ON RELATIVE PROJECTIVITY OF SOME CLASSES OF MODULES

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ABSTRACT

ON RELATIVE PROJECTIVITY OF SOME CLASSES OF MODULES

The main purpose of this thesis is to study R-projectivity and max-projectivity of some classes of modules, and module classes related to max-projective modules. A right R-module M is called max-projective provided that each homomorphism $f : M \to R/I$ where I is any maximal right ideal, factors through the canonical projection $\pi : R \to R/I$. We call a ring R right almost-QF (resp. right max-QF) if every injective right R-module is R-projective (resp. max-projective). In this thesis we attempt to understand the class of right almost-QF (resp. right max-QF) rings. Among other results, we prove that a right Hereditary right Noetherian ring R is right almost-QF if and only if R is right max-QF if and only if $R = S \times T$, where S is semisimple Artinian and T is right small. A right Hereditary ring is max-QF if and only if every injective simple right R-module is projective. Furthermore, a commutative Noetherian ring R is almost-QF if and only if R is max-QF if and only if $R = A \times B$, where A is QF and B is a small ring. Moreover, we introduced and studied some homological objects related with max-projective modules.

ÖZET

BAZI MODÜL SINIFLARININ BAĞIL PROJEKTİFLİĞİ ÜZERİNE

Bu tezde bazı modül sınıflarının R-projektifliği ve max-projektifliğinin ve max-projektiflikle bağlantılı modül sınıflarının çalışılması amaçlanmaktadır. Bir sağ modül M'ye max-projektif modül denir eğer her maksimal sağ ideal için, her $f : M \to R/I$ homomorfizması, $\pi : R \to R/I$ kanonik projeksiyonu üzerinden taşınabiliyorsa. Her injektif sağ R-modülü R-projektif (max-projektif) olan halkalara sağ almost-QF (max-QF) halka denir. Bu tezde sağ almost-QF (max-QF) halka sınıflarını anlamaya çalışacağız. Diğer sonuçlar arasında, sağ kalıtsal sağ Noether bir R halkasının sağ almost-QF olması ancak ve ancak R halkasının sağ max-QF olması ancak ve ancak S yarıbasit Artin halka ve T sağ küçük halka olmak üzere $R = S \times T$ şeklinde olmasıdır. Sağ kalıtsal R halkanın max-QF olması ancak ve ancak her injektif basit sağ R-modülün projektif olmasıdır. Dahası, değişmeli Noether bir R halkasının almost-QF olması ancak ve ancak halkanın max-QF olması ancak ve ancak A halkası QF halka ve B halkası küçük halka olmak üzere $R = A \times B$ şeklinde olmasıdır. Bunların yanında, max-projektif modüller ile ilgili bazı homolojik nesneleri tanımladık ve inceledik.

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LIST OF ABBREVIATIONS

R	an associative ring with unit unless otherwise stated
$\mathbb{N}$, $\mathbb{Z}^+$	the set of all non-negative integers, the set of all positive integers
$\mathbb{Z}$, $\mathbb{Q}$	the ring of integers, the field of rational numbers
$\mathbb{Z}_{p^\infty}$	the Prüfer (divisible) group for the prime p (the p-primary part of the torsion group $\mathbb{Q}/\mathbb{Z}$)
$R\text{-}\mathcal{M}od$	the category of *left* R-modules
$\mathcal{M}od\text{-}R$	the category of *right* R-modules
$\mathrm{Hom}_R(M, N)$	all R-module homomorphisms from M to N
$M \otimes_R N$	the tensor product of the *right* R-module M and the *left* R-module N
$\mathrm{Ker}\, f$	the kernel of the map f
$\mathrm{Im}\, f$	the image of the map f
M^+	the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$
$\mathrm{Soc}\, M$	the socle of the R-module M
$\mathrm{Rad}\, M$	the radical of the R-module M
$\mathrm{E}(M)$	the injective envelope (hull) of a module M
$\mathrm{T}(M)$	the torsion submodule of a module M
$Z(M)$	the singular submodule of a module M
$\mathrm{J}(R)$	the Jacobson radical of the ring R
$cl(M)$	the composition lengt of the R-module M
$\mathrm{Ext}_R(C, A) = \mathrm{Ext}^1_R(C, A)$	set of all equivalence classes of short exact sequences starting with the R-module A and ending with the R-module C
$\cong$	isomorphic
$\subseteq$	submodule
$\ll$	small (=superfluous) submodule

CHAPTER 1

INTRODUCTION

Throughout, we shall assume that all rings are associative with identity and all modules are unitary modules. Let R be any ring. Given right R-modules M and N, M is said to be injective relative to N (or M is N-injective) if, for any submodule K of N, any R-homomorphism $f : K \to M$ extends to an R-homomorphism $g : N \to M$. A right module M is injective relative to every right R-module is called an injective right R-module. A right R-module E is said to be max-injective (or m-injective for short) in case for any maximal right ideal I of R, each homomorphism $f : I \to E$ can be extended to a homomorphism $g : R \to E$, i.e. $Ext^1_R(R/I, E) = 0$ for every maximal right ideal I of R_R. A ring R is said to be right max-injective in case the right regular module R_R is max-injective. Obviously, any injective module is max-injective. For the properties of max-injective modules (see, (Wang and Zhao, 2005)).

Let M and N be right R-modules. M is called N-projective (projective relative to N) if every R-homomorphism from M into an image of N can be lifted to an R-homomorphism from M into N. M is called R-projective if it is projective relative to the right R-module R_R. The module M is called projective if M is N-projective, for every R-module N. In recent years, there is an appreciably interest to R-projective modules and to the rings defined via these modules. We call a right R-module M is *max-projective* provided that each homomorphism $f : M \to R/I$ where I is any maximal right ideal, factors through the canonical projection $\pi : R \to R/I$. This notion properly generalizes the notions R-projective modules and rad-projective modules studied in (Amin, Ibrahim and Yousif, 2013).

Characterizing rings by projectivity of some classes of their modules is a classical problem in ring and module theory. A result of Bass ((Anderson and Fuller, 1992), Theorem 28.4) states that a ring R is right perfect if and only if each flat right R-module is projective. On the other hand, the ring R is QF if and only if each injective right R-module is projective (Faith, 1976). Recently, the notion of R-projectivity and its generalizations are considered in (Alhilali et al., 2017), (Amin, Ibrahim and Yousif, 2011), (Amin, Ibrahim and Yousif, 2013), (Amini, Ershad and Sharif, 2008), (Sandomierski, 1964). The rings whose flat right R-modules are R-projective and max-projective are characterized in (Amini, Amini, and Ershad, 2009), (Amini, Ershad and Sharif, 2008) and (Büyükaşık, 2012), respectively.

We call a ring R right almost-QF (resp. right max-QF) in case all injective right R-modules are R-projective (resp. max-projective). Right almost QF-rings are max-QF. The ring of integers is almost-QF, since $\mathrm{Hom}(E, \mathbb{Z}/n\mathbb{Z}) = 0$ for each injective $\mathbb{Z}$-module E.

In Chapter 3, we investigate some properties of max-projective R-modules and give some characterizations of almost-QF and max-QF rings. We obtain that R-projectivity and max-projectivity coincide over the ring of integers and over right perfect rings. Characterizations of semiperfect, perfect and QF rings in terms of max-projectivity are given. As an application, we show that a ring R is right (semi)perfect if and only if every (finitely generated) right R-module has a max-projective cover if and only if every (simple) semisimple right R-module has a max-projective cover. By ((Alhilali et al., 2017), Lemma 2.1) any finitely generated R-projective right R-module is projective. This result is not true when R-projectivity is replaced with max-projectivity. We prove that if R is either a semiperfect or nonsingular self-injective ring, then finitely generated max-projective right R-modules are projective. We show that any max-projective right R-module of finite length is projective.

We also give some characterizations of almost-QF and max-QF rings. Every right small ring is right max-QF, while a right small ring is right almost-QF provided R is right Hereditary or right Noetherian. A right Hereditary right Noetherian ring R is right almost-QF if and only if R is right max-QF if and only if $R = S \times T$, where S is a semisimple Artinian and T is a right small ring. A right Hereditary ring R is right max-QF if and only if every simple injective right R-module is projective. A commutative Noetherian ring R is almost-QF if and only if R is max-QF if and only if $R = A \times B$, where A is QF and B is a small ring. A right Noetherian local ring is almost-QF if and only if R is QF or right small.

In Chapter 4, some homological objects which is related to the max-projective modules are studied. Namely, max-injective and max-flat modules. A right R-module M is called max-flat (or m-flat for short) if $Tor_1^R(M, R/I) = 0$ for any maximal left ideal I of R (see (Wang, 2005)). A right R-module M is m-flat if and only if M^+ is m-injective by the standard isomorphism $Ext_R^1(R/I, M^+) \cong Tor_1^R(M, R/I)^+$ for any maximal left ideal I of R. Also, the concept of m-cotorsion modules is introduced. A right R-module M is said to be m-cotorsion if $Ext_R^1(N, M) = 0$ for any m-flat right R-module N. Several elementary properties of m-flat, m-injective and m-cotorsion modules are obtained in this chapter. In what follows, we write $\mathfrak{MI}$, $\mathfrak{MF}$, $\mathfrak{MC}$ for the class of all m-injective left, all m-flat right and all m-cotorsion right R-modules, respectively. We introduce the concept of m-flat dimensions of modules and rings in terms of right derived functors of $-\otimes-$. For a left max-coherent ring R, we prove that R is left max-hereditary (i.e. if every maximal left ideal is projective) if and only if every quotient of an m-injective left R-module is m-injective if and only if every submodule of an m-flat right R-module is m-flat if and only if every left R-module has a monic m-injective cover if and only if R is a left strongly max-coherent ring and gl left $\mathfrak{MF}$-dim ${}_R\mathfrak{M} \leq 1$ (gl right $\mathfrak{MI}$-dim ${}_R\mathfrak{M} \leq 1$). For a left max-coherent ring R, it is also shown that, R is left SF-ring if and only if gl right $\mathfrak{MI}$-dim ${}_R\mathfrak{M} = 0$ if and only if every cotorsion left R-module is m-injective if and only if every cotorsion right R-module is m-flat if and only if R is left strongly max-coherent ring and every m-cotorsion right R-module is m-flat. Similar to well known notion of pure modules, in (Crivei, 2014), an exact sequence of left R-modules $0 \to A \to B \to C \to 0$ is called s-pure exact provided that $0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ is exact for any simple right R-module. In this case, A is said to be an s-pure submodule of B. In (Büyükaşık and Durğun, 2015), the authors introduced that a left R-module N is s-pure injective (in short sp-injective), (in (Hamid, 2019) is called coneat injective) if it is injective with respect to s-pure short exact sequences. Clearly, every SP-injective module is pure-injective. Motivated by SP-injective modules, we introduce the concept of SP-flat modules. We call a right R-module M SP-flat if for every s-pure exact sequence $0 \to A \to B \to C \to 0$ of left R-modules, the sequence $0 \to M\otimes A \to M\otimes B \to M\otimes C \to 0$ is exact. Some preliminary properties of SP-injective and SP-flat modules are obtained. We then give several characterizations of s-purity and m-flat modules in terms of SP-injective modules. Finally we prove that a ring R left SF if and only if every m-cotorsion right (SP-injective right) R-module is injective if and only if every SP-flat left R-module is flat if and only if gl left $\mathfrak{MF}$-dim $\mathfrak{M}_R = 0$.

CHAPTER 2

PRELIMINARIES

Throughout this thesis, R will denote an associative ring with identity. If not stated otherwise, the symbol R, stands for a general ring and modules will be unital R-modules.

Essentially, we assume the fundamentals of module and ring theory and homological algebra are known. All definitions which are not given here can be found in (Anderson and Fuller, 1992), (Rotman, 1979), (Goodearl, 1976), (Lam, 1999) and (Enochs and Jenda, 2000).

In this chapter we introduce our basic terminology for rings and modules, as well as the fundamental results to be used in this thesis.

2.1. Injective Modules and Noetherian Rings

In this section we give some properties of relative injective modules which can be found in (Anderson and Fuller, 1992) and (Lam, 1999). We also give the characterizations of (commutataive) Noetherian rings in terms of (indecomposable) injective modules.

Definition 2.1 *Given right modules M and N, M is said to be injective relative to N (or M is N-injective) if, for any submodule K of N, any R-homomorphism $f : K \to M$ extends to an R-homomorphism $g : N \to M$. A right module M is injective relative to every right R-module is called an injective right R-module.*

Lemma 2.1 *Let p be a prime integer and $m, n \in \mathbb{Z}^+$. If $m \leq n$, then $\mathbb{Z}_{p^n}$ is $\mathbb{Z}_{p^m}$-injective.*

Proposition 2.1 *(**Baer's Criterion**) A right module M is injective if and only if for any right ideal I of R, any R-homomorphism $f : I \to M$ can be extended to $g : R \to M$.*

A family of subsets $\{A_i : i \in I\}$ in a set $\mathfrak{A}$ is said to satisfy the *Ascending Chain Condition (ACC)* if, for any ascending chain $A_{i_1} \subseteq A_{i_2} \subseteq A_{i_3} \subseteq \cdots$ in the family, there exists an integer n such that $A_{i_n} = A_{i_{n+k}}$ for each $k \in \mathbb{N}$. A family of subsets $\{A_i : i \in I\}$ in a set $\mathfrak{A}$ is said to satisfy the *Descending Chain Condition (DCC)* if, for any descending chain $A_{i_1} \supseteq A_{i_2} \supseteq A_{i_3} \supseteq \cdots$ in the family, there exists an integer n such that $A_{i_n} = A_{i_{n+k}}$ for each $k \in \mathbb{N}$.

A module M is called *Artinian (Noetherian)* if the family of all submodules of M satisfies *DCC (ACC)*. A ring R is called right *Artinian (Noetherian)* if R_R is *Artinian (Noetherian)*. A similar definition can be made on the left. R is Artinian (Noetherian) if it is both right and left Artinian (Noetherian). M is Noetherian if and only if every submodule of M is finitely generated. The Artinian and Noetherian properties are inherited by submodules and factor modules. Finitely generated modules over a right Artinian (Noetherian) ring are Artinian (Noetherian). If R is right Noetherian, then every finitely generated module is finitely presented.

Direct summands and direct product of injective modules are injective. On the other hand, it is not true that the direct sum of injective modules is injective.

Theorem 2.1 *((Lam, 1999), Theorem 3.46 and 3.48) The following are equivalent for a ring R.*

(1) R is right Noetherian.

(2) Every direct limit (directed index set) of injective right modules is injective.

(3) Every direct sum of injective right modules is injective.

(4) Any injective module M_R is a direct sum of indecomposable (injective) submodules.

We now proceed to understand the structure of a typical indecomposable injective R-modules over a commutative noetherian rings.

Proposition 2.2 *(See (Matlis, 1958)) Let R be a commutative Noetherian ring, P be a prime ideal of R, $E = E(R/P)$, and $A_i = \{x \in E : P^i x = 0\}$. Then:*

(1) *A_i is a submodule of E, $A_i \subseteq A_{i+1}$, and $E = \bigcup A_i$.*

(2) *If P is a maximal ideal of R, then $A_i \subseteq E(R/P)$ is a finitely generated R-module for every integer i.*

(3) *$E(R/P)$ is Artinian.*

2.2. Projective Modules and Hereditary Rings

In this section we give some properties of relative projective modules which can be found in (Anderson and Fuller, 1992) and (Lam, 1999). We also give some characterizations of right hereditary, semihereditary and quasi-Frobenius rings in terms of injective and projective modules.

Definition 2.2 *Let M and N be right R-modules. M is called N-projective (projective relative to N) if for each epimorphism $g : N \to K$ and each homomorphism $f : M \to K$ there is an R-homomorphism $h : M \to N$ such that $gh = f$. M is called R-projective if it is projective relative to the right R-module R_R. The module M is called projective if M is N-projective, for every right R-module N.*

Direct sums and direct summands of (R-)projective modules are (R-)projective. A ring is a projective module over itself. Every free module is a projective module.

Proposition 2.3 *((Anderson and Fuller, 1992), Proposition 17.2) The following properties hold for a right R-module P.*

(1) P is projective;

(2) Every epimorphism $M \to P \to 0$ splits;

(3) P is isomorphic to a direct summand of a free R-module.

Proposition 2.4 *((Anderson and Fuller, 1992), Proposition 17.14) Every non-zero projective module contains a maximal submodule.*

Definition 2.3 *A ring R is called right hereditary if each right ideal of R is projective. A ring R is called right semihereditary if each finitely generated right ideal of R is projective.*

Dedekind domains and Prüfer domains are hereditary and semihereditary rings, respectively. The following Theorem shows that if R is hereditary, projective modules are closed under submodules and injective modules are closed under quotient modules.

Theorem 2.2 *((Rotman, 1979), Theorem 4.23) The following are equivalent for a ring R.*

(1) R is right hereditary.

(2) Every submodule of a projective module is projective.

(3) Every factor module of an injective module is injective.

The ring R is called a QF (quasi-Frobenius) ring if R is right and left self-injective and Artinian. Equivalently, R is a right self injective ring which is right or left Noetherian. The class of QF rings is one of the most interesting classes of non-semisimple rings.

Theorem 2.3 *((Lam, 1999), §15) The following are equivalent for a ring R.*

(1) R is QF

(2) R is left or right artinian, and R is left or right self-injective.

(3) R is left or right noetherian, and R is left or right self-injective.

(4) R has ACC on left or right annihilators, and R is left or right self-injective.

(5) Every injective right (or left) module is projective.

(6) Every projective right (or left) R-module is injective.

One can formulate the Dual Baer Criterion as follows: a module M is projective if and only if it is R-projective. In general, the dual to the Baer Criterion is not true as there are examples of R-projective modules that are not projective. From this point of view, it is natural to consider the rings over which dual Baer Criteria hold. Dualizations are often possible over perfect rings. In (Sandomierski, 1964), Sandomierski proved that, dual Baer Criteria hold over right perfect rings. Later on, C. Faith conjectured that, the rings over which dual Baer Criteria is hold are exactly the right perfect rings, (Faith, 1976). Recently, in (Trlifaj, 2019) it is proved that the Faith conjecture is undecidable.

2.3. Flat Modules and Regular Rings

The purpose of this section is to give some properties of flat modules which can also be found in (Lam, 1999) and (Rotman, 1979).

Definition 2.4 *Given a left module M and a right module N, M is N-flat if for every submodule K of N the map* $1_M \otimes i : K \otimes M \to N \otimes M$ *is a monomorphism, where* $i : K \to N$ *is the inclusion map and* 1_M *is the identity map on* M*. A left R-module M that is flat relative to every right R-module is called a flat left R-module.*

Proposition 2.5 *((Rotman, 1979)) The following statements are equivalent for a right R-module* M*.*

(1) M is flat;

(2) The sequence $0 \to M \otimes I \to M \otimes R$ *is exact for every left ideal I of R;*

(3) The sequence $0 \to M \otimes I \to M \otimes R$ is exact for every finitely generated left ideal I of R.

(4) $Tor_1(M, N) = 0$ for any left R-module N.

(5) $Tor_1(M, R/I) = 0$ for any maximal left ideal I of R.

We arrive now at the following remarkable connection between injective modules and flat modules.

Theorem 2.4 *((Lam, 1999), Theorem 4.9) A right R-module M is flat if and only if its character module M^+ is injective left R-module.*

Let R be a ring. A left R-module M is called finitely presented if there is an exact sequence $0 \to K \to F \to M \to 0$ where F is finitely generated free and K is finitely generated. An exact sequence of left R-modules $0 \to A \to B \to C \to 0$ is called pure exact provided that $0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ is exact for any right R-module M (Warfield Jr., 1969). A is said to be a pure submodule of B. M is called *absolutely pure* (or FP-injective) if it is pure in every module containing it as a submodule. Any split short exact sequence is pure.

For any family of right R-modules $\{B_i\}$, $\oplus_{i \in I} B_i$ is a pure submodule of $\prod_{i \in I} B_i$ for any index set I. Also it is known that injective modules are FP-injective, but the converse is not true in general.

Example 2.1 *Let F_i be a field for each $i \in I$, where I is an infinite set and $M = \oplus_{i \in I} F_i$ and $R = \prod_{i \in I} F_i$. Then M_R is FP-injective but not injective.*

Theorem 2.5 *((Megibben, 1970), Theorem 3) R is right Noetherian if and only if each FP-injective right module is injective.*

FP-injective modules play precisely the same role relative to semihereditary rings that injectives play relative to hereditary rings.

Theorem 2.6 *((Megibben, 1970), Theorem 2) A ring R is right semihereditary if and only if the homomorphic image of an FP-injective R-module is FP-injective.*

Theorem 2.7 *((Lam, 1999), Theorem 4.30) Let M be a finitely presented module over any ring R. Then M is flat if and only if M is projective.*

The relationship between flat modules and pure exact sequences is given in the following theorem.

Theorem 2.8 *((Lam, 1999), Proposition 4.14) Let* $0 \to K \to P \to M \to 0$ *be a short exact sequence, where P is a projective module. Then M is flat if and only if K is a pure submodule of P.*

Similar to well known notion of pure exact sequences, an exact sequence of left R-modules $0 \to A \to B \to C \to 0$ is called s-pure exact provided that $0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ is exact for any simple right R-module M (Crivei, 2014). A is said to be an s-pure submodule of B. M is called *absolutely s-pure* if it is s-pure in every module containing it as a submodule.

Definition 2.5 *A ring R is called a von Neumann regular ring if for each* $x \in R$*, there exists* $y \in R$ *such that* $xyx = x$.

Theorem 2.9 *((Goodearl, 1979), Theorem 1.1) The following are equivalent for a ring R.*

(1) R is von Neumann regular;

(2) Every principal right (left) ideal of R is generated by an idempotent;

(3) Every finitely generated right (left) ideal of R is generated by an idempotent.

The following Theorem gives relation between flat modules and von Neumann regular rings.

Theorem 2.10 *((Goodearl, 1979), Corollary 1.13) For any ring R, the following are equivalent.*

(1) R is von Neumann regular;

(2) Every right module is flat;

(3) Every cyclic right module is flat.

2.4. Singular Submodules And Small Rings

In this section we recall the definition of a singular module and state some results about singular and nonsingular modules. Also we recall the definition of a small ring and give some characterizations of it.

A submodule N of a right R-module M is said to be an *essential (or a large)* submodule of M, written $N \trianglelefteq M$, if $N \cap K \neq 0$ for each nonzero submodule K of M.

Given any right module M, the singular submodule of M is the set

$$Z(M) = \{m \in M : mI = 0 \text{ for some essential right ideal } I \text{ of } R\}.$$

An R-module M is said to be *singular (nonsingular)* if $Z(M) = M$ $(Z(M) = 0)$. A ring R is called a *right nonsingular* ring if R is nonsingular as a right R-module. $Z_r(R)$ will be used for $Z(R_R)$. Similarly, we say that R is *left nonsingular ring* if $Z_l(R) = 0$. Right and left nonsingular rings are not equivalent ((Goodearl, 1976), Exercise 1).

Proposition 2.6 *(Goodearl, 1976) The following hold for any ring R.*

(1) A module N is nonsingular if and only if $Hom(M, N) = 0$ for all singular modules M.

(2) If R is a right semihereditary ring, then $Z_r(R) = 0$.

(3) If $Z_r(R) = 0$, then $Z(M/Z(M)) = 0$ for all right R-modules M.

(4) If $N \leq M$, then $Z(N) = N \cap Z(M)$.

(5) Suppose that $Z_r(R) = 0$. A right module M is singular if and only if $Hom(M, N) = 0$ for all nonsingular right modules N.

Let M be an R-module and $N \leq M$. If N is an essential submodule of M, then M/N is singular. Converse is not true in general. For example, let $M = \mathbb{Z}/2\mathbb{Z}$ and $N = 0$. M/N is singular but N is not an essential submodule of M. The following Proposition shows when the converse true.

Proposition 2.7 *((Goodearl, 1976), Proposition 1.21) Let M be a nonsingular module and $N \leq M$. Then M/N is singular if and only if N is an essential submodule of M.*

The class of all singular right modules is closed under submodules, factor modules and direct sums. On the other hand, the class of all nonsingular right modules is closed under submodules, direct products, essential extensions and module extensions.

Proposition 2.8 *((Goodearl, 1976), Proposition 1.24) If M is any simple right R-module, then M is either singular or projective, but not both.*

Let M be an R-module and $N \subseteq M$. N is called *small* in M, written $N \ll M$, if, for every submodule $L \subseteq M$, the equality $N + L = M$ implies $L = M$.

Let M be an R-module. The *jacobson radical* of M is defined by

$$Rad(M) = \bigcap\{K \subseteq M \mid K \text{ is a maximal submodule in } M\}$$

$$= \sum\{L \subseteq M \mid L \text{ is a small submodule in } M\}$$

If M has no maximal submodule we set $Rad(M) = M$. For a ring R, $J(R_R)$, $J({}_RR)$ are equal, and we denote both of them by $J(R)$.

In (Ramamurthi, 1982), a ring R is called *right small* if R_R is small in its injective hull $E(R_R)$.

Lemma 2.2 *(See (Ramamurthi, 1982), 3.3) For a ring R the following are equivalent.*

(1) *R is a right small ring.*

(2) $\text{Rad}(E) = E$ *for every injective right R-module E.*

(3) $\text{Rad}(E(R)) = E(R)$.

2.5. Local, Semilocal, Perfect and Semiperfect Rings

A nonzero ring R is called *local* if R has a unique maximal left ideal or equivalently R has a unique maximal right ideal.

Proposition 2.9 *((Lam, 2001), Theorem 19.1) For a ring R, the following are equivalent.*

(1) R is local.

(2) R has a unique maximal left ideal.

(3) R has a unique maximal right ideal.

(4) $R/J(R)$ is a division ring.

(5) $J(R)$ is the set of all non-invertible elements of R.

(6) the sum of two non-invertible elements of R is non-invertible.

Definition 2.6 *A ring R is said to be semilocal if $R/J(R)$ is semisimple Artinian.*

Let R be a semilocal ring. Then, for every right module M, $RadM = MJ(R)$. Thus $M/RadM$ is a semisimple $R/J(R)$- module. Hence, $M/RadM$ is a semisimple R-module (Anderson and Fuller, 1992).

Definition 2.7 *(1) Two idempotents $e_1, e_2 \in R$ are said to be orthogonal if $e_1e_2 = e_2e_1 =$* 0.
(2) An idempotent e is said to be a local idempotent if eRe is a local ring.

Proposition 2.10 *((Lam, 2001), Proposition 21.18) Let e be an idempotent in R, and let $\overline{R} = R/J(R)$. Then e is a local idempotent in R if and only if $eR/eJ(R)$ is a simple right R-module.*

Now, we give the definitions of perfect and semiperfect rings.

Definition 2.8 *A ring R is called semiperfect if R is semilocal and idempotents of $R/J(R)$ can be lifted to R.*

Theorem 2.11 *((Lam, 2001), §23 and §24) The following are equivalent for any ring R.*

(1) R is semiperfect.

(2) The identity element 1 can be decomposed into $e_1 + e_2 + ... + e_n$, where the $e_i^{\prime}s$ are mutually orthogonal local idempotents.

(3) Every finitely generated right R-module has a projective cover.

(4) Every cyclic right R-module has a projective cover.

(5) Every simple right R-module has a projective cover.

Theorem 2.12 *((Lam, 2001), Theorem 23.11) A commutative ring R is semiperfect if and only if it is a finite direct product of (commutative) local rings.*

Our next goal is to introduce the notion of left and right perfect rings. For this, we need a new notion of nilpotency called T-nilpotency.

Definition 2.9 *A subset A of a ring R is called left (right) T-nilpotent if, for any sequence of elements $\{a_1, a_2, \cdots\} \subseteq A$, there exists an integer $n \geq 1$ such that $a_1a_2...a_n = 0$ ($a_n...a_2a_1 = 0$).*

Definition 2.10 *A ring R is called right (left) perfect if $R/J(R)$ is semisimple and $J(R)$ is right (left) T-nilpotent. If R is right and left perfect, we call R a perfect ring.*

Semiperfect rings are left-right symmetric, while left (right) perfect rings are always semiperfect. Both of these notions are generalizations of one-sided Artinian rings.

The following result offers various other characterizations for right perfect rings. This Theorem says that one sided Artinian rings are right and left perfect rings.

Theorem 2.13 *((Lam, 2001), §23 and §24) The following are equivalent for any ring R.*

(1) R is right perfect;

(2) R satisfies DCC on principal left ideals;

(3) Any left module M satisfies DCC on cyclic submodules;

(4) R does not contain an infinite orthogonal set of nonzero idempotents, and any nonzero left module M contains a simple module.

(5) Every right R-module has a projective cover.

(6) Every flat right R-module is projective.

From the definitions of above, it is clear that a ring R is left perfect if and only if R is semiperfect and right semiartinian.

A module M is said to be *uniserial* if the lattice of submodules of M is totally ordered by inclusion. A ring R is called a right (left) *uniserial ring* (or *chain ring*) if R_R ($_RR$) is a uniserial module. Any direct sum of uniserial modules called a *serial module*. A ring R is said to be right (left) *serial ring* if the module R_R ($_RR$) is serial. A ring R is called a serial ring if R is both left as well as right serial. If R is a right or left serial ring, then $R/J(R)$ is a semisimple Artinian ring.

2.6. m-injective and m-flat Modules

Definition 2.11 *A right R-module E is said to be max-injective (or m-injective for short) in case for any maximal right ideal I of R, each homomorphism $f : I \to E$ can be extended to a homomorphism $g : R \to E$, i.e. $Ext^1_R(R/I, E) = 0$ for every maximal right ideal I of R_R. A ring R is said to be right max-injective in case the right regular module R_R is max-injective.*

Obviously, any injective module is max-injective. Max-injectivity was first introduced in (Wang and Zhao, 2005), which was used to characterize QF-rings.

Definition 2.12 *(Wang, 2005) A right R-module M is called max-flat (or m-flat for short) if $Tor^R_1(M, R/I) = 0$ for any maximal left ideal I of R.*

A right R-module M is max-flat if and only if M^+ is max-injective by the standard isomorphism $Ext^1_R(R/I, M^+) \cong Tor^R_1(M, R/I)^+$ for any maximal left ideal I of R.

For convenience in concepts, the max-injective and max-flat modules are called m-injective and m-flat in this thesis.

2.7. Covers, Envelopes and Cotorsion Pairs

Definition 2.13 *Let R be a ring and $\mathfrak{C}$ a class of R-modules. Then for an R-module M, a morphism $\varphi : M \to F$, where $F \in \mathfrak{C}$, is called $\mathfrak{C}$-envelope of M if*

(1) any diagram with $F' \in \mathfrak{C}$

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & F \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle g} & \\ F' & & \end{array}$$

can be completed such that $g\varphi = f$ and

(2)

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & F \\ {\scriptstyle \varphi}\downarrow & \swarrow{\scriptstyle g} & \\ F & & \end{array}$$

can be completed only by automorphisms of F such that $g\varphi = \varphi$.

If $\varphi : M \to F$ satisfies (1) *but may be not* (2)*, then it is called an $\mathfrak{C}$-preenvelope of M.*

If envelopes exist, they are unique up to isomorphism.

Definition 2.14 *Let R be a ring and $\mathfrak{C}$ a class of R-modules. Then, for an R-module M, a morphism $\varphi : C \to M$, where $C \in \mathfrak{C}$ is called an $\mathfrak{C}$-cover of M if*

(1) any diagram with $C' \in \mathfrak{C}$

$$\begin{array}{ccc} C' & & \\ {\scriptstyle g}\downarrow & \searrow{\scriptstyle f} & \\ C & \xrightarrow{\varphi} & M \end{array}$$

can be completed to a commutative diagram such that $\varphi g = f$ and

(2) the diagram

$$\begin{array}{ccc} C & & \\ {\scriptstyle g}\downarrow & \searrow{\scriptstyle \varphi} & \\ C & \xrightarrow{\varphi} & M \end{array}$$

can be completed only by automorphisms of Csuch that $\varphi g = \varphi$.

If $\varphi : C \to M$ satisfies (1) *but may be not* (2)*, then it is called an $\mathfrak{C}$-precover of M.*

If a $\mathfrak{C}$-cover exists, then it is unique up to isomorphism.

Following (Enochs and Jenda, 2000), a monomorphism $\alpha : M \to C$ with $C \in \mathfrak{C}$ is said to be a special $\mathfrak{C}$-preenvelope of M if $coker(\alpha) \in^{\perp} \mathfrak{C}$. Dually, we have the definitions of a (special) $\mathfrak{C}$-precover and a $\mathfrak{C}$-cover. $\mathfrak{C}$-envelopes ($\mathfrak{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

Given a class $\mathfrak{C}$ of right R-modules, we will denote by $\mathfrak{C}^{\perp} = \{X : Ext^1_R(C, X) = 0$ for all $C \in \mathfrak{C}\}$ the right orthogonal class of $\mathfrak{C}$, and by ${}^{\perp}\mathfrak{C} = \{X : Ext^1_R(X, C) = 0$ for all $C \in \mathfrak{C}\}$ the left orthogonal class of $\mathfrak{C}$.

Definition 2.15 *(Enochs and Jenda, 2000) A pair $(\mathfrak{F}, \mathfrak{C})$ of classes of right R-modules is called a cotorsion theory (for the category of R-modules) if $\mathfrak{F}^{\perp} = \mathfrak{C}$ and ${}^{\perp}\mathfrak{C} = \mathfrak{F}$.*

$(\mathfrak{M}, \mathfrak{Inj})$ and $(\mathfrak{Proj}, \mathfrak{M})$ are cotorsion theories where $\mathfrak{M}$ denotes the class of right R-modules and $\mathfrak{Inj}$ and $\mathfrak{Proj}$ denotes the classes of injective and projective modules.

We note that $(\mathfrak{F}, \mathfrak{C})$ is a cotorsion theory, then $\mathfrak{F}$ and $\mathfrak{C}$ are both closed under extensions and summand, and $\mathfrak{F}$ contains all the projective modules while $\mathfrak{C}$ contains all the injective modules. Also, $\mathfrak{F}$ is closed under arbitrary direct sums and $\mathfrak{C}$ is closed under arbitrary direct products.

A cotorsion theory $(\mathfrak{F}, \mathfrak{C})$ is called perfect (complete) if every right R-module has a $\mathfrak{C}$-envelope and an $\mathfrak{F}$-cover (a special $\mathfrak{C}$-preenvelope and a special $\mathfrak{F}$-precover). A cotorsion theory $(\mathfrak{F}, \mathfrak{C})$ is said to be hereditary ((Enochs, Jenda and Lopez-Ramos, 2004)) if whenever $0 \to L' \to L \to L'' \to 0$ is exact with $L, L'' \in \mathfrak{F}$, then L' is also in $\mathfrak{F}$. By (Enochs, Jenda and Lopez-Ramos, 2004), $(\mathfrak{F}, \mathfrak{C})$ is hereditary if and only if $0 \to C' \to C \to C'' \to 0$ is exact with $C, C' \in \mathfrak{C}$, then C'' is also in $\mathfrak{C}$.

The following result is useful while proving whether a class of modules is (pre)enveloping or (pre)covering.

Lemma 2.3 *(1) ((Rada and Saorin, 1998), Corollary 3.5(c))) If a class $\mathcal{M}$ of modules over a ring is closed under pure submodules, then $\mathcal{M}$ is preenveloping if and only if it is closed under direct products.*

(2) ((Holm and Jørgensen, 2008), Theorem 2.5) If a class $\mathcal{M}$ of modules over a ring is closed under pure quotients, then $\mathcal{M}$ is precovering if and only if it is covering if and only if it is closed under direct sums.

CHAPTER 3

ALMOST-QF AND MAX-QF RINGS

The purpose of this chapter is to characterize the rings whose injective R-modules are R-projective and max-projective, respectively.

3.1. Max-projective Module

In this section, we study the properties of max-projective R-modules and give some characterizations of semiperfect, perfect and QF rings in terms of max-projectivity.

Definition 3.1 *A right R-module M is said to be* max-projective *if for every epimorphism $f : R \to R/I$ where I is a maximal right ideal of R and every homomorphism $g : M \to R/I$, there exists a homomorphism $h : M \to R$ such that $fh = g$.*

Example 3.1

(a) Every projective R-module is max-projective.

(b) The $\mathbb{Z}$-module $\mathbb{Q}$ is max-projective since $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}_p) = 0$ *for each simple $\mathbb{Z}$-module $\mathbb{Z}_p$.*

(c) Every simple max-projective R-module is projective. For if S is a simple right R-module and $1_S : S \to S$ is the identity map, then by max-projectivity of S there is a homomorphism $f : S \to R$ such that $\pi f = 1_S$, where $\pi : R \to S$ is the natural epimorphism. Then $R \cong K \oplus S$, so S is projective.

(d) Any R-module M with $\mathrm{Rad}(M) = M$ *is max-projective since M has no simple factors.*

Given modules M and N, M is said to be N-subprojective if for every homomorphism $f : M \to N$ and for every epimorphism $g : B \to N$, there exists a homomorphism $h : M \to B$ such that $gh = f$ (see (Holston et al., 2015)).

Lemma 3.1 *For an R-module M, the following are equivalent.*

(1) *M is max-projective.*

(2) *M is S-subprojective for each simple R-module S.*

(3) *For every epimorphism $f : N \to S$ with S simple, and homomorphism $g : M \to S$, there exists a homomorphism $h : M \to N$ such that $fh = g$.*

Proof (2) ⇔ (3) By definition. (3) ⇒ (1) is clear.
(1) ⇒ (3) Let $f : N \to S$ be an epimorphism where S is a simple R-module and $g : M \to S$ a homomorphism. Since S is simple, there exists an epimorphism $\pi : R \to S$. By the hypothesis there exists a homomorphism $h : M \to R$ such that $\pi h = g$. Since R is projective, there exists a homomorphism $h' : R \to N$ such that $fh' = \pi$. Then $f(h'h) = \pi h = g$, so M is max-projective. □

We need the following result in the sequel.

Lemma 3.2 *The following conditions are true.*

(1) *A direct sum $\oplus_{i \in I} A_i$ of modules is max-projective (resp. R-projective) if and only if each A_i is max-projective (resp. R-projective).*

(2) *If $0 \to A \to B \to C \to 0$ is an exact sequence and M is B-projective, then M is projective relative to both A and C.*

Proof (1) Since it is similar to the one provided in ((Anderson and Fuller, 1992), Proposition 16.10) for R-projective modules, the proof is omitted for max-projective modules.

(2) is clear by ((Anderson and Fuller, 1992), Proposition 16.12). □

Next we characterize semisimple rings in terms of max-projective modules.

Corollary 3.1 *For a ring R, the following are equivalent.*

(1) *R is semisimple.*

(2) *Every right R-module is max-projective.*

(3) *Every finitely generated right R-module is max-projective.*

(4) *Every cyclic right R-module is max-projective.*

(5) *Every simple right R-module is max-projective.*

Proof (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) are clear.

(5) ⇒ (1) Example 3.1(c) and the hypothesis implies that each simple right R-module is projective. Thus R is semisimple. □

In (Amin, Ibrahim and Yousif, 2013), the module M is called *rad-projective* if, for any epimorphism $\sigma : R \to K$ where K is an image of $R/J(R)$ and any homomorphism $f : M \to K$, there exists a homomorphism $g : M \to R$ such that $f = \sigma g$. We have the following implications:
projective ⇒ R-projective ⇒ rad-projective ⇒ max-projective

Proposition 3.1 *Let R be a semilocal ring and M an R-module. Then the following are equivalent.*

(1) *M is rad-projective.*

(2) *M is max-projective.*

(3) *Every homomorphism $f : M \to R/J(R)$ can be lifted to a homomorphism $g : M \to R$.*

Proof (1) ⇒ (2) Clear. (3) ⇒ (1) By ((Amin, Ibrahim and Yousif, 2011), Proposition 3.14).

(2) ⇒ (3) Since $R/J(R)$ is semisimple, $R/J(R) = \oplus_{i=1}^{n} K_i$, with each K_i simple as an R-module. Let $\pi_i : \oplus_{i=1}^{n} K_i \to K_i$, and $\pi : R \to \oplus_{i=1}^{n} K_i$. Set $h := \pi_i \pi$. By the hypothesis, there exists a homomorphism $g : M \to R$ such that $hg = \pi_i f$. Since $R/J(R)$ is semisimple, each π_i splits and there exists a homomorphism $\varepsilon_i : K_i \to \oplus_{i=1}^{n} K_i$ such that $\varepsilon_i \pi_i = 1_{R/J(R)}$. Then $\pi g = \varepsilon_i hg = \varepsilon_i \pi_i f = f$. □

In the next Proposition we provide a sufficient condition for an R-module to be max-projective. We establish a converse for self-injective rings.

Proposition 3.2 *If M is a right R-module such that $Ext_R^1(M, I) = 0$ for every maximal right ideal I of R, then M is max-projective. The converse is true when R is a right self-injective ring.*

Proof By applying $\mathrm{Hom}(M, -)$ to the short exact sequence $0 \to I \to R \to R/I \to 0$, with I being a maximal right ideal of R, we obtain the following exact sequence:
$0 \to \mathrm{Hom}(M, I) \to \mathrm{Hom}(M, R) \to \mathrm{Hom}(M, R/I) \to Ext_R^1(M, I) \to Ext_R^1(M, R) \to ....$ If $Ext_R^1(M, I) = 0$ for every maximal right ideal I of R, it follows that M is max-projective. Conversely, since R is right self injective, $Ext_R^1(M, R) = 0$. If M is a max-projective right R-module, then the map $\mathrm{Hom}(M, R) \to \mathrm{Hom}(M, R/I)$ is onto, so $Ext_R^1(M, I) = 0$ for any maximal right ideal I of R. □

Proposition 3.3 *Let $0 \to A \to B \to C \to 0$ be a short exact sequence. If M is A-subprojective and C-subprojective, then M is B-subprojective.*

Proof Let $\gamma : F \to B$ be an epimorphism with F projective. Then using the pullback diagram of $\gamma : F \to B$ and $\beta : A \to B$ and applying Hom$(M, -)$, we get a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Hom(M,K) & \longrightarrow & Hom(M,X) & \xrightarrow{\theta} & Hom(M,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \beta^*} & & \\
0 & \longrightarrow & Hom(M,K) & \longrightarrow & Hom(M,F) & \xrightarrow{\gamma^*} & Hom(M,B) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Hom(M,C) & \xrightarrow{\phi} & Hom(M,C) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since M is A-subprojective and C-subprojective, θ and ϕ are epic. Hence, γ^* is epic by ((Anderson and Fuller, 1992), Five Lemma 3.15).

□

Proposition 3.4 *Let M be an R-module. M is max-projective if and only if M is N-subprojective for any R-module N with composition length $cl(N) < \infty$.*

Proof Let M be a max-projective R-module and N an R-module with $cl(N) = n$. Then there exists a composition series $0 = S_0 \subset S_1 ... \subset S_n = N$ with each composition factor S_{i+1}/S_i simple. Consider the short exact sequence $0 \to S_1 \to S_2 \to S_2/S_1 \to 0$. Since M is max-projective, by Lemma 3.1, M is S_1-subprojective and S_2/S_1-subprojective. So, by Proposition 3.3, M is S_2-subprojective. Continuing in this way, M is S_i-subprojective for each $0 \leq i \leq n$. Hence, M is N-subprojective. Conversely, since each simple right R-module has finite length, M is max-projective by Lemma 3.1. □

In the following corollary we obtain that R-projectivity and max-projectivity coincide over the ring of integers.

Corollary 3.2 *A $\mathbb{Z}$-module M is max-projective if and only if M is $\mathbb{Z}$-projective.*

Proof By the Fundamental Theorem of Abelian Groups, a cyclic $\mathbb{Z}$-module M is isomorphic either to $\mathbb{Z}$ or to a finite direct sum of $\mathbb{Z}$-modules of finite length. Now the proof is clear by Proposition 3.4. □

Corollary 3.3 *Let M be an R-module with finite composition length. If M is max-projective, then it is projective.*

Proof Let $f : R^n \to M$ be an epimorphism. The module M is M-subprojective by Proposition 3.4. That is, there is a homomorphism $g : M \to R^n$ such that $1_M = fg$. Thus the map f splits, so M is projective. □

Submodules of max-projective R-modules need not be max-projective. Consider the ring $R = \mathbb{Z}/p^2\mathbb{Z}$, for some prime integer p. R is max-projective, whereas the simple ideal pR is not max-projective, since the epimorphism $R \to pR \to 0$ does not split.

Recall that a ring R is called right V-*ring* (resp. *right* GV-*ring*) if all simple (resp. all singular simple) right R-modules are injective.

Proposition 3.5 *Consider the following conditions for a ring R:*

(1) *R is a right GV-ring.*

(2) *Submodules of max-projective right R-modules are max-projective.*

(3) *Submodules of projective right R-modules are max-projective.*

(4) *Every right ideal of R is max-projective.*

Then, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). *Also, if R is a right self injective ring, then* (4) $\Rightarrow$ (1).

Proof (1) $\Rightarrow$ (2) Let N be a submodule of a max-projective right R-module M. Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{i} & M \\ & & \downarrow{\scriptstyle f} & & \\ R & \xrightarrow{\pi} & S & \longrightarrow & 0 \end{array}$$

where S is a simple right R-module, $i : N \to M$ is the inclusion map and $\pi : R \to S$ is the canonical quotient map. Since, by Proposition 2.8, the simple module S is either projective or singular, the former implies $\pi : R \to S$ splits and there exists a homomorphism $\varepsilon : S \to R$ such that $\varepsilon\pi = 1_R$. In the latter one, S is singular, so it is injective by the hypothesis. Thus, there is a homomorphism $g : M \to S$ such that $gi = f$. Since M is max-projective, there is a homomorphism $h : M \to R$ such that $\pi h = g$. Hence, $\pi(hi) = gi = f$. In either case, there exists a homomorphism from N to R that makes the diagram commute. This implies that N is max-projective.

(2) ⇒ (3) ⇒ (4) Clear. (4) ⇒ (1) Let I be a right ideal of R and J a maximal right ideal of R. Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \xrightarrow{i} & R \\ & & \downarrow{\scriptstyle f} & & \\ R & \xrightarrow{\pi} & R/J & \longrightarrow & 0 \end{array}$$

where R/J is a simple right R-module, $i : I \to R$ is the inclusion map and $\pi : R \to R/J$ is the canonical quotient map. Since I is max-projective, there is a homomorphism $h : I \to R$ such that $\pi h = f$. Since R is injective, there exists a homomorphism $\lambda : R \to R$ such that $\lambda i = h$. Now the map $\beta = \pi\lambda : R \to R/J$ satisfies $\beta i = \pi\lambda i = \pi h = f$, as required. □

Proposition 3.6 *Let R be a commutative or semilocal ring. Then pure submodules of max-projective R-modules are max-projective.*

Proof Let M be a max-projective (right) module and N a pure submodule of M. Let S be a simple (right) module and $f : N \to S$ be a homomorphism. Since S is pure-injective and N is a pure submodule of M, there is $g : M \to S$ such that $gi = f$, where $i : N \to M$ is the inclusion map. By max-projectivity of M, there is a homomorphism $h : M \to R$ such that $g = \pi h$, where $\pi : R \to S$ is the natural epimorphism. Now we have $f = gi = \pi hi$, i.e. $hi : N \to R$ lifts f. This proves that N is max-projective. □

Lemma 3.3 *Let R be a ring and τ a preradical with $\tau(R) = 0$. If M is a max-projective R-module, then $M/\tau(M)$ is max-projective.*

Proof Let M be a max-projective R-module and $f : M/\tau(M) \to S$ a homomorphism with S simple R-module. Consider the following diagram:

$$\begin{array}{ccccc} M & \xrightarrow{\pi} & M/\tau(M) & & \\ & & \downarrow{\scriptstyle f} & & \\ R & \xrightarrow{\eta} & S & \longrightarrow & 0 \end{array}$$

Since M is max-projective, there exists a homomorphism $g : M \to R$ such that $f\pi =$

ηg. Since $g(\tau(M)) \subseteq \tau(R) = 0$, $\tau(M)) \subseteq \mathrm{Ker}(g)$, and so there exists a homomorphism $h : M/\tau(M) \to R$ such that $h\pi = g$. Now, since $\eta h\pi = \eta g = f\pi$ and π is an epimorphism, $\eta h = f$, and so $M/\tau(M)$ is max-projective. □

Remark 3.1 *Recall that any finitely generated R-projective module is projective, ((Alhilali et al., 2017), Lemma 2.1). This is not true for max-projective modules in general. Let R be a right V-ring which is not right semihereditary. Then R has a finitely generated right ideal which is not projective. By Proposition 3.5, each right ideal of R is max-projective.*

Proposition 3.7 *Let R be a right nonsingular right self-injective ring. Every finitely generated max-projective right R-module is projective.*

Proof Let M be a finitely generated max-projective right R-module. As R is a right nonsingular ring, by Lemma 3.3, $M/Z(M)$ is max-projective. Since $M/Z(M)$ is finitely generated, there exists an epimorphism $f : F \to M/Z(M)$ such that F is finitely generated free. This means $\mathrm{Ker}(f)$ is closed in F. By the injectivity of F, $\mathrm{Ker}(f)$ is a direct summand of F, so $M/Z(M)$ is projective. Then, $M = Z(M) \oplus K$ for some projective submodule K of M. We claim that $Z(M) = 0$. Assume to the contrary that $Z(M) \neq 0$. Since, $Z(M)$ is a finitely generated submodule of M, there exists a nonzero epimorphism $g : Z(M) \to S$ for some simple right R-module S. Then, by Lemma 3.2, $Z(M)$ is max-projective, so there exists a nonzero homomorphism $h : Z(M) \to R$ such that $\pi h = g$, where $\pi : R \to S$ is the natural epimorphism. But then $h(Z(M)) \subseteq Z(R_R) = 0$, a contradiction. Thus we must have $Z(M) = 0$, whence M is projective. □

A ring R is called *right max-ring* if every nonzero right R-module M has a maximal submodule i.e. $\mathrm{Rad}(M) \neq M$.

Proposition 3.8 *The following conditions are true.*

(1) *Over a semiperfect ring R, every max-projective right R-module with small radical is projective.*

(2) *A ring R is right perfect if and only if R is semilocal and every max-projective right R-module is projective.*

Proof (1) Let M be a max-projective right R-module with $\mathrm{Rad}(M) \ll M$. Since R is semilocal, M is rad-projective by Proposition 3.1. Hence M is projective by ((Amin, Ibrahim and Yousif, 2011), Theorem 4.7).

(2) Since over a right perfect ring R every right R-module has small radical, it follows from (1) that every max-projective right R-module is projective. Conversely, assume that R is semilocal and every max-projective right R-module is projective. Let M be a nonzero right R-module. We claim that $\mathrm{Rad}(M) \neq M$. Assume to the contrary that M has no maximal submodule, i.e. $\mathrm{Rad}(M) = M$. Since $\mathrm{Hom}(M, S) = 0$ for any simple right R-module, M is max-projective. Thus M is projective, by the hypothesis. Since,

by Proposition 2.4, projective modules have maximal submodules, this is a contradiction. Hence, every right R-module has a maximal submodule. Since R is semilocal, R is right perfect by ((Anderson and Fuller, 1992), Theorem 28.4). □

Recall that if R is a right perfect ring, every R-projective right R-module is projective, (Sandomierski, 1964). Thus the following result follows from Proposition 3.8(2).

Corollary 3.4 *Let R be a right perfect ring and M be a right R-module. Then the following are equivalent.*

(1) *M is projective.*

(2) *M is R-projective.*

(3) *M is max-projective.*

The following Remark is an example of a right nonperfect ring R such that every max-projective module is R-projective.

Remark 3.2 *Let K be a field, and R the subalgebra of K^ω consisting of all eventually constant sequences in K^ω. For each $i < \omega$, we let e_i be the idempotent in K^ω whose ith component is 1 and all the other components are 0. Notice that $\{e_i : i < \omega\}$ a set of pairwise orthogonal idempotents in R, so R is not perfect, ((Trlifaj, 2019), Lemma 2.3). By ((Trlifaj, 2019), Lemma 2.3 and Lemma 2.4), $R/Soc(R)$ is simple R-module and a module M is R-projective if and only if it is projective with respect to the projection $\pi : R \to R/Soc(R)$. Thus, an R-module M is max-projective if and only if M is R-projective.*

The following Corollary follows from ((Trlifaj, 2019), Theorem 3.3) and Remark 3.2.

Corollary 3.5 *Let K be a field of cardinality $\leq 2^\omega$ and R the subalgebra of K^ω consisting of all eventually constant sequences in K^ω. Assume Gödel's Axiom of Constructibility ($V = L$). Then all max-projective R-modules are projective.*

Lemma 3.4 *If M_R is max-projective right R-module and $\bar{R} = R/J(R)$, then $(M/\operatorname{Rad}(M))_{\bar{R}}$ is max-projective.*

Proof Let $\pi : \bar{R}_{\bar{R}} \to K_{\bar{R}}$ be an $\bar{R}$-epimorphism with $K_{\bar{R}}$ simple $\bar{R}$-module. Consider the following diagram:

$$\begin{array}{ccccc} M & \xrightarrow{\eta} & M/Rad(M) & & \\ & & \downarrow f & & \\ \bar{R}_{\bar{R}} & \xrightarrow{\pi} & K_{\bar{R}} & \longrightarrow & 0 \end{array}$$

Since M is max-projective, there exists a homomorphism $\lambda : M_R \to \bar{R}_R$ such that $\pi\lambda = f\eta$. Since $\lambda(\mathrm{Rad}(M)) \subseteq \mathrm{Rad}(R/J(R)) = 0$, $\mathrm{Rad}(M) \subseteq \mathrm{Ker}(\lambda)$, so there exists a homomorphism $\delta : (M/\mathrm{Rad}(M))_R \to \bar{R}_R$ such that $\delta\eta = \lambda$. Now, since $\pi\delta\eta = \pi\lambda = f\eta$ and η is an epimorphism, $\pi\delta = f$, so $(M/\mathrm{Rad}(M))_{\bar{R}}$ is a max-projective $\bar{R}$-module. □

It is well-known that a ring R is *semiperfect* if and only if every simple R-module has a projective cover. In the next Proposition, we extend this result by replacing projective covers with max-projective covers. Let R be a ring and Ω a class of right R-modules which is closed under isomorphisms. A homomorphism $f : P \to M$ is called an Ω-cover of the right R-module M if $P \in \Omega$ and f is an epimorphism with small kernel. That is to say, if Ω is the class of max-projective right R-modules, the homomorphism $f : P \to M$ is called max-projective cover of M. With the help of an argument similar to the one provided in ((Amin, Ibrahim and Yousif, 2013), Theorem 18), we can establish the next Proposition.

Proposition 3.9 *For a ring R, the following are equivalent.*

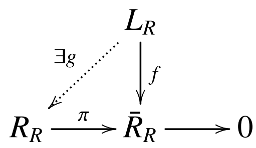
(1) *R is semiperfect.*

(2) *Every finitely generated right R-module has a max-projective cover.*

(3) *Every cyclic right R-module has a max-projective cover.*

(4) *Every simple right R-module has a max-projective cover.*

Proof (1) ⇒ (2) ⇒ (3) ⇒ (4) are clear.
(4) ⇒ (1) We first show that $\bar{R} = R/J(R)$ is a semisimple ring. Let S be a simple right $\bar{R}$-module. By the hypothesis S_R has a max-projective cover P_R, say $f : P \to S$ with $Rad(P) = \mathrm{Ker}(f) \ll P$. Since S is simple and $P/\mathrm{Rad}(P) \cong S$, $P/\mathrm{Rad}(P)$ is a simple $\bar{R}$-module. So, $(P/\mathrm{Rad}(P))_{\bar{R}}$ is max-projective by Lemma 3.4, whence $(P/\mathrm{Rad}(P))_{\bar{R}}$ is projective. Consider the map $\tilde{f} : P/\mathrm{Rad}(P) \to S$. This map induces an isomorphism. Since $P/\mathrm{Rad}(P)$ is projective $\bar{R}$-module, $P/\mathrm{Rad}(P)$ is the projective cover of $S_{\bar{R}}$. Hence, $\bar{R}$ is a semiperfect ring. Therefore, $\bar{R}$ is semisimple as an $\bar{R}$-module, and hence semisimple as an R-module. Write $\bar{R} = R/J(R) = \oplus_{i=1}^{n} K_i$, with each K_i simple as a right R-module, and let L_i be a max-projective cover of K_i, $1 \leq i \leq n$, as right R-modules. Now, in order to prove that R is a semiperfect ring, it is enough to show that each L_i, $1 \leq i \leq n$, is projective as a right R-module. Clearly, $L = \oplus_{i=1}^{n} L_i$, as a right R-module, is a max-projective cover of $\bar{R}_R$. Consider the diagram



with f being the max-projective cover of $\bar{R}_R$ and π the canonical R-epimorphism. By the max-projectivity of L_R, f can be lifted to a map $g : L_R \to R_R$ such that $\pi g = f$. Since $R = \mathrm{Im}(g) + J(R)$ and $J(R) \ll R$, we infer that $R = \mathrm{Im}(g)$, and g is onto. By the projectivity of R, the map g splits, and $L_R = \mathrm{Ker}(g) \oplus A$ for a submodule A of L_R. Since $\mathrm{Ker}(g) \subseteq \mathrm{Ker}(f) \ll L_R$, $\mathrm{Ker}(g) = 0$, and $L_R \cong R_R$ is projective. Therefore, each L_i, $1 \leq i \leq n$, is projective as a right R-module, and R is semiperfect. □

Proposition 3.10 *For a ring R, the following conditions are equivalent.*

(1) *R is right perfect.*

(2) *Every right R-module has a max-projective cover.*

(3) *Every semisimple right R-module has a max-projective cover.*

Proof (1) ⇒ (2) ⇒ (3) are clear.

(3) ⇒ (1) By Proposition 3.9, R is a semiperfect ring. Let M be a semisimple right R-module and $f : P \to M$ a max-projective cover of M. Since $Rad(P) = \mathrm{Ker}(f) \ll P$, P is projective by Proposition 3.8(1). Thus every semisimple right R-module has a projective cover, so R is right perfect. □

Let R be any ring and M an R-module. A submodule N of M is called radical submodule if N has no maximal submodules, i.e. $N = \mathrm{Rad}(N)$. By $P(M)$ we denote the sum of all radical submodules of a module M. For any module M, $P(M)$ is the largest radical submodule of M, so $\mathrm{Rad}(P(M)) = P(M)$. Moreover, P is an idempotent radical with $P(M) \subseteq \mathrm{Rad}(M)$ and $P(M/P(M)) = 0$, (see (Büyükaşık, Mermut and Özdemir, 2010)).

In ((Dinh, Holston, and Huynh, 2012), Lemma 1), the authors prove that over a right nonsingular right V-ring, max-projective right R-modules are nonsingular. Regarding the converse of this fact, we have the following.

Proposition 3.11 *If every max-projective right R-module is nonsingular, then R is right nonsingular and right max-ring.*

Proof Clearly the ring R is right nonsingular. If R is a right V-ring, then $\mathrm{Rad}(M) = 0$ for any right R-module M. Thus R is a max-ring. Suppose R is not a right V-ring, and let S be a noninjective simple right R-module. We shall first see that $P(E(S)) = 0$. Suppose $\text{Rad}(P(E(S))) = P(E(S)) \neq 0$. Then $P(E(S))/S$ is singular. Furthermore, since $\text{Rad}(P(E(S))/S) = P(E(S))/S$, $P(E(S))/S$ is max-projective. This contradicts with the hypothesis. Therefore, for every simple right R-module S, $P(E(S)) = 0$. Let M be a nonzero right R-module. We claim that $\text{Rad}(M) \neq M$. Assume to the contrary that $\text{Rad}(M) = M$. Let $0 \neq x \in M$ and K be a maximal submodule of xR. Then the simple right R-module $S = xR/K$ is noninjective because S small. Now, the obvious map $xR \to E(S)$ extends to a nonzero map $f : M \to E(S)$. Since $P(\text{Im}(f)) \subseteq P(E(S)) = 0$, $P(M/\text{Ker}(f)) = 0$. This contradicts with $P(M) = M$. Hence $\text{Rad}(M) \neq M$ for every right R-module M, so R is a right max-ring. □

Corollary 3.6 *For a ring R, the following are equivalent.*

(1) *R is semilocal and every max-projective right R-module is nonsingular.*

(2) *R is right perfect and right nonsingular.*

3.2. Almost-QF and max-QF rings

In this section, we give some characterizations of almost-QF and max-QF rings. Recall that a ring R is QF if and only if every injective (right) R-module is projective (see, (Faith, 1976)). We slightly weaken this condition and obtain the following definition.

Definition 3.2 *A ring R is called right* almost-QF *if every injective right R-module is R-projective. We call R* right max-QF *if every injective right R-module is max-projective. Left almost-QF and left max-QF rings are defined similarly.*

We note that our notion of almost-QF ring differs from the other definition of almost-QF ring introduced by Harada (Harada, 1993).

Clearly, we have the following inclusion relationship:

$$\{QF \text{ rings}\} \subseteq \{\text{right almost-}QF \text{ rings}\} \subseteq \{\text{ right max-}QF \text{ rings}\}.$$

Example 3.2 *The ring of integers $\mathbb{Z}$ is a right almost-QF but not QF: For every injective $\mathbb{Z}$-module E, we have* $\text{Rad}(E) = E$. *Thus* $\text{Hom}(E, \mathbb{Z}/n\mathbb{Z}) = 0$, *for each cyclic $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$. This means that each injective $\mathbb{Z}$-module is $\mathbb{Z}$-projective, so $\mathbb{Z}$ is almost-QF.*

Remark 3.3 *Sandomierski (Sandomierski, 1964) proved that if R is a right perfect ring, then every R-projective right module is projective. Thus a ring R is right perfect and right almost-QF if and only if R is QF.*

Now we investigate some properties of almost-QF rings. The next result shows that being almost-QF is preserved by Morita equivalence.

Proposition 3.12 *Let R and S be Morita equivalent rings. Then, R is right almost-QF if and only if S is right almost-QF.*

Proof An R-module M is R-projective if and only if M is N-projective for any finitely generated projective R-module N. Now, by ((Anderson and Fuller, 1992), Propositions 21.6 and 21.8), since injectivity, relative projectivity and being finitely generated are preserved by Morita equivalence, the proof is clear. □

Lemma 3.5 *Let R_1 and R_2 be rings. Then $R = R_1 \times R_2$ is right almost-QF (resp. right max-QF) if and only if R_1 and R_2 are both right almost-QF (resp. right max-QF).*

Proof Let M be an injective right R_1-module. Then M is an injective right R-module, as well as an R-projective module by the hypothesis. Hence, by Lemma 3.2, M is R_1-projective, so R_1 is right almost-QF. Similarly, R_2 is right almost-QF. Conversely, let M be an injective right R-module. Since we have the decomposition $M = MR_1 \oplus MR_2$, MR_1 is an injective right R-module, whence it is an injective right R_1-module. On the other hand, since $(MR_2)R_1 = 0$, MR_2 is an R_1-module, so it is an injective R_1-module. This means that MR_1 and MR_2 are R_1-projective by the hypothesis. Then, by Lemma 3.2, $M = MR_1 \oplus MR_2$ is R_1-projective. Similarly, M is R_2-projective. Therefore, M is R-projective. Since it is similar to the one provided for almost-QF rings, the proof is omitted for max-QF rings. □

To prove the main result of this section we need the following Proposition.

Proposition 3.13 *Let R be a right Hereditary ring and E an indecomposable injective right R-module. Then the following are equivalent.*

(1) *E is R-projective.*

(2) *E is max-projective.*

(3) *Either* $\mathrm{Rad}(E) = E$ *or E is projective.*

Proof (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) Assume that $\mathrm{Rad}(E) \neq E$. Then E has a simple factor module isomorphic to R/I. Let $f : E \to R/I$ be a nonzero homomorphism. Since E is max-projective,

there exists a homomorphism $g : E \to R$ such that $\mathrm{Im}(g) \neq 0$. By the fact that R is right Hereditary, $\mathrm{Im}(g)$ is projective, whence $E \cong \mathrm{Im}(g) \oplus K$ for some right R-module K. Since E is indecomposable, either $K = 0$ or $\mathrm{Im}(g) = 0$, where the latter case implies that $g = 0$ which is a contradiction. In the former case $K = 0$ implying that E is projective.
(3) $\Rightarrow$ (1) Conversely, if E is projective, then E is clearly R-projective. Now suppose $\mathrm{Rad}(E) = E$ and let $f : E \to R/I$ be a homomorphism. Then $f(E) = f(\mathrm{Rad}(E)) \subseteq \mathrm{Rad}(R/I) \ll R/I$. Moreover $f(E)$ is a direct summand of R/I since R is right Hereditary. Therefore $f(E) = 0$, so f can be lifted to R. □

Corollary 3.7 *If R is a right Hereditary right small ring, then R is right almost-QF.*

Proposition 3.14 *If R is a right semihereditary right small ring, then* $\mathrm{Hom}(E, R) = 0$, *for any injective right R-module E. In particular, R is right almost-QF if and only if* $\mathrm{Hom}(E, R/I) = 0$ *for any right ideal I of R.*

Proof Let E be an injective right R-module and $f \in \mathrm{Hom}(E, R)$. Then $f(E) = f(\mathrm{Rad}(E)) \subseteq J(R)$. Since R is right semihereditary, $f(E)$ is FP-injective by Theorem 2.6. This means that $R/f(E)$ is flat by Theorem 2.8. Then, by ((Lam, 1999), §4 Exercise 20), $f(E) = 0$, i.e. $\mathrm{Hom}(E, R) = 0$. Hence, the rest is clear. □

Recall that by Example 3.1(d), any right small ring R is right max-QF. Moreover, if R is right Noetherian, we have the following.

Proposition 3.15 *If R is a right Noetherian and right small ring, then R is right almost-QF.*

Proof Let E be an injective right R-module. Then, by Lemma 2.2, $\mathrm{Rad}(E) = E$. Now let $f : E \to R/I$ be a homomorphism for any right ideal I of R. This implies that $f(E) \subseteq R/I$, and since $\mathrm{Rad}(E) = E$, we have $\mathrm{Rad}(f(E)) = f(E)$. By the right Noetherian assumption, R/I is a Noetherian right R-module, and its submodule $f(E)$ is finitely generated, i.e. $\mathrm{Rad}(f(E)) \neq f(E)$. Also since $\mathrm{Rad}(f(E)) = f(E)$, this means that $f(E) = 0$, whence $f : E \to R/I$ can be lifted to R. Consequently, E is R-projective. □

Now we give a characterization of almost-QF and max-QF rings over right Hereditary and right Noetherian rings.

Theorem 3.1 *Let R be a right Hereditary and right Noetherian ring. Then the following are equivalent.*

(1) *R is right almost-QF.*

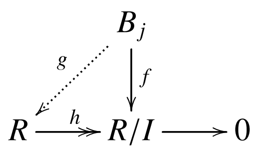
(2) *R is right max-QF.*

(3) *Every injective right R-module E has a decomposition* $E = A \oplus B$*, where* $\mathrm{Rad}(A) = A$ *and B is projective and semisimple.*

(4) $R = S \times T$*, where S is a semisimple Artinian ring and T is a right small ring.*

Proof (1) ⇒ (2) Clear.

(2) ⇒ (3) Let E be an injective right R-module. Then E has an indecomposable decomposition $E = \oplus_{i\in\Gamma} A_i$ where A_i's are either projective or $\mathrm{Rad}(A_i) = A_i$ by Proposition 3.13. Let $\Lambda = \{j \in \Gamma : A_j \text{ is projective}\}$. So the decomposition of E can be written as $E = (\oplus_{j\in\Lambda} A_j)\oplus(\oplus_{i\in\Gamma-\Lambda} A_i)$. We claim that each A_j is simple for $j \in \Lambda$. Since A_j is projective for $j \in \Lambda$, $\mathrm{Rad}(A_j) \neq A_j$. So, there exists a simple factor B_j of A_j i.e. $B_j \cong A_j/N \cong R/I$ for some maximal submodule N of A_j and for some maximal right ideal I of R. Since B_j is injective, by (2), the following diagram commutes.

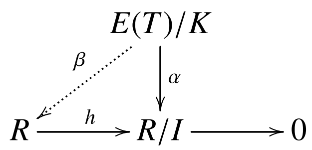


With the Hereditary assumption on R, $\mathrm{Im}(g) \cong B_j$ is projective, so $A_j \cong N \oplus B_j$. However A_j is indecomposable, whence $N = 0$. Consequently, each A_j is simple for $j \in \Lambda$.

(3) ⇒ (1) Let E be an injective right R-module. By the assumption, $E = A \oplus B$, where $\mathrm{Rad}(A) = A$ and B is semisimple and projective. Since B is R-projective, we only need to show that A is R-projective. By the Noetherian assumption, the injective R-module A has a decomposition $A = \oplus_{i\in\Gamma} A_i$, where each A_i is indecomposable injective with $\mathrm{Rad}(A_i) = A_i$. Proposition 3.13 implies that each A_i is R-projective, whence A is R-projective by Lemma 3.2. Therefore, $M = A \oplus B$ is R-projective by Lemma 3.2.

(2) ⇒ (4) Let S be the sum of minimal injective right ideals of R. Then S is injective since R is right Noetherian. Thus, we have the decomposition $R = S \oplus T$ for some right ideal T of R such that $\mathrm{Soc}(S) = S$, and T has no simple injective submodule. If $f : S \to T$ is a nonzero homomorphism, then $f(\mathrm{Soc}(S)) = f(S) \subseteq \mathrm{Soc}(T)$, where $f(S)$ is injective by the Hereditary assumption, so $\mathrm{Soc}(T)$ contains a semisimple injective direct summand $f(S)$. This means that $f(S) = 0$, a contradiction. Thus, we have $\mathrm{Hom}(S, T) = 0$, so S is a two sided ideal. On the other hand, if $g : T \to S$ is a nonzero homomorphism, then $T/\mathrm{Ker}(g) \cong \mathrm{Im}(g) \subseteq S$, so $T/\mathrm{Ker}(g)$ is projective by Hereditary assumption. Also, since S is a semisimple injective R-module, $T/\mathrm{Ker}(g)$ is semisimple injective, whence $K/\mathrm{Ker}(g)$ is semisimple injective for any maximal submodule $K/\mathrm{Ker}(g)$ of $T/\mathrm{Ker}(g)$.

This implies that $T/\operatorname{Ker}(g) \cong K/\operatorname{Ker}(g) \oplus T/K$. Then the simple R-module T/K is injective and projective, so T contains an isomorphic copy of a simple injective R-module T/K, yielding a contradiction. Therefore, $\operatorname{Hom}(T, S) = 0$, so T is a two sided ideal. Consequently, $R = S \oplus T$ is a ring decomposition. Now, let $E(T)$ be the injective hull of T as an R-module. The injective hull $E(T)$ is also a T-module by the fact that $E(T)S = 0$. We claim that $\operatorname{Rad}(E(T)) = E(T)$. Suppose the contrary, and let K be a maximal submodule of $E(T)$. Then $E(T)/K$ is injective by the Hereditary assumption, and it is max-projective by (2). Since $E(T)/K$ is a simple right R-module, it is isomorphic to R/I for some maximal right ideal I of R, so R/I is injective. Then, the isomorphism $\alpha : E(T)/K \to R/I$ lifts to $\beta : E(T)/K \to R$ i.e. the following diagram commutes.



Since β is monic and $E(T)/K$ injective, $U = \beta(E(T)/K)$ is a direct summand of R. It is easy to see that U is also a right T-module, so $U \subseteq T$. On the other hand, since U is minimal and injective, U is also contained in S, a contradiction. So we must have $\operatorname{Rad}(E(T)) = E(T)$, whence $T \ll E(T)$ by Lemma 2.2. This proves (4).

(4) $\Rightarrow$ (1) Clear, by Lemma 3.5 and Proposition 3.15.

□

Theorem 3.2 *Let R be a right Hereditary ring. Then the following are equivalent.*

(1) *R is right max-QF.*

(2) *Every simple injective right R-module is projective.*

(3) *Every singular injective right R-modules is R-projective.*

(4) *Every singular injective right R-modules is max-projective.*

(5) $\operatorname{Rad}(E) = E$ *for every singular injective right R-module E.*

(6) *Every injective right R-module E can be decomposed as $E = Z(E) \oplus F$ with* $\operatorname{Rad}(Z(E)) = Z(E)$.

Proof (1) $\Rightarrow$ (4), (3) $\Rightarrow$ (4) and (6) $\Rightarrow$ (5) are clear.

(4) $\Rightarrow$ (2) Let S be a simple injective right R-module. We claim that S is projective. Assume that S is not projective. Then it is singular by Proposition 2.8. This implies,

by our hypothesis, that S is max-projective, hence S is projective, this is a contradiction. The conclusion now follows.

(2) $\Rightarrow$ (1) Let E be an injective right R-module and $f : E \to S$ where S is a simple right R-module. If $f = 0$, there is nothing to prove. We may assume that f is a nonzero homomorphism, so f is an epimorphism. Since R is right Hereditary, S is injective, and so by (2), S is projective. Hence, the natural epimorphism $\pi : R \to S$ splits, i.e. there exists a homomorphism $\eta : S \to R$ such that $\pi\eta = 1_S$. Then, $\pi\eta f = f$, so E is max-projective.

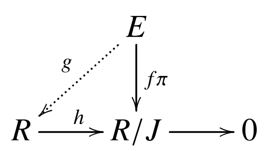
(4) $\Rightarrow$ (5) Let E be a singular injective right R-module. Assume to the contrary that E has a maximal submodule K such that $E/K \cong R/I$ for some maximal right ideal I of R. So, there is a nonzero homomorphism $f : E \to R/I$, and by (4), there exists a nonzero homomorphism $g : E \to R$ such that $\pi g = f$, where $\pi : R \to R/I$ is the canonical epimorphism. Since E is singular, $\mathrm{Im}(g)$ is singular. Moreover, $\mathrm{Im}(g) \subseteq R$, so $\mathrm{Im}(g)$ is nonsingular. This implies that $g(E) = 0$, yielding a contradiction.

(5) $\Rightarrow$ (6) Let E be an injective right R-module. Since R is a right nonsingular ring, $Z(E)$ is a closed submodule of E, so $E = Z(E) \oplus F$ for some submodule F of E. Then, by (5), $\mathrm{Rad}(Z(E)) = Z(E)$.

(5) $\Rightarrow$ (3) Let E be a singular injective right R-module. This implies, by our hypothesis, that $\mathrm{Rad}(E) = E$. Let $f : E \to R/I$ be homomorphism for some right ideal I of R. Since $\mathrm{Rad}(E) = E$ and $\mathrm{Rad}(R/I) \neq R/I$, $f : E \to R/I$ is not an epimorphism. By the right Hereditary assumption, $f(E)$ is injective, so $f(E)$ is a direct summand of R/I. But since $f(E) \subseteq \mathrm{Rad}(R/I)$, we must have $f(E) \ll R/I$. This means, $f(E) = 0$, whence $\mathrm{Hom}(E, R/I) = 0$ for each right ideal I of R. Therefore, E is R-projective. □

Proposition 3.16 *Let R be a local right max-QF ring. Then R is either right self-injective or right small.*

Proof Let J be the unique maximal right ideal of R and E the injective hull of the ring R. Assume first that R is not a small ring i.e. $\mathrm{Rad}(E) \neq E$. Then E has a maximal submodule K such that E/K is isomorphic to R/J and denote this isomorphism by f. Consider the composition $f\pi$ where $\pi : E \to E/K$ is the canonical projection. Since R is right max-QF, there is a nonzero homomorphism $g : E \to R$ such that



commutes. Furthermore, h is a small epimorphism and $f\pi$ is an epimorphism, which means $g : E \to R$ is also an epimorphism and splits. Thus, $E \cong R \oplus T$ for some T. Hence, R is a right self injective ring. □

Corollary 3.8 *Let R be a right Noetherian local ring. Then the following are equivalent.*

(1) *R is right almost-QF.*

(2) *R is right max-QF.*

(3) *R is QF or right small.*

Proof (1) ⇒ (2) Clear. (3) ⇒ (1) Follows from Proposition 3.15.

(2) ⇒ (3) Clear by Proposition 3.16 since right Noetherian right self-injective rings are QF. □

Proposition 3.17 *Let R be a semiperfect ring. Then the following are equivalent.*

(1) *R is right almost-QF and direct sum of small right R-modules is small.*

(2) *R is right max-QF and direct sum of small right R-modules is small.*

(3) *R is right almost-QF and* $\mathrm{Rad}(Q) \ll Q$ *for each injective right R-module Q.*

(4) *R is right max-QF and* $\mathrm{Rad}(Q) \ll Q$ *for each injective right R-module Q.*

(5) *R is QF.*

Proof (1) ⇒ (2) and (3) ⇒ (4) Clear. (2) ⇒ (3) and (3) ⇒ (1) By ((Rayar, 1967), Lemma 9).

(4) ⇒ (5) Let M be an injective right R-module. Since M is max-projective with $\mathrm{Rad}(M) \ll M$, by Proposition 3.8(1), M is projective. Hence R is QF by Theorem 2.3.

(5) ⇒ (3) Let M be an injective right R-module. By the hypothesis, M is projective. Since R is right Artinian, every right R-module has a small radical, whence $\mathrm{Rad}(M) \ll M$. □

In (Crivei, 2014), a submodule N of a right R-module M is called *coneat* in M if $\mathrm{Hom}(M, S) \to \mathrm{Hom}(N, S)$ is epic for every simple right R-module S. In (Büyükaşık and Durğun, 2015), N is called *s-pure* in M if $N \otimes S \to M \otimes S$ is monic for every simple left R-module S. M is *absolutely coneat* (resp., *absolutely s-pure*) if M is coneat (resp., s-pure) in every extension of it. If R is commutative, then s-pure short exact sequences coincide with coneat short exact sequences, ((Fuchs, 2012), Proposition 3.1).

Proposition 3.18 *Consider the following conditions for a ring R:*

(1) *R is right max-QF.*

(2) *Every absolutely coneat right R-module is max-projective.*

(3) *Every absolutely s-pure right R-module is max-projective.*

(4) *Every absolutely pure right R-module is max-projective.*

Then (3) $\Rightarrow$ (4) $\Rightarrow$ (1) $\Rightarrow$ (2)*. Also, if R is a commutative ring, then* (2) $\Rightarrow$ (3)*.*

Proof (3) $\Rightarrow$ (4) $\Rightarrow$ (1) Clear.

(1) $\Rightarrow$ (2) Let M be an absolutely coneat right R-module. Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{i} & E(M) \\ & & \downarrow{\scriptstyle f} & & \\ R & \xrightarrow{\pi} & S & \longrightarrow & 0 \end{array}$$

where S is a simple right R-module, $i : M \to E(M)$ is the inclusion map and $\pi : R \to S$ is the canonical quotient map. Since M coneat in $E(M)$, there is a homomorphism $g : E(M) \to S$ such that $gi = f$. Also, by (1), there exists a homomorphism $h : E(M) \to R$ such that $\pi h = g$. Hence, $(\pi h)i = gi = f$.

(2) $\Rightarrow$ (3) Let M be an absolutely s-pure right R-module. Then M is s-pure in $E(M)$. Since R is commutative, M is coneat in $E(M)$. Hence, M is max-projective by (2). □

In ((Nicholson, 1976), Lemma 1.16), it was shown that for a projective module M, if $M = P + K$, where P is a summand of M and $K \subseteq M$, then there exists a submodule $Q \subseteq K$ with $M = P \oplus Q$. By using the same method in the proof of ((Amini, Amini, and Ershad, 2009), Theorem 2.8), one can prove the following result.

Proposition 3.19 *A ring R is right almost-QF if and only if for every injective right R-module E, if* $E = P + L$*, where P is a finitely generated projective summand of E and* $L \subseteq E$*, then* $E = P \oplus K$ *for some* $K \subseteq L$*.*

Let R be a ring and Ω a class of R-modules which is closed under isomorphic copies. Following Enochs, a homomorphism $\varphi : G \to M$ with $G \in \Omega$ is called an Ω-precover of the R-module M if for each homomorphism $\psi : H \to M$ with $H \in \Omega$, there exists $\lambda : H \to G$ such that $\varphi\lambda = \psi$.

Lemma 3.6 *Let R be a right self-injective ring. Then the following are equivalent.*

(1) *R is right almost-QF.*

(2) *Every finitely generated right R-module has injective precover which is R-projective.*

(3) *Every cyclic right R-module has an injective precover which is R-projective.*

Proof (1) $\Rightarrow$ (2) Let M be a finitely generated right R-module and $g : R^n \to M$ an epimorphism. For any homomorphism $f : E \to M$ with E is injective, there exists $h : E \to R^n$ such that $gh = f$. Since R^n is injective, g is an injective precover of M.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Let E be an injective right R-module and I a right ideal of R. Suppose that $f : E \to R/I$ is a homomorphism, $\pi : R \to R/I$ is the natural epimorphism and $g : G \to R/I$ be an injective cover of R/I. So, there is $h : E \to G$ such that $gh = f$. By hypothesis, G is R-projective, and hence there is $k : G \to R$ such that $\pi k = g$. Let $\overline{f} = kh$. So $\pi\overline{f} = \pi kh = gh = f$. Therefore, E is R-projective, so R is right almost-QF. □

In (Enochs and Jenda, 1991), a module M is said to be *copure-injective* if $\text{Ext}^1_R(E, M) = 0$ for any injective module E. Now we give the characterization of almost-QF rings in terms of copure-injective modules.

Proposition 3.20 *Let R be a ring. Then the following are equivalent.*

(1) *R is right almost-QF and R_R is copure-injective.*

(2) *Every right ideal of R is copure-injective.*

(3) *Every submodule of a finitely generated projective right R-module is copure injective.*

Proof (1) $\Rightarrow$ (2) Let E be an injective right R-module and I a right ideal of R. By applying $\text{Hom}(E, -)$ to the short exact sequence $0 \to I \to R \to R/I \to 0$, we obtain the following exact sequence: $0 \to \text{Hom}(E, I) \to \text{Hom}(E, R) \to \text{Hom}(E, R/I) \to \text{Ext}^1_R(E, I) \to \text{Ext}^1_R(E, R) \to ....$ Since R_R is copure-injective, $\text{Ext}^1_R(E, R) = 0$. Then the map $\text{Hom}(E, R) \to \text{Hom}(E, R/I)$ is onto since E is R-projective. Hence, $\text{Ext}^1_R(E, I) = 0$ for any injective R-module E.

(2) $\Rightarrow$ (3) Suppose that every right ideal of R is copure-injective. First, by induction, we show that every submodule of R^n is copure-injective. The case $n = 1$ follows by the hypothesis. Now suppose that $n > 1$ and every submodule of R^{n-1} is copure-injective. Let N be a submodule of R^n, and consider the exact sequence $0 \to N \cap R^{n-1} \to N \to N/(N \cap R^{n-1}) \to 0$. By induction hypothesis, $N \cap R^{n-1}$ is copure-injective, and

$N/(N \cap R^{n-1}) \cong (N + R^{n-1})/R^{n-1} \subseteq R^n/R^{n-1} \cong R$ is also copure-injective. Therefore, for any injective right R-module E, consider the exact sequence $\operatorname{Ext}^1_R(E, N \cap R^{n-1}) \to \operatorname{Ext}^1_R(E, N) \to \operatorname{Ext}^1_R(E, N/(N \cap R^{n-1}))$. Since $\operatorname{Ext}^1_R(E, N \cap R^{n-1}) = \operatorname{Ext}^1_R(E, N/(N \cap R^{n-1})) = 0$, we have $\operatorname{Ext}^1_R(E, N) = 0$. Therefore, N is copure-injective. Now if M is a submodule of a finitely generated projective right R-module P, then there is $n \geq 1$ such that $M \subseteq P \subseteq R^n$. By the above observation, M is also copure-injective. (3) $\Rightarrow$ (2) is clear. (2) $\Rightarrow$ (1) by Proposition 3.2. □

Proposition 3.21 *Let R be a ring. Then the following are equivalent.*

(1) *R is semisimple.*

(2) *R is right almost-QF right V-ring.*

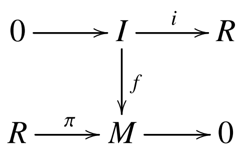
(3) *R is right almost-QF and every submodule of an R-projective right module is R-projective.*

(4) *R is right self-injective and every submodule of an R-projective right module is R-projective.*

Proof (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) are clear.

(2) $\Rightarrow$ (1) Since R is a right V-ring, every simple right R-module is injective. By the hypothesis, every simple right R-module is R-projective, whence projective.

(4) $\Rightarrow$ (1) Let M be a cyclic right R-module and I a right ideal of R. Consider the following diagram:



where $i : I \to R$ is the inclusion map and $\pi : R \to M$ is the canonical quotient map. Since I is R-projective, there exists $h : I \to R$ such that $\pi h = f$. By the injectivity of R, there exists $\lambda : R \to R$ such that $\lambda i = h$. Then $(\pi\lambda)i = \pi h = f$, and $\pi\lambda : R \to M$ is the required map.

(3) $\Rightarrow$ (1) Since every simple right R-module can be embedded in an injective R-module, every simple right R-module is R-projective, and so every simple right R-module is projective. Hence, R is semisimple. □

3.3. Almost-QF Rings Over Commutative Rings

In this section we deal with almost-QF and max-QF rings over commutative rings. We also give a complete characterization of almost-QF and max-QF rings over commutative Noetherian rings.

Let R be a commutative semiperfect ring, then by Theorem 2.12, $R = R_1 \times ... \times R_n$, where R_i is a local ring ($1 \leq i \leq n$). Let m_i be the maximal ideal of R_i ($1 \leq i \leq n$). Then every maximal ideal of R is of the form: $M_i = R_1 \times ...R_{i-1} \times m_i \times R_{i+1} \times ... \times R_n$. Since $(0, ..., 0, 1, 0, ..., 0) \in R \setminus M_i$ (1 is in the ith coordinate), we can easily show that $R_{M_i} \cong (R_i)_{m_i} \cong R_i$. Therefore, by Lemma 3.5 we can prove the following Proposition.

Proposition 3.22 *Let R be a commutative semiperfect ring. Then R is almost-QF (resp. max-QF) if and only if for any maximal ideal M of R, the ring R_M is almost-QF (resp. max-QF).*

The following example shows that the semiperfect condition in Proposition 3.22, is essential.

Example 3.3 *Let $\{F_n\}_{n\geq 1}$ be a family of fields and let $R = \prod_{n=1}^{\infty} F_n$. Then R is a von Neumann regular ring. By ((Lam, 2001), Exercise 4.15), for any maximal ideal M of R, the ring R_M is a field (and hence almost-QF). But R is not max-QF, otherwise, since every simple R-module is injective, every simple R-module would be projective, so R would be a semisimple ring.*

Corollary 3.9 *Let R be a commutative semiperfect ring. If R is max-QF, then $R = S \times T$ where S is self-injective and T is small.*

Proof Let R be a commutative semiperfect ring, then by ((Lam, 2001), Theorem 23.11), $R = R_1 \times ... \times R_n$, where R_i is a local ring ($1 \leq i \leq n$). Hence, by Lemma 3.5 and Proposition 3.16, R can be written as a direct product of local max-QF rings and every local max-QF ring either self-injective or small. □

We do not know whether every right chain ring is almost-QF. But the following result will imply that each right chain ring with $P(R) = 0$ is right almost QF.

Proposition 3.23 *Let R be a right chain ring and $J = J(R)$. Then $P(R) = \bigcap_{n\geq 1} J^n$.*

Proof Assume first that $J^n = 0$ for some $n \in \mathbb{Z}^+$. Then $\bigcap_{n\geq 1} J^n = 0$, and so by ((Facchini, 1998), Proposition 5.3(b)), R is a right Noetherian ring with $P(R) = 0$. On the other hand if we suppose that $J^n \neq 0$ for all $n \in \mathbb{Z}^+$, then, by ((Facchini, 1998), Proposition 5.2(d)), $A = \bigcap_{n\geq 1} J^n$ is a completely prime ideal. Let us now look at the case $A \neq AJ$. Then A/AJ simple right R-module and $AJ \ll A$. Let $a \in A \setminus AJ$. If we have $A = aR + AJ$, then $A = aR$, whence either $A = J(R)$ or $A = 0$, by ((Facchini, 1998), Proposition 5.2(f)). If $A = \bigcap_{n\geq 1} J^n = 0$, then R is a right Noetherian ring with $P(R) = \bigcap_{n\geq 1} J^n = 0$. Otherwise, if $A = J(R) = \bigcap_{n\geq 1} J^n$, then $J = J^2$, but since $A \neq AJ$, this is not the case. If we look at the case $A = AJ$, then $A \subseteq P(R)$. Since $P(R) = P^2(R)$, $P(R)$ is a completely prime ideal of R, and so, by ((Facchini, 1998), Lemma 5.1), $P(R) \subseteq A$. Hence, $P(R) = A = \bigcap_{n\geq 1} J^n$. □

Corollary 3.10 *Let R be a right chain ring. Then $R/P(R)$ is a right almost-QF ring.*

Proof Since $P(R)$ is an ideal of R and every factor ring of a right chain ring is a right chain ring, without loss of generality we may assume that $P(R) = 0$. Then, by Proposition 3.23 and ((Facchini, 1998), Proposition 5.3), R is a right Noetherian ring. We have two cases for $J = J(R)$: if J is nilpotent, then R is Artinian. This implies that R is right self-injective by ((Facchini, 1998), Lemma 5.4) which then yields, R is QF. So, now assume that J is not nilpotent. Then R is a domain by ((Facchini, 1998), Proposition 5.2(d)), whence R is right small. So, R is right almost-QF by Proposition 3.15. Thus in any case R is right almost-QF. □

We shall characterize commutative Noetherian max-QF rings.

Lemma 3.7 *Let R be a commutative Noetherian ring, and let $E = E(R/Q)$ for a maximal ideal Q of R. The following are equivalent.*

(1) *E is R-projective.*

(2) *E is max-projective.*

(3) $\mathrm{Rad}(E) = E$ *or E is projective, local and isomorphic to an ideal of R.*

Proof (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) Assume that $\mathrm{Rad}(E) \neq E$. Since R is commutative, $\mathrm{Rad}(E) = \bigcap_{i\in\wedge} IE$, where $\wedge$ is the set of all maximal ideals of R, ((Anderson and Fuller, 1992), Exercises 15.(5)). Now we will show that $IE = E$ for any maximal ideal I distinct from Q. Let I be a maximal ideal distinct from Q. The fact $I + Q = R$ implies $I + Q^n = R$ for any $n \in \mathbb{N}$. Let $x \in E$. Then $Q^n x = 0$ for some $n \in \mathbb{N}$ by Proposition 2.2. We have $1 = y + z$, where $y \in I$, $z \in Q^n$, and then $x = yx \in IE$. Hence, $\mathrm{Rad}(E) = \bigcap_{i\in\wedge} IE = QE \neq E$. Since R is commutative and $(E/QE)Q = 0$, E/QE is a semisimple R/Q-module, so E/QE semisimple as an R-module. Then E/QE is finitely generated by Artinianity of E, and

hence $QE + K = E$ for some finitely generated submodule K of E. Since K is finitely generated, K is a submodule of A_n for some n, by Proposition 2.2. Thus $Q^n K = 0$. Since $QE + K = E$, $Q^{n+1}E = Q^n E$, implying $Q^n E \subseteq P(E)$. On the other hand, $Q^2E + QK = QE$, and so $Q^2E + K = E$. Continuing in this manner $Q^n E + K = E$, whence $E/Q^n E$ is finitely generated. Since R is Noetherian, $P(E/Q^n E) = 0$, so $P(E) = Q^n E$. Since $E/P(E)$ is finitely generated, $E/P(E)$ has finite composition length by Proposition 2.2(3). By max-projectivity of E and Lemma 3.3, $E/P(E)$ is max-projective. Thus $E/P(E)$ is projective by Corollary 3.3. Then, $E = P(E) \oplus L$ for some projective submodule L of E. Since E is indecomposable and $P(E) \neq E$, $E = L$. Therefore E is projective. Furthermore, since E is indecomposable, the endomorphism ring of E is local by ((Facchini, 1998), Lemma 2.25). By ((Ware, 1971), Theorem 4.2), E is a local module, so it is cyclic and $R \cong E \oplus I$ for some ideal I of R. Hence E is isomorphic to an ideal of R. This proves (3).

(3) $\Rightarrow$ (1) is obvious. □

Lemma 3.8 *((Kasch, 1982), [9.7]) Suppose R is a commutative Noetherian or semilocal right Noetherian ring, and let $\{M_i\}_{i\in I}$ be a class of right R-modules. Then* $\mathrm{Rad}(\prod_{i\in I} M_i) = \prod_{i\in I} \mathrm{Rad}(M_i)$.

Lemma 3.9 *Let R be a commutative Noetherian ring. Then the following are equivalent.*

(1) *R is a small ring, i.e., $R \ll E(R)$.*

(2) $\mathrm{Rad}(E(S)) = E(S)$ *for each simple R-module S.*

Proof (1) $\Rightarrow$ (2): Clear by Lemma 2.2.

(2) $\Rightarrow$ (1): Let Δ be a complete set of representatives of simple R-modules. Then $C = \oplus_{S\in\Delta} E(S)$ is an injective cogenerator. Then, for some index set I, the injective hull $E(R)$ of R is a direct summand of C^I. By Lemma 3.8, $\mathrm{Rad}(C^I) = C^I$. Since $E(R)$ is a direct summand of C^I, we have $\mathrm{Rad}(E(R)) = E(R)$. Thus R is a small ring by Lemma 2.2. □

Theorem 3.3 *Let R be a commutative Noetherian ring. Then the following are equivalent.*

(1) *R is almost-QF.*

(2) *R is max-QF.*

(3) *$R = A \times B$, where A is QF and B is small.*

Proof (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) First suppose that $\mathrm{Rad}(E(S)) = E(S)$ for all simple R-module S. Then R is a small ring by Lemma 3.9. On the other hand, if $\mathrm{Rad}(E(S)) \neq E(S)$ for some simple R-module S, then $E(S)$ is isomorphic to a direct summand of R by Lemma 3.7. Let X be sum of minimal ideals U of R with $\mathrm{Rad}(E(U)) \neq E(U)$. Then $E(U)$ is isomorphic to an ideal of R. Thus without loss of generality we can assume that $E(U)$ is an ideal of R. Since R is Noetherian, X is finitely generated, and so $A = E(X) = E(U_1) \oplus \cdots \oplus E(U_n)$ where each $E(U_i)$ is an ideal of R. Thus $R = A \oplus B$ for some ideal B of R. Now A is injective and Noetherian, so A is a QF ring. On the other hand, let V be a simple B-module, then V is a simple R-module. Let $E(V)$ be the injective hull of V. As V is a B-module, $VA = 0$. If $\mathrm{Rad}(E(V)) \neq E(V)$, then this would imply $V \subseteq A$, by the same arguments above. Thus $\mathrm{Rad}(E(V)) = E(V)$, and so B is a small ring by Lemma 3.9.

(3) $\Rightarrow$ (1) Clear, by Proposition 3.15 and Lemma 3.5. □

CHAPTER 4

ON M-FLAT AND M-COTORSION MODULES

The purpose of this chapter is to mention the study of some homological objects which is related to the max-projective modules. Namely, m-injective and m-flat modules. Recall that a right R-module E is said to be m-injective if $Ext^1_R(R/I, E) = 0$ for every maximal right ideal I of R_R (see, (Wang and Zhao, 2005)). A right R-module M is called m-flat if $Tor^R_1(M, R/I) = 0$ for any maximal left ideal I of R (see (Wang, 2005)). Also, the concept of m-cotorsion modules is introduced. A right R-module M is said to be m-cotorsion if $Ext^1_R(N, M) = 0$ for any m-flat right R-module N. Several elementary properties of m-flat, m-injective and m-cotorsion modules are obtained in this chapter.

4.1. m-flat and m-cotorsion Modules

Recall that R is called a *left coherent* ring if every finitely generated left ideal of R is finitely presented. Following (Xiang, 2010), a ring R is said to be *left max-coherent* if every maximal left ideal is finitely presented. Obviously, Noetherian rings are max-coherent. But, left coherent rings need not be max-coherent.

Lemma 4.1 *((Xiang, 2010)) Let R be a left max-coherent ring, the following are true.*

(1) *Any maximal left ideal is finitely generated, and any direct product of m-flat right R-modules is m-flat.*

(2) *A left R-module M is m-injective if and only if M^+ is m-flat.*

(3) *A right R-module N is m-flat if and only if N^{++} is m-flat.*

(4) $\mathfrak{MI}$ *is closed under pure submodules, pure quotients and direct sums.*

(5) *Every right R-module has an $\mathfrak{MF}$-preenvelope.*

Recall that an exact sequence of left R-modules $0 \to A \to B \to C \to 0$ is called s-pure exact provided that $0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ is exact for any simple right R-module M, (Crivei, 2014). In this case, A is said to be s-pure submodule of B. In (Büyükaşık and Durğun, 2015), a submodule N of a left R-module M is called neat in M

if for every simple R-module S, every homomorphism $f : S \to M/N$ can be lifted to a homomorphism $g : S \to M$.

Lemma 4.2 *The following are true for a ring R.*

(1) $\mathfrak{MI}$ *is closed under extensions, direct products, direct summands and neat submodules.*

(2) $\mathfrak{MF}$ *is closed under s-pure quotients.*

(3) $\mathfrak{MF}$ *is closed under extensions, direct sums, direct summands, pure submodules and pure quotients.*

Proof (1) The class of m-injective left R-modules is closed under extensions, direct products, direct summands by ((Xiang, 2010), Proposition 2.4(1)). Let N be a neat submodule of a m-injective left R-module M. For any maximal left ideal I of R, we have the exact sequence $Hom(R/I, M) \to Hom(R/I, M/N) \to Ext^1(R/I, N) \to Ext^1(R/I, M) = 0$. Since $Hom(R/I, M) \to Hom(R/I, M/N)$ is epic by neatness, $Ext^1(R/I, N) = 0$. So N is m-injective.

(2) Let N be an s-pure submodule of a m-flat left R-module M. For any maximal left ideal I of R, we have the exact sequence
$0 = Tor_1(M, R/I) \to Tor_1(M/N, R/I) \to N \otimes R/I \to M \otimes R/I$. Since, $0 \to N \otimes R/I \to M \otimes R/I$ is exact, $Tor_1(M/N, R/I) = 0$. So M/N is m-flat.

(3) The class of m-flat right R-modules is closed under extensions, direct sums, direct summands by ((Xiang, 2010), Proposition 2.4(2)). Let $0 \to N \to M \to M/N \to 0$ be a pure exact sequence of right R-modules where M is m-flat. Then $Tor_1(M, R/I)^+ = 0 = Ext(R/I, M^+)$ for any maximal left ideal I of R. Since M^+ is m-injective and $0 \to (M/N)^+ \to M^+ \to N^+ \to 0$ a split exact sequence, N^+ and $(M/N)^+$ is m-injective. Hence N and M/N is m-flat. □

Definition 4.1 *A right R-module M is said to be m-projective (resp. m-cotorsion) if* $Ext^1_R(M, N) = 0$ *(resp.* $Ext^1_R(N, M) = 0$*) for any m-injective (resp. m-flat) right R-module N. The left version can be defined similarly.*

Remark 4.1 *By the definitions, any simple right R-module is m-projective and any SP-injective right R-module is m-cotorsion. Moreover, any m-cotorsion right R-module is cotorsion. (a right R-module C is called cotorsion provided that* $Ext^1_R(F, C) = 0$ *for any flat right R-module F (Enochs and Jenda, 2000)).*

In what follows, $\mathfrak{MP}$ (resp. $\mathfrak{MC}$) stands for the class of all m-projective left (resp. all m-cotorsion right) R-modules.

As is known to all, every R-module has a cotorsion envelope and a flat cover. Now we have the following Lemma.

Lemma 4.3 *The following are true for a ring R.*

(1) *($\mathfrak{MF}$,$\mathfrak{MC}$) is a perfect cotorsion theory.*

(2) *Every module has a special $\mathfrak{MF}$-precover and a special $\mathfrak{MC}$-preenvelope.*

(3) *($\mathfrak{MP}$,$\mathfrak{MI}$) is a complete cotorsion theory.*

Proof (1) follows from Lemma 4.2(3) and ((Holm and Jørgensen, 2008), Theorem 3.4).

(2) A perfect cotorsion theory is always complete by Wakamatsu′s Lemmas ((Xu, 1996), §2.1), that is every module has a special $\mathfrak{MF}$-precover and a special $\mathfrak{MC}$-preenvelope.

(3) Let $\mathfrak{C}$ be the set of representatives of simple left R-modules. Thus $\mathfrak{MI} = \mathfrak{C}^{\perp}$. Since $\mathfrak{MP} = ^{\perp} (\mathfrak{C}^{\perp})$, the result follows from ((Enochs and Jenda, 2000), Definition 7.1.5) and ((Eklof and Trlifaj, 2001), Theorem 10). □

Corollary 4.1 *Let R be a ring. Then the following are equivalent.*

(1) *Every left R-module is m-projective.*

(2) *Every cyclic left R-module is m-projective.*

(3) *Every m-injective left R-module is injective.*

(4) *($\mathfrak{MP}$,$\mathfrak{MI}$) is hereditary and every m-injective left R-module is m-projective.*

In this case, if R is left max-coherent, R is left Noetherian.

Proof (1) ⇒ (2) and (1) ⇒ (4) are clear. (3) ⇒ (1) holds by Lemma 4.3(3).

(2) ⇒ (3) Let M be any m-injective left R-module and I any left ideal of R. Then $Ext^1_R(R/I, M) = 0$ by (2). Thus M is injective, as desired.

(4) ⇒ (1) By Lemma 4.3(3), for any left R-module M, there is a short exact sequence $0 \to M \to F \to L \to 0$, where F is m-injective and L is m-projective. So (1) follows from (4).

In this case, if R is a left max-coherent ring, every FP-injective left R-module is m-injective and so injective. This means R is left Noetherian by Theorem 2.5. □

Corollary 4.2 *Let R be a left max-coherent ring. Then the following are equivalent.*

(1) *Every m-flat right R-module is flat.*

(2) *Every cotorsion right R-module is m-cotorsion.*

(3) *Every m-injective left R-module is FP-injective.*

(4) *Every finitely presented left R-module is m-projective.*

In this case, R is a left coherent ring.

Proof (1) ⇔ (2) and (3) ⇔ (4) follow from Lemma 4.3.

(1) ⇒ (3) Let M be any m-injective left R-module. Then M^+ is m-flat by Lemma 4.1(2) and so M^+ is flat by (1). On the other hand, for any finitely presented left R-module N, $0 = Tor_1^R(M^+, N) \cong (Ext_R^1(N, M))^+$. Thus M is FP-injective.

(3) ⇒ (1) Let M be any m-flat right R-module. Then M^+ is m-injective and so M^+ is FP-injective by (3). Hence M is flat.

To prove the last statement, let M be an FP-injective left R-module with N a pure submodule, then M/N is m-injective by ((Xiang, 2010), Proposition 2.6) since R is left max-coherent. Therefore M/N is FP-injective by (3) and hence R is a left coherent ring by ((Moradzadeh-Dehkordi and Shojaee, 2015), Theorem 3.7). □

In general, $\mathfrak{MI}$-cover need not be an epimorphism and $\mathfrak{MF}$-preenvelope need not be a monomorphism. In the following theorem, we will extend the result in ((Xiang, 2010), Theorem 2.11).

Theorem 4.1 *Let R be a left max-coherent ring. Then the following are equivalent.*

(1) *R is left m-injective.*

(2) *Every left R-module has an epic $\mathfrak{MI}$-cover.*

(3) *Every right R-module has a monic $\mathfrak{MF}$-preenvelope.*

(4) *Every injective right R-module is m-flat.*

(5) *Every flat left R-module is m-injective.*

(6) *($\mathfrak{MI}$,$\mathfrak{MI}^{\perp}$) is a perfect cotorsion theory.*

Proof (1) ⇔ (2) ⇔ (3) follow from ((Xiang, 2010), Theorem 2.11).

(3) ⇒ (4) is clear since by (3), every injective right R-module can be embedded in an m-flat right R-module.

(4) ⇒ (5) Let M be a flat left R-module. Then M^+ is injective, so M^+ is m-flat by (4). Thus M is m-injective by Lemma 4.1(2).

(5) ⇒ (6) Note that $\mathfrak{MI}$ is closed under extensions and by Lemma 4.1(4) is closed under pure submodules, pure quotients and direct sums over a left max-coherent ring. Hence by (5) and ((Holm and Jørgensen, 2008), Theorem 3.4), $(\mathfrak{MI},\mathfrak{MI}^{\perp})$ is a perfect cotorsion theory.

(6) ⇒ (1) is clear. □

We conclude this section with the following Theorem.

Theorem 4.2 *Let R be a ring. Then the following are equivalent.*

(1) *Every quotient of an m-cotorsion right R-module is m-cotorsion.*

(2) *All m-flat right R-modules are of projective dimension* ≤ 1.

(3) *For any s-pure exact sequence* $0 \to N \to M \to L \to 0$ *of right R-modules with M projective, N is projective.*

Proof (1) ⇒ (3) Let $0 \to N \to M \to L \to 0$ be an s-pure exact sequence of right R-modules with M projective. Then L is m-flat Lemma 4.2(2). Let A be any right R-module. Then there is an exact sequence $0 \to A \to E \to B \to 0$ with E injective. Note that B is m-cotorsion by (1), and hence $Ext^2(L, A) = Ext^1(L, B) = 0$. Thus, $pd(L) \leq 1$, so N is projective.

(3) ⇒ (2) Let M be any m-flat right R-module. There exists an exact sequence $0 \to N \to P \to M \to 0$ with P projective. Since for any maximal left ideal I of R, $0 = Tor_1(M, R/I) \to N \otimes R/I \to P \otimes R/I \to M \otimes R/I \to 0$ is exact, the sequence is s-pure, so N is projective by (3). It follows that $pd(M) \leq 1$.

(2) ⇒ (1) Let M be any m-cotorsion right R-module and K a submodule of M. For any m-flat right R-module N, the exactness of the sequence $0 \to K \to M \to M/K \to 0$ induces the exact sequence $0 = Ext^1(N, M) \to Ext^1(N, M/K) \to Ext^2(N, K)$. Note that $Ext^2(N, K) = 0$ by (2), so $Ext^1(N, M/K) = 0$. □

4.2. m-flat Dimensions

Since every right R-module over a left max-coherent ring R has a $\mathfrak{MF}$-preenvelope by Lemma 4.1, every right R-module M has a right $\mathfrak{MF}$-resolution, that is, there is a $Hom(-, \mathfrak{MF})$ exact complex $0 \to M \to F^0 \to F^1 \to ...$ (not necessarily exact) with each

F^i m-flat. On the other hand, every left R-module M over a left max-coherent ring R, has a $\mathfrak{MI}$-preenvelope by (Xiang, 2010). So M has a right $\mathfrak{MI}$-resolution, that is, there is a $Hom(-, \mathfrak{MI})$ exact complex $0 \to M \to E^0 \to E^1 \to ...$ with each E^i m-injective. Obviously, this complex is exact since injective R-modules are m-injective (Xiang, 2010).

Proposition 4.1 *Let R be a left max-coherent ring, then $-\otimes-$ is right balanced on $\mathfrak{M}_R \times_R \mathfrak{M}$ by $\mathfrak{MF} \times \mathfrak{MI}$.*

Proof Let M be any right R-module and $0 \to M \to F^0 \to F^1 \to ...$ a right $\mathfrak{MF}$-resolution of M. Suppose K is any m-injective left R-module. Note that K^+ is a m-flat right R-module by Lemma 4.1. Hence, the sequence $... \to Hom(F^1, K^+) \to Hom(F^0, K^+) \to Hom(M, K^+) \to 0$ is exact. Thus the sequence $... \to (F^1 \otimes K)^+ \to (F^0 \otimes K)^+ \to (M \otimes K)^+ \to 0$ is exact by ((Rotman, 1979), Theorem 2.11). So the sequence $0 \to M \otimes K \to F^0 \otimes K \to F^1 \otimes K$ is exact. On the other hand, let N be any left R-module and $0 \to N \to E^0 \to E^1 \to ...$ a right $\mathfrak{MI}$-resolution of N. We need to show that the sequence $0 \to F \otimes N \to F \otimes E^0 \to F \otimes E^1 \to ...$ is exact for any m-flat right R-module F. This follows from the proof above by noting that F^+ is a m-injective left R-module. □

Lemma 4.4 *Let R be a ring. Then the following are equivalent.*

(1) *R is left max-coherent and if $0 \to A \to B \to C \to 0$ is an exact sequence of left R-modules with A and B m-injective, C is m-injective.*

(2) *R is left max-coherent and if $0 \to N \to M \to L \to 0$ is an exact sequence of right R-modules with M and L m-flat, N is m-flat.*

(3) *R is left max-coherent and $Ext^i(R/I, M) = 0$ for every m-injective left R-module M, every maximal left ideal I of R and every $i \geq 1$.*

(4) *R is left max-coherent and $Tor_i(N, R/I) = 0$ for every m-flat right R-module N, every maximal left ideal I of R and every $i \geq 1$.*

Proof (1) $\Rightarrow$ (2) Let $0 \to N \to M \to L \to 0$ be an exact sequence of right R-modules with M and L m-flat. Then we get an exact sequence $0 \to L^+ \to M^+ \to N^+ \to 0$. Since L^+ and M^+ is m-injective, so is N^+ by (1). Thus, N is m-flat.

(2) $\Rightarrow$ (1) Let $0 \to A \to B \to C \to 0$ be an exact sequence of left R-modules with A and B m-injective. Then we get an exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$. Note that A^+ and B^+ are m-flat by Lemma 4.1. Thus C^+ is m-flat by (2), so C is m-injective by Lemma 4.1.

(2) $\Rightarrow$ (4) Let N be a m-flat right R-module. Then there is an exact sequence $0 \to K \to F \to N \to 0$ with F projective, so K is m-flat by (2). Thus, $Tor_2(N, R/I) \cong Tor_1(K, R/I) = 0$ for every maximal left ideal I of R, hence (4) holds by induction.

(4) $\Rightarrow$ (2) is easy. (1) $\Leftrightarrow$ (3) is dual to that of (2) $\Leftrightarrow$ (4). □

We will call R a *left strongly max-coherent* ring if it satisfies the equivalent conditions of Lemma 4.4.

Example 4.1

(a) Recall that a ring R is said to be left semiartinian if every nonzero left R-module has a nonzero left socle. It is shown that if R is a left semiartinian ring, then every m-injective left R-module is injective, (Wang and Zhao, 2005). Hence, every left max-coherent left semiartinian ring is left strongly max-coherent by Lemma 4.4.

(b) If R is a left max-coherent left SF-ring (i.e, every simple left R-module is flat), then it is left strongly max-coherent.

If R is a left max-coherent ring, then $- \otimes -$ is right balanced on $\mathfrak{M}_R \times_R \mathfrak{M}$ by $\mathfrak{MF} \times \mathfrak{MI}$ by Proposition 4.1. Let $Tor_R^n(-,-)$ denote the nth right derived functor of $- \otimes -$ with respect to $\mathfrak{MF} \times \mathfrak{MI}$. Then, for any right R-module M and any left R-module N, $Tor_R^n(M, N)$ can be computed using either a right $\mathfrak{MF}$-resolution of M or a right $\mathfrak{MI}$-resolution of N by ((Enochs and Jenda, 2000), Exercise 18, p. 177).

Let $0 \to M \xrightarrow{\epsilon} F^0 \xrightarrow{d^0} F^1 \xrightarrow{d^1} ...$ be a right $\mathfrak{MF}$-resolution of a right R-module M. Applying $- \otimes N$, we obtain the deleted complex $0 \to F^0 \otimes N \xrightarrow{d^0 \otimes 1} F^1 \otimes N \to F^2 \otimes N \to$ ... Then $Tor_R^n(M, N)$ is exactly the nth homology of the complex above. There exists a canonical map $\beta : M \otimes N \to Tor_R^0(M, N) = \ker(d^0 \otimes 1)$
defined by $\beta(\sum(m_i \otimes n_i)) = \sum(\epsilon(m_i) \otimes n_i)$ for any $\sum(m_i \otimes n_i) \in M \otimes N$.

Proposition 4.2 *Let R be a left strongly max-coherent ring and M a right R-module. Then the following are equivalent.*

(1) *M is m-flat.*

(2) $\beta : M \otimes N \to Tor_R^0(M, N)$ *is monic for any left R-module N.*

(3) $\beta : M \otimes N \to Tor_R^0(M, N)$ *is monic for any finitely presented left R-module N.*

(4) $\beta : M \otimes R/I \to Tor_R^0(M, R/I)$ *is monic for any maximal left ideal I of R.*

Proof (1) $\Rightarrow$ (2) holds by letting $F^0 = M$.
(2) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear.

(4) ⇒ (1) Let $0 \to M \xrightarrow{\epsilon} F^0 \xrightarrow{d^0} F^1 \xrightarrow{d^1} ...$ be a right $\mathfrak{MF}$-resolution of M. For any maximal left ideal I of R, we have the following commutative diagram:

$$\begin{array}{ccccc} M \otimes R/I & \xrightarrow{\epsilon\otimes 1} & F^0 \otimes R/I & \xrightarrow{d^0\otimes 1} & F^1 \otimes R/I \\ & \searrow^{\beta} & \uparrow{\scriptstyle i} & & \\ & & Tor_R^0(M, R/I) & & \end{array}$$

By (4), $\epsilon \otimes 1$ is monic. Thus $0 \to M \xrightarrow{\epsilon} F^0$ is s-pure exact and hence, M is m-flat by Lemma 4.4(2) and Lemma 4.2(2). □

The right $\mathfrak{MF}$-dimension of a right R-module M, denoted by right $\mathfrak{MF}$-dim M, is defined as inf{n: there is a right $\mathfrak{MF}$-resolution of M of the form $0 \to M \to F^0 \to F^1 \to ... \to F^n \to 0$}. If there is no such n, set right $\mathfrak{MF}$-dim M=∞. The global right $\mathfrak{MF}$-dimension of $\mathfrak{M}_R$, denoted by gl right $\mathfrak{MF}$-dim $\mathfrak{M}_R$,is defined to be the sup {right $\mathfrak{MF}$-dim M: $M \in \mathfrak{M}_R$ } and is infinite otherwise.

Proposition 4.3 *Let R be a left strongly max-coherent ring and M a right R-module. Consider the following conditions.*

(1) *right $\mathfrak{MF}$-dim $M \leq 1$.*

(2) $\beta : M \otimes N \to Tor_R^0(M, N)$ *is epic for any left R-module N.*

(3) $\beta : M \otimes N \to Tor_R^0(M, N)$ *is epic for any finitely presented left R-module N.*

(4) $\beta : M \otimes R/I \to Tor_R^0(M, R/I)$ *is epic for any maximal left ideal I of R.*

Then, (2) ⇒ (3) ⇒ (4) ⇒ (1). *The converse hold if R is a left m-injective ring.*

Proof (2) ⇒ (3) ⇒ (4) trivial.

(4) ⇒ (1) Consider the exact sequence $0 \to M \to F^0 \to D^1 \to 0$, where $0 \to M \to F^0$ is a $\mathfrak{MF}$-preenvelope. We only need to show that D^1 is m-flat. For any maximal left ideal I of R, we have the commutative diagram with exact rows by (Enochs and Jenda, 2000):

$$\begin{array}{ccccccc} & M \otimes R/I & \longrightarrow & F^0 \otimes R/I & \longrightarrow & D^1 \otimes R/I & \longrightarrow 0 \\ & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_3} & \\ 0 \longrightarrow & Tor_R^0(M, R/I) & \longrightarrow & Tor_R^0(F^0, R/I) & \longrightarrow & Tor_R^0(D^1, R/I) & \end{array}$$

Note that β_2 is monic by Proposition 4.2 and β_1 is epic by (4). Hence β_3 is monic by the Snake Lemma. Thus D^1 is m-flat by Proposition 4.2.

(1) ⇒ (2) By (1), M has a right $\mathfrak{MF}$-resolution $0 \to M \to F^0 \to F^1 \to 0$. Since R is left m-injective ring, the above $\mathfrak{MF}$-resolution is exact by the proof of Proposition 4.1. Thus we get an exact sequence $M \otimes N \to F^0 \otimes N \to F^1 \otimes N \to 0$ for any left R-module N. Hence β is epic. □

Note that every right R-module over any ring R has an m-flat cover by Lemma 4.3. So M has a left $\mathfrak{MF}$-resolution, that is, there is a $Hom(\mathfrak{MF}, -)$ exact complex $... \to F_1 \to F_0 \to M \to 0$ with each F_i m-flat. Obviously, this complex is exact. The left $\mathfrak{MF}$-dimension of a right R-module M, denoted by left $\mathfrak{MF}$-dim M, is defined as inf{n: there is a left $\mathfrak{MF}$-resolution of M of the form $0 \to F_n ... \to F_1 \to F_0 \to M \to 0\}$. If no such n exists, set left $\mathfrak{MF}$-dim $M = \infty$. The global left $\mathfrak{MF}$-dimension of $\mathfrak{M}_R$, denoted by gl left $\mathfrak{MF}$-dim $\mathfrak{M}_R$, is defined to be sup{left $\mathfrak{MF}$-dim M: $M \in \mathfrak{M}_R\}$ and is infinite otherwise.

Proposition 4.4 *Let R be a left strongly max-coherent ring, n a nonnegative integer and M a right R-module. The following are equivalent.*

(1) *left $\mathfrak{MF}$-dim $M \leq n$.*

(2) $Tor_{n+k}(M, R/I) = 0$ *for every maximal left ideal I of R and every* $k \geq 1$.

(3) $Tor_{n+1}(M, R/I) = 0$ *for every maximal left ideal I of R.*

(4) *If* $0 \to K \to F_{n-1} \to ... \to F_1 \to F_0 \to M \to 0$ *is exact with each F_i m-flat, then K is m-flat.*

Proof (2) ⇒ (3) is trivial.

(1) ⇒ (2) Since left $\mathfrak{MF}$-dim $M \leq n$, there is a left $\mathfrak{MF}$-resolution of the form $0 \to F^n \to F^{n-1} ... \to F^1 \to F^0 \to M \to 0$. So, $Tor_{n+k}(M, R/I) \cong Tor_n(F^n, R/I) = 0$ for every maximal left ideal I of R and every $k \geq 1$ by Lemma 4.4(4).

(3) ⇒ (4) Let $0 \to K \to F_{n-1} \to ... \to F_1 \to F_0 \to M \to 0$ be an exact sequence with each F_i m-flat. Then $Tor_1(K, R/I) \cong Tor_{n+1}(M, R/I) = 0$ for for every maximal left ideal I of R by (3). So K is m-flat.

(4) ⇒ (1) Let $... \to F_{n-1} \to ... \to F_1 \to F_0 \to M \to 0$ be a partial left $\mathfrak{MF}$-resolution of M. Then we get an exact sequence $0 \to K \to F_{n-1} \to ... \to F_1 \to F_0 \to M \to 0$. By (4), K is m-flat. Thus left $\mathfrak{MF}$-dim $M \leq n$. □

Proposition 4.5 *Let R be a left strongly max-coherent ring, n a nonnegative integer and N a right R-module. The following are equivalent.*

(1) *right $\mathfrak{MI}$-dim $N \leq n$.*

(2) $Ext^{n+k}(R/I, N) = 0$ *for every maximal left ideal I of R and every* $k \geq 1$.

(3) $Ext^{n+1}(R/I, N) = 0$ *for every maximal left ideal I of R.*

(4) *If* $0 \to N \to E^0 \to E^1 \to ... \to E^{n-1} \to L \to 0$ *is exact with each* E^i *m-injective, then K is m-injective.*

Proof The proof is analogous to that of Proposition 4.4 by Lemma 4.4(3). □

Theorem 4.3 *Let R be a left strongly max-coherent ring, n a nonnegative integer. The following are equivalent.*

(1) *gl right* $\mathfrak{MI}$*-dim* ${}_R\mathfrak{M} \leq n$.

(2) *gl left* $\mathfrak{MF}$*-dim* $\mathfrak{M}_R \leq n$.

(3) *left* $\mathfrak{MF}$*-dim* $M \leq n$ *for every m-cotorsion right R-module M.*

(4) $Ext^{n+1}(R/I, N) = 0$ *for every maximal left ideal of R and every left R-module N.*

(5) $Tor_{n+1}(M, R/I) = 0$ *for every maximal left ideal of R and every right R-module M.*

(6) *Every simple left R-module has projective dimension* $\leq n$.

(7) *Every simple left R-module has flat dimension* $\leq n$.

In this case, every m-cotorsion right R-module has injective dimension $\leq n$.

Proof (2) ⇔ (5) and (1) ⇔ (4) follows from Propositions 4.4 and 4.5, respectively.

(2) ⇒ (3), (4) ⇔ (6) and (5) ⇔ (7) are obvious.

(3) ⇒ (2) Let M be any right R-module. Then, by Lemma 4.3(2), there is an exact sequence $0 \to M \to N \to K \to 0$, where N is m-cotorsion and K is m-flat. Thus we get an induced exact sequence $0 = Tor_{n+2}(K, R/I) \to Tor_{n+1}(M, R/I) \to Tor_{n+1}(N, R/I) = 0$ for every left ideal I of R by (3) and Proposition 4.4. So, left $\mathfrak{MF}$-dim $M \leq n$ and (2) follows.

(4) ⇒ (5) holds because $Tor_{n+1}(M, R/I)^+ \cong Ext^{n+1}(R/I, M^+)$ for every maximal left ideal I of R and every right R-module M.

(5) ⇒ (4) holds because $Ext^{n+1}(R/I, N)^+ \cong Tor_{n+1}(N^+, R/I)$ for every maximal left ideal I of R and every left R-module N.

Next we prove the last statement. Let M be an m-cotorsion right R-module and N any right R-module. Then, by (5), there is an exact sequence $0 \to F_n \to P_{n-1} \to ... \to P_1 \to P_0 \to N \to 0$ with F_n m-flat and each P_i projective and so $Ext^{n+1}(N, M) \cong Ext^1(F_n, M) = 0$. Thus M has injective dimension $\leq n$. □

Let $\mathfrak{C}$ be a class of left R-modules and M a left R-module. Recall that a $\mathfrak{C}$-cover $f : C \to M$ is said to have the unique mapping property if for any homomorphism $g : D \to M$ with $D \in \mathfrak{C}$, there is a unique homomorphism $h : D \to C$ such that $fh = g$, (Ding 1996).

Corollary 4.3 *Let R be a left strongly max-coherent ring. The following are equivalent.*

(1) *gl right* $\mathfrak{MI}$*-dim* ${}_R\mathfrak{M} \leq 2$.

(2) *gl left* $\mathfrak{MF}$*-dim* $\mathfrak{M}_R \leq 2$.

(3) *Every left R-module has a* $\mathfrak{MI}$*-cover with the unique mapping property.*

Proof (1) ⇔ (2) holds by Theorem 4.3. (3) ⇒ (1) by ((Xiang, 2010), Theorem 4.6)

(1) ⇒ (3) Let M be a left R-module. Then M has a $\mathfrak{MI}$-cover $f : F \to M$ by ((Xiang, 2010), Remark 2.10). It is enough to show that, for any m-injective left R-module N and any homomorphism $g : N \to F$ such that $fg = 0$, we have $g = 0$. In fact, there exists $\beta : F/Im(g)$ such that $\beta\pi = f$ since $Im(g) \subseteq \ker(f)$, where $\pi : F \to F/Im(g)$ is the natural map. Consider the exact sequence $0 \to \ker(g) \to G \to F \to F/Im(g) \to 0$. Note that $F/Im(g)$ is m-injective by (1) and Proposition 4.5. Thus there exists $\alpha : F/Im(g) \to F$ such that $\beta = f\alpha$, and so $f\alpha\pi = \beta\pi = f$. Hence $\alpha\pi$ is an isomorphism since f is a cover. Therefore π is monic, and so $g = 0$. □

A ring R will be called left max-hereditary if every maximal left ideal is projective. Recall that a ring R is said to be left PP if every principal left ideal of R is projective. Then any left PP-ring with every maximal left ideal principal is left max-hereditary. Now we have the following characterizations of left max-hereditary rings.

Proposition 4.6 *Let R be a ring. The following are equivalent.*

(1) *R is a left max-hereditary.*

(2) *Every quotient of an m-injective left R-module is m-injective.*

(3) *Every m-projective left R-module has projective dimension at most 1.*

Proof (1) ⇒ (2) Let M be an m-injective left R-module and N a submodule of M. We shall show that M/N is m-injective. To this end, let I be a maximal left ideal of R and $i : I \to R$ the inclusion and $\pi : M \to M/N$ the canonical map. For any $f : I \to M/N$, then there exists $g : I \to M$ such that $\pi g = f$ since I is projective by (1). Hence there is $h : R \to M$ such that $hi = g$ since M is m-injective. It follows that $(\pi h)i = f$, and so M/N is m-injective.

(2) $\Rightarrow$ (3) Let M be an m-projective left R-module and N a left R-module, then there is a short exact sequence $0 \to N \to E \to L \to 0$ with E injective. Note that L is m-injective by (2), and so we have the exact sequence $0 = Ext_R^1(M, L) \to Ext_R^2(M, N) \to Ext_R^2(M, E) = 0$. Thus $Ext_R^2(M, N) = 0$ and hence M has projective dimension at most 1.

(3) $\Rightarrow$ (1) holds since every simple left R-module is m-projective. □

A left R-module M is said to be MI-injective (Xiang, 2010) if $Ext_R^1(N, M) = 0$ for any m-injective left R-module N. Next we characterize left max-hereditary rings over a left max-coherent ring.

Theorem 4.4 *Let R be a left max-coherent ring. The following are equivalent.*

(1) *R is a left max-hereditary.*

(2) *Every MI-injective left R-module is injective.*

(3) *R is a left strongly max-coherent ring and gl right $\mathfrak{MI}$-dim $_R\mathfrak{M} \leq 1$.*

(4) *R is a left strongly max-coherent ring and gl left $\mathfrak{MF}$-dim $_R\mathfrak{M} \leq 1$.*

(5) *R is a left strongly max-coherent ring and left $\mathfrak{MF}$-dim $M \leq 1$ for every m-cotorsion right R-module M.*

(6) *Every submodule of an m-flat right R-module is m-flat.*

(7) *Every left R-module has a monic m-injective cover.*

(8) *($\mathfrak{MP}$,$\mathfrak{MI}$) is hereditary and every m-projective left R-module has a monic m-injective cover.*

Proof

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) follows from Theorem 4.3. (1) $\Rightarrow$ (3) is clear by Proposition 4.6.

(1) $\Rightarrow$ (2) is clear by ((Xiang, 2010), Proposition 3.4).

(2) $\Rightarrow$ (1) Let N be a quotient of an m-injective left R-module M. Suppose $f : F \to N$ is a m-injective cover of N by ((Xiang, 2010), Remark 2.10(1)). Then there exists a homomorphism $h : M \to F$ such that $fh = \pi$, where $\pi : M \to N$. Hence f is an epimorphism. By ((Xiang, 2010), Remark 3.2(1)), ker(f) is MI-injective, and so it is injective by (2). So, N is m-injective.

(3) $\Rightarrow$ (1) Let M be any m-injective left R-module and N a submodule of M. By (3), there is a right $\mathfrak{MI}$-resolution $0 \to N \to K \to L \to 0$. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & K & \longrightarrow & H & \longrightarrow & M/N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since M and L are m-injective, H is m-injective by Lemma 4.2(1). So M/N is m-injective by Lemma 4.4(1), as desired.

(6) $\Rightarrow$ (1) Let N be a quotient of an m-injective left R-module M. Then the exact sequence $0 \to K \to M \to N \to 0$ induces the exactness of $0 \to N^+ \to M^+ \to K^+ \to 0$. Since M^+ is m-flat, N^+ is m-flat by (6). Hence N is m-injective.

(1) $\Rightarrow$ (6) Let A be a submodule of an m-flat right R-module B. Then the exactness of $0 \to A \to B \to B/A \to 0$ induces the exact sequence $0 \to B/A^+ \to B^+ \to A^+ \to 0$. Note that B^+ is m-injective, so A^+ is m-injective by (1), and hence A is m-flat.

(1) $\Leftrightarrow$ (7) holds by ((García Rozas and Torrecillas, 1994), Proposition 4) since the class of m-injective left R-modules is closed under direct sums by Lemma 4.1(4).

(1) $\Rightarrow$ (8) is clear by the equivalence of (1) and (7).

(8) $\Rightarrow$ (1) Let M be any m-injective left R-module and N any submodule of M. We have to prove that M/N is m-injective. In fact, there exists an exact sequence $0 \to N \to E \xrightarrow{\pi} L \to 0$ with E is m-injective and L is m-projective by Lemma 4.3(3). Since L has a monic m-injective cover $\phi : F \to L$ by (8), there is $\alpha : E \to F$ such that $\pi = \phi\alpha$. Thus ϕ is epic and hence it is an isomorphism. So L is m-injective. For any simple left R-module S, we have the exact sequence $0 = Ext^1_R(S, L) \to Ext^2_R(S, N) \to Ext^2_R(S, E)$. Note that $Ext^2_R(S, E) = 0$ by ((Enochs, Jenda and Lopez-Ramos, 2004), Proposition 1.2) since $(\mathfrak{MP}, \mathfrak{MI})$ is hereditary, and hence $Ext^2_R(S, N) = 0$. On the other hand, the short exact sequence $0 \to N \to M \to M/N \to 0$ induces the exactness of the sequence $0 = Ext^1_R(S, M) \to Ext^1_R(S, M/N) \to Ext^2_R(S, N) = 0$. Therefore $Ext^1_R(S, M/N) = 0$, as desired. □

Finally, we give some new characterizations of left SF-rings. Recall that a ring R is called a left SF-ring (Ramamurthi, 1975) if each simple right R-module is flat. This is equivalent to saying that every right R-module is m-flat.

Corollary 4.4 *Let R be a left max-coherent ring. The following are equivalent.*

(1) *R is left SF-ring.*

(2) *gl right* $\mathfrak{M}\mathfrak{I}$*-dim* ${}_R\mathfrak{M} = 0$.

(3) *Every m-projective left R-module is projective.*

(4) *Every cotorsion left R-module is m-injective.*

(5) *Every cotorsion right R-module is m-flat.*

(6) *R is left strongly max-coherent ring and every m-cotorsion right R-module is m-flat.*

(7) *R is left strongly max-coherent ring and every m-projective left R-module is m-injective.*

Proof (2) ⇒ (4) is trivial and (2) ⇔ (6) comes from Theorem 4.3 and Lemma 4.4.

(2) ⇒ (3) Let M be any m-projective left R-module. Since every left R-module is m-injective by (2), $Ext^1_R(M, N) = 0$ for any left R-module N. Hence M is projective.

(3) ⇒ (2) Let M be a left R-module. There exists an exact sequence $0 \to M \to E \to P \to 0$ with E m-injective and P m-projective by Lemma 4.3(3). By (3), P is projective and so M is m-injective.

(1) ⇒ (5) is clear since over a left SF-ring, every right R-module is m-flat.

(2) ⇒ (7) is clear by Lemma 4.4, since R is left max-coherent ring.

(7) ⇒ (2) Let M be any left R-module. By Lemma 4.3(3), there is a short exact sequence $0 \to K \to F \to M \to 0$ with F m-projective and K m-injective. Then F is m-injective by (7) and hence M is m-injective since R is left strongly max-coherent ring.

(4) ⇒ (1) Let M be any right R-module. Since M^+ is pure-injective and hence cotorsion, M^+ is m-injective by (4). So M is m-flat.

(5) ⇒ (2) Let M be any left R-module. Then M^+ is m-flat by (5). Thus M^{++} is m-injective. Note that M is a pure submodule of M^{++}, so M is m-injective by Lemma 4.1(4).

□

4.3. SP-flat Modules

In (Büyükaşık and Durğun, 2015), the authors introduced that a left R-module N is s-pure injective (in short sp-injective), (in (Hamid, 2019) is called coneat injective) if it is injective with respect to s-pure short exact sequences. Clearly, every SP-injective module is pure-injective. Motivated by this, we first introduce the concept of SP-flat modules.

Definition 4.2 *Let R be a ring. A right R-module M is called SP-flat if for every s-pure exact sequence $0 \to A \to B \to C \to 0$ of left R-modules, the sequence $0 \to M \otimes A \to M \otimes B \to M \otimes C \to 0$ is exact.*

Remark 4.2 *(1) By the definition, any simple right R-module is SP-flat.*

(2) Flat right R-modules are clearly SP-flat. But the converse is not true in general. For example, $\mathbb{Z}_p$ is an SP-flat $\mathbb{Z}$-module for a prime integer p since $\mathbb{Z}_p$ is a simple $\mathbb{Z}$-module. But it is not a flat $\mathbb{Z}$-module.

Lemma 4.5 *Let R be a ring. Then*

(1) *A right R-module M is SP-flat if and only if M^+ is SP-injective.*

(2) *The class of SP-flat right R-modules is closed under pure submodules and pure quotient modules.*

Proof (1) Let $0 \to A \to B \to C \to 0$ be an s-pure exact sequence of left R-modules and M a right R-module. Then the sequence $0 \to M \otimes A \to M \otimes B \to M \otimes C \to 0$ is exact if and only if the sequence $0 \to (M \otimes C)^+ \to (M \otimes B)^+ \to (M \otimes A)^+ \to 0$ is exact if and only if $0 \to Hom(C, M^+) \to Hom(B, M^+) \to Hom(A, M^+) \to 0$ is exact. So M is SP-flat if and only if M^+ is SP-injective.

(2) Let $0 \to A \to B \to C \to 0$ be a pure exact sequence of right R-modules with B SP-flat. Then we get the split exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$. Since B^+ is SP-injective by (1), A^+ and C^+ are SP-injective. So A and C are SP-flat. □

Remark 4.3 *(1) Every R-module is embedded as an s-pure submodule of an SP-injective module by ((Hamid, 2019), Corollary 2.4).*

(2) Every right R-module has an SP-flat cover by Lemma 4.5 and ((Holm and Jørgensen, 2008), Theorem 2.5).

(3) If R is a left max-coherent ring, then every SP-injective right R-module has an injective cover. In fact let M be an SP-injective left R-module. By ((Büyükaşık and Durğun, 2015), Proposition 5.1), M has an absolutely s-pure cover $f : A \to M$. Hence by ((Büyükaşık and Durğun, 2015), Proposition 5.2), A is injective.

Corollary 4.5 *Let R be a ring. The following are equivalent.*

(1) *Every right R-module is SP-flat.*

(2) *Every s-pure exact sequence* $0 \to A \to B \to C \to 0$ *of left R-modules is pure.*

(3) *Every pure-injective left R-module is SP-injective.*

Proof (1) ⇒ (2) is clear. (2) ⇔ (3) by ((Hamid, 2019), Proposition 3.15).

(3) ⇒ (1) Let M be a right R-module. Then M^+ is pure-injective and so SP-injective by (3). Thus M is SP-flat by Lemma 4.5(1). □

Now we give further characterizations of s-pure exact sequences.

Lemma 4.6 *The following are equivalent for an exact sequence* $0 \to A \to B \to C \to 0$ *of left R-modules.*

(1) $0 \to A \to B \to C \to 0$ *is s-pure.*

(2) *The sequence* $0 \to Hom(C, N) \to Hom(B, N) \to Hom(A, N) \to 0$ *is exact for any SP-injective left R-module N.*

(3) *Every simple right R-module is projective relative to the exact sequence* $0 \to C^+ \to B^+ \to A^+ \to 0$.

(4) *The sequence* $0 \to F \otimes A \to F \otimes B \to F \otimes C \to 0$ *is exact for any SP-flat right R-module F.*

Proof (1) ⇒ (2) and (1) ⇒ (4) are clear by the definition.

(4) ⇒ (1) is clear since every simple right R-module is SP-flat.

(2) ⇒ (1) Let S be a simple right R-module. Then S^+ is SP-injective. Thus by (2), $0 \to Hom(C, S^+) \to Hom(B, S^+) \to Hom(A, S^+) \to 0$ is exact. Hence $0 \to (S \otimes C)^+ \to (S \otimes B)^+ \to (S \otimes A)^+ \to 0$ is exact. So we get the exact sequence $0 \to S \otimes A \to S \otimes B \to S \otimes C \to 0$ and (1) follows. (1) ⇔ (3) Let S be a simple right R-module. Then the exact sequence $0 \to S \otimes A \to S \otimes B \to S \otimes C \to 0$ is exact if and only if $0 \to (S \otimes C)^+ \to (S \otimes B)^+ \to (S \otimes A)^+ \to 0$ is exact if and only if $0 \to Hom(S, C^+) \to Hom(S, B^+) \to Hom(S, C^+) \to 0$ is exact. So (1) ⇔ (3) holds. □

Proposition 4.7 *The following are equivalent for a left R-module M.*

(1) *M is absolutely s-pure.*

(2) *Every exact sequence* $0 \to M \to B \to C \to 0$ *is s-pure.*

(3) *There exists an s-pure exact sequence* $0 \to M \to E \to N \to 0$ *with E absolutely s-pure.*

(4) *For every SP-injective left R-module N, every homomorphism* $f : M \to N$ *factors through an injective left R-module.*

Proof (1) ⇔ (2) ⇔ (3) by ((Büyükaşık and Durğun, 2015), Lemma 3.3).
(2) ⇒ (4) is easy since M can be embedded in an injective left R-module.
(4) ⇒ (2) Let $0 \to M \xrightarrow{i} B \to C \to 0$ be an exact sequence. For any SP-injective left module N and any homomorphism $f : M \to N$, there are an injective left module E, $g : M \to E$ and $h : E \to N$ such that $f = hg$ by (4). Since E is injective, there is $\alpha : B \to E$ such that $\alpha i = g$. Thus $f = h\alpha i$. So the sequence $0 \to M \to B \to C \to 0$ is s-pure by Lemma 4.6. □

The following Proposition gives some interesting characterizations of m-flat modules in terms of s-purity.

Proposition 4.8 *The following are equivalent for a right R-module N.*

(1) *N is m-flat.*

(2) *Every exact sequence* $0 \to K \to L \to N \to 0$ *is s-pure.*

(3) $Ext^1(N, M) = 0$ *for any SP-injective right R-module M.*

(4) *There exists an s-pure exact sequence* $0 \to M \to E \to N \to 0$ *with E m-flat.*

Proof (1) ⇒ (2) Let $0 \to K \to L \to N \to 0$ be an exact sequence. Since N is m-flat, for any maximal left ideal I of R, we have the exact sequence $0 = Tor_1(N, R/I) \to K \otimes R/I \to L \otimes R/I \to N \otimes R/I \to 0$. So the exact sequence $0 \to K \to L \to N \to 0$ is s-pure.

(2) ⇒ (3) There is an s-pure exact sequence $0 \to K \to P \to N \to 0$ with P projective by (2). Thus, by Lemma 4.6, $Hom(P, M) \to Hom(K, M) \to 0$ is exact for any SP-injective left R-module M. Consider the induced exact sequence: $Hom(P, M) \to Hom(K, M) \to Ext^1(N, M) \to Ext^1(P, M) = 0$. So $Ext^1(N, M) = 0$.

(3) ⇒ (4) Let $0 \to K \to F \to N \to 0$ be an exact sequence with F (m-)flat. For any SP-injective right R-module M, by (3), we have the exact sequence $0 \to Hom(N, M) \to Hom(F, M) \to Hom(K, M) \to Ext^1(N, M) = 0$. Thus, $0 \to K \to F \to N \to 0$ is s-pure by Lemma 4.6.

(4) ⇒ (1) Let $0 \to M \to F \to N \to 0$ be an s-pure exact sequence with F m-flat. For any maximal left ideal I of R, we have the exact sequence $0 = Tor_1(F, R/I) \to Tor_1(N, R/I) \to M \otimes R/I \to F \otimes R/I$. Since by (4), $M \otimes R/I \to F \otimes R/I$ is monic, $Tor_1(N, R/I) = 0$. Hence, N is m-flat. □

The following corollary clarifies the relationship between SP-injective (resp. SP-flat) modules and injective (resp. flat) modules.

Corollary 4.6 *The following are true for any ring R:*

(1) *Any absolutely s-pure SP-injective left R-module is injective.*

(2) *If R is a left max-coherent ring, any neat flat SP-flat right R-module is flat.*

Proof (1) Let M be any absolutely s-pure SP-injective left R-module. By Proposition 4.7, there exists an s-pure exact sequence $0 \to M \to E \to N \to 0$ with E injective. So the exact sequence is splits and hence M is injective.

(2) Let M be any neat flat SP-flat right R-module. Then M^+ is absolutely s-pure by ((Büyükaşık and Durğun, 2015), Proposition 4.3) and SP-injective by Lemma 4.5, so is injective by (1). Thus M is flat. □

Theorem 4.5 *The following are equivalent for a ring R and integer* $n \geq 0$.

(1) *gl left* $\mathfrak{M}\mathfrak{F}$*-dim* $\mathfrak{M}_R \leq n$

(2) *Every m-cotorsion right R-module has injective dimension* $\leq n$.

(3) *Every SP-injective right R-module has injective dimension* $\leq n$.

(4) *Every SP-flat left R-module has flat dimension* $\leq n$.

Proof (1) ⇒ (2) Let M be an m-cotorsion right R-module and N any right R-module. Since left $\mathfrak{M}\mathfrak{F}$-dim $N \leq n$, there is an exact sequence $0 \to K_n \to K_{n-1} \to ... \to K_1 \to K_0 \to N \to 0$ with each K_i m-flat. So $Ext^{n+1}(N, M) = Ext^1(K_n, M) = 0$. It follows that M has injective dimension $\leq n$.

(2) ⇒ (3) is trivial by Proposition 4.8.

(3) ⇒ (4) For any SP-flat left R-module M, M^+ is SP-injective. By (3), for every left R-module N, we have $Tor_{n+1}(N, M)^+ \cong Ext^{n+1}(N, M^+) = 0$. So, $Tor_{n+1}(M, N) = 0$, and hence M has flat dimension $\leq n$.

(4) ⇒ (1) Let $... \to F_{n-1} \to ... \to F_1 \to F_0 \to M \to 0$ be a partial left $\mathfrak{M}\mathfrak{F}$-resolution of M. Then we get an exact sequence $0 \to K \to F_{n-1} \to ... \to F_1 \to F_0 \to M \to 0$. Since every simple left R-module is SP-flat, by (4), $Tor_1(K, R/I) = Tor_{n+1}(M, R/I) = 0$ for any maximal left ideal I of R. Hence K is m-flat. □

As a consequences of Theorem 4.5 and ((Hamid, 2019), Theorem 3.16), we obtain new characterization of left SF-rings.

Corollary 4.7 *Let R be a ring. Then the following are equivalent.*

(1) *R is left SF-ring.*

(2) *gl left* $\mathfrak{M}\mathfrak{F}$*-dim* $\mathfrak{M}_R = 0$*.*

(3) *Every m-cotorsion right R-module is injective.*

(4) *Every SP-injective right R-module is injective.*

(5) *Every SP-injective right R-module is absolutely s-pure.*

(6) *Every SP-flat left R-module is flat.*

(7) *All exact sequences of right R-modules is s-pure.*

(8) *All right R-modules are absolutely s-pure.*

Remark 4.4 *The class of SP-injective modules need not be closed under extensions. Note that for each simple right R-module S,* S^+ *is an SP-injective left R-module by the standard adjoint isomorphism. Consider the short exact sequence* $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$*. The simple* $\mathbb{Z}$*-modules* $\mathbb{Z}_2$ *are SP-injective, but* $\mathbb{Z}_4$ *is not SP-injective.*

Proposition 4.9 *Let R be a ring. Then the following are equivalent.*

(1) *The class of SP-injective left R-module is closed under extensions.*

(2) *Every m-cotorsion left R-module is SP-injective.*

In this case, the class of SP-flat right R-modules is closed under extensions.

Proof (1) ⇒ (2) Let M be an m-cotorsion left R-module. By Remark 4.3(1), we have an s-pure exact sequence $0 \to M \to N \to L \to 0$ with N is SP-injective. By (1) and ((Xu, 1996), Lemma 2.1.2) $Ext^1(L, C) = 0$ for every SP-injective left R-module C, so L is m-flat by Proposition 4.8. Therefore $Ext^1(L, M) = 0$ and hence the sequence $0 \to M \to N \to L \to 0$ is split. Thus M is isomorphic to a direct summand of N and so is SP-injective.

(2) ⇒ (1) is obvious since m-cotorsion modules closed under extensions.

In this case, if $0 \to M \to N \to L \to 0$ is an exact sequence of right R-modules with M and L SP-flat, then we get the exact sequence $0 \to L^+ \to N^+ \to M^+ \to 0$. By Lemma 4.5(1), L^+ and M^+ are SP-injective. Thus N^+ is SP-injective and hence N is SP-flat by Lemma 4.5(1). □

Recall by Lemma 4.3(1) that all R-modules have m-flat covers and all R-modules have m-cotorsion envelopes for an arbitrary ring R. In (Rothmaler, 2002), Rothmaler considered when the cotorsion envelope of every flat R-module is pure-injective. Motivated by this idea, we next study when the m-cotorsion envelope of every m-flat R-module is SP-injective.

Theorem 4.6 *Let R be a ring. Then the following are equivalent.*

(1) *Every m-flat m-cotorsion left R-module is SP-injective.*

(2) *If $0 \to A \to B \to C \to 0$ is an exact sequence of left R-modules, where A is SP-injective and C is an SP-injective envelope of an m-flat left R-module, then B is SP-injective.*

(3) *The m-flat cover of every m-cotorsion left R-module is SP-injective.*

(4) *The m-flat cover of every SP-injective left R-module is SP-injective.*

(5) *The SP-injective envelope of every m-flat left R-module is m-flat.*

(6) *The m-cotorsion envelope of every m-flat left R-module is SP-injective.*

Proof (1) $\Rightarrow$ (3) Let $f : X \to M$ be an m-flat cover of a m-cotorsion module M. Since m-flat modules are closed under extensions, $\ker(f)$ is m-cotorsion by ((Xu, 1996), Lemma 2.1.1). Hence, X is m-cotorsion implies that X is SP-injective by (1).

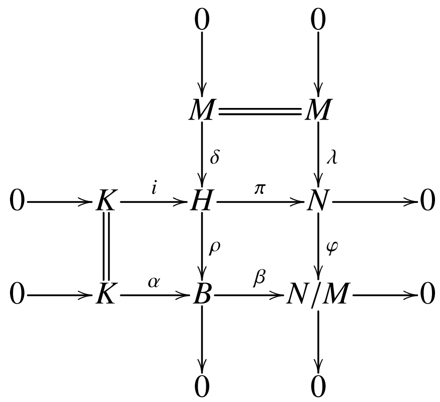
(3) $\Rightarrow$ (4) and (6) $\Rightarrow$ (1) are trivial.

(4) $\Rightarrow$ (5) Let M be an m-flat left R-module, $f : M \to N$ the SP-injective envelope, and $g : F \to N$ the m-flat cover of N. Then there exists $h : M \to F$ such that $gh = f$. On the other hand, since F is SP-injective by (4), there exists $\beta : N \to F$ such that $\beta f = h$. Thus $(g\beta)f = gh = f$, and so $g\beta$ is an isomorphism since f is an envelope. It follows that N is m-flat.

(5) $\Rightarrow$ (1) Let M be a m-flat m-cotorsion left R-module. By Remark 4.3(1), we have an exact sequence $0 \to M \xrightarrow{i} N \to L \to 0$ where $i : M \to N$ is a SP-injective envelope of M, and the sequence is s-pure. By (5), N is m-flat, so L is m-flat by Proposition 4.8. Therefore $Ext^1(L, M) = 0$ and hence the sequence $0 \to M \xrightarrow{i} N \to L \to 0$ is split. Thus M is SP-injective.

(1) $\Rightarrow$ (6) Let $f : M \to X$ be an m-cotorsion envelope of an m-flat module M. Since m-cotorsion modules are closed under extensions, $coker(f)$ is m-flat by ((Xu, 1996), Lemma 2.1.2). Hence, X is m-flat implies that X is SP-injective by (1).

(2) ⇒ (5) Let N be the SP-injective envelope of an m-flat left R-module M and $\lambda : M \to N$ be the inclusion map. We will first show that $Ext^1(N/M, K) = 0$ for any SP-injective left R-module K. In fact, let $0 \to K \to B \to N/M \to 0$ be any exact sequence. Then we have the following pullback diagram:



By (2), H is SP-injective. So there exists $\gamma : N \to H$ such that $\delta = \gamma\lambda$. Note that $\lambda = \pi\delta = \pi\gamma\lambda$, thus $\pi\gamma$ is an isomorphism since λ is an envelope. So $(\pi\gamma)^{-1}\lambda = \lambda$. It follows that $\rho\gamma(\pi\gamma)^{-1}(M) = \rho\gamma(\pi\gamma)^{-1}\lambda(M) = \rho\gamma\lambda(M) = \rho\delta(M) = 0$. Thus we get an induced map $\psi : N/M \to B$ such that $\psi\varphi = \rho\gamma(\pi\gamma)^{-1}$. Hence $\beta\psi\varphi = \beta\rho\gamma(\pi\gamma)^{-1} = \varphi\pi\gamma(\pi\gamma)^{-1}\varphi$. So $\beta\psi = 1$ since φ is epic. Thus the sequence $0 \to K \to B \to N/M \to 0$ is split, so $Ext^1(N/M, K) = 0$. By Proposition 4.8, N/M is m-flat. Hence N is m-flat by Lemma 4.2(3).

(5) ⇒ (2) If $0 \to A \to B \to C \to 0$ is an exact sequence of left R-modules, where A is SP-injective and C is an SP-injective envelope of a m-flat left R-module, then C is m-flat by (5). So the sequence $0 \to A \to B \to C \to 0$ is split. Thus B is SP-injective. □

Next we characterize SP-injective and SP-flat modules in terms of s-pure exact sequences.

Proposition 4.10 *Let R be a ring. The following are equivalent for a left R-module M.*

(1) *M is an SP-injective left R-module.*

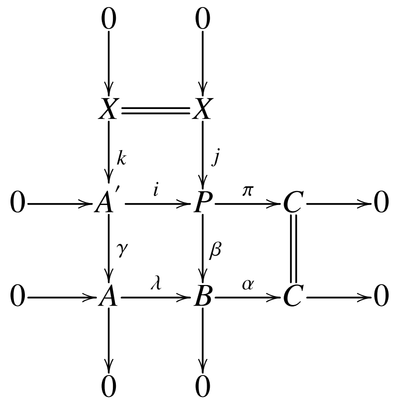
(2) *Every s-pure exact sequence $0 \to M \to N \to L \to 0$ of left R-modules is split.*

(3) *M is injective relative to every s-pure exact sequence $0 \to A \to B \to C \to 0$ of left R-modules with B pure-projective.*

Proof (1) ⇒ (2) and (1) ⇒ (3) are obvious.

(2) $\Rightarrow$ (1) By ((Hamid, 2019), Corollary 2.4), there is an s-pure exact sequence $0 \to M \to N \to L \to 0$ with N SP-injective. So M is SP-injective by (2).

(3) $\Rightarrow$ (1) Let $0 \to A \to B \to C \to 0$ be an s-pure exact sequence of left R-modules. By ((Enochs and Jenda, 2000), Example 8.3.2), there is an (s-)pure exact sequence $0 \to X \to P \to B \to 0$ with P pure-projective. Then we have the following pullback diagram:



Thus, $j = ik$ and $\pi = \alpha\beta$. Since α and β are s-pure epimorphisms, $\pi = \alpha\beta$ is an s-pure epimorphism. Hence, $0 \to A^{'} \to P \to C \to 0$ is s-pure. Let $f : A \to M$ be any homomorphism. By (3), there exists $g : P \to M$ such that $gi = f\gamma$. Since $gik = f\gamma k = 0$, we have $\ker(\beta) = Im(j) = Im(ik) \subseteq \ker(g)$. So there exists an induced map $h : B \to M$ such that $h\beta = g$. Thus, $f\gamma = h\beta i = h\lambda\gamma$, and so $f = h\lambda$ since γ is epic. Hence M is SP-injective. □

Corollary 4.8 *Let R be a ring. The following are equivalent for a right R-module N.*

(1) *N is an SP-flat right R-module.*

(2) *For every s-pure exact sequence $0 \to A \to B \to C \to 0$ of left R-modules with B pure-projective, the sequence $0 \to N \otimes A \to N \otimes B \to N \otimes C \to 0$ is exact.*

Proof (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Let $0 \to A \to B \to C \to 0$ be any s-pure exact sequence of left R-modules with B pure-projective. By (2), we get the exact sequence $0 \to N \otimes A \to N \otimes B \to N \otimes C \to 0$, which induces the exact sequence $0 \to Hom(C, N^{+}) \to Hom(B, N^{+}) \to Hom(A, N^{+}) \to 0$. So N^{+} is SP-injective by Proposition 4.10. Thus N is SP-flat by Lemma 4.5(1). □

In (Crivei, 2014), a submodule N of a right R-module M is called *coneat* in M if $\mathrm{Hom}(M, S) \to \mathrm{Hom}(N, S)$ is epic for every simple right R-module S. In ((Durğun 2015), Definition 3.1), a right R-module M is called *coneat-injective* if it is injective with respect to the coneat monomorphisms. If R is commutative, then s-pure short exact sequences coincide with coneat short exact sequences, ((Fuchs, 2012), Proposition 3.1).

Proposition 4.11 *Let R be a commutative ring. The following are equivalent for an R-module M.*

(1) *M is an SP-injective R-module.*

(2) *M is coneat-injective R-module.*

(3) *$Hom(F, M)$ is an SP-injective R-module for any flat R-module F.*

Proof (1) $\Leftrightarrow$ (2) is clear. (3) $\Rightarrow$ (1) is clear by letting $F = R$.

(1) $\Rightarrow$ (3) Let $0 \to A \to B \to C \to 0$ be an s-pure exact sequence of left R-modules. For any simple R-module S, we get the exact sequence $0 \to S \otimes A \to S \otimes B \to S \otimes C \to 0$. It follows that, for any flat R-module F, we get the exact sequence $0 \to F \otimes S \otimes A \to F \otimes S \otimes B \to F \otimes S \otimes C \to 0$. Hence the sequence $0 \to S \otimes (F \otimes A) \to S \otimes (F \otimes B) \to S \otimes (F \otimes C) \to 0$ is exact. So the exact sequence $0 \to F \otimes A \to F \otimes B \to F \otimes C \to 0$ is s-pure. Since M is SP-injective, we obtain $0 \to Hom(F \otimes C, M) \to Hom(F \otimes B, M) \to Hom(F \otimes A, M) \to 0$ which gives the exactness of the sequence $0 \to Hom(C, Hom(F, M)) \to Hom(B, Hom(F, M)) \to Hom(A, Hom(F, M)) \to 0$. Thus, $Hom(F, M)$ is an SP-injective R-module. □

Proposition 4.12 *Let R be a commutative ring. The following are equivalent for an R-module N.*

(1) *M is an SP-flat R-module.*

(2) *$Hom(N, E)$ is an SP-injective R-module for any injective R-module E.*

(3) *$N \otimes F$ is an SP-flat R-module for any flat R-module F.*

Proof (1) $\Rightarrow$ (2) Let E be an injective R-module. Then there is a split exact sequence $0 \to E \to \prod R^+$. So, we get the split exact sequence $0 \to Hom(N, E) \to Hom(N, \prod R^+) \cong \prod(Hom(N, R^+)) \cong \prod N^+$. By (1), N^+ is SP-injective, so $\prod N^+$ is SP-injective. Thus, $Hom(N, E)$ is SP-injective.

(2) $\Rightarrow$ (3) Let F be any flat module. Then F^+ is injective. So, $(N \otimes F)^+ \cong Hom(N, F^+)$ is SP-injective by (2). Thus, $N \otimes F$ is SP-flat.

(3) $\Rightarrow$ (1) is clear by letting $F = R$. □

CHAPTER 5

CONCLUSION

In this thesis, motivated by the recent works on R-projective modules, we introduce the max-projective modules. The aim of this study is to investigate the almost-QF and max-QF rings and the homological objects related with max-projective modules. We characterized the almost-QF and max-QF rings over right hereditary right Noetherian rings and over the commutative Noetherian rings. Connections between some homological objects related with max-projective modules such as m-injective, m-flat and m-cotorsion modules are given.

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