-\phi(x-u)\big|\,du.$$

Let $u_i \in \mathbb{R}$ denote the ith coordinate of u and let $\overline{u} \in \mathbb{R}^{d-1}$ be obtained from u by removing u_i. Since ϕ is a Schwartz function, we can write

$$\begin{aligned} I &\le B\cdot \int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}} \big|\phi(x-u+k_i^{-1}\cdot e_i)-\phi(x-u)\big|\,du_i\,d\overline{u} \\ &= B\cdot \int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}\int_0^{k_i^{-1}} |\phi_i(x-u+v_i\cdot e_i)|\,dv_i\,du_i\,d\overline{u} \\ &= B\cdot \int_{\mathbb{R}^{d-1}}\int_0^{k_i^{-1}}\int_{\mathbb{R}} |\phi_i(x-u+v_i\cdot e_i)|\,du_i\,dv_i\,d\overline{u}. \end{aligned}$$

Observing that the inner integral is independent of v_i, we can write

$$B \cdot \int_{\mathbb{R}^{d-1}} \int_0^{k_i^{-1}} \int_{\mathbb{R}} |\phi_i(x-u)|\, du_i\, dv_i\, d\overline{u} = B \cdot k_i{}^{-1} \cdot \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} |\phi_i(x-u)|\, du_i\, d\overline{u},$$

and obviously this last term is $B \cdot k_i{}^{-1} \cdot \|\phi_i\|_1$. Since we have property (2.2), we can apply the same process to obtain, for any $n_i > 0$ and $(x, y) \in \mathbb{R}^{2d}$,

$$\big|Z(\widehat{g} * \psi)(y + n_i{}^{-1} \cdot e_i, -x) - Z(\widehat{g} * \psi)(y, -x)\big| \le B \cdot n_i{}^{-1} \cdot \|\psi_i\|_1. \tag{4.1}$$

If we choose $k_i, n_i \ge 8$ to be the smallest integers that satisfy

$$k_i \ge \frac{8B}{A}(1 + \|\phi_i\|_1) \quad \text{and} \quad n_i \ge \frac{24\pi B}{A}(1 + \|\psi_i\|_1),$$

we have

$$\begin{aligned} &\big|Z(g * \phi)(x + k_i^{-1} \cdot e_i, y) - Z(g * \phi)(x, y)\big| \le A/8, \\ &\big|Z(\widehat{g} * \psi)(y + n_i{}^{-1} \cdot e_i, -x) - Z(\widehat{g} * \psi)(y, -x)\big| \le A/72. \end{aligned}$$

We will use the properties of quasiperiodic functions derived in the last section to obtain another estimate, which, combined with the last two, will suffice to complete the proof. To this end, we introduce a slight modification of $Zg(x, y)$ as follows. Let $G(x, y) := Zg(x, y)$ when $A \le |Zg(x, y)| \le B$, and when this is not the case, let $G(x, y) = B$ for $(x, y) \in [0, 1)^{2d}$, and extend it so that it will be quasiperiodic. This G is a measurable, complex-valued, quasiperiodic function, and $A \le |G(x, y)| \le B$ everywhere. Now we define $G_{\overline{x},\overline{y}}(x_i, y_i) = G(x, y)$, with $(\overline{x}, \overline{y})$ denoting an element of $\mathbb{R}^{2d-2}$ obtained by removing x_i, y_i from $(x, y) \in \mathbb{R}^{2d}$. To this $G_{\overline{x},\overline{y}}(x_i, y_i)$ we wish to apply Lemma 3.2. We see that for any $(\overline{x}, \overline{y})$, by definition, it is complex valued, quasiperiodic, and satisfies $A \le |G_{\overline{x},\overline{y}}(x, y)| \le B$ for all (x_i, y_i). Also, by applying the Fubini–Tonelli theorem for complete measures (see [4, Theorem 2.39], or [11, Theorem 8.12]) to $G\chi_{[0,1)^{2d}}$, we see that for almost all $(\overline{x}, \overline{y}) \in \mathbb{R}^{2d-2}$, the function $G_{\overline{x},\overline{y}}\chi_{[0,1)^2}$ is measurable, and hence, by quasiperiodicity, for almost all $(\overline{x}, \overline{y}) \in \mathbb{R}^{2d-2}$, the function $G_{\overline{x},\overline{y}}(x, y)$ is measurable. Thus, we can apply Lemma 3.2 to this function for almost all $(\overline{x}, \overline{y}) \in [0, 1)^{2d-2}$. Let $S_{\overline{x},\overline{y},i}$ be the set described in Lemma 3.2 for such a point $(\overline{x}, \overline{y})$ with k_i, n_i as chosen above. Then for $(x_i, y_i) \in S_{\overline{x},\overline{y},i}$, we have

$$\begin{aligned} |G_{\overline{x},\overline{y}}(x_i + k_i^{-1}, y_i) - G_{\overline{x},\overline{y}}(x_i, y_i)| \ge A/3 \quad &\text{or} \quad |G_{\overline{x},\overline{y}}(x_i, y_i + n_i^{-1}) - G_{\overline{x},\overline{y}}(x_i, y_i)| \ge A/3, \\ |G(x + k_i^{-1} \cdot e_i, y) - G(x, y)| \ge A/3 \quad &\text{or} \quad |G(x, y + n_i^{-1} \cdot e_i) - G(x, y)| \ge A/3. \end{aligned}$$

Then if $(x_i, y_i) \in S_{\overline{x},\overline{y},i}$, we have $(x, y) \in S_i''$. Hence, again by the Fubini–Tonelli theorem for complete measures, this set has measure at least $k_i^{-1} \cdot n_i^{-1}$. Since the set F of points for which $Zg \ne G$ has measure zero, if we remove from S_i'' the set F and its translations by $-k_i^{-1} \cdot e_i$ and, $-n_i^{-1} \cdot e_i$, the remainder has the same measure as S_i'', and for (x, y) in this remainder, which we denote by S_i', we have

$$|Zg(x + k_i^{-1} \cdot e_i, y) - Zg(x, y)| \ge A/3 \quad \text{or} \quad |Zg(x, y + n_i^{-1} \cdot e_i) - Zg(x, y)| \ge A/3.$$

We let $S_i' = U_i' \cup V_i'$, where for elements of U_i', the first of these inequalities holds, and for elements of V_i', the second one holds. Then for $(x, y) \in U_i'$, we have

$$\begin{aligned} &|Zg(x, y) - Z(g * \phi)(x, y)| + \big|Zg(x + k_i^{-1} \cdot e_i, y) - Z(g * \phi)(x + k_i^{-1} \cdot e_i, y)\big| \\ &\quad \ge \big|Zg(x + k_i^{-1} \cdot e_i, y) - Zg(x, y) - Z(g * \phi)(x + k_i^{-1} \cdot e_i, y) + Z(g * \phi)(x, y)\big| \\ &\quad \ge |Zg(x + k_i^{-1} \cdot e_i, y) - Zg(x, y)| - \big|Z(g * \phi)(x + k_i^{-1} \cdot e_i, y) - Z(g * \phi)(x, y)\big| \\ &\quad \ge A/6. \end{aligned}$$

Therefore, one of the following is certainly true for any element $(x, y) \in U_i'$:

$$\begin{aligned} &|Zg(x, y) - Z(g * \phi)(x, y)| \ge A/12, \\ &\big|Zg(x + k_i^{-1} \cdot e_i, y) - Z(g * \phi)(x + k_i^{-1} \cdot e_i, y)\big| \ge A/12. \end{aligned}$$

We thus have $U_i' = U_{i_1}' \cup U_{i_2}'$, with the element (x, y) belonging to the set U_{i_1}' if the first inequality holds, and to U_{i_2}' if the second holds, and to both if both inequalities hold. Obviously, at least one of these sets has a measure not less than half the measure of U_i'. Thus, if we consider the union of U_{i_1}' and the translate of U_{i_2}' by $(k_i^{-1} \cdot e_i, 0)$, then its measure is not less than half the measure of U_i'. But it may be that some of the elements of this union are not in $[0, 1]^{2d}$ due to the translation. Therefore, we define U_i to be the union of the set of all elements in $[0, 1]^{2d}$ and the set of all elements outside $[0, 1]^{2d}$ translated by $(-e_i, 0)$. This U_i then has at least a quarter of the measure of U_i', and any element of this set satisfies the first inequality of our lemma.

We now turn to $(x, y) \in V_i'$. We have equation (4.1). We also have

$$\begin{aligned}
|Z\widehat{g}(y + n_i^{-1} \cdot e_i, -x) - Z\widehat{g}(y, -x)| &\geq \big|e^{-2\pi i(x\cdot y + n_i^{-1}x_i)} Zg(x, y + n_i^{-1} \cdot e_i) - e^{-2\pi i x\cdot y} Zg(x, y)\big| \\
&\geq \big|e^{-2\pi i n_i^{-1}x_i} Zg(x, y + n_i^{-1} \cdot e_i) - Zg(x, y)\big| \\
&\geq \big|e^{-2\pi i n_i^{-1}x_i} [Zg(x, y + n_i^{-1} \cdot e_i) - Zg(x, y)] - Zg(x, y)[1 - e^{-2\pi i n_i^{-1}x_i}]\big| \\
&\geq |Zg(x, y + n_i^{-1} \cdot e_i) - Zg(x, y)| - |Zg(x, y)[1 - e^{-2\pi i n_i^{-1}x_i}]| \\
&\geq |Zg(x, y + n_i^{-1} \cdot e_i) - Zg(x, y)| - B \cdot |1 - e^{-2\pi i n_i^{-1}x_i}|.
\end{aligned}$$

We can easily estimate, using the unit circle, that $|1 - e^{-2\pi i n_i^{-1}x_i}| \leq A/12B$, thus the last term is not less than $A/4$. Combining this with (4.1), as we did in the case of elements of U_i', we have for $(x, y) \in V_i'$ that one of the following is certainly true:

$$\begin{aligned}
&|Z\widehat{g}(y, -x) - Z(\widehat{g} * \psi)(y, -x)| \geq A/12, \\
&\big|Z\widehat{g}(y + n_i^{-1} \cdot e_i, -x) - Z(\widehat{g} * \psi)(y + n_i^{-1} \cdot e_i, -x)\big| \geq A/12.
\end{aligned}$$

We let $V_i' = V_{i_1}' \cup V_{i_2}'$ as before, and obviously at least one of these sets has a measure not less than half the measure of V_i'. Thus, if we consider the union of V_{i_1}' and the translate of V_{i_2}' by $(0, n_i^{-1} \cdot e_i)$, then its measure is not less than half the measure of V_i'. But it may be that some of the elements of this set are not in $[0, 1]^{2d}$ due to the translation applied. We therefore take the union of elements in $[0, 1]^{2d}$ with $(0, -e_i)$ translates of those that are not in $[0, 1]^{2d}$, and if we set V_i to be the set of points $(y, -x)$ such that (x, y) is in this last union, its measure is not less than a quarter of that of V_i', it lies entirely in $[0, 1]^{2d}$ and any element of it satisfies the second inequality of our lemma. We finally define $S_i = U_i \cup V_i$ and easily observe that it satisfies all of the required properties. □

We will use what we have learned from the study of the Zak transform of Riesz basis generators to prove Theorem 1.1. Let g be a function as in the theorem. We pick a Schwarz class function ρ on $\mathbb{R}^d$ such that $\widehat{\rho}$ is radially symmetric, satisfies $|\widehat{\rho}| \leq 1$ everywhere, and

$$\widehat{\rho}(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq 1, \\ 0 & \text{if } |\xi| \geq 2. \end{cases}$$

We define two Schwarz functions ϕ, ψ by

$$\begin{aligned}
\phi(x_1, x_2, \ldots, x_d) &:= R_1 R_2 \cdots R_d \cdot \rho(R_1 x_1, R_2 x_2, \ldots, R_d x_d), \\
\psi(x_1, x_2, \ldots, x_d) &:= L_1 L_2 \cdots L_d \cdot \rho(L_1 x_1, L_2 x_2, \ldots, L_d x_d).
\end{aligned}$$

Therefore, we have $\|\phi_i\|_1 = R_i \|\rho_i\|_1$ and $\|\psi_i\|_1 = L_i \|\rho_i\|_1$. Since ρ is a fixed radial Schwarz function, $\|\rho_i\|_1$ is a fixed constant for every i, which we denote by Γ. We apply Lemma 4.1 to obtain, for any chosen $1 \leq i \leq d$, a set $S_i \subseteq [0, 1]^{2d}$ with measure at least $A/4000B(1 + R_i\Gamma)(1 + L_i\Gamma)$ such that all $(x, y) \in S_i$ satisfy

$$A/144 \leq |Z\widehat{g}(x, y) - Z(\widehat{g} * \phi)(x, y)|^2 + |Zg(x, y) - Z(g * \psi)(x, y)|^2.$$

Since we have assumed in the theorem $R_i, L_i \geq 1$ for every $1 \leq i \leq d$, we have $A/4000B(1 + R_i\Gamma)(1 + L_i\Gamma) \geq A/2^4 10^3 B\Gamma^2 R_i L_i$. Thus, if we integrate over S_i, then

$$\begin{aligned}
A^2/10^8 B\Gamma^2 R_i L_i &\leq \big\| |Z\widehat{g} - Z(\widehat{g} * \phi)|^2 + |Zg - Z(g * \psi)|^2 \big\|_{L^1([0,1]^{2d})} \\
&\leq \|Z\widehat{g} - Z(\widehat{g} * \phi)\|^2_{L^2([0,1]^{2d})} + \|Zg - Z(g * \psi)\|^2_{L^2([0,1]^{2d})} \\
&\leq \|Z[\widehat{g} - (\widehat{g} * \phi)]\|^2_{L^2([0,1]^{2d})} + \|Z[g - (g * \psi)]\|^2_{L^2([0,1]^{2d})}.
\end{aligned}$$

As the Zak transform is a unitary operator from $L^2([0,1]^{2d})$ to $L^2(\mathbb{R}^d)$, we have

$$A^2/10^8 B\Gamma^2 R_i L_i \le \|\widehat{g} - (\widehat{g} * \phi)\|^2_{L^2(\mathbb{R}^d)} + \|g - (g * \psi)\|^2_{L^2(\mathbb{R}^d)}.$$

We apply the Plancherel theorem and use the assumption that $\widehat{\rho}$ is radially symmetric to obtain

$$A^2/10^8 B\Gamma^2 R_i L_i \le \|g(1-\widehat{\phi})\|^2_{L^2(\mathbb{R}^d)} + \|\widehat{g}(1-\widehat{\psi})\|^2_{L^2(\mathbb{R}^d)} \le \int\limits_{\mathbb{R}^d\setminus\mathcal{R}} |g(x)|^2\,dx + \int\limits_{\mathbb{R}^d\setminus\mathcal{L}} |\widehat{g}(\xi)|^2\,d\xi,$$

which proves our theorem with the constant C in the theorem being not more than $A^2/10^8 B\Gamma^2$.

Our theorem can be extended without much effort in two different directions. The first is to take the rectangles $\mathcal{R}$, $\mathcal{L}$ directed along not the canonical basis but a different orthonormal basis. As long as we take both rectangles directed along the same orthonormal basis, our theorem generalizes easily by employing rotations. The second extension is to more general Gabor systems that are produced by simple scaling of the canonical lattice: for real numbers a, b, we define

$$G(g,a,b) := \{e^{2\pi i bnx} g(x - ma)\}_{(m,n)\in\mathbb{Z}^{2d}}.$$

It is possible for such a system to be a Riesz basis if $ab = 1$. Our result also holds for these more general systems, and this can be seen by employing appropriate dilations.

We now present the counterexample showing that our estimate is sharp. Nitzan and Olsen observed that a function $f \in L^2(\mathbb{R})$ constructed in [2] satisfies $|Zf| = 1$ on all of $\mathbb{R}^2$, and for $R, L \ge 1$,

$$\int\limits_{|x|\ge R} |f(x)|^2\,dx + \int\limits_{|\xi|\ge L} |\widehat{f}(\xi)|^2\,d\xi \le \frac{1}{R^2} + \frac{\log L}{L^2}.$$

Since the Zak transform is unitary, we have $\|f\|_{L^2(\mathbb{R})} = 1$, and hence from the Plancherel theorem, we further have $\|\widehat{f}\|_{L^2(\mathbb{R})} = 1$. This function is a counterexample showing that the result of Nitzan and Olsen, which is the case where $d = 1$ of our result, is sharp. We will construct a counterexample from this function to show that our result cannot be improved in any dimension d.

Let x denote $(x_1, x_2, \dots, x_d)$ and define on $\mathbb{R}^d$ the function $g \in L^2(\mathbb{R}^d)$ by $g(x) := f(x_1)f(x_2)\cdots f(x_d)$. Since $\|g\|_{L^2(\mathbb{R}^d)} = \|\widehat{g}\|_{L^2(\mathbb{R}^d)} = 1$, we then have the same relation between the Fourier transforms of f and g: $\widehat{g}(\xi) = \widehat{f}(\xi_1)\widehat{f}(\xi_2)\cdots\widehat{f}(\xi_d)$, and between the Zak transforms, we have $Zg(x,y) = Zf(x_1,y_1)Zf(x_2,y_2)\cdots Z(x_d,y_d)$. Therefore, $|Zg| = 1$ everywhere on $\mathbb{R}^{2d}$, and this means that g generates a Gabor frame and satisfies the hypothesis of our theorem. On the other hand, for the rectangles $\mathcal{R} = (-R_1, R_1) \times \cdots \times (-R_d, R_d)$ and $\mathcal{L} = (-L_1, L_1) \times \cdots \times (-L_d, L_d)$, observe that by the definition of g, we have

$$\int\limits_{|x_i|\ge R_i} |g(x)|^2\,dx = \int\limits_{|x_i|\ge R_i} |f(x_i)|^2\,dx_i, \qquad \int\limits_{|\xi_i|\ge L_i} |\widehat{g}(\xi)|^2\,d\xi = \int\limits_{|\xi_i|\ge L_i} |\widehat{f}(\xi_i)|^2\,d\xi_i$$

for any index i. Therefore,

$$\begin{aligned}\int\limits_{\mathbb{R}^d\setminus\mathcal{R}} |g(x)|^2\,dx + \int\limits_{\mathbb{R}^d\setminus\mathcal{L}} |\widehat{g}(\xi)|^2\,d\xi &\le \sum_{i=1}^d \int\limits_{|x_i|\ge R_i} |g(x)|^2\,dx + \int\limits_{|\xi_i|\ge L_i} |\widehat{g}(\xi)|^2\,d\xi \\ &= \sum_{i=1}^d \int\limits_{|x_i|\ge R_i} |f(x_i)|^2\,dx_i + \int\limits_{|\xi_i|\ge L_i} |\widehat{f}(\xi_i)|^2\,d\xi_i = \sum_{i=1}^d \frac{1}{R_i^2} + \frac{\log L_i}{L_i^2}.\end{aligned}$$

If we choose $R_1 = R_2 = \cdots = R_d = R$, $L_1 = L_2 = \cdots = L_d = L$ and $L = R\log^{1/2} R$, we obtain

$$\int\limits_{\mathbb{R}^d\setminus\mathcal{R}} |g(x)|^2\,dx + \int\limits_{\mathbb{R}^d\setminus\mathcal{L}} |\widehat{g}(\xi)|^2\,d\xi \le d\cdot\Big(\frac{1}{R^2} + \frac{\log(R\log^{1/2} R)}{R^2\log R}\Big) \le 3d\cdot\frac{1}{R^2},$$

whereas if we could improve the right-hand side of our estimate as mentioned we would, with such choices of R_i, L_i, have

$$\int\limits_{\mathbb{R}^d\setminus\mathcal{R}} |g(x)|^2\,dx + \int\limits_{\mathbb{R}^d\setminus\mathcal{L}} |\widehat{g}(\xi)|^2\,d\xi \ge C\cdot\frac{\log(R^2\log^{1/2} R)}{R^2\log^{1/2} R} \ge C\cdot\frac{\log^{1/2} R}{R^2},$$

with a constant C independent of R, which is a clear contradiction.

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