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Max-projective modules

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Weakening the notion of R-projectivity, a right R-module M is called *max-projective* provided that each homomorphism $f : M \to R/I$, where I is any maximal right ideal, factors through the canonical projection $\pi : R \to R/I$. We study and investigate properties of max-projective modules. Several classes of rings whose injective modules are R-projective (respectively, max-projective) are characterized. For a commutative Noetherian ring R, we prove that injective modules are R-projective if and only if $R = A \times B$, where A is QF and B is a small ring. If R is right hereditary and right Noetherian then, injective right modules are max-projective if and only if $R = S \times T$, where S is a semisimple Artinian and T is a right small ring. If R is right hereditary then, injective right modules are max-projective if and only if each injective simple right module is projective. Over a right perfect ring max-projective modules are projective. We discuss the existence of non-perfect rings whose max-projective right modules are projective.

Keywords: Injective modules; R-projective modules; max-projective modules; QF rings.

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1. Introduction and Preliminaries

Baer's Criteria for injectivity says that, a right module M is injective if and only if each homomorphism from any right ideal I of R into M extends to R. This criteria is very crucial in characterization and classification of injective modules over certain rings. Dual Baer Criterion formulated as follows: a right module M is projective if and only if M is R-projective, that is, each homomorphism from M into R/I where I is any right ideal, factors through the canonical projection $\pi : R \to R/I$.

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In [15], Faith asked, when does R-projectivity imply projectivity for all right R-modules? In other words, Faith asked for which rings does the dual Baer Criterion hold. Recently, this problem was considered by several authors in [1, 12, 31, 32]. In [1], the rings whose R-projective modules are projective are called right *right testing* rings. In [29], it was shown that right perfect rings are right testing, and in [1] examples of local rings that are not right testing are provided. Using old results of Trlifaj, the authors in [1] showed that the statement 'each right testing ring is right perfect' is consistent with ZFC, and recently, in [31], Trlifaj showed that the existence of a right testing ring that is not right perfect is also consistent with ZFC. Thus the answer to Faith's question above is undecidable in ZFC.

Weakening the notion of R-projectivity, we call a right module M *max-projective* if each homomorphism $f : M \to R/I$ where I is any maximal right ideal, factors through the canonical projection $\pi : R \to R/I$. This notion properly generalizes the notion R-projective modules. This is also a dualization of the notion of mininjective rings and modules that was introduced by Nicholson and Yousif in [27] and a generalization of the work by Amin *et al.* in [3].

Characterizing rings by projectivity of some classes of their modules is a classical problem in ring and module theory. A result of Bass [6, Theorem 28.4] states that a ring R is right perfect if and only if each flat right R-module is projective. On the other hand, the ring R is QF if and only if each injective right R-module is projective [15]. Recently, the notion of R-projectivity is considered in [1, 4, 5, 29]. The rings whose flat right R-modules are R-projective and max-projective are characterized in [4, 5, 7], respectively.

At this point it is natural to ask, "what are the rings over which each injective right module is R-projective (respectively, max-projective)"?

We call a ring R *right almost-QF* (respectively, *right max-QF*) in case all injective right R-modules are R-projective (respectively, max-projective). Trivially, almost-QF rings are max-QF. The ring of integers is almost-QF, since $\mathrm{Hom}(E, \mathbb{Z}/n\mathbb{Z}) = 0$ for each injective $\mathbb{Z}$-module E.

In this paper, we investigate some properties and give some characterizations of max-projective modules. Several characterizations of almost-QF and max-QF rings are given.

We organize the paper as follows. In Sec. 2, some properties of max-projective modules are investigated. We obtain that R-projectivity and max-projectivity coincide over the ring of integers and over right perfect rings. Submodules of max-projective modules are max-projective, provided that singular simple right modules are injective. Pure submodules of max-projective modules are max-projective over commutative rings and over semilocal rings. Characterizations of semiperfect, perfect and QF rings in terms of max-projectivity are given. Every max-projective right module of finite length is projective. By [1, Lemma 2.1] any finitely generated R-projective right R-module is projective. This result is not true when R-projectivity is replaced by max-projectivity. We prove that if R is right hereditary right Noetherian, or semiperfect, or right nonsingular right self-injective ring, then finitely generated max-projective right R-modules are projective. We prove that, R is semilocal and max-projective right modules are nonsingular if and only if R is right nonsingular and right perfect.

In Sec. 3, we give some characterizations of almost-QF and max-QF rings. Every right small ring is right max-QF, while a right small ring is right almost-QF provided R is right hereditary or right Noetherian. A right hereditary right Noetherian ring R is right almost-QF if and only if R is right max-QF if and only if $R = S \times T$, where S is a semisimple Artinian and T is a right small ring. A right hereditary ring R is right max-QF if and only if every simple injective right R-module is projective. A commutative Noetherian ring R is almost-QF if and only if R is max-QF if and only if $R = A \times B$, where A is QF and B is a small ring. A right Noetherian local ring is almost-QF if and only if R is QF or right small.

To keep in line with the terminology used in [1], we call in Sec. 4 of this paper, a ring R right max-testing if every right max-projective R-module is projective. In Proposition 6, we show that if R is a semilocal ring, then R is right max-testing if and only if R is right perfect. If $R = \mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at any prime element $p \in \mathbb{Z}$, then the field of fractions of R is the field of rational numbers $\mathbb{Q}_R$. Since $\mathrm{Rad}(\mathbb{Q}_R) = \mathbb{Q}_R$, $\mathbb{Q}_R$ is max-projective which is not projective. This shows that semiperfect (indeed local) rings need not be right max-testing.

Finally, we close the paper by highlighting some questions that are partially solved or left unanswered within the paper.

As usual, we denote by $\mathrm{Mod}-R$ the category of right R-modules. For a module M, $E(M)$, $Z(M)$, $\mathrm{Rad}(M)$ and $\mathrm{Soc}(M)$ denote the injective hull, singular submodule, Jacobson radical and socle of M, respectively. The notation $K \ll M$ means that K is a small submodule of M in the sense that $K + L \neq M$ for any proper submodule L of M.

2. Max-Projective Modules

In this section, several properties and characterization of max-projective modules are given. First we recall the definition.

Definition 1. A right R-module M is said to be *max-projective* if for every epimorphism $f : R \to R/I$ with I is a maximal right ideal of R, and every homomorphism $g : M \to R/I$, there exists a homomorphism $h : M \to R$ such that $fh = g$.

Example 1.

(a) Projective modules are max-projective.
(b) Any R-module M with $\mathrm{Rad}(M) = M$ is max-projective, since M has no simple factors. In particular, the $\mathbb{Z}$-module $\mathbb{Q}$ is max-projective but not a projective module.
(c) Every simple max-projective R-module is projective. For if S is a simple right R-module and $1_S : S \to S$ is the identity map, then by max-projectivity of S

there is a homomorphism $f : S \to R$ such that $\pi f = 1_S$, where $\pi : R \to S$ is the natural epimorphism. Then $R \cong K \oplus S$, and so S is projective.

The next corollary is now an immediate consequence of Example 1(c).

Corollary 1. *For a ring R, the following are equivalent.*

(1) *R is semisimple.*
(2) *Every right R-module is max-projective.*
(3) *Every finitely generated right R-module is max-projective.*
(4) *Every cyclic right R-module is max-projective.*
(5) *Every simple right R-module is max-projective.*

Lemma 1. *The following conditions are true.*

(1) *A direct sum $\bigoplus_{i \in I} A_i$ of modules is max-projective (respectively, R-projective) if and only if each A_i is max-projective (respectively, R-projective).*
(2) *If $0 \to A \to B \to C \to 0$ is an exact sequence and M is B-projective, then M is projective relative to both A and C.*

Proof. (1) The proof is similar to the one provided in [6, Proposition 16.10] for R-projective modules.

(2) is clear by [6, Proposition 16.12]. □

Following [18], given modules M and N, M is said to be N-subprojective if for every homomorphism $f : M \to N$ and for every epimorphism $g : B \to N$, there exists a homomorphism $h : M \to B$ such that $gh = f$.

Max-projective modules have the following investigation in terms of subprojectivity.

Lemma 2. *For an R-module M, the following are equivalent.*

(1) *M is max-projective.*
(2) *M is S-subprojective for each simple R-module S.*
(3) *For every epimorphism $f : N \to S$ with S simple, and homomorphism $g : M \to S$, there exists a homomorphism $h : M \to N$ such that $fh = g$.*

Proof. (1) ⇒ (3) Let $f : N \to S$ be an epimorphism with S is simple R-module and $g : M \to S$ a homomorphism. Since S is simple, there exists an epimorphism $\pi : R \to S$. By the hypothesis there exists a homomorphism $h : M \to R$ such that $\pi h = g$. Since R is projective, there exists a homomorphism $h' : R \to N$ such that $fh' = \pi$. Then $f(h'h) = \pi h = g$. The conclusion now follows.

(3) ⇒ (1) is clear.

(2) ⇔ (3) By definition. □

Proposition 1. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence. If M is A-subprojective and C-subprojective, then M is B-subprojective.*

Proof. Let $\gamma : F \to B$ be an epimorphism with F projective. Then using the pullback diagram of $\gamma : F \to B$ and $\beta : A \to B$, and applying $\mathrm{Hom}(M, -)$, we get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}(M,K) & \to & \mathrm{Hom}(M,X) & \xrightarrow{\theta} & \mathrm{Hom}(M,A) & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \beta^*} & & \\
0 & \to & \mathrm{Hom}(M,K) & \to & \mathrm{Hom}(M,F) & \xrightarrow{\gamma^*} & \mathrm{Hom}(M,B) & \to & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{Hom}(M,C) & \xrightarrow{\phi} & \mathrm{Hom}(M,C) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & .
\end{array}$$

Since M is A-subprojective and C-subprojective, θ and ϕ are epic. Hence, γ^* is epic by [6, Five Lemma 3.15]. □

Proposition 2. *Let M be an R-module. M is max-projective if and only if M is N-subprojective for any R-module N with composition length* $\mathrm{cl}(N) < \infty$.

Proof. Let M be a max-projective R-module and N be an R-module with $cl(N) = n$. Then there exists a composition series $0 = S_0 \subset S_1 \subset \cdots \subset S_n = N$ with each composition factor S_{i+1}/S_i simple. Consider the short exact sequence $0 \to S_1 \to S_2 \to S_2/S_1 \to 0$. Since M is max-projective, by Lemma 2, M is S_1-subprojective and S_2/S_1-subprojective. So, by Proposition 1, M is S_2-subprojective. Continuing in this way, M is S_i-subprojective for each $0 \leq i \leq n$. Hence, M is N-subprojective. Conversely, since each simple right R-module has finite length, M is max-projective by Lemma 2. □

Corollary 2. *A $\mathbb{Z}$-module M is max-projective if and only if M is $\mathbb{Z}$-projective.*

Proof. By the Fundamental Theorem of Abelian Groups, a cyclic $\mathbb{Z}$-module M is isomorphic either to $\mathbb{Z}$ or to a finite direct sum of $\mathbb{Z}$-modules of finite length (see, [16, Theorem 15.5]). Now, the proof is clear by Proposition 2. □

Corollary 3. *Let M be an R-module with finite composition length. If M is max-projective, then it is projective.*

Proof. Let $f : R^n \to M$ be an epimorphism. The module M is M-subprojective by Proposition 2. That is, there is a homomorphism $g : M \to R^n$ such that $1_M = fg$. Thus the map f splits, and so M is projective. □

Submodules of max-projective R-modules need not be max-projective. Consider the ring $R = \mathbb{Z}/p^2\mathbb{Z}$, for some prime integer p. R is max-projective, whereas the simple ideal pR is not max-projective, since the epimorphism $R \to pR \to 0$ does not split.

Recall that a ring R is called right *V-ring* (respectively, *right GV-ring*) if all simple (respectively, all singular simple) right R-modules are injective.

Proposition 3. *Consider the following conditions for a ring R:*

(1) *R is a right GV-ring.*
(2) *Submodules of max-projective right R-modules are max-projective.*
(3) *Submodules of projective right R-modules are max-projective.*
(4) *Every right ideal of R is max-projective.*

Then, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). *Also, if R is a right self injective ring, then* (4) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) Let N be a submodule of a max-projective right R-module M. Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{\ i\ } & M \\ & & \big\downarrow f & & \\ R & \xrightarrow{\ \pi\ } & S & \longrightarrow & 0, \end{array}$$

where S is a simple right R-module, $i : N \to M$ is the inclusion map and $\pi : R \to S$ is the canonical quotient map. Since the simple module S is either projective or singular, the former implies $\pi : R \to S$ splits and there exists a homomorphism $\varepsilon : S \to R$ such that $\varepsilon\pi = 1_R$. In the latter one, S is singular, so it is injective by the hypothesis. Thus, there is a homomorphism $g : M \to S$ such that $gi = f$. Since M is max-projective, there is a homomorphism $h : M \to R$ such that $\pi h = g$. Hence, $\pi(hi) = gi = f$. In either case, there exists a homomorphism from N to R that makes the diagram commute. This implies that N is max-projective.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) Clear. (4) $\Rightarrow$ (1) Let I be a right ideal of R and J a maximal right ideal of R. Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \xrightarrow{\ i\ } & R \\ & & \big\downarrow f & & \\ R & \xrightarrow{\ \pi\ } & R/J & \longrightarrow & 0, \end{array}$$

where R/J is a simple right R-module, $i : I \to R$ is the inclusion map and $\pi : R \to R/J$ is the canonical quotient map. Since I is max-projective, there is a homomorphism $h : I \to R$ such that $\pi h = f$. Since R is injective, there exists a homomorphism $\lambda : R \to R$ such that $\lambda i = h$. Now, the map $\beta = \pi\lambda : R \to R/J$ satisfies $\beta i = \pi\lambda i = \pi h = f$, as required. □

Over commutative or semilocal rings each simple module is pure-injective (see [10, Corollary 4; 23, Proposition 4.3], respectively). This leads the following.

Proposition 4. *Over a commutative or semilocal ring, pure submodules of max-projective modules are max-projective.*

Proof. Let M be a max-projective (right) module and N a pure submodule of M. Let S be a simple (right) module and $f : N \to S$ be a homomorphism. Since S is pure-injective and N is a pure submodule of M, there is $g : M \to S$ such that $gi = f$, where $i : N \to M$ is the inclusion map. By max-projectivity of M, there is a homomorphism $h : M \to R$ such that $g = \pi h$, where $\pi : R \to S$ is the natural epimorphism. Now, we have $f = gi = \pi hi$, i.e. the map $hi : N \to R$ lifts f. This shows that N is max-projective. □

Now, we shall prove that certain factors of max-projective modules are max-projective.

Lemma 3. *Let R be a ring and τ be a preradical with $\tau(R) = 0$. If M is a max-projective R-module, then $M/\tau(M)$ is max-projective.*

Proof. Let M be a max-projective R-module and $f : M/\tau(M) \to S$ a homomorphism with S simple R-module. Consider the following diagram:

$$\begin{array}{ccccc} M & \xrightarrow{\pi} & M/\tau(M) & & \\ & & \downarrow f & & \\ R & \xrightarrow{\eta} & S & \longrightarrow & 0. \end{array}$$

Since M is max-projective, there exists a homomorphism $g : M \to R$ such that $f\pi = \eta g$. Since $g(\tau(M)) \subseteq \tau(R) = 0$, $\tau(M)) \subseteq \mathrm{Ker}(g)$, and so there exists a homomorphism $h : M/\tau(M) \to R$ such that $h\pi = g$. Now, since $\eta h\pi = \eta g = f\pi$ and π is an epimorphism, $\eta h = f$, and so $M/\tau(M)$ is max-projective. □

Unlike the case for R-projective modules, finitely generated max-projective modules need not be projective. For example, let R be a right V-ring which is not right semihereditary. Then R has a non-projective finitely generated right ideal, say I. By Proposition 3, the finitely generated right ideal I is max-projective but not projective.

Proposition 5. *Over the following rings, finitely generated max-projective right modules are projective.*

(1) *Right hereditary right noetherian rings,*
(2) *Semiperfect rings,*
(3) *Right nonsingular right self-injective rings.*

Proof. (1) Let M be a finitely generated module. Then there is a simple right module S and a nonzero $f : M \to S$. By max-projectivity of M, there is a $g : M \to R$ that is a lift of f. Since R is right hereditary, $g(M)$ is projective. Thus $M = M_1 \oplus N$ where $M_1 \cong g(M)$. Now, N is max-projective and finitely generated. By the same arguments, $N = M_2 \oplus K$ for some projective submodule M_2 of M. By the Noetherian assumption M has finite uniform dimension. Thus after finitely many steps, we get $M = M_1 \oplus \cdots \oplus M_k$, where M_i projective for $i = 1, \dots, k$.

(2) Since R is semilocal, $M/\operatorname{Rad}(M)$ is semisimple. Thus $M/\operatorname{Rad}(M)$ has a projective cover by the hypothesis. Let $f : P \to M/\operatorname{Rad}(M)$ be a projective cover. Note that, P is finitely generated, and whence M is P-projective. Then the canonical map $\pi : M \to M/\operatorname{Rad}(M)$ lifts to a homomorphism, $g : M \to P$. By [34, 19.2], g is an epimorphism. Thus g splits, and so $M = K \oplus \operatorname{Ker}(g)$ for some submodule K of M with $K \cong P$. Clearly, $\operatorname{Ker}(g) \subseteq \operatorname{Ker}(\pi) = \operatorname{Rad}(M) \ll M$. Thus $\operatorname{Ker}(g) \ll M$, and so $M = K$ i.e. M is projective.

(3) Let M be a finitely generated max-projective right R-module. As R is a right nonsingular ring, by Lemma 3, $M/Z(M)$ is max-projective. Since $M/Z(M)$ is finitely generated, there exists an epimorphism $f : F \to M/Z(M)$ such that F is finitely generated free. This means $\operatorname{Ker}(f)$ is closed in F. By the injectivity of F, the kernel of f is a direct summand of F, and so $M/Z(M)$ is projective. Then, $M = Z(M) \oplus K$ for some projective submodule K of M. We claim that $Z(M) = 0$. Assume to the contrary that $Z(M) \neq 0$. Since, $Z(M)$ is a finitely generated submodule of M, there exists a nonzero epimorphism $g : Z(M) \to S$ for some simple right R-module S. Then, by Lemma 1, $Z(M)$ is max-projective, and so there exists a nonzero homomorphism $h : Z(M) \to R$ such that $\pi h = g$, where $\pi : R \to S$ is the natural epimorphism. But then $h(Z(M)) \subseteq Z(R_R) = 0$, a contradiction. Thus we must have $Z(M) = 0$, whence M is projective. □

A ring R is called *right max-ring or right Bass ring* if every nonzero right R-module M has a maximal submodule i.e. $\operatorname{Rad}(M) \neq M$.

By [20, Theorem 1], every R-projective right module with a small radical over a semiperpect ring is projective. The same result still holds when R-projective replaced by max-projective.

Proposition 6. *The following conditions are true.*

(1) *Over a semiperfect ring R, every max-projective right R-module with small radical is projective.*
(2) *A ring R is right perfect if and only if R is semilocal and every max-projective right R-module is projective.*

Proof. (1) Let M be a max-projective right R-module with $\operatorname{Rad}(M) \ll M$ and $f : M \to R/J(R)$ a homomorphism. Since $R/J(R)$ is semisimple, $R/J(R) = \bigoplus_{i=1}^{n} K_i$, with each K_i simple as an R-module. Let $\pi_i : \bigoplus_{i=1}^{n} K_i \to K_i$, and $\pi : R \to \bigoplus_{i=1}^{n} K_i$. Set $h := \pi_i \pi$. By the hypothesis, there exists a homomorphism

$g : M \to R$ such that $hg = \pi_i f$. Since $R/J(R)$ is semisimple, π_i splits and there exists a homomorphism $\varepsilon_i : K_i \to \bigoplus_{i=1}^n K_i$ such that $\varepsilon_i \pi_i = 1_{R/J(R)}$. Then, $\pi g = \varepsilon_i hg = \varepsilon_i \pi_i f = f$. Hence M is projective by [2, Proposition 3.14, Theorem 4.7].

(2) Since over a right perfect ring R every right R-module has small radical, it follows from (1) that every max-projective right R-module is projective. Conversely, assume that R is semilocal and every max-projective right R-module is projective. Let M be a nonzero right R-module. We claim that $\mathrm{Rad}(M) \neq M$. Assume to the contrary that M has no maximal submodule, i.e. $\mathrm{Rad}(M) = M$. Since $\mathrm{Hom}(M, S) = 0$ for any simple right R-module, M is max-projective. Thus M is projective, by the hypothesis. Since projective modules have maximal submodules, this is a contradiction. Hence, every right R-module has a maximal submodule. Since R is semilocal, R is right perfect by [6, Theorem 28.4]. □

Recall that if R is a right perfect ring then, every R-projective right R-module is projective, [29]. Thus the following result follows from Proposition 6(2).

Corollary 4. *Let R be a right perfect ring and M be a right R-module. Then the following are equivalent.*

(1) *M is projective.*
(2) *M is R-projective.*
(3) *M is max-projective.*

Let R be any ring and M be an R-module. A submodule N of M is called radical submodule if N has no maximal submodules, i.e. $N = \mathrm{Rad}(N)$. By $P(M)$ we denote the sum of all radical submodules of a module M. Then $P(M)$ is the largest radical submodule of M, and so $\mathrm{Rad}(P(M)) = P(M)$. Moreover, P is an idempotent radical with $P(M) \subseteq \mathrm{Rad}(M)$ and $P(M/P(M)) = 0$, (see [9]).

In [12, Lemma 1], the authors prove that over a right nonsingular right V-ring, max-projective right R-modules are nonsingular. Regarding the converse of this fact, we have the following.

Proposition 7. *If every max-projective right R-module is nonsingular, then R is right nonsingular and right max-ring.*

Proof. Clearly the ring R is right nonsingular. If R is right V-ring, then $\mathrm{Rad}(M) = 0$ for any right R-module M. Thus R is a max-ring. Suppose R is not right V-ring and let S be a noninjective simple right R-module. We shall first see that, $E(S)$ has no nonzero radical submodule i.e. $P(E(S)) = 0$. Suppose $\mathrm{Rad}(P(E(S))) = P(E(S)) \neq 0$. Then $P(E(S))/S$ is singular. Furthermore, since $\mathrm{Rad}(P(E(S))/S) = P(E(S))/S$, $P(E(S))/S$ is max-projective. This contradicts with the hypothesis. Therefore, for every simple right R-module S, $P(E(S)) = 0$. Let M be a nonzero right R-module. We claim that $\mathrm{Rad}(M) \neq M$. Assume to the contrary that $\mathrm{Rad}(M) = M$. Let $0 \neq x \in M$ and K be a maximal submodule of xR. Then the simple right R-module $S = xR/K$ is noninjective, because S small.

Now, the obvious map $xR \to E(S)$ extends to a nonzero map $f : M \to E(S)$. Since $P(\mathrm{Im}(f)) \subseteq P(E(S)) = 0$, $P(M/\mathrm{Ker}(f)) = 0$. This contradicts with $P(M) = M$. Hence $\mathrm{Rad}(M) \neq M$ for every right R-module M, and so R is a right max-ring. □

For semilocal rings we are able to see when all max-projective modules are nonsingular.

Corollary 5. *For a ring R, the following are equivalent.*

(1) *R is semilocal and every max-projective right R-module is nonsingular.*
(2) *R is right perfect and right nonsingular.*

By a well-known result of Faith and Walker [15], R is QF iff every projective right R-module is injective.

Proposition 8. *A ring R is QF if and only if every max-projective R-module is injective.*

Proof. Let M be a max-projective module, since $\mathrm{Rad}(M) \ll M$, M is projective by Proposition 6(1). So, M is injective by the hypothesis. The converse is clear since every projective module is max-projective. □

By adapting the proofs in [1, Propositions 2.5 and 2.6] and choosing the right ideal I in their proofs to be maximal, we can establish the next result.

Proposition 9. *If M is a right R-module such that $\mathrm{Ext}^1_R(M, I) = 0$ for every maximal right ideal I of R, then M is max-projective. The converse is true when R is a right self-injective ring.*

3. Rings Whose Injective Modules are *R*-Projective

Recall that a ring R is QF if and only if every injective (right) R-module is projective (see, [15]). We slightly weaken this condition and obtain the following definition.

Definition 2. A ring R is called *right almost-QF* if every injective right R-module is R-projective. We call *R right max-QF*, if every injective right R-module is max-projective. Left almost-QF and left max-QF rings are defined similarly.

Clearly, we have the following containment relations:
$\{QF \text{ rings}\} \subseteq \{\text{right almost-}QF \text{ rings}\} \subseteq \{\text{right max-}QF \text{ rings}\}$.

Example 2. The ring of integers $\mathbb{Z}$, is a right almost-QF but not QF: For every injective $\mathbb{Z}$-module E, we have $\mathrm{Rad}(E) = E$. Thus $\mathrm{Hom}(E, \mathbb{Z}/n\mathbb{Z}) = 0$, for each cyclic $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$. This means that each injective $\mathbb{Z}$-module is $\mathbb{Z}$-projective, and so $\mathbb{Z}$ is almost-QF.

Note that, right perfect rings are right testing. Thus a ring R is right perfect and right almost-QF if and only if R is QF.

Now, we investigate some properties of almost-QF and max-QF rings. The next result shows that being almost-QF ring is preserved by Morita equivalence.

Proposition 10. *Let R and S be Morita equivalent rings. Then, R is right almost-QF if and only if S is right almost-QF.*

Proof. An R-module M is R-projective if and only if M is N-projective for any finitely generated projective R-module N. Now, by [6, Propositions 21.6 and 21.8], since injectivity, relative projectivity and being finitely generated are preserved by Morita equivalence, the proof is clear. □

Lemma 4. *Let R_1 and R_2 be rings. Then $R = R_1 \times R_2$ is right almost-QF (respectively, right max-QF) if and only if R_1 and R_2 are both right almost-QF (respectively, right max-QF).*

Proof. Let M be an injective right R_1-module. Then M is an injective right R-module by [21, Example 3.11A], as well as an R-projective module by the hypothesis. Hence, by Lemma 1, M is R_1-projective, and so R_1 is right almost-QF. Similarly, R_2 is right almost-QF. Conversely, let M be an injective right R-module. Since we have the decomposition $M = MR_1 \oplus MR_2$, MR_1 is an injective right R-module, whence it is an injective right R_1-module by [21, Example 3.11A]. On the other hand, since $(MR_2)R_1 = 0$, MR_2 is an R_1-module, so it is an injective R_1-module again by [21, Example 3.11A]. This means that MR_1 and MR_2 are R_1-projective by the hypothesis. Then, by Lemma 1, $M = MR_1 \oplus MR_2$ is R_1-projective. Similarly, M is R_2-projective. Therefore, M is R-projective by [6, Proposition 16.12]. Since it is similar to the one provided for almost-QF rings, the proof is omitted for max-QF rings. □

Before proving our first major result we need the following proposition.

Proposition 11. *Let R be a right hereditary ring and E be an indecomposable injective right R-module. Then the following are equivalent.*

(1) *E is R-projective.*
(2) *E is max-projective.*
(3) $\mathrm{Rad}(E) = E$ *or E is projective.*

Proof. (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) Assume that $\mathrm{Rad}(E) \neq E$. Then E has a simple factor module isomorphic to R/I. Let $f : E \to R/I$ be a nonzero homomorphism. Since E is max-projective, there exists a homomorphism $g : E \to R$ such that $\mathrm{Im}(g) \neq 0$. By the fact that R is right hereditary, $\mathrm{Im}(g)$ is projective, whence $E \cong \mathrm{Im}(g) \oplus K$ for some right R-module K. Since E is indecomposable, either $K = 0$ or $\mathrm{Im}(g) = 0$, where the latter case implies that $g = 0$ which is a contradiction. In the former case $K = 0$, implying that E is projective.

(3) $\Rightarrow$ (1) Conversely, if E is projective then E is clearly R-projective. Now, suppose $\mathrm{Rad}(E) = E$ and let $f : E \to R/I$ be a homomorphism. Then $f(E) = f(\mathrm{Rad}(E)) \subseteq \mathrm{Rad}(R/I) \ll R/I$. Moreover, $f(E)$ is a direct summand of R/I since R is right hereditary. Therefore, $f(E) = 0$, and so f can be lifted to R. □

Recall that a ring R is called right small if R_R is small in $E(R_R)$, [28]. The next Lemma is due to Ramamurthi [28, 3.3].

Lemma 5. *For a ring R the following are equivalent.*

(1) *R is a right small ring.*
(2) $\mathrm{Rad}(E) = E$ *for every injective right R-module E.*
(3) $\mathrm{Rad}(E(R)) = E(R)$.

By Example 1(b) we have the following.

Corollary 6. *Every right small ring is right max-QF.*

Corollary 7. *Let R be a right small ring. If R is either right hereditary or right Noetherian then, R is right almost-QF.*

Proof. If R is right hereditary or right Noetherian small ring then, it is easy to see that $\mathrm{Hom}(E, R/I) = 0$ for each injective right module E and right ideal I of R. □

Now, we give a characterization of right semihereditary right small almost-QF rings.

Proposition 12. *Let R be a right semihereditary right small ring. Then* $\mathrm{Hom}(E, R) = 0$, *for any injective right R-module E. In particular, R is right almost-QF if and only if* $\mathrm{Hom}(E, R/I) = 0$ *for any right ideal I of R.*

Proof. Let E be an injective right R-module and $f \in \mathrm{Hom}(E, R)$. Then $f(E) = f(\mathrm{Rad}(E)) \subseteq J(R)$. Since R is right semihereditary, $f(E)$ is absolutely pure by [25, Theorem 2]. This means that $R/f(E)$ is flat by [21, Corollary 4.86]. Then, by [21, §4, Exercise 20], $f(E) = 0$, i.e. $\mathrm{Hom}(E, R) = 0$. Now, the rest is clear. □

Theorem 1. *Let R be a right hereditary and right Noetherian ring. The following statements are equivalent.*

(1) *R is right almost-QF.*
(2) *R is right max-QF.*
(3) *Every injective right R-module E has a decomposition $E = A \oplus B$ where* $\mathrm{Rad}(A) = A$ *and B is projective and semisimple.*
(4) *$R = S \times T$, where S is a semisimple Artinian ring and T is a right small ring.*

Proof. (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) Let E be an injective right R-module. Then E has an indecomposable decomposition $E = \bigoplus_{i\in\Gamma} A_i$ where A_i's are either projective or $\mathrm{Rad}(A_i) = A_i$ by Proposition 11. Let $\Lambda = \{j \in \Gamma : A_j \text{ is projective}\}$. So the decomposition of E can be written as $E = (\bigoplus_{j\in\Lambda} A_j) \bigoplus (\bigoplus_{i\in\Gamma-\Lambda} A_i)$. We claim that each A_j is simple for $j \in \Lambda$. Since A_j is projective for $j \in \Lambda$, $\mathrm{Rad}(A_j) \neq A_j$. So there exists a simple factor B_j of A_j i.e. $B_j \cong A_j/N \cong R/I$ for some maximal submodule N of A_j and for some maximal right ideal I of R. Since B_j is injective, by (2), the following diagram commutes:

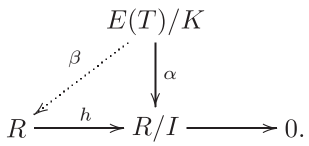
$$\begin{array}{ccccc} & & B_j & & \\ & {\scriptstyle g}\swarrow & \downarrow{\scriptstyle f} & & \\ R & \xrightarrow{h} & R/I & \longrightarrow & 0. \end{array}$$

With the hereditary assumption on R, $\mathrm{Im}(g) \cong B_j$ is projective and so $A_j \cong N \oplus B_j$. However, A_j is indecomposable, whence $N = 0$. Consequently, each A_j is simple for $j \in \Lambda$.

(3) $\Rightarrow$ (1) Let E be an injective right R-module. By the assumption, $E = A \oplus B$ where $\mathrm{Rad}(A) = A$ and B is semisimple and projective. Since B is R-projective, we only need to show that A is R-projective. By the Noetherian assumption, the injective R-module A has a decomposition $A = \bigoplus_{i\in\Gamma} A_i$ where each A_i is indecomposable injective with $\mathrm{Rad}(A_i) = A_i$. Proposition 11 implies that each A_i is R-projective, whence A is R-projective by Lemma 1. Therefore, $M = A \oplus B$ is R-projective by Lemma 1.

(2) $\Rightarrow$ (4) Let S be the sum of minimal injective right ideals of R. Then S is injective since R is right Noetherian. Thus we have the decomposition $R = S \oplus T$ for some right ideal T of R such that $\mathrm{Soc}(S) = S$ and T has no simple injective submodule. If $f : S \to T$ is a nonzero homomorphism, then $f(\mathrm{Soc}(S)) = f(S) \subseteq \mathrm{Soc}(T)$, where $f(S)$ is injective by the hereditary assumption, and so $\mathrm{Soc}(T)$ contains a semisimple injective direct summand $f(S)$. This means that $f(S) = 0$, a contradiction. Thus, we have $\mathrm{Hom}(S, T) = 0$, and so S is a two sided ideal. On the other hand, if $g : T \to S$ is a nonzero homomorphism, then $T/\mathrm{Ker}(g) \cong \mathrm{Im}(g) \subseteq S$, and so $T/\mathrm{Ker}(g)$ is projective by hereditary assumption. Also since S is a semisimple injective R-module, $T/\mathrm{Ker}(g)$ is semisimple injective, whence $K/\mathrm{Ker}(g)$ is semisimple injective for any maximal submodule $K/\mathrm{Ker}(g)$ of $T/\mathrm{Ker}(g)$. This implies that $T/\mathrm{Ker}(g) \cong K/\mathrm{Ker}(g) \oplus T/K$. Then the simple R-module T/K is injective and projective, and so T contains an isomorphic copy of a simple injective R-module T/K, yielding a contradiction. Therefore, $\mathrm{Hom}(T, S) = 0$, and so T is a two sided ideal. Consequently, $R = S \oplus T$ is a ring decomposition. Now, let $E(T)$ be the injective hull of T as an R-module. The injective hull $E(T)$ is also a T-module by the fact that $E(T)S = 0$. We claim that $\mathrm{Rad}(E(T)) = E(T)$. Suppose the contrary and let K be a maximal submodule of $E(T)$. Then $E(T)/K$ is injective by the hereditary assumption and it is max-projective by (2). Since $E(T)/K$ is a simple right

R-module, it is isomorphic to R/I for some maximal right ideal I of R, and so R/I is injective. Then, the isomorphism $\alpha : E(T)/K \to R/I$ lifts to $\beta : E(T)/K \to R$ i.e. the following diagram commutes:



Since β is monic and $E(T)/K$ injective, $U = \beta(E(T)/K)$ is a direct summand of R. It is easy to see that U is also a right T-module and so $U \subseteq T$. On the other hand, since U is minimal and injective, U is also contained in S, a contradiction. So, we must have $\mathrm{Rad}(E(T)) = E(T)$, whence $T \ll E(T)$ by Lemma 5. This proves (4).

(4) $\Rightarrow$ (1) Clear, by Lemma 4 and Corollary 7. □

The next result essentially shows that over a right hereditary ring, right max-QF rings are precisely the rings whose simple injective right modules are projective.

Theorem 2. *Let R be a right hereditary ring. Then the following are equivalent.*

(1) *R is right max-QF.*
(2) *Every simple injective right R-module is projective.*
(3) *Every singular injective right R-module is R-projective.*
(4) *Every singular injective right R-module is max-projective.*
(5) $\mathrm{Rad}(E) = E$ *for every singular injective right R-module E.*
(6) *Every injective right R-module E can be decomposed as $E = Z(E) \oplus F$ with* $\mathrm{Rad}(Z(E)) = Z(E)$.

Proof. (1) $\Rightarrow$ (4), (3) $\Rightarrow$ (4) and (6) $\Rightarrow$ (5) are clear.

(4) $\Rightarrow$ (2) Let S be a simple injective right R-module. We claim that S is projective. Assume that S is not projective. Then it is singular and injective. This implies, by our hypothesis that S is max-projective, hence S is projective, this is a contradiction. The conclusion now follows.

(2) $\Rightarrow$ (1) Let E be an injective right R-module and $f : E \to S$ with S is a simple right R-module. If $f = 0$, there is nothing to prove. We may assume that f is a nonzero homomorphism, and so f is an epimorphism. Since R is right hereditary, S is injective, and so by (2), S is projective. Hence, the natural epimorphism $\pi : R \to S$ splits, i.e. there exists a homomorphism $\eta : S \to R$ such that $\pi\eta = 1_S$. Then, $\pi\eta f = f$, and so E is max-projective.

(4) $\Rightarrow$ (5) Let E be a singular injective right R-module. Assume to the contrary that E has a maximal submodule K such that $E/K \cong R/I$ for some maximal right ideal I of R. So, there is a nonzero homomorphism $f : E \to R/I$, and by (4), there exists a nonzero homomorphism $g : E \to R$ such that $\pi g = f$, where $\pi : R \to R/I$ is the canonical epimorphism. Since E is singular, $\mathrm{Im}(g)$ is singular.

Moreover, $\operatorname{Im}(g) \subseteq R$, and so $\operatorname{Im}(g)$ is nonsingular. This implies that $g(E) = 0$, yielding a contradiction.

(5) $\Rightarrow$ (6) Let E be an injective right R-module. Since R is a right nonsingular ring, $Z(E)$ is a closed submodule of E, and so $E = Z(E) \oplus F$ for some submodule F of E. Then, by (5), $\operatorname{Rad}(Z(E)) = Z(E)$.

(5) $\Rightarrow$ (3) Let E be a singular injective right R-module. This implies, by our hypothesis, that $\operatorname{Rad}(E) = E$. Let $f : E \to R/I$ be homomorphism for some right ideal I of R. Since $\operatorname{Rad}(E) = E$ and $\operatorname{Rad}(R/I) \neq R/I$, $f : E \to R/I$ is not an epimorphism. By the right hereditary assumption, $f(E)$ is injective, and so $f(E)$ is a direct summand of R/I. But since $f(E) \subseteq \operatorname{Rad}(R/I)$, we must have $f(E) \ll R/I$. This means, $f(E) = 0$, whence $\operatorname{Hom}(E, R/I) = 0$ for each right ideal I of R. Therefore, E is R-projective. □

We shall characterize commutative Noetherian max-QF rings. First, we recall the following.

Proposition 13 (See [24]). *Let R be a commutative Noetherian ring, P be a prime ideal of R, $E = E(R/P)$, and $A_i = \{x \in E : P^i x = 0\}$. Then*

(1) *A_i is a submodule of E, $A_i \subseteq A_{i+1}$, and $E = \bigcup A_i$.*
(2) *If P is a maximal ideal of R, then $A_i \subseteq E(R/P)$ is a finitely generated R-module for every integer i.*
(3) *$E(R/P)$ is Artinian.*

In the following lemma, we characterize when injective hull of simple modules are R-projective over a commutative Noetherian ring.

Lemma 6. *Let R be a commutative Noetherian ring, and let $E = E(R/Q)$ for a maximal ideal Q of R. The following are equivalent.*

(1) *E is R-projective.*
(2) *E is max-projective.*
(3) $\operatorname{Rad}(E) = E$ *or E is a projective local module isomorphic to an ideal of R.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) Assume that $\operatorname{Rad}(E) \neq E$. Since R is commutative, $\operatorname{Rad}(E) = \bigcap_{i\in\wedge} IE$, where $\wedge$ is the set of all maximal ideals of R, [6, Exercises 15(5)]. Now, we will see that $IE = E$ for any maximal ideal I distinct from Q. Let I be a maximal ideal distinct from Q. The fact $I + Q = R$ implies $I + Q^n = R$ for any $n \in \mathbb{N}$. Let $x \in E$. Then $Q^n x = 0$ for some $n \in \mathbb{N}$, by Proposition 13. We have $1 = y + z$, where $y \in I$, $z \in Q^n$, and then $x = yx \in IE$. Hence, $\operatorname{Rad}(E) = \bigcap_{i\in\wedge} IE = QE \neq E$. Since R is commutative and $(E/QE)Q = 0$, E/QE is a semisimple R/Q-module, and so E/QE semisimple as an R-module. Then E/QE is finitely generated by Artinianity of E, and hence $QE + K = E$ for some finitely generated submodule K of E. Since K is finitely generated, K is a submodule of

A_n for some n, by Proposition 13. Thus $Q^nK = 0$. Since $QE + K = E$, $Q^{n+1}E = Q^nE$, implies that $Q^nE \subseteq P(E)$. On the other hand, $Q^2E + QK = QE$, and so $Q^2E+K = E$. Continuing in this manner $Q^nE+K = E$, whence E/Q^nE is finitely generated. Since R is Noetherian, $P(E/Q^nE) = 0$ and so $P(E) = Q^nE$. As $E/P(E)$ is finitely generated, $E/P(E)$ has finite composition length by Proposition 13(3). By max-projectivity of E and Lemma 3, $E/P(E)$ is max-projective. Thus $E/P(E)$ is projective by Corollary 3. Then, $E = P(E) \oplus L$ for some projective submodule L of E. Since E is indecomposable and $P(E) \neq E$, $E = L$. Therefore, E is projective. Furthermore, since E is indecomposable, the endomorphism ring of E is local by [14, Lemma 2.25]. By [33, Theorem 4.2], E is a local module, so it is cyclic and $R \cong E \oplus I$ for some ideal I of R. Hence E is isomorphic to an ideal of R. This proves (3).

(3) ⇒ (1) is obvious. □

Lemma 7 (See [19, 9.7]). *Suppose R commutative Noetherian or semilocal right Noetherian ring and $\{M_i\}_{i\in I}$ be a class of right R-modules. Then* $\mathrm{Rad}(\prod_{i\in I} M_i) = \prod_{i\in I} \mathrm{Rad}(M_i)$.

Lemma 8. *Let R be a commutative Noetherian ring. Then the following are equivalent.*

(1) *R is a small ring, i.e. $R \ll E(R)$.*
(2) $\mathrm{Rad}(E(S)) = E(S)$ *for each simple R-module S.*

Proof. (1) ⇒ (2): Clear by Lemma 5.

(2) ⇒ (1): Let Δ be a complete set of representatives of simple R-modules. Then $C = \bigoplus_{S\in\Delta} E(S)$ is an injective cogenerator. Then, for some index set I, the injective hull $E(R)$ of R is a direct summand of C^I. By Lemma 7, $\mathrm{Rad}(C^I) = C^I$. Since $E(R)$ is a direct summand of C^I, we have $\mathrm{Rad}(E(R)) = E(R)$. Thus R is a small ring by Lemma 5. □

Now, we are ready to give and prove a characterization of almost-QF commutative Noetherian rings.

Theorem 3. *Let R be a commutative Noetherian ring. The following are equivalent.*

(1) *R is almost-QF.*
(2) *R is max-QF.*
(3) *$R = A \times B$, where A is QF and B is small.*

Proof. (1) ⇒ (2) is clear.

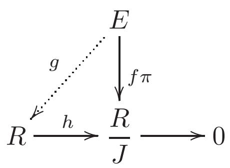
(2) ⇒ (3) First suppose that $\mathrm{Rad}(E(S)) = E(S)$ for all simple R-module S. Then R is a small ring by Lemma 8. On the other hand, if $\mathrm{Rad}(E(S)) \neq E(S)$ for some simple R-module S, then $E(S)$ is isomorphic to a direct summand of R

by Lemma 6. Let X be sum of minimal ideals U of R with $\text{Rad}(E(U)) \neq E(U)$. Then $E(U)$ is isomorphic to an ideal of R. Thus without loss of generality we can assume that $E(U)$ is an ideal of R. Since R is Noetherian, X is finitely generated, and so $A = E(X) = E(U_1) \oplus \cdots \oplus E(U_n)$ where each $E(U_i)$ is an ideal of R. Thus $R = A \oplus B$ for some ideal B of R. Now, A is injective and Noetherian, so A is a QF ring. On the other hand, let V be a simple B-module, then V is a simple R-module. Let $E(V)$ be the injective hull of V. As V is a B-module, $VA = 0$. If $\text{Rad}(E(V)) \neq E(V)$, then this would imply $V \subseteq A$, by the same arguments above. Thus $\text{Rad}(E(V)) = E(V)$, and so B is a small ring by Lemma 8.

(3) $\Rightarrow$ (1) Clear, by Corollary 7 and Lemma 4. □

Proposition 14. *Let R be a local right max-QF ring. Then R is either right self-injective or right small.*

Proof. Let J be the unique maximal right ideal of R and E be the injective hull of the ring R. Assume first that R is not a small ring i.e. $\text{Rad}(E) \neq E$. Then E has a maximal submodule K such that $\frac{E}{K}$ is isomorphic to $\frac{R}{J}$ and denote this isomorphism by f. Consider the composition $f\pi$ where $\pi : E \to \frac{E}{K}$ is the canonical projection. Since R is right max-QF, there is a nonzero homomorphism $g : E \to R$ such that



commutes. Furthermore, h is a small epimorphism and $f\pi$ is an epimorphism, which means $g : E \to R$ is also an epimorphism and splits. Thus, $E \cong R \oplus T$ for some T. Hence, R is a right self injective ring. □

Corollary 8. *Let R be a commutative semiperfect ring. If R is max-QF, then $R = S \times T$ where S is self-injective and T is small.*

Proof. Let R be a commutative semiperfect ring, then by [22, Theorem 23.11], $R = R_1 \times \cdots \times R_n$, where R_i is a local ring ($1 \leq i \leq n$). Hence, by Lemma 4 and Proposition 14, R can be written as a direct product of local max-QF rings and every local max-QF ring either self-injective or small. □

Corollary 9. *Let R be a right Noetherian local ring. Then the following are equivalent.*

(1) *R is right almost-QF.*
(2) *R is right max-QF.*
(3) *R is QF or right small.*

Proof. (1) ⇒ (2) Clear. (3) ⇒ (1) Follows by Corollary 7.

(2) ⇒ (3) Clear by Proposition 14, since right Noetherian right self-injective rings are QF. □

We do not know whether every right chain ring is almost-QF. But the following result will imply that each right chain ring with $P(R) = 0$ is right almost QF.

Proposition 15. *Let R be a right chain ring and $J = J(R)$. Then $P(R) = \bigcap_{n\geq 1} J^n$.*

Proof. Assume first that $J^n = 0$ for some $n \in \mathbb{Z}^+$. Then $\bigcap_{n\geq 1} J^n = 0$, and so, by [14, Proposition 5.3(b)], R is a right Noetherian ring with $P(R) = 0$. On the other hand if we suppose that $J^n \neq 0$ for all $n \in \mathbb{Z}^+$, then, by [14, Proposition 5.2(d)], $A = \bigcap_{n\geq 1} J^n$ is a completely prime ideal. Let us now look at the case $A \neq AJ$. Then $\frac{A}{AJ}$ simple right R-module and $AJ \ll A$. Let $a \in A \setminus AJ$. If we have $A = aR + AJ$, then $A = aR$, whence either $A = J(R)$ or $A = 0$, by [14, Proposition 5.2(f)]. If $A = \bigcap_{n\geq 1} J^n = 0$, then R is a right Noetherian ring with $P(R) = \bigcap_{n\geq 1} J^n = 0$. Otherwise, if $A = J(R) = \bigcap_{n\geq 1} J^n$, then $J = J^2$, but since $A \neq AJ$, this is not the case. If we look at the case $A = AJ$, then $A \subseteq P(R)$. Since $P(R) = P^2(R)$, $P(R)$ is a completely prime ideal of R, and so, by [14, Lemma 5.1], $P(R) \subseteq A$. Hence, $P(R) = A = \bigcap_{n\geq 1} J^n$. □

Corollary 10. *Let R be a right chain ring. Then $R/P(R)$ is a right almost-QF ring.*

Proof. Since $P(R)$ is an ideal of R, and every factor ring of a right chain ring is a right chain ring, without loss of generality we may assume that $P(R) = 0$. Then by Proposition 15 and [14, Proposition 5.3], R is a right Noetherian ring. We have two cases for $J = J(R)$: if J is nilpotent, then R is Artinian. This implies that R is right self-injective by [14, Lemma 5.4] which then yields, R is QF. So now assume that J is not nilpotent. Then R is a domain by [14, Proposition 5.2(d)], whence R is right small. So, R is right almost-QF by Corollary 7. Thus in any case R is right almost-QF. □

In [11], a submodule N of a right R-module M is called *coneat* in M if $\mathrm{Hom}(M, S) \to \mathrm{Hom}(N, S)$ is epic for every simple right R-module S. In [8], N is called *s-pure* in M if $N \otimes S \to M \otimes S$ is monic for every simple left R-module S. M is *absolutely coneat* (respectively, *absolutely s-pure*) if M is coneat (respectively, s-pure) in every extension of it. If R is commutative, then s-pure short exact sequences coincide with coneat short exact sequences, [17, Proposition 3.1].

Proposition 16. *Consider the following conditions for a ring R:*

(1) *R is right max-QF.*
(2) *Every absolutely coneat right R-module is max-projective.*

(3) *Every absolutely s-pure right R-module is max-projective.*
(4) *Every absolutely pure right R-module is max-projective.*

Then (3) $\Rightarrow$ (4) $\Rightarrow$ (1) $\Rightarrow$ (2). *Also, if R is a commutative ring, then* (2) $\Rightarrow$ (3).

Proof. (3) $\Rightarrow$ (4) $\Rightarrow$ (1) Clear.

(1) $\Rightarrow$ (2) Let M be an absolutely coneat right R-module. Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{i} & E(M) \\ & & \big\downarrow f & & \\ R & \xrightarrow{\pi} & S & \longrightarrow & 0, \end{array}$$

where S is a simple right R-module, $i : M \to E(M)$ is the inclusion map and $\pi : R \to S$ is the canonical quotient map. Since M coneat in $E(M)$, there is a homomorphism $g : E(M) \to S$ such that $gi = f$. Also, by (1), there exists a homomorphism $h : E(M) \to R$ such that $\pi h = g$. Hence, $(\pi h)i = gi = f$.

(2) $\Rightarrow$ (3) Let M be an absolutely s-pure right R-module. Then M is s-pure in $E(M)$. Since R is commutative, M is coneat in $E(M)$. Hence, M is max-projective by (2). □

In [26, Lemma 1.16], it was shown that for a projective module M, if $M = P + K$, where P is a summand of M and $K \subseteq M$, then there exists a submodule $Q \subseteq K$ with $M = P \oplus Q$. By using the same method in the proof of [4, Theorem 2.8], one can prove the following result.

Proposition 17. *A ring R is right almost-QF if and only if for every injective right R-module E, if* $E = P + L$, *where P is a finitely generated projective summand of E and* $L \subseteq E$, *then* $E = P \oplus K$ *for some* $K \subseteq L$.

Let R be a ring and Ω be a class of R-modules which is closed under isomorphic copies. Following Enochs, a homomorphism $\varphi : G \to M$ with $G \in \Omega$ is called an Ω-precover of the R-module M if for each homomorphism $\psi : H \to M$ with $H \in \Omega$, there exists $\lambda : H \to G$ such that $\varphi\lambda = \psi$.

Lemma 9. *Let R be a right self-injective ring. Then the following are equivalent.*

(1) *R is right almost-QF.*
(2) *Every finitely generated right R-module has an injective precover which is R-projective.*
(3) *Every cyclic right R-module has an injective precover which is R-projective.*

Proof. (1) $\Rightarrow$ (2) Let M be a finitely generated right R-module and $g : R^n \to M$ be an epimorphism. For any homomorphism $f : E \to M$ with E is injective, there exists $h : E \to R^n$ such that $gh = f$. Since R^n is injective, g is an injective precover of M.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Let E be an injective right R-module and I be a right ideal of R. Suppose that $f : E \to R/I$ is a homomorphism, $\pi : R \to R/I$ is the natural epimorphism and $g : G \to R/I$ be an injective cover of R/I. So, there is $h : E \to G$ such that $gh = f$. By hypothesis, G is R-projective and hence there is $k : G \to R$ such that $\pi k = g$. Let $\overline{f} = kh$. So $\pi\overline{f} = \pi kh = gh = f$. Therefore, E is R-projective, and so R is right almost-QF. □

In [13], a module M is said to be *copure-injective* if $\mathrm{Ext}^1_R(E, M) = 0$ for any injective module E. Now, we give the characterization of almost-QF rings in terms of copure-injective modules.

Proposition 18. *Let R be a ring. Then the followings are equivalent.*

(1) *R is right almost-QF and R_R is copure-injective.*
(2) *Every right ideal of R is copure-injective.*
(3) *Every submodule of a finitely generated projective right R-module is copure injective.*

Proof. (1) $\Rightarrow$ (2) Let E be an injective right R-module and I be a right ideal of R. By applying $\mathrm{Hom}(E, -)$ to the short exact sequence $0 \to I \to R \to R/I \to 0$, we obtain the following exact sequence:

$$0 \to \mathrm{Hom}(E, I) \to \mathrm{Hom}(E, R) \to \mathrm{Hom}(E, R/I) \to \mathrm{Ext}^1_R(E, I) \to \mathrm{Ext}^1_R(E, R) \to \cdots .$$

Since R_R is copure-injective, $\mathrm{Ext}^1_R(E, R) = 0$. Then the map $\mathrm{Hom}(E, R) \to \mathrm{Hom}(E, R/I)$ is onto since E is R-projective. Hence, $\mathrm{Ext}^1_R(E, I) = 0$ for any injective R-module E.

(2) $\Rightarrow$ (3) Suppose that every right ideal of R is copure-injective. First, by induction, we show that every submodule of R^n is copure-injective. The case $n = 1$ follows by the hypothesis. Now suppose that $n > 1$ and every submodule of R^{n-1} is copure injective. Let N be a submodule of R^n, and consider the exact sequence $0 \to N \cap R^{n-1} \to N \to N/(N \cap R^{n-1}) \to 0$. By induction hypothesis, $N \cap R^{n-1}$ is copure-injective, and $N/(N \cap R^{n-1}) \cong (N + R^{n-1})/R^{n-1} \subseteq R^n/R^{n-1} \cong R$ is also copure-injective. Therefore, for any injective right R-module E, consider the exact sequence

$$\mathrm{Ext}^1_R(E, N \cap R^{n-1}) \to \mathrm{Ext}^1_R(E, N) \to \mathrm{Ext}^1_R(E, N/(N \cap R^{n-1})).$$

Since $\mathrm{Ext}^1_R(E, N \cap R^{n-1}) = \mathrm{Ext}^1_R(E, N/(N \cap R^{n-1})) = 0$, we have $\mathrm{Ext}^1_R(E, N) = 0$. Therefore, N is copure-injective. Now, if M is a submodule of a finitely generated projective right R-module P, then there is $n \geq 1$ such that $M \subseteq P \subseteq R^n$. By the above observation, M is also copure injective.

(3) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (1) By [1, Proposition 2.5]. $\square$

Proposition 19. *Let R be a ring. Then the following are equivalent.*

(1) *R is semisimple.*
(2) *R is right almost-QF right V-ring.*
(3) *R is right almost-QF and every submodule of an R-projective right module is R-projective.*
(4) *R is right self-injective and every submodule of an R-projective right module is R-projective.*

Proof. (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) are clear.

(2) $\Rightarrow$ (1) Since R is a right V-ring, every simple right R-module is injective. By the hypothesis, every simple right R-module is R-projective, whence projective.

(4) $\Rightarrow$ (1) Let M be a cyclic right R-module and I a right ideal of R. Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \xrightarrow{i} & R \\ & & \downarrow{\scriptstyle f} & & \\ R & \xrightarrow{\pi} & M & \longrightarrow & 0, \end{array}$$

where $i : I \to R$ is the inclusion map and $\pi : R \to M$ is the canonical quotient map. Since I is R-projective there exists $h : I \to R$ such that $\pi h = f$. By the injectivity of R, there exists $\lambda : R \to R$ such that $\lambda i = h$. Then $(\pi\lambda)i = \pi h = f$, and $\pi\lambda : R \to M$ is the required map.

(3) $\Rightarrow$ (1) Since every simple right R-module can be embedded in an injective R-module, every simple right R-module is R-projective, and so every simple right R-module is projective. Hence, R is semisimple. $\square$

4. Max-Testing Rings

In [15], Faith asked for which rings does the dual Baer Criterion hold. In [29], it was shown that right perfect rings are right testing. The authors in [1] showed that the statement "each right testing ring is right perfect" is consistent with ZFC, and recently, in [31], Trlifaj showed that the existence of a right testing ring that is not right perfect is also consistent with ZFC. Thus the answer to Faiths question above is undecidable in ZFC. In this section, by using Trlifaj's arguments we obtain that, the answer to the question "when does max-projectivity imply projectivity for all right R-modules?" is undecidable in ZFC.

Definition 3. A ring R is called right max-testing if every right max-projective module is projective.

In Proposition 6 we show that if R is a semilocal ring, then R is right max-testing if and only if R is right perfect. If $R = \mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at any prime element $p \in \mathbb{Z}$, then the field of fractions of R is the field of rational numbers $\mathbb{Q}_R$. Since $\mathrm{Rad}(\mathbb{Q}_R) = \mathbb{Q}_R$, $\mathbb{Q}_R$ is max-projective which is not projective (since projective modules have maximal submodules). This shows that semiperfect (indeed local) rings need not be right max-testing.

By using [1, Lemma 2.4], Puninski *et al.* showed in [1, Theorem 2.7] how to use Shelah's uniformization principle UP_κ to prove that the statement each right testing ring is right perfect. For a definition of Shelah's uniformization principle UP_κ, where κ is a cardinal of cardinality $\aleph_0$, (see [30]). With the help of an argument similar to the one provided in [1, Theorem 2.7], we can establish the following Corollary.

Corollary 11. *Let R be a ring of cardinality λ and let $k = (\lambda_\omega^+)$. If UP_κ is assumed, then R is right max-testing if and only if R is right perfect.*

Proof. The sufficiency is clear. To prove the necessity suppose that R is right max-testing. If R is not right perfect, then by [1, Lemma 2.4], there exists a non-projective right R-module M such that $\mathrm{Ext}(M, I) = 0$ for every maximal right ideal I of R. So by Proposition 9, M is max-projective. Since R is right max-testing, M is projective, a contradiction. □

The following Remark is an example of a right non-perfect ring R over which every max-projective module is R-projective.

Remark 1. Let K be a field, and R the subalgebra of K^ω consisting of all eventually constant sequence in K^ω. For each $i < \omega$, we let e_i be the idempotent in K^ω whose ith component is 1 and all the other components are 0. Note that $\{e_i : i < \omega\}$ a set of pairwise orthogonal idempotents in R, so R is not perfect. By [31, Lemma 2.3(3)], $\{e_i : i < \omega\} \cup \{R/\mathrm{Soc}(R)\}$ is a representative set of all simple modules and all but $R/\mathrm{Soc}(R)$ are projective. By [31, Lemma 2.4], a module M is R-projective if and only if it is projective with respect to the projection $\pi : R \to R/\mathrm{Soc}(R)$. Thus, an R-module M is max-projective if and only if M is R-projective.

In [31], Trlifaj used Gödel's Axiom of Constructibility ($V = L$) to show that the existence of a right testing ring that is not right perfect is also consistent with ZFC. The following Corollary is an immediate consequence of [31, Theorem 3.3] and Remark 1.

Corollary 12. *Let K be a field of cardinality $\leq 2^\omega$, R the subalgebra of K^ω consisting of all eventually constant sequence in K^ω and Assume Gödel's Axiom of Constructibility ($V = L$). Then all max-projective R-modules are projective.*

In Corollary 11, we show that the statement "each right max-testing ring is right perfect" is consistent with ZFC, and Corollary 12 says that the existence of a right max-testing ring that is not right perfect is also consistent with ZFC. By adapting the proof in [31, Corollary 3.4], we can establish the following Corollary.

Corollary 13. *Let K be a field of cardinality $\leq 2^\omega$ and R the subalgebra of K^ω consisting of all eventually constant sequence in K^ω. Then the statement "R is a max-testing ring" is independent of ZFC+ GCH. Hence, the answer to the question: "when does max-projectivity imply projectivity for all R-modules?" is undecidable in ZFC+ GCH.*

Proof. Assume UP_κ for some κ such that $\mathrm{card}(R) < \kappa$ and $\mathrm{cf}(\kappa) = \aleph_0$. Since R is not perfect, R is not max-testing by Corollary 11. If we assume $(V = L)$, then R is max-testing by Corollary 12. The conclusion now follows. □

5. Questions

We would like to close by highlighting the following questions some of which are partially answered within the paper.

Q1. Characterize the local almost-QF rings. An almost-QF local ring is either small or self injective by Proposition 14. Noetherian and local almost-QF-rings are characterized in Corollary 9.

Q2. Every right small ring is max-QF by Corollary 6. Is every right small ring almost-QF? A negative answer to this question provide an example of a max-QF ring which is not almost-QF.

Q3. Every right chain ring R having $P(R) = 0$ is almost-QF by Corollary 10. Is every right chain ring almost-QF? Is it possible to characterize right chain rings that are right almost-QF or max-QF?

Q4. Let R be a right semihereditary right small ring. Motivated by Proposition 12, it is natural to ask: When does $\mathrm{Hom}(E, R/I) = 0$ hold for every injective right module E and right ideal I of R? An answer to this question gives a characterization of right almost-QF rings over such rings.

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