

# Hirota Method for Solving Reaction-Diffusion Equations with Generalized Nonlinearity

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**Abstract.** The Hirota Method is applied to find an exact solitary wave solution to evolution equation with generalized nonlinearity. By introducing the power form of Hirota ansatz the bilinear representation for this equation is derived and the traveling wave solution is constructed by Hirota perturbation. We show that velocity of this solution is naturally fixed by truncating the Hirota's perturbation expansion. So in our approach, this truncate on works similarly to the way Ablowitz and Zeppetella obtained an exact travelling wave solution of Fisher's equation by finding the special wave speed for which the resulting ODE is of the Painleve type. In the special case the model admits N shock soliton solution and the reduction to Burgers' equation.

**Keywords:** Solitary wave, Nonlinear evolution equation, Hirota Method.

## 1 Introduction

In this paper we consider the following general reaction-diffusion equation

$$U_t + \alpha U^\delta U_x - m \frac{U_x^2}{U} = U_{xx} + h_1 U + h_2 U^{\delta+1} + h_3 U^{2\delta+1} \quad (1.1)$$

where  $\delta, \alpha, m, h_i, (i = 1, 2, 3)$  are real constants. This equation contains as particular cases many well-known evolution equations of physics. For instance, it can be reduced to generalized Burgers-Huxley [1, 15] for the limiting cases,  $h_1 = -as, h_2 = -a(s+1), h_3 = -a, m = 0,$

$$U_t + \alpha U^\delta U_x = U_{xx} + aU(1 - U^\delta)(U^\delta - s). \quad (1.2)$$

The Kolmogorov-Petrovskii-Piskunov [11] equation can be obtained when  $\alpha = m = 0, \delta = 1,$

$$U_t = U_{xx} + aU(1 - U)(U - s). \quad (1.3)$$

For  $\alpha = m = 0, \delta = 1, h_1 = 1 - a^2, h_2 = 2a, h_3 = -1, \phi = a - U,$  Eq. (1.1) becomes the Fitzhugh-Nagumo equation [9, 10]

$$\frac{\partial \phi}{\partial t} = U_{xx} - (\phi - a)(\phi^2 - 1) \quad -1 < a < 0. \quad (1.4)$$

For  $\alpha = 0, m = 0,$  it reduces to the generalized model considered in [14]. In mathematics, a number of techniques has been developed to obtain the travelling wave solution for such nonlinear evolution equations [3, 4, 5]. One of the most popular classical methods is the Lie method, based on using the Lie symmetry of a given PDE in order to construct exact solution, and its generalizations as nonclassical method of group-invariant solutions, direct method for finding symmetry reductions, [3]. On the other hand, it is known that

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some non-integrable nonlinear PDEs (for example, the well known Fisher equation) have poor Lie symmetry, being invariant only under the time and space translations. The Lie method is not efficient for such PDEs and become cumbersome [4]. The other method, based on travelling wave ansatz [5], reduces nonlinear PDE to a nonlinear ODE. But in ODE system, the travelling wave speed is an unknown parameter that must be fixed by the analysis and choosing special trial trajectory [2]. It appears that if resulting ODE is of the Painleve type then it can be solved explicitly, leading to exact solution of the original equation. This way Ablowitz and Zeppetella [6] obtained an exact travelling wave solution of Fisher’s equation by finding the special wave speed for which the resulting ODE is of the Painleve type. All above mentioned methods has natural extensions to nonintegrable equations in higher dimensions and multi component order parameter [12, 13].

From another site, during the last 30 years the direct method proposed by Hirota becomes a powerful tool for constructing multisoliton solutions to integrable nonlinear evolution equations [7]. This, relatively simple and algebraic rather than analytic method, allows one to avoid many analytic difficulties of more sophisticated the inverse scattering method. Moreover, it is deeply related with Plücker coordinates of Grassmanians, quantum theory of fermions,  $\tau$  functions and vertex operator representation of infinite-dimensional algebras [8]. The general idea of the method is first to transform the nonlinear equation under consideration into a bilinear equation or system of equations, and then use the formal power series expansion to solve it. For integrable systems the series admits exact truncation for an arbitrary number of solitons. While for periodic solutions it includes an infinite number of terms.

In this paper, we propose Hirota Bilinear Method to construct solitary wave solution for Eq. (1.1). We show that truncation of Hirota’s perturbation series for one soliton solution similarly to the Painleve reduction [6], fixes the velocity of soliton. Moreover the method we proposed allows us extend results for multicomponent system with vector solitons [12, 13]. For the special case when  $h_1 = h_2 = h_3$  the model admits N shock soliton solution and reduction to the Burgers’s equation.

## 2 Solution of the Problem

To reduce Eq.(1.1) to the bilinear form, we have to modify the standard rational form of the Hirota ansatz by assuming solution in the form

$$U = \left(\frac{g}{f}\right)^{1/\delta} \tag{2.5}$$

where  $g(x, t), f(x, t)$  are real functions of  $x$  and  $t$ . All derivatives with respect to the dependent variables in Eq. (1.1) are expressed as

$$U_t = \frac{1}{\delta} \left(\frac{g}{f}\right)^{1/\delta-1} \frac{D_t(g \cdot f)}{f^2} \tag{2.6a}$$

$$U_{xx} = \frac{1}{\delta} \left[ \left(\frac{1}{\delta} - 1\right) \left(\frac{g}{f}\right)^{1/\delta-2} \left(\frac{D_x(g \cdot f)}{f^2}\right)^2 + \left(\frac{D_x^2(g \cdot f)}{f^2} - \frac{g D_x^2(f \cdot f)}{f^2}\right) \left(\frac{g}{f}\right)^{1/\delta-1} \right] \tag{2.6b}$$

where the Hirota derivatives according to  $x$  and  $t$  are defined as

$$D_x^n(a \cdot b) = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)^n(a(x_1)b(x_2))|_{x=x_1=x_2}. \tag{2.7}$$

The Derivatives given by Eqs. (2.6) are substituted into Eq. (1.1), then obtained equation is multiplied by  $\left(\frac{g}{f}\right)^{1/\delta-1} \delta$ . The quadratic Bilinear derivatives coefficient must be zero in order to obtain a Bilinear Equation from this expression , this gives us a constraint on  $m$  and  $\delta$ , i.e.,  $m = \delta - 1$ . Thus, the following Equation is obtained :

$$\frac{D_t(g \cdot f)}{f^2} - \frac{D_x^2(g \cdot f)}{f^2} + \frac{g D_x^2(f \cdot f)}{f^2} + \alpha \frac{g D_x(g \cdot f)}{f^2} = \delta h_1 \left(\frac{g}{f}\right) + \delta h_2 \left(\frac{g}{f}\right)^2 + \delta h_3 \left(\frac{g}{f}\right)^3 \tag{2.8}$$

We then deduce the following system of bilinear form after collecting the first two terms with  $1/f^2$  and the rest of the terms with  $\left(\frac{g}{f}\right)\left(\frac{1}{f^2}\right)$  parentheses in Eq. (2.8) and equating them zero separately.

$$(D_t - D_x^2)(g \cdot f) = 0 \quad (2.9a)$$

$$D_x^2(f \cdot f) + \alpha D_x(g \cdot f) = \delta h_1 f^2 + \delta h_2 g f + \delta h_3 g^2. \quad (2.9b)$$

To solve this system by Hirota method, the functions  $f$  and  $g$  suppose to have form of the formal perturbation series in a parameter  $\epsilon$

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \dots \quad (2.10a)$$

$$g = g_0 + \epsilon g_1 + \epsilon^2 g_2 + \dots \quad (2.10b)$$

Substituting Eq. (2.10) into Eq. (2.9) and equating coefficients of like powers of  $\epsilon$ , converts [2.9] into a sequence of the zeroth, first, second order bilinear equations, given in Eqs. (2.11),(2.12) and (2.13), respectively,

$$(D_t - D_x)(g_0 \cdot 1) = 0 \quad (2.11a)$$

$$D_x^2(1 \cdot 1) + \alpha D_x(g_0 \cdot 1) = \delta h_1 + \delta h_2 g_0 + \delta h_3 g_0^2, \quad (2.11b)$$

$$(D_t - D_x^2)(g_0 \cdot f_1 + g_1 \cdot 1) = 0 \quad (2.12a)$$

$$D_x^2(2 \cdot f_1) + \alpha D_x(g_0 \cdot f_1 + g_1 \cdot 1) = 2\delta h_1 f_1 + \delta h_2(g_0 f_1 + g_1) + 2\delta h_3 g_0 g_1, \quad (2.12b)$$

$$(D_t - D_x^2)(g_0 \cdot f_2 + g_1 \cdot f_1 + g_2 \cdot 1) = 0 \quad (2.13a)$$

$$D_x^2(2 \cdot f_2 + f_1 \cdot f_1) + \alpha D_x(g_0 \cdot f_2 + g_1 \cdot f_1 + g_2 \cdot 1) = \delta h_1(2f_0 f_2 + f_1 f_1) + \delta h_2(g_0 f_2 + g_1 f_1 + g_2 f_0) + \delta h_3(2g_0 g_2 + g_1 g_1).$$

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We assume that all components of  $g_0$  are constants, then the first equation in Eq. (2.11) is satisfied automatically. From the second equation we get

$$h_3 g_0^2 + h_2 g_0 + h_1 = 0 \quad (2.14)$$

which is the quadratic equation of  $g_0$ . The solution of this algebraic equation is

$$g_0^{1,2} = \frac{-h_2 \mp \sqrt{\mathcal{D}}}{2h_3} \quad \text{where} \quad \mathcal{D} = h_2^2 - 4h_1 h_3. \quad (2.15)$$

There are three cases related to the discriminate : if  $\mathcal{D} < 0$ , the complex solutions are obtained, we are not interested in such solutions, if  $\mathcal{D} = 0$ , the constant solution is obtained thus we assume that  $\mathcal{D} > 0$ . The following relation can be easily found :

$$g_0^1 + g_0^2 = -\frac{h_2}{h_3}, \quad (2.16a)$$

$$g_0^1 - g_0^2 = -\frac{\sqrt{\mathcal{D}}}{h_3} \quad (2.16b)$$

As a next step we find the first order solutions,  $g_1$  and  $f_1$ . First we assume that  $g_0 = g_0^{(1)}$  or  $g_0^{(2)}$ . Eqs. (2.12) may be rewritten as a linear system

$$g_0(-\partial_t - \partial_x^2)f_1 + (\partial_t - \partial_x^2)g_1 = 0, \quad (2.17a)$$

$$(2\partial_x^2 - \alpha g_0 \partial_x - 2\delta h_1 - \delta h_2 g_0)f_1 + (\alpha \partial_x - \delta h_2 - 2\delta h_3 g_0)g_1 = 0. \quad (2.17b)$$

The simplest nontrivial solution of this system has the form

$$g_1 = e^{\eta_1}, \quad f_1 = be^{\eta_1} \quad \text{where } \eta_1 = k_1x + \omega_1t + \gamma_1. \quad (2.18)$$

The unknown constants  $k_1$ ,  $\omega_1$  and  $b$  will be found. The constant  $\gamma_1$  can be absorbed in  $b$ . The following linear system of equations are obtained after substituting Eq. (2.18) into Eq. (2.17)

$$-g_0(\omega_1 + k_1^2)b + (\omega_1 - k_1^2) = 0, \quad (2.19a)$$

$$(2k_1^2 - \alpha g_0 k_1 - 2\delta h_1 - \delta h_2 g_0)b + \alpha k_1^2 - \delta h_2 - 2\delta h_3 g_0 = 0 \quad (2.19b)$$

where  $g_0$  is the zeroth order solution given in Eq. (2.15). The nontrivial solution of this system of equations exists if the following dispersion relation on parameters  $k_1$  and  $\omega_1$  is satisfied

$$\omega_1 = k_1^2 - \alpha g_0 k_1 - \delta(2h_1 + g_0 h_2) \quad (2.20)$$

so that the free parameter  $b$  is fixed as

$$b = \frac{-\alpha g_0 k_1 - \delta(2h_1 + g_0 h_2)}{g_0(2k_1^2 - \alpha g_0 k_1 - \delta(2h_1 + g_0 h_2))}. \quad (2.21)$$

Substituting  $g_1, f_1$  to the system (2.13) and using property of bilinear Hirota operators

$$(D_t - D_x^2)(g_1 \cdot f_1) = b(D_t - D_x^2)(e^{\eta_1} \cdot e^{\eta_1}) = 0 \quad (2.22)$$

for  $g_2, f_2$ , we find the following system of equation similar to the system (2.17)

$$g_0(-\partial_t - \partial_x^2)f_2 + (\partial_t - \partial_x^2)g_2 = 0, \quad (2.23a)$$

$$(2\partial_x^2 - \alpha g_0 \partial_x)f_2 + \alpha \partial_x g_2 = \quad (2.23b)$$

$$\delta h_1(2f_0 f_2 + f_1 f_1) + \delta h_2(g_0 f_2 + g_1 f_1 + g_2 f_0) + \delta h_3(2g_0 g_2 + g_1 g_1).$$

The simplest solution for these equations is the trivial one,  $g_2 = 0$  and  $f_2 = 0$  (2.13). Then from the last Eq. (2.23) we find the additional constraint on  $g_1, f_1$  :

$$h_1(f_1 f_1) + h_2(g_1 f_1) + h_3(g_1 g_1) = 0 \quad (2.24)$$

Next substitute Eq.(2.18) into Eq. (2.24) to find the arbitrary constant  $b$ . These substitutions result in

$$h_1 + h_2 \left(\frac{1}{b}\right) + h_3 \left(\frac{1}{b^2}\right) = 0. \quad (2.25)$$

The roots of quadratic equation can be found as

$$b = \frac{h_3}{h_1} g_0^{(2)} \quad \text{if we assume } g_0 = g_0^{(1)}, \quad (2.26a)$$

$$b = \frac{h_3}{h_1} g_0^{(1)} \quad \text{if we assume } g_0 = g_0^{(2)} \quad (2.26b)$$

by comparing Eqs. (2.14) and (2.25) ( $b = 1/g_0$ ) and using the relation  $g_0^{(1)} g_0^{(2)} = \frac{h_1}{h_3}$ . Combining Eqs. (2.21) and (2.26) and using the relations given in Eq. (2.16) we find restrictions on allowed values of the length for the wave number  $k$

$$k_1^{\mp} = \frac{\alpha}{4}(g_0^{(1)} - g_0^{(2)}) \mp \frac{|g_0^{(1)} - g_0^{(2)}|}{4} \sqrt{\alpha^2 - 8\delta h_3 \delta}. \quad (2.27)$$

Eq. (2.27) can be written as follows:

$$k_{1,I}^{\pm} = \frac{|g_0^{(1)} - g_0^{(2)}|}{4} \left( \alpha \epsilon \mp \sqrt{\alpha^2 - 8\delta h_3 \delta} \right), \quad (2.28a)$$

$$k_{1,II}^{\mp} = -\frac{|g_0^{(1)} - g_0^{(2)}|}{4} \left( \alpha \epsilon \mp \sqrt{\alpha^2 - 8\delta h_3 \delta} \right) \quad (2.28b)$$

where

$$\epsilon = \text{sgn}(g_0^{(1)} - g_0^{(2)}) = \begin{cases} 1, & (g_0^{(1)} - g_0^{(2)}) > 0 \\ -1, & (g_0^{(1)} - g_0^{(2)}) < 0 \end{cases}$$

From Eq. (2.28), it can be seen that  $k_{1,I}^{\pm} = -k_{1,II}^{\mp}$ . Next, after substituting Eq. (2.28) into Eq. (2.20), the following explicit form for the frequency is obtained

$$\omega_{1,I}^{\pm} = -\frac{(g_0^{(1)} + g_0^{(2)})}{2}(-\alpha k_{1,I}^{\mp} + \delta h_3(g_0^{(1)} - g_0^{(2)})) = -\omega_{1,II}^{\mp}. \quad (2.29)$$

Finally, the velocity vector is given by the following formula:

$$\mathbf{V} = \left( -\frac{\omega_1}{k_1} \right). \quad (2.30)$$

With the wave number  $k$  given by Eq. (2.27) and frequency  $\omega$  given by Eq.(2.29) for the speed of solitary wave we have the expression

$$\mathbf{V}_{I^{\mp}} = \frac{h_2}{2h_3} \left[ \frac{(-\alpha^2 + 4\delta h_3)\epsilon \pm \alpha \sqrt{\alpha^2 - 8\delta h_3}}{-\alpha \epsilon \pm \sqrt{\alpha^2 - 8\delta h_3}} \right] = \mathbf{V}_{II^{\pm}}. \quad (2.31)$$

It is easy to show that each bilinear equation, which has order greater than 2, has simple solution as,  $g_i = 0$  and  $f_i = 0$ , for  $i > 2$ . Therefore, we have only finite number of terms in the expansion (2.10). After substituting  $f$  and  $g$  in the Eq. (2.5), we find the following exact solution to our problem

$$U = \left( \frac{h_1 g_0^{(1)} + h_1 e^{\eta_1}}{h_1 + h_3 g_0^{(2)} e^{\eta_1}} \right)^{1/\delta} \quad (2.32)$$

or

$$U = \left( \frac{h_1 g_0^{(2)} + h_1 e^{\eta_1}}{h_1 + h_3 g_0^{(1)} e^{\eta_1}} \right)^{1/\delta}. \quad (2.33)$$

We note that the constant  $b$  appearing only in front of exponential terms can be absorbed by the arbitrary constant  $\gamma_1$  in Eq.(2.18) and leads just to shift of the soliton's origin.

In the special case  $h_1 = h_2 = h_3 = 0$  the solution (2.32) or (2.33) has no meaning so that we should return back to bilinear system (2.9) which in this case becomes:

$$(D_t - D_x^2)(g \cdot f) = 0, \quad (2.34a)$$

$$D_x^2(f \cdot f) + \alpha D_x(g \cdot f) = 0. \quad (2.34b)$$

Applying Hirota perturbation in this case we have  $f_0 = 1, g_0 = \text{const}$  for the zero order solutions. For the first order perturbation we have the system:

$$g_0(-\partial_t - \partial_x^2)f_1 + (\partial_t - \partial_x^2)g_1 = 0, \quad (2.35a)$$

$$2\partial_x^2 - \alpha g_0 \partial_x + \alpha \partial_x f_1 = 0 \quad (2.35b)$$

with the solution  $g_1 = A_1 e^{\eta_1}, f_1 = B_1 e^{\eta_1}$  where  $\eta_1 = k_1 x + \omega_1 t + \delta_0$ . Substituting  $g_1 = f_1$  into Eqn. (2.35) yields the following linear equation:

$$A_1(\omega_1 - k_1^2) - B_1 g_0(\omega_1 + k_1^2) = 0, \quad (2.36a)$$

$$A_1 \alpha + B_1(2k_1 - \alpha g_0) = 0. \quad (2.36b)$$

This gives dispersion relation  $\omega_1 = k_1^2 - \alpha g_0 k_1$  and  $A_1 = (\alpha g_0 - 2k_1)C_1, B_1 = \alpha C_1$ . Due to the fact that  $D_t(e^{\eta_1} \cdot e^{\eta_1}) = D_t(e^{\eta_1} \cdot e^{\eta_1}) = D_x^2(e^{\eta_1} \cdot e^{\eta_1}) = 0$  in the second order perturbation we have

$$g_0(-\partial_t - \partial_x^2)f_2 + (\partial_t - \partial_x^2)g_2 = 0, \quad (2.37a)$$

$$(2\partial_x^2 - \alpha g_0 \partial_x)f_2 + \alpha \partial_x g_2 = 0. \quad (2.37b)$$

The simplest solution is  $g_2 = f_2 = 0$ . It provides shock soliton solution:

$$g = g_0 + (\alpha g_0 - 2k_1)e^{\eta_1} \quad \text{and} \quad f = 1 + \alpha e^{\eta_1}$$

where free parameters  $C_1$  is absorbed by  $\delta$  constant. This way we have solution

$$U(x, t) = \left( \frac{g_0 + (\alpha g_0 - 2k_1)e^{\eta_1}}{1 + \alpha e^{\eta_1}} \right)^{1/\delta}.$$

For  $k_1 > 0$ , the solution has asymptotics

$$U \rightarrow (g_0)^{1/\delta} \text{ as } \eta_1 \rightarrow -\infty \text{ and } U \rightarrow (g_0 - \frac{2k_1}{\alpha})^{1/\delta} \text{ as } \eta_1 \rightarrow -\infty.$$

It is convenient to use notations  $g_0 = a_2$ ,  $g_0 - \frac{2k_1}{\alpha} = a_1$  so that  $a_2 - a_1 = \frac{2k_1}{\alpha}$  and

$$U \rightarrow a_2^{1/\delta} \text{ as } \eta_1 \rightarrow -\infty \text{ and } U \rightarrow a_1^{1/\delta} \text{ as } \eta_1 \rightarrow -\infty.$$

For this parametrization our solution is

$$U(x, t) = \left( \frac{a_2 + \alpha a_1 e^{\eta_1}}{1 + \alpha e^{\eta_1}} \right)^{1/\delta} \tag{2.38}$$

and  $\eta_1 = k_1 x + (k_1^2 - \alpha g_0 k_1)t + \delta_0 = \frac{\alpha}{2}(a_2 - a_1)(x - vt - x_0)$  where for velocity of our shock soliton, we have  $v = \alpha \frac{a_1 + a_2}{2}$ . The form of the shock soliton solution (2.38) suggests relation with Burgers equation. In fact in the limit  $h_1 = h_2 = h_3 = 0$ , Eq. (1.1) reduces to the generalized Burgers' equation. Now we represent  $U$  in terms of the function  $f(x, t)$  as  $U(x, t) = (f(x, t))^{1/\delta}$ , then we find that  $f$  satisfies the Burgers equation

$$f_t + \alpha f f_x = f_{xx}$$

which admits linearization in terms of the heat equation by the Cole-Hopf transformation. Thus the solution of the linear heat equation provides N-shocks soliton solution of Equation (2.39),

$$U_t + \alpha U^\delta U_x - (\delta - 1) \frac{U_x^2}{U} = U_{xx}. \tag{2.39}$$

Asymptotic analysis of this solution shows that it describes the process of fusion of N-shock solitons to the one shock soliton.

### 3 Conclusion and discussion

In the present paper, the modified Hirota method is applied to find exact analytic soliton solution of the class of the nonlinear differential equation with generalized potential. To write equations in the bilinear form we proposed a special form of the Hirota ansatz. Truncation of the perturbation series in our second order calculations, restricts value of wave number and velocity of the travelling wave. In this sense, this truncation works similarly to the way Ablowitz and Zeppetella [6] obtained an exact travelling wave solution of Fisher's equation by finding the special wave speed for which the resulting ODE is of the Painleve type. To our knowledge, nobody has given a bilinear form and any exact analytic solution by using the Hirota method for this equation.

Finally, we showed that Hirota method is very efficient and systematic procedure to obtain exact solutions of such nonlinear equations. We hope that the method allows one to find solutions of other extensions of the model to higher dimensions and multicomponent order parameters similar to [12, 13]. These questions are under our study now.

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