



Rugged modules: The opposite of flatness

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ABSTRACT

Relative notions of flatness are introduced as a mean to gauge the extent of the flatness of any given module. Every module is thus endowed with a flatness domain and, for every ring, the collection of flatness domains of all of its modules is a lattice with respect to class inclusion. This lattice, the flatness profile of the ring, allows us, in particular, to focus on modules which have a smallest flatness domain (namely, one consisting of all regular modules.) We establish that such modules exist over arbitrary rings and we call them *Rugged Modules*. Rings all of whose (cyclic) modules are rugged are shown to be precisely the von Neumann regular rings. We consider rings without a flatness middle class (i.e., rings for which modules must be either flat or rugged.) We obtain that, over a right Noetherian ring every left module is rugged or flat if and only if every right module is poor or injective if and only if $R = S \times T$, where S is semisimple Artinian and T is either Morita equivalent to a right PCI-domain, or T is right Artinian whose Jacobson radical properly contains no nonzero ideals. Character modules serve to bridge results about flatness and injectivity profiles; in particular, connections between rugged and poor modules are explored. If R is a ring whose regular left modules are semisimple, then a right module M is rugged if and only if its character left module M^+ is poor. Rugged Abelian groups are fully characterized and shown to coincide precisely with injectively poor and projectively poor Abelian groups. Also, in order to get a feel for the class of rugged modules over an arbitrary ring, we consider the homological ubiquity of rugged modules in the category of all modules in terms of the feasibility of rugged precovers and covers for arbitrary modules.

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1. Introduction

Our study follows a pattern which has been somewhat established in previous studies about injectivity and projectivity profiles and their corresponding opposite notions to injectivity and projectivity (the so-called injectively and projectively poor modules.) Even though there is a bit of template to investigate notions of these types, the nature of projectivity, injectivity and flatness are sufficiently different that invariably the technical details for each case are remarkably unique. We will resist the temptation to mimic the terminology and use the expression “flatly poor” (too many *flat broke* jokes come to mind!) and take advantage of the fact that the, unlike with the case of Projective and Injective modules where no word is a natural denomination for an opposite notion, the word Flat does have Rugged as a natural antonym.

Throughout, R will denote an associative ring with identity, and modules will be unital modules. By $R\text{-Mod}$ and $\text{Mod-}R$ we denote the category of left modules and right modules, respectively. $\text{rad}(R)$ will stand for the Jacobson radical of the ring R , and for a right module M , $E(M)$ denotes the injective hull of M . Given right modules M and N , M is said to be injective relative to N (or M is N -injective) if, for any submodule K of N , any R -homomorphism $f : K \rightarrow M$ extends to an R -homomorphism $g : N \rightarrow M$. The class of modules N such that M is N -injective is called the injectivity domain of M and denoted by $\mathcal{I}n^{-1}(M)$. Clearly $\mathcal{I}n^{-1}(M)$ contains the class of semisimple right modules. As an opposite notion of injectivity, the authors of [1], defined a right module M to be poor if its injectivity domain is exactly the class of semisimple right modules. In recent years, there is an appreciable interest to poor modules and to the rings defined via this modules (see, [2, 3, 5, 9, 10]). A ring R is said to have no right middle class if every right module is poor or injective.

Similar studies have been carried regarding the so-called projective profile of a ring and projectively poor (or simply p-poor) modules (see, [16, 17, 19]).

Recall that a right module M is called *regular* if every submodule is pure in the sense of Cohn (see, [6]). That is, for every submodule K of M and left module N , the map $K \otimes N \rightarrow M \otimes N$ is a monomorphism. Given a left module M and a right module N , M is N -flat if for every submodule K of N the map $1_M \otimes i : K \otimes M \rightarrow N \otimes M$ is a monomorphism, where $i : K \rightarrow N$ is the inclusion map and 1_M is the identity map on M . The *flat domain* $\mathcal{F}^{-1}(M)$ of a left module M is defined to be the collection of right modules N such that M is N -flat (see, [4, p. 232, Question 15]). It is evident from the definitions that regular right modules are contained in $\mathcal{F}^{-1}(M)$ for each left module M .

In this paper, we study the modules whose flat domain is as small as possible. We call a left module M *rugged* if $\mathcal{F}^{-1}(M)$ is exactly the class of regular right modules. Every ring has a rugged module. The ring is von Neumann regular if and only if every left module is rugged. Any left module that contains a pure and rugged submodule is itself rugged. If R is a ring such that regular right modules are semisimple, then a left module M is rugged if and only if its character right module M^+ is poor.

Let R be a right Noetherian ring. We prove that a left module M is rugged if and only if the character right module M^+ is poor. Every right module is poor or injective if and only if every left module is rugged or flat if and only if $R = S \times T$, where S is semisimple Artinian and T is Morita equivalent to a right PCI-domain (i.e. nonsemisimple Artinian rings over which proper cyclic right modules are injective), or T is right Artinian whose Jacobson radical properly contains no nonzero ideals.

A ring R is said to be right simple-destitute if every simple right module is poor. Simple-destitute rings are studied in [1, 5], where several examples of simple-destitute rings are given. The structure of simple-destitute general rings is not known. For a commutative ring R , we prove that R is simple-destitute if and only if R is local or semisimple Artinian if and only if every simple module is rugged and regular modules are semisimple. For a right semiartinian ring R , we prove that if every simple right module is poor or injective, then R is a right V -ring, or $R = S \times T$, where S is semisimple Artinian and T is a ring with a unique simple right module. In addition, if R is commutative, then every simple module is poor or injective if and only if $R = S \times T$, where S is semisimple Artinian and T is a local ring.

We also give a characterization of rugged abelian groups. An abelian group G is rugged if and only if the torsion part of G contains a direct summand isomorphic to $\bigoplus_p \mathbb{Z}_p$, where p ranges over all primes and \mathbb{Z}_p is the simple abelian group of order p . We obtain that the notions of poor, rugged and p -poor coincide over the ring of integers.

Finally we study (pre)covers and (pre)envelopes of modules relative to the class of rugged modules.

2. Relative flatness of modules

This section is devoted to prove the basic properties about relative flatness of modules that will be needed later in the paper. We start by recalling what is understood by a relative flat module.

Definition 2.1. Given a right R -module N , a left R -module M is said to be flat relative to N , relatively flat to N , or N -flat if the canonical morphism $K \otimes_R M \rightarrow N \otimes_R M$ is a monomorphism for every submodule K of N .

Proposition 2.2. Let M be a left R -module, N be any right R -module and $K \leq N$ be any submodule. If M is N -flat then M is K -flat and N/K -flat. If, in addition, K is pure in N , then if M is K -flat and N/K -flat then M is N -flat.

Proof. If M is N -flat then it is clearly K -flat so we only have to prove that M is N/K -flat.

If $A \leq N/K$ is any submodule, tensoring with M the pullback diagram of the injection $A \hookrightarrow N/K$ and the projection $N \rightarrow N/K$ we get a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & K \otimes M & \xlongequal{\quad} & K \otimes M & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & P \otimes M & \longrightarrow & N \otimes M & \longrightarrow & C \otimes M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 & & A \otimes M & \longrightarrow & \frac{N}{K} \otimes M & \longrightarrow & C \otimes M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

so $A \otimes M \rightarrow N/K \otimes M$ is a monomorphism.

Suppose now that M is K -flat and N/K -flat.

Given any submodule $A \leq N$ we know $A \cap K$ (together with the injection maps) is the pullback diagram of the injection maps of K and A into N , so the quotient module $A/(A \cap K)$ is, up to an isomorphism, a submodule of N/K . But M is N/K -flat so the map $\frac{A}{A \cap K} \otimes M \rightarrow \frac{N}{K} \otimes M$ is monic and we get the commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & (A \cap K) \otimes M & \longrightarrow & A \otimes M & \longrightarrow & \frac{A}{A \cap K} \otimes M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & K \otimes M & \longrightarrow & N \otimes M & \longrightarrow & \frac{N}{K} \otimes M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \frac{K}{A \cap K} \otimes M & \longrightarrow & \frac{N}{A} \otimes M & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

From the diagram we immediately see that $A \otimes M \rightarrow N \otimes M$ is monic and so that M is N -flat. \square

Now we know how the relative flatness of a module behaves with respect to the modules of a short exact sequence, but, on the other hand, what can we say about the relative flatness of the modules of a short exact sequence respect to a given module?

Proposition 2.3. *Let N be any right R -module, M be any left R -module and K be any pure submodule of M . If K and M/K are both N -flat modules then M is N -flat.*

Proof. For any submodule $A \leq N$ we have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & A \otimes K & \longrightarrow & A \otimes M & \longrightarrow & A \otimes \frac{M}{K} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & N \otimes K & \longrightarrow & N \otimes M & \longrightarrow & N \otimes \frac{M}{K} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \frac{N}{A} \otimes K & \longrightarrow & \frac{N}{A} \otimes M & \longrightarrow & \frac{N}{A} \otimes \frac{M}{K} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

□

From Proposition 2.2 we know that the relative flatness of a module preserves finite direct sums, that is, if M is N_i -flat then M is $\bigoplus_{i=1}^n N_i$ -flat. Let us now prove that this property does not only hold for finite direct sums.

Proposition 2.4. *If $\{N_i; i \in I\}$ is any family of right R -modules and M is a left R -module, then M is $\bigoplus_i N_i$ -flat if and only if M is N_i -flat for every i .*

Proof. The necessary condition becomes clear from Proposition 2.2.

If we well order I we can write our family of N_i 's as $\{N_\alpha; \alpha < \omega\}$ for some ordinal number ω .

For any $\mu < \omega$ call $A_\mu = \sum_{\alpha < \mu} N_\alpha$. Then $\bigoplus_i N_i = A_\omega$ and so, using induction and Proposition 2.2, we get that if $\mu < \omega$ is any successor ordinal number and M is A_α -flat for every $\alpha < \mu$, then M is A_μ -flat.

Let μ be now any limit ordinal, $\mu < \omega$. If $K \leq A_\mu$ is any submodule, since A_μ is a direct union we see that $K = \sum_{\alpha < \mu} (K \cap A_\alpha)$ so we get that $K \otimes M \rightarrow A_\mu \otimes M$ is a monomorphism if and only if $(\sum_{\alpha < \mu} (K \cap A_\alpha)) \otimes M \rightarrow (\sum_{\alpha < \mu} A_\alpha) \otimes M$ is a monomorphism.

But the square

$$\begin{array}{ccc}
 (\sum_{\alpha < \mu} (K \cap A_\alpha)) \otimes M & \longrightarrow & (\sum_{\alpha < \mu} A_\alpha) \otimes M \\
 \cong \downarrow & & \downarrow \cong \\
 \sum_{\alpha < \mu} ((K \cap A_\alpha) \otimes M) & \longrightarrow & \sum_{\alpha < \mu} (A_\alpha \otimes M)
 \end{array}$$

is commutative so we get that $(\sum_{\alpha < \mu} (K \cap A_\alpha)) \otimes M \rightarrow (\sum_{\alpha < \mu} A_\alpha) \otimes M$ is a monomorphism if and only if $(K \cap A_\alpha) \otimes M \rightarrow A_\alpha \otimes M$ is a monomorphism for each $\alpha < \mu$.

Now, M is A_α -flat for every $\alpha < \mu$ by the induction hypothesis, so each $(K \cap A_\alpha) \otimes M \rightarrow A_\alpha \otimes M$ is a monomorphism. \square

Given a left R -module M , following [7] we say that a right R -module N is absolutely M -pure provided that N is M -pure in any of its extensions, that is, the canonical map $N \otimes M \rightarrow A \otimes M$ is a monomorphism for every extension A of N .

As in the non-relative case, N is absolutely M -pure if and only if N is M -pure in its injective envelope. From this fact it is easy to see the following relation between absolutely M -pure modules and N -flat modules.

Proposition 2.5. *If every submodule of N is absolutely M -pure then M is N -flat.*

3. Rugged modules

In this section we introduce and study the opposite concept to that of a flat module. We call these new modules rugged modules.

Definition 3.1. The flatness extent of a left R -module M , $\mathcal{F}(M)$, is defined as the class of all M -flat right R -modules. The flatness domain of M , $\mathcal{F}^{-1}(M)$, is the class of all right R -modules relative to which M is flat.

We will call a module rugged if its flatness domain is as small as it can be, which begs the question of how small the flatness domain of a module can be.

It is clear that if a right R -module N is such that any of its submodules is pure in it then every left R -module is N -flat, that is, $\mathcal{F}(N) = R\text{-Mod}$, and vice versa. The types of modules for which every submodule is pure were defined as regular modules by Fieldhouse in his Ph.D. Thesis in 1967 [11] and have been studied by several authors (see for instance [6, 12]).

Thus, if N_R is regular then $\mathcal{F}(N) = R\text{-Mod}$ and then $N \in \mathcal{F}^{-1}(M)$ for every left R -module M . And conversely, if $N_R \in \bigcap_{R\text{-Mod}} \mathcal{F}^{-1}(M)$ then $R\text{-Mod} = \mathcal{F}(N)$ and then N is regular. So, answering the question above, we see that $\bigcap_{R\text{-Mod}} \mathcal{F}^{-1}(M)$ is the class of all regular right R -modules and so the answer to the question of what the domain of flatness of a rugged module should be is automatically answered.

Definition 3.2. A left R -module M is said to be rugged if its flatness domain is the class of all regular right R -modules.

A trivial example of the existence of rugged modules is that of a semisimple ring. Over these rings every right (left) module is regular and so every left (right) module is rugged.

Another example of rings over which every module is rugged is that of regular rings. Over these rings every module is flat and so every submodule of a given module is pure in it. This means that every module is regular and so again every module is rugged.

But of course, over an arbitrary ring not all modules are rugged in general. So for instance in the category of abelian groups $\mathcal{F}^{-1}(\mathbb{Z}) = \mathbb{Z}\text{-Mod}$ since \mathbb{Z} is flat, but not every abelian group is regular since \mathbb{Z} itself is not. Thus \mathbb{Z} is not a rugged abelian group.

So the first question that comes to mind at this point is, what are the rings over which every module is rugged?

Proposition 3.3. *The following statements are equivalent for a ring R .*

- (1) R is von Neumann regular.
- (2) ${}_R R$ is rugged.
- (3) Every cyclic left module is rugged.
- (4) Every left module is rugged.
- (5) Every right module is rugged.

Proof.

(1) \Rightarrow (4) Since the ring is regular, every left (right) module is flat. Therefore every right (left) module is regular. This implies that every left module is rugged.

(2) \Rightarrow (1) Since ${}_R R$ is flat, $\mathcal{F}^{-1}({}_R R) = \text{Mod-}R$. On the other hand, by (2), ${}_R R$ is rugged. Therefore every right module is regular, and so R is a von Neumann regular ring.

(4) \Rightarrow (3) \Rightarrow (2) is clear.

(1) \Leftrightarrow (5) By left-right symmetry. □

But in general, do nontrivial rugged R -modules exist for any ring R ?

Theorem 3.4. *Rugged left (right) R -modules always exist for any ring R .*

Proof. Let \mathcal{S} be a set of representatives of all finitely presented left R -modules.

If a right R -module N is $\bigoplus_{M \in \mathcal{S}} M$ -flat then, for any submodule $S \leq N$, the morphism $S \otimes (\bigoplus_{M \in \mathcal{S}} M) \rightarrow N \otimes (\bigoplus_{M \in \mathcal{S}} M)$ is a monomorphism and so $S \otimes M \rightarrow N \otimes M$ is a monomorphism for every $M \in \mathcal{S}$. But every left R -module is a direct limit of modules in \mathcal{S} , and \otimes commutes with direct limits, so we have that $S \otimes U \rightarrow N \otimes U$ is a monomorphism for every left R -module U , and this means that S is pure in N . □

Our next step will be to see how rugged modules behave with respect to submodules. This behavior will be used later.

Proposition 3.5. *If a module has a pure and rugged submodule then it is rugged itself. In particular, every module having a rugged direct summand is itself rugged and, as a consequence, direct summands of rugged modules need not be rugged and direct sums of rugged modules are rugged.*

Proof. Suppose that M is a left R -module and that $P \leq M$ is a pure and rugged submodule. Then, for any right R -module N in the flatness domain of M and any submodule $S \leq N$, we have a commutative diagram

$$\begin{array}{ccc} S \otimes P & \longrightarrow & N \otimes P \\ \downarrow & & \downarrow \\ S \otimes M & \longrightarrow & N \otimes M \end{array}$$

in which the bottom row and both columns are monomorphisms.

Then the upper row of the diagram is a monomorphism too and this means that $N \in \mathcal{F}^{-1}(P)$. But P is rugged so N is regular and then M is rugged. □

It is clear that if M is an N -flat module and $K \leq M$ is any pure submodule, then K is also N -flat, that is, $\mathcal{F}^{-1}(M) \subseteq \mathcal{F}^{-1}(K)$. This will be used in the following.

Proposition 3.6. *Let K be a submodule of a right module M . If M/K is flat for some $K \leq M$, then K is rugged if and only if M is rugged.*

Proof. Since M/K is flat, K is pure in M and $\mathcal{F}^{-1}(M/K) = R\text{-Mod}$. Then $\mathcal{F}^{-1}(K) = \mathcal{F}^{-1}(K) \cap \mathcal{F}^{-1}(M/K) \subseteq \mathcal{F}^{-1}(M)$ by Proposition 2.3, so by the comment above we get that $\mathcal{F}^{-1}(K) = \mathcal{F}^{-1}(M)$. This implies that K is rugged if and only if M is rugged. □

Semisimple left modules are left regular. There are left regular modules which are not semisimple. Coincidence of semisimple and regular modules leads to the following. Note that if the ring is right Noetherian or semilocal then every regular right module is semisimple (see [6]).

Proposition 3.7. *Let R be a ring such that every regular left module is semisimple. Then a right module M is rugged if and only if its character module M^+ is poor.*

Proof. The equality $\mathcal{F}^{-1}(M) = \mathcal{I}n^{-1}(M^+)$ follows immediately by the adjoint isomorphism theorem without any further assumption on R . Now, with our hypothesis on R this implies that M is rugged if and only if M^+ is poor. \square

Theorem 3.8 ([8, Theorem 3.2.11]). *Let M and N be right R -modules. If M is finitely presented then $M \otimes N^+ \cong \text{Hom}_R(M_R, N_R)^+$.*

Proposition 3.9. *Let R be a right Noetherian ring. Then $\mathcal{I}n^{-1}(M) = \mathcal{F}^{-1}(M^+)$ for each right module M . In particular, M is poor if and only if M^+ is rugged.*

Proof. By [20, Proposition 1.4] we know that a right R -module N holds in the class $\mathcal{I}n^{-1}(M)$ if and only if $nR \in \mathcal{I}n^{-1}(M)$ for every $n \in N$. Thus, if we prove that $C \in \mathcal{F}^{-1}({}_R M^+)$ if and only if $C \in \mathcal{I}n^{-1}(M)$ for each cyclic right R -module C we will have that $\mathcal{I}n^{-1}(M) = \mathcal{F}^{-1}({}_R M^+)$: if a right R -module N holds in $\mathcal{F}^{-1}(M^+)$ then $nR \in \mathcal{F}^{-1}(M^+) \forall n \in N$. But this means that $nR \in \mathcal{I}n^{-1}(M)$ by [20, Proposition 1.4] and so that $N \in \mathcal{I}n^{-1}(M)$.

On the other hand, if $N \in \mathcal{I}n^{-1}(M)$ then clearly $nR \in \mathcal{I}n^{-1}(M)$ for all $n \in N$. Again we have that $nR \in \mathcal{F}^{-1}(M^+)$ and then $\oplus nR \in \mathcal{F}^{-1}(M^+)$ (Proposition 2.4). But $\mathcal{F}^{-1}(M^+)$ is closed under quotients so $N = \sum_{n \in N} nR \in \mathcal{F}^{-1}(M^+)$ and we are done.

So let us now prove that $N \in \mathcal{F}^{-1}({}_R M^+) \Leftrightarrow N \in \mathcal{I}n^{-1}(M)$ for any cyclic N .

Let K be a submodule of N . Since K is finitely presented, we get the following commutative diagram whose columns are isomorphisms by Theorem 3.8

$$\begin{array}{ccccccc}
 K \otimes M^+ & \xrightarrow{\alpha} & N \otimes M^+ & \longrightarrow & N/K \otimes M^+ & \longrightarrow & 0 \\
 \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
 \text{Hom}(K, M)^+ & \xrightarrow{\beta} & \text{Hom}(N, M)^+ & \longrightarrow & \text{Hom}(N/K, M)^+ & \longrightarrow & 0
 \end{array}$$

Clearly α is monic if and only if β is monic. This implies that $N \in \mathcal{F}^{-1}(M^+)$ if and only if $N \in \mathcal{I}n^{-1}(M)$.

Finally, regular right modules are semisimple over right Noetherian rings so in particular M is poor if and only if M^+ is rugged. \square

Summing up Propositions 3.7 and 3.9 we get the following.

Corollary 3.10. *Let R be a right Noetherian ring. The following are hold.*

- (1) *A left module N is rugged if and only if N^{++} is rugged.*
- (2) *A right module M is poor if and only if M^{++} is poor.*

4. The flatness profile of a ring

Intuitively, the flatness domain of a module somehow tells us how far (or how close) such a module is from being flat. We shall see now that for any ring we can construct, by means of the flatness domain of all its modules, a lattice that shows the “levels of flatness” that the category of modules over such a ring can have.

Definition 4.1. The (left) flat profile of any ring R is defined as the class of flatness domains of all (left) R -modules,

$$f\mathcal{P}(R) = \{\mathcal{F}^{-1}(M); M \in R\text{-Mod}\}.$$

Recall (see [19] for instance) that the injectivity domain of a right R -module M is defined as $\mathcal{I}n^{-1}(M) = \{N \in \text{Mod-}R; M \text{ is } N\text{-injective}\}$, that the right injective profile of R is $i\mathcal{P}(R) = \{\mathcal{I}n^{-1}(M); M \in \text{Mod-}R\}$ (which is in bijective correspondence with a set), and that $\bigcap \mathcal{I}n^{-1}(M_i) = \mathcal{I}n^{-1}(\prod M_i)$ for any family of right R -modules $\{M_i; i \in I\}$.

Now, we know that the flat domain of a left module is the same as the injectivity domain of its character module. So there is a one to one correspondence from the left flat profile of a ring to its right injective profile, and so the flat profile of a ring can be considered as a set.

Since $f\mathcal{P}(R)$ is ordered by the inclusion, to have a lattice structure we only need to find the minimum of every subset of $f\mathcal{P}(R)$. This becomes clear from the following.

Proposition 4.2. $\bigcap_{i \in I} \mathcal{F}^{-1}(M_i) = \mathcal{F}^{-1}(\oplus_{i \in I} M_i)$ for any family $\{M_i; i \in I\}$ of left R -modules. Therefore $f\mathcal{P}(R)$ is always a lattice.

Proof. $\oplus M_i$ is N_R -flat if and only if $0 \rightarrow A \otimes (\oplus M_i) \rightarrow N \otimes (\oplus M_i)$ is exact for every submodule $A \leq N$, that is, if and only if $0 \rightarrow A \otimes M_i \rightarrow N \otimes M_i$ is exact for every $i \in I$ and every submodule $A \leq N$. But this just means that every M_i is N -flat, so we are done. □

Therefore we now see that $f\mathcal{P}(R)$ is a lattice, and by the comments above, the map

$$\begin{aligned} \varphi : f\mathcal{P}(R) &\rightarrow i\mathcal{P}(R) \\ \mathcal{F}^{-1}(M) &\mapsto \mathcal{I}n^{-1}(M^+) \end{aligned}$$

is one to one. Indeed φ is the canonical inclusion, and $\varphi(\bigcap \mathcal{F}^{-1}(M_i)) = \varphi(\mathcal{F}^{-1}(\oplus M_i)) = \mathcal{I}n^{-1}((\oplus M_i)^+) = \mathcal{I}n^{-1}(\prod M_i^+) = \bigcap \mathcal{I}n^{-1}(M_i)$ so this inclusion φ is a monomorphism of lattices and then we see that $f\mathcal{P}(R)$ is a sublattice of $i\mathcal{P}(R)$.

Rings for which every module is either injective or poor, that is, its injective profile consists exactly of two elements (the whole category of modules and the class of all semisimple modules) were introduced in [1] and named as rings having no middle class. We shall call rings having a similar flat profile as rings having no flat middle class.

Definition 4.3. A ring is said to have no flat middle class on the left if its left flat profile contains exactly two classes of modules, that is, every left R -module is either flat or rugged (and these two classes of modules are different).

Now, can we determine, at least in some cases, the shape of the lattice $f\mathcal{P}(R)$? Here there are some examples.

Examples.

- (1) The flat profile of a ring has a unique element if and only if all modules are rugged, so we have that $|f\mathcal{P}(R)| = 1$ if and only if R is regular.
- (2) By [19, Proposition 2.13] we know that the right injective profile of any right Artinian uniserial ring has exactly $\ell(R) - 1$ elements, where $\ell(R)$ denotes the length of its composition series. Since we the left flat profile is a sublattice of the right injective profile, it is easy to find rings with flat profile consisting of precisely 2 elements: non regular, right Artinian rings with composition length 3 ($\mathbb{Z}/4\mathbb{Z}$ is one such ring). Then, the flat profile of these rings consists of the whole category $\text{Mod-}R$ and the class of all regular right R -modules

(3) By the same argument of example 2 we see that if we find a non-flat and non-rugged $\mathbb{Z}/8\mathbb{Z}$ -module then we will have $|f\mathcal{P}(\mathbb{Z}/8\mathbb{Z})| = 3$.

$2\mathbb{Z}/8\mathbb{Z}$ is not a flat module so we only have to check that it is not rugged.

It is clear that $2\mathbb{Z}/8\mathbb{Z} \in \mathcal{F}^{-1}(2\mathbb{Z}/8\mathbb{Z})$ since the canonical map

$$4 \frac{\mathbb{Z}}{8\mathbb{Z}} \otimes 2 \frac{\mathbb{Z}}{8\mathbb{Z}} \longrightarrow 2 \frac{\mathbb{Z}}{8\mathbb{Z}} \otimes 2 \frac{\mathbb{Z}}{8\mathbb{Z}}$$

is a monomorphism, and on the other side the canonical map

$$4 \frac{\mathbb{Z}}{8\mathbb{Z}} \otimes 4 \frac{\mathbb{Z}}{8\mathbb{Z}} \longrightarrow 2 \frac{\mathbb{Z}}{8\mathbb{Z}} \otimes 4 \frac{\mathbb{Z}}{8\mathbb{Z}}$$

is not a monomorphism so $2\mathbb{Z}/8\mathbb{Z}$ is not a regular $\mathbb{Z}/8\mathbb{Z}$ -module.

We see that the flat and the injective profiles of the rings in examples 2 and 3 above coincide. We now prove that over general Noetherian rings we do always have an isomorphism between these two types of profiles.

Proposition 4.4. *If R is right Noetherian, then its left flat profile and its right injective profile coincide. In particular, R has no left flat middle class if and only if R has no right middle class.*

Proof. We know that

$$\begin{aligned} \varphi : f\mathcal{P}({}_R R) &\longrightarrow i\mathcal{P}(R_R) \\ \mathcal{F}^{-1}(M) &\longmapsto \mathcal{I}n^{-1}(M^+) \end{aligned}$$

is one to one. On the other hand, by Proposition 3.9, $\mathcal{I}n^{-1}(N) = \mathcal{F}^{-1}(N^+)$ for every right R -module N . Then $\mathcal{I}n^{-1}(N) = \mathcal{F}^{-1}(N^+) = \mathcal{I}n^{-1}(N^{++})$ for every N , by [4, Lemma 19.14]. This means that, the map φ is onto, and so φ is a bijection. □

The rings with no right middle class are characterized in [5, 9]. The question whether a ring with no right middle class is right Noetherian or not is not known. Now Proposition 4.4 on hand, we shall characterize the right Noetherian rings with no left flat middle class. First we recall the following result.

Theorem 4.5 ([5, Theorem 3]). *Let R be any ring. Then R has no right middle class if and only if $R = S \times T$, where S is semisimple Artinian and T satisfies one of the following conditions:*

- (i) T is Morita equivalent to a right PCI-domain, or
- (ii) T is a right SI right V -ring with the following properties:
 - (a) T has essential homogeneous right socle and
 - (b) for any submodule A of Q_T which does not contain the right socle of T properly, $QA = Q$, where Q is the maximal right quotient ring of T , or
- (iii) T is a right Artinian ring whose Jacobson radical properly contains no nonzero ideals.

Theorem 4.6 ([15, Theorem 3.11]). *A ring R is right SI if and only if R is right non-singular and $R = K \oplus R_1 \oplus R_2 \oplus \dots \oplus R_n$ where $K/\text{Soc}(K_K)$ is semisimple and each R_i is Morita equivalent to a right SI-domain.*

For domains the concepts of SI and PCI are equivalent.

Corollary 4.7. *Let R be a right Noetherian, right SI and right V -ring. Then R has no right middle class if and only if $R = S \oplus T$, where S is semisimple Artinian and T is Morita equivalent to a right PCI-domain.*

Proof. By Theorem 4.6, $R = K \oplus R_1 \oplus R_2 \oplus \cdots \oplus R_n$ where $K/\text{Soc}(K_K)$ is semisimple and each R_i is Morita equivalent to a right *SI*-domain for each $i = 1, \dots, n$. By the right Noetherian and right *V*-ring assumptions $\text{Soc}(K_K)$ is injective. This together with the fact that $K/\text{Soc}(K_K)$ is semisimple implies that K is semisimple. On the other hand, by [9, Lemmas 2 and 3], we must have $i = 1$. Setting $S = K$ and $T = R_1$, we get the desired decomposition. This proves the necessity. Sufficiency holds by [5, Lemma 2.4]. \square

Corollary 4.8. *Let R be a right Noetherian ring. The following are equivalent.*

- (1) R has no left flat middle class.
- (2) R has no right middle class.
- (3) $R = S \times T$, where S is semisimple Artinian and T satisfies one of the following conditions:
 - (i) T is Morita equivalent to a right *PCI*-domain, or
 - (ii) T is right Artinian ring whose Jacobson radical properly contains no nonzero ideals.

Proof.

(1) \Leftrightarrow (2) By Proposition 4.4.

(2) \Leftrightarrow (3) By Theorem 4.5 and Corollary 4.7. \square

Remark 4.9. Any von Neumann regular ring is a ring with no flat middle class. There are von Neumann regular rings which are not with no right (left) middle class (see, [9, Example 8]). We do not know whether any ring which is not regular and with no left flat middle class is right Noetherian. If this is the case, then we would have a complete characterization of the rings with no left flat middle class by Corollary 4.8.

5. Rings whose simple right modules are poor

In this section, we characterize the commutative simple-destitute rings. Also some results in [5] are generalized.

The following is a generalization of [1, Theorem 3.3], and it also shows that [5, Lemma 4.6] holds without the commutativity assumption on R .

Proposition 5.1. *Let R be a semilocal ring. The right R -module $R/J(R)$ is poor.*

Proof. Set $S = R/J(R)$ and suppose S is B -injective for some cyclic right R -module B . We claim that $\text{Rad}(B) = 0$. Suppose that there is a nonzero element $x \in \text{Rad}(B)$. Let $f : xR \rightarrow R/J(R)$ be a nonzero homomorphism. Then f can be extended to a homomorphism $g : B \rightarrow R/J(R)$. This implies that $f(xR) = g(xR) \subseteq g(\text{Rad}(B)) \subseteq \text{Rad}(R/J(R)) = 0$, a contradiction. Therefore $\text{Rad}(B) = 0$, i.e., $B.J(R) = 0$. Hence B is semisimple, because R is semilocal. \square

For convenience and the sake of self-containment, we include here the following known result.

Lemma 5.2. *Let R be a commutative ring and S a simple module. Then $S \cong S^+$.*

Proof. Let S be a simple module and P be the maximal ideal of R such that $S \cong R/P$. Let $\{S_i\}_{i \in I}$ be the complete set of non isomorphic simple modules. Then $\prod_{i \in I} E(S_i)$ is an injective cogenerator of R . $S^+ = \text{Hom}(S, \prod_{i \in I} E(S_i)) \cong \prod_{i \in I} \text{Hom}(S, E(S_i)) \cong \text{Hom}(S, E(S))$. By [14, p. 30], $\text{Hom}(S, E(S)) \cong S$. The proof is complete. \square

Proposition 5.3. *Let R be a commutative ring. The following are equivalent.*

- (1) Every simple R -module is either poor or injective.

(2) R satisfies one of the following two conditions:

- (a) Simple R -modules are either rugged or flat and regular R -modules are semisimple, or
- (b) R is a von Neumann regular ring.

Proof. Let S be a simple module. Then $S = S^+$ by Lemma 5.2, and so $\mathcal{F}^{-1}(S) = \mathcal{I}n^{-1}(S)$ by [4, Lemma 19.14]. (1) \Rightarrow (2) Suppose R is not von Neumann regular. Let S be a simple module. If S is injective, then $\mathcal{F}^{-1}(S) = R\text{-Mod}$. That is, S is flat. If S is poor, then $\mathcal{F}^{-1}(S)$ contains only the semisimple modules. Thus S is rugged. Since regular modules are always contained in $\mathcal{F}^{-1}(S)$ and S is poor, regular modules are semisimple. This proves (2).

(2) \Rightarrow (1) If the ring is von Neumann regular, then every simple is injective. So (1) holds in this case. Let S be a noninjective simple module. Then S is rugged by (2). That is, $\mathcal{F}^{-1}(S)$ consists of exactly the regular modules. By (2) again, every regular module is semisimple. This fact together with $\mathcal{F}^{-1}(S) = \mathcal{I}n^{-1}(S)$ implies that S is poor. \square

Lemma 5.4 ([21, Proposition 1.1]). *Let R be a commutative ring and I a finitely generated ideal of R . If $I^2 = I$, then I is a direct summand of R .*

By [5, Lemma 4.6], commutative local rings are simple-destitute. The following theorem gives a characterization of commutative simple-destitute rings.

Theorem 5.5. *Let R be a commutative ring. The following are equivalent.*

- (1) R is simple-destitute.
- (2) Every simple module is rugged, and regular modules are semisimple.
- (3) R is semisimple Artinian, or R is local.

Proof.

(1) \Rightarrow (3) Suppose R is not local and let us prove that R is semisimple Artinian. If R has an injective simple module, then being poor and injective implies R is semisimple Artinian (see, [1, Remark 2.3]). Now assume that all simple modules are noninjective. Let I be an ideal which is not properly contained in $\text{rad}(R)$. We shall prove that R/I is semisimple. First suppose $I \neq \text{rad}(R)$. Then there is a maximal ideal P of R such that $R = P + I$. Set $S = R/P$. Then $S = S.R = S(P + I) = SP + SI = SI$. We shall prove that S is R/I -injective. Let $X/I \leq R/I$ and $f : R/I \rightarrow S$. Then $S = SI$ and $f(R/I) \subseteq S$ together implies that $f(R/I) \subseteq f(R/I).I = 0$. Thus S is R/I -injective, and so R/I is semisimple by the hypothesis.

Now for $I = \text{rad}(R)$, let us prove that R/I is semisimple. We shall prove this by showing that S is R/I -injective. Let $0 \neq X/I \leq R/I$ and $0 \neq f : X/I \rightarrow S$. If $\text{Ker}(f) = 0$, then X/I is simple and so a direct summand of R/I , because $\text{rad}(R/I) = 0$. This, clearly, implies that f extends to R/I . Now suppose $0 \neq \text{Ker } f = K/I$. Then K/I is a maximal submodule of X/I and I is properly contained in K . Thus R/K is semisimple by the previous paragraph, and so $R/K = X/K \oplus L/K$ for some $X/K \leq R/K$. Let $\bar{f} : X/K \rightarrow S$ be the homomorphism induced by f , $\pi : R/I \rightarrow R/K$ the natural epimorphism, $p : R/K \rightarrow X/K$ projection homomorphism. Then $\bar{f}p\pi$ extends f . Thus S is R/I -injective, and so R/I is semisimple, by the hypothesis again.

Now let B be an ideal of R properly containing $J = \text{rad}(R)$. Since R/J is semisimple and cyclic, B/J is finitely generated and semisimple as a direct summand of R/J . Then there is a finitely generated ideal A of R such that $B = A + J$. Since A is not contained in J , R/A is semisimple again by the arguments above. This implies that $J \subseteq A$, and so $B = A + J = A$ is finitely generated. We conclude that every maximal ideal of R is finitely generated. Now let P, Q be distinct maximal ideals of R . Since $P + Q^2 = R$, R/P is (R/Q^2) -injective. Then R/Q^2 is semisimple by our hypothesis, and hence $Q = Q^2$. In the same way, we get that for every maximal ideal X of R , $X^2 = X$. Being idempotent and finitely generated implies every maximal ideal is a direct summand by Lemma 5.4. Therefore every simple module is projective, and so R is semisimple Artinian.

(3) \Rightarrow (1) If R is semisimple Artinian, then every module is poor, in particular simple modules are poor. If the ring is local, then its unique simple is poor by Proposition 5.1.

(1) \Rightarrow (2) By Proposition 5.3. (2) \Rightarrow (1) By Proposition 3.7 and Lemma 5.2. \square

In [5], the authors characterize the commutative Noetherian rings with no simple middle class. The following result is a slight generalization of [5, Theorem 4.7].

Proposition 5.6. *Let R be a commutative ring with $\text{Soc}(R) = 0$. Suppose every maximal ideal is finitely generated. The following are equivalent.*

- (1) R has no simple middle class.
- (2) R is simple-destitute.
- (3) R is local.

Proof.

(2) \Leftrightarrow (3) By Theorem 5.5 and the fact that $\text{Soc}(R) = 0$.

(2) \Rightarrow (1) is clear.

(1) \Rightarrow (2) Let S be a simple module. Suppose S is injective. Then S is flat by [22, Lemma 2.6]. Since maximal ideals are finitely generated, S is finitely presented. Then S is projective by [18, Theorem 4.30]. This implies that R has a direct summand isomorphic to S . This contradicts with the fact that $\text{Soc}(R) = 0$. Therefore S is not injective, and so every simple module is poor by (1). This completes the proof. \square

In [5, Theorem 3.7], the right Artinian rings with no right simple middle class are characterized. For right semiartinian rings we have the following.

Proposition 5.7. *Let R be a right semiartinian ring with no simple middle class. Then R is a right V -ring or, there is a ring direct sum $R = S \oplus T$, where S is semisimple Artinian and T has a unique noninjective simple right module up to isomorphism, and $\text{Soc}(T)$ is homogeneous.*

Proof. Suppose R is not a V -ring. Let U be a noninjective simple right module. Then U is poor by the hypothesis. By similar arguments used in [9], R has a unique noninjective simple right module, up to isomorphism, under the stated hypothesis. Let S be the sum of the injective simple right ideals of R . We claim that, S is injective. Suppose the contrary and let E be the injective hull of S . By the right semiartinian condition the socle of E/S is nonzero. Let X/S be a simple submodule of E/S . We shall prove that U is X -injective. Let A be a nonzero submodule of X and $f : A \rightarrow U$ be a homomorphism. If $A \leq S$, then $\text{Hom}(A, U) = 0$ because U is noninjective and A is semisimple. So f extends to X , trivially. Suppose A is not contained in S . Then $A + S = X$, because S is a maximal submodule of X and A is nonzero. Since S is semisimple, $S = A \cap S \oplus S'$ for some $S' \leq S$. Then $X = A \oplus S'$, and clearly $f\pi : X \rightarrow U$ extends f , where $\pi : X \rightarrow A$ is the natural projection. This implies that U is X -injective. But U is poor, so X is semisimple, and then S is a direct summand of X . This contradicts with the fact that S is essential in X . Therefore S must be injective. Hence $R = S \oplus T$ for some right ideal T of R . By the choice of S , we have $\text{Hom}(S, T) = 0$ and $\text{Hom}(T, S) = 0$. Thus S and T are two-sided ideals, and so $R = S \oplus T$ is a ring direct sum. As R has a unique noninjective simple right module, T has the same property as well. This completes the proof. \square

We do not know whether the converse of Proposition 5.7 is true or not. Regarding this, we have the following.

Proposition 5.8. *Let R be a right semiartinian ring with a unique noninjective simple right module U . Then $U^{(\mathbb{N})}$ is poor.*

Proof. Suppose $U^{(\mathbb{N})}$ is B -injective for some (nonzero) cyclic right module B . The socle $\text{Soc}(B)$ is essential in B by the semiartinian condition. Let $\text{Soc}(B) = U^{(I)}$ for some index set I . Let us show that I must be finite. If I is infinite, then B has a (non finitely generated) semisimple submodule, say A , isomorphic to $U^{(\mathbb{N})}$. This implies that, A is B -injective, and so $B = A \oplus A'$ for some $A' \leq B$. This contradicts with the fact that B is cyclic. Therefore I must be finite.

If I is finite, then $\text{Soc}(B) = U^{(I)}$ is B -injective. So $\text{Soc}(B)$ is a direct summand of B . Thus B is semisimple, because $\text{Soc}(B)$ is essential in B . Therefore $U^{(\mathbb{N})}$ is poor. □

The converse of Proposition 5.7 is true for commutative semiartinian rings. To see this, we need the following lemma.

Lemma 5.9. *Let R be a commutative ring and U be a simple R -module. If U is B -injective for some module B , then $U^{(I)}$ is B -injective for every index set I .*

Proof. Let $P = \text{ann}_R(U)$ and I be an index set. U is B -injective by the hypothesis, so U^I is B -injective by [20, Proposition 1.6]. Since R is commutative and $U^I.P = 0$, the module U^I is a semisimple R/P -module. So U^I is also semisimple as an R -module. This implies $U^{(I)}$ is a direct summand of U^I , and so it is B -injective. □

The following is a consequence of Proposition 5.7 and Lemma 5.9.

Corollary 5.10. *Let R be a commutative semiartinian ring. Then R has no simple middle class if and only if $R = S \times T$, where S is semisimple Artinian and T is local.*

6. Rugged abelian groups

In this section, we characterize the rugged abelian groups. It turns out that, the notions of rugged, poor and p -poor coincide over the ring of integers.

Proposition 6.1. *Let R be a commutative hereditary Noetherian ring. Then a module N is rugged if and only if the singular submodule $Z(N)$ of N is rugged.*

Proof. Since R is a commutative hereditary ring, $N/Z(N)$ is flat by [15, Proposition 2.3]. Thus $Z(N)$ is a pure submodule of N . Now the proof follows by Proposition 3.6. □

Example 6.2. The socle $\bigoplus_p \mathbb{Z}_p$ of \mathbb{Q}/\mathbb{Z} , where p ranges over all primes and \mathbb{Z}_p is the simple abelian group of order p , is a rugged \mathbb{Z} -module.

Proof. We have $(\bigoplus_p \mathbb{Z}_p)^+ = \text{Hom}(\bigoplus_p \mathbb{Z}_p, \mathbb{Q}/\mathbb{Z}) \cong \prod_p \text{Hom}(\mathbb{Z}_p, \mathbb{Q}/\mathbb{Z}) \cong \prod_p \mathbb{Z}_p$. The group $\prod_p \mathbb{Z}_p$ is poor by [2, Theorem 3.1]. Since $\bigoplus_p \mathbb{Z}_p$ is the torsion part of $\prod_p \mathbb{Z}_p$, $\bigoplus_p \mathbb{Z}_p$ is pure in $\prod_p \mathbb{Z}_p$. Then $\bigoplus_p \mathbb{Z}_p$ is rugged by Proposition 3.7. □

The following lemma is well known. We include it for easy reference.

Lemma 6.3. *Let p be a prime integer and $m, n \in \mathbb{Z}^+$. If $m \leq n$, then \mathbb{Z}_{p^n} is \mathbb{Z}_{p^m} -injective.*

Theorem 6.4. *An abelian group G is rugged if and only if its torsion part $T(G)$ has a direct summand isomorphic to $\bigoplus_p \mathbb{Z}_p$.*

Proof. To prove the necessity, let G be a rugged group. Then $T(G)$ is rugged by Proposition 6.1. If $T_p(G) = 0$ for some prime p , then $pT(G) = T(G)$. Whence $p\mathbb{Z}_{p^2} \otimes T(G) \cong \mathbb{Z}/p\mathbb{Z} \otimes T(G) \cong T(G)/pT(G) = 0$. This implies that $T(G)$ is \mathbb{Z}_{p^2} -flat, a contradiction. Thus $T_p(G) \neq 0$ for each prime p . Fix a prime p , and let B be the p -basic subgroup of $T_p(G)$ (see, [13, Chapter VI]). B is a direct sum of cyclic p -groups i.e. groups isomorphic \mathbb{Z}_{p^n} . We claim that B has a direct summand isomorphic to \mathbb{Z}_p . Suppose the contrary, and let $B = \bigoplus_{i \in I} \langle a_i \rangle$ where each $\langle a_i \rangle$ is a cyclic group isomorphic to \mathbb{Z}_{p^n} for some $n \geq 2$. Then, for each $i \in I$, $\langle a_i \rangle$ is \mathbb{Z}_{p^2} -flat by Lemma 6.3. So B is \mathbb{Z}_{p^2} -flat. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & B \otimes \mathbb{Z}_p & \xrightarrow{\theta} & T_p(G) \otimes \mathbb{Z}_p & \longrightarrow & (T_p(G)/B) \otimes \mathbb{Z}_p \longrightarrow 0 \\
 & & \downarrow \alpha & & \downarrow \gamma & & \downarrow \\
 0 & \longrightarrow & B \otimes \mathbb{Z}_{p^2} & \xrightarrow{\beta} & T_p(G) \otimes \mathbb{Z}_{p^2} & \longrightarrow & (T_p(G)/B) \otimes \mathbb{Z}_{p^2} \longrightarrow 0
 \end{array}$$

Since B is \mathbb{Z}_{p^2} -flat and pure in $T_p(G)$, α, β and θ are monomorphisms. Then $\beta\alpha = \gamma\theta$ is a monomorphism. $T_p(G)/B$ is divisible, so $T_p(G)/B \otimes \mathbb{Z}_p = (T_p(G)/B) \otimes \mathbb{Z}_{p^2} = 0$. Thus θ is an isomorphism. This clearly implies that γ is a monomorphism, and so $T_p(G)$ is \mathbb{Z}_{p^2} -flat. Since $T(G) = \bigoplus_p T_p(G)$ and $T_q(G) \otimes \mathbb{Z}_p = 0$ for all primes $q \neq p$, we have $T(G) \otimes \mathbb{Z}_p \cong T_p(G) \otimes \mathbb{Z}_p$ and $T(G) \otimes \mathbb{Z}_{p^2} \cong T_p(G) \otimes \mathbb{Z}_{p^2}$. Thus $T(G)$ is \mathbb{Z}_{p^2} -flat. This contradicts with the fact that, $T(G)$ is rugged. So B must have a direct summand, say A_p , isomorphic to \mathbb{Z}_p . Now A_p is bounded and pure in $T_p(G)$, so A_p is a direct summand of $T_p(G)$ by [13, Theorem 27.5]. Write $T_p(G) = A_p \oplus C_p$. Then $T(G) = \bigoplus_p T_p(G) = \bigoplus_p (A_p \oplus C_p) = (\bigoplus_p A_p) \oplus (\bigoplus_p C_p)$. Note that $\bigoplus_p A_p \cong \bigoplus_p \mathbb{Z}_p$. This completes the proof of the necessity.

For the sufficiency, suppose that $T(G)$ contains a direct summand isomorphic to $\bigoplus_p \mathbb{Z}_p$. Then $\bigoplus_p \mathbb{Z}_p$ is a pure submodule of $T(G)$. On the other hand $T(G)$ is pure in G , whence $\bigoplus_p \mathbb{Z}_p$ is a pure submodule of G . Since $\bigoplus_p \mathbb{Z}_p$ is rugged, G is rugged by Proposition 3.5. □

By [2, Theorem 3.1], [3, Theorem 4.1] and Theorem 6.4, we get the following.

Corollary 6.5. *For an abelian group G , the following are equivalent.*

- (1) G is poor.
- (2) G is rugged.
- (3) G is p -poor.

7. Homological properties

Let us now study how often rugged (pre)covers and (pre)envelopes (see [8] for the definitions) happen in any category of modules.

Proposition 7.1. *Every left (right) R -module has a surjective rugged precover for any ring R .*

Proof. We know there exists at least a rugged left R -module P , so for $M \in R\text{-Mod}$ the module $M \oplus P$ is rugged. Thus the first canonical projection $M \oplus P \rightarrow M$ is a surjective rugged precover of M . □

But what about rugged covers? Are they as easy to find as rugged precovers? The answer is no. Indeed, as we shall see in the next result, rugged covers become really rare in general. In fact every ring is very poor in its supply of rugged covers. They only occur in the trivial case when the module is itself rugged.

Theorem 7.2. *A left (right) module has a rugged cover if and only if it is rugged.*

Proof. If M is any left R -module we know $p_1 : M \oplus P \rightarrow M$ (P is any rugged left module) is a rugged precover, so if M has a rugged cover, say $\varphi : F \rightarrow M$, we get a commutative diagram

$$\begin{array}{ccc} & M \oplus P & \\ \nearrow f & \downarrow p_1 & \\ F & \xrightarrow{\varphi} & M \end{array}$$

and we know that φ is surjective and that $f(F)$ is a direct summand of $M \oplus P$.

But since $f(F)$ is a direct summand of $M \oplus P$, $p_1 f(F) = M$ means that $f(F) = M \oplus T$ for some direct summand T of P .

Now, the diagram

$$\begin{array}{ccc} & M \oplus T & \\ \nwarrow & \downarrow p_1 & \\ M \oplus T & \xrightarrow{p_1} & M \end{array}$$

can be completed commutatively by $id_M \oplus 0$, so we see that $p_1 : M \oplus T \rightarrow M$ cannot be a rugged cover unless $T = 0$ and so M should be rugged and its rugged cover should be the trivial one, $M \overset{=} M$. □

The same applies to rugged preenvelopes and envelopes, so rugged preenvelopes always exist but nontrivial rugged envelopes never happen.

Proposition 7.3. *If P is any nontrivial rugged module, then for any module M the canonical injection $M \rightarrow M \oplus P$ is an injective rugged preenvelope. Furthermore, M has a rugged envelope if and only if M is rugged.*

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