KALEIDOSCOPE OF QUANTUM COHERENT STATES AND UNITS OF QUANTUM INFORMATION

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ABSTRACT

KALEIDOSCOPE OF QUANTUM COHERENT STATES AND UNITS OF QUANTUM INFORMATION

In the present thesis, we study superposition of coherent states as the kaleidoscope of quantum coherent states, associated with regular n-polygon symmetry and the roots of unity $q^{2n} = 1$. These states are generalizations of the Schrödinger cat states, corresponding to the roots of unity $q^2 = -1$. To describe physical characteristics of kaleidoscope states, we introduce new type of mod n exponential functions as a superposition of exponential functions in the form of discrete Fourier transform. These functions are also known as generalized hyperbolic functions, satisfying ordinary differential equations with proper initial conditions.

Kaleidoscope states are eigenstates of n-th order eigenvalue problem for annihilation operator and are not minimal uncertainty states. These states are described as quantum Fourier transform of Glauber coherent states. Normalization factors, uncertainty relations, average number of photons and coordinate representation for these states are found in a compact form by mod n exponential functions. The set of kaleidoscope states, as orthonormal computatitonal basis of quantum states, describes generic qudit unit of quantum information. Relations of kaleidoscope states with quantum group symmetry are discussed. The special cases of trinity and quartet states, corresponding to qutrit and ququat units of quantum information are treated in details.

ÖZET

EŞ UYUMLU KUANTUM DURUMLARININ KALEYDOSKOBU VE KUANTUM BİLGİSİNİN BİRİMLERİ

Bu tezde, eş uyumlu durumların kuantum birleştirimi olarak n kenarlı düzgün çokgen ve $q^{2n} = 1$ birimin kökleriyle ilişkili eş uyumlu kuantum durumlarının kaleydoskobu çalışılmıştır. Bu durumlar $q^2 = -1$ birim köküne karşılık gelen Schrödinger'in kedisi durumlarının genelleştirilmesidir. Kaleydoskop durumlarının fiziksel özelliklerini oluşturmak için, ayrık Fourier dönüşümü ile üstel fonksiyonların kombinasyonu olan yeni tipteki mod n üstel fonksiyonları tanımladık. Ayrıca, bu tipteki fonksiyonlar uygun başlangıç koşullarıyla adi diferansiyel denklemlerin çözümleri olan genelleştirilmiş hiperbolik fonksiyonlar olarak da bilinir.

Kaleydoskop durumları yok etme operatörünün n-inci dereceden özdeğer probleminin özdurumudur ve minimum belirsiz kuantum durumları değildir. Bu durumlar Glauber eş uyumlu durumların kuantum Fourier dönüşümü olarak tanımlanır. Mod n fonksiyonları ile bu durumların normalleştirme faktörleri, belirsizlik ilişkileri, fotonların ortalama değerleri ve koordinat temsilleri kompakt bir formda bulunmuştur. Ortonormal hesaplama bazı olarak kaleydoskop durumların kümesi, kuantum bilgisinin küdit(qudit) birimini tanımlar. Kaleydoskop durumların kuantum grup simetrisi ile ilişkisi çalışılmıştır. Özel olarak, kutirit(qutrit) ve kukuat(ququqat) kuantum bilgi birimlerine karşılık gelen üçlü(trinity) ve dörtlü(quartet) durumları detaylı bir şekilde inceledik.

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CHAPTER 1

INTRODUCTION

The symmetry as a harmony of propotions, like the mirror symmetry, symmetry of crystals, symmetry of elementary particles and their interactions, has a long history. Starting from ancient times, it appears in art works, decorations, paintings and architecture design (Weyl, 1952). Then, it was formalized mathematically as the concept of group and its representation. This mathematical formulation leads to symmetry classification of elementary particles in modern physics, crystal structures and fundamental interactions. The concept of beauty is directly related with symmetry, as the golden ratio propotion and Albrecht Dürer followed this to establish standards of human body propotions (Weyl, 1952). Vitruvius gave definition of symmetry, appearing as a harmony, composed by parts of unity with unity itself. More detalized relation between symmetry and mathematics are described in Birkhoff's Aesthetic Measure. Pythagoras considered circle as a most perfect geometrical object, due to complete rotational symmetry. As Hermann Weyl expressed, the symmetry is an idea by which the human during centuries is trying to understand and create order, beauty and perfection (Weyl, 1952).

From variety of different kind of symmetries, the symmetry related with rotation in plane for angle $\frac{2\pi}{n}$ corresponds to discrete group of rotations. In this case, the elementary rotation around the fixed axes to angle $\frac{2\pi}{n}$, $Q = e^{i\frac{2\pi}{n}}$ can generate by iterations, all terms in the group $Q, Q^2, ..., Q^{n-1}, Q^n = Q^0 = 1$. This group is completely characterised by order n. It was explored in architecture for decoration of columns, cells and buildings (Weyl, 1952). This type of symmetry also appears in nature as symmetry of leaves of flowers and snowflakes. The group, consisting of application of one rotation $e^{i\frac{2\pi}{n}}$, where n is an integer number, is called the cyclic group C_n. For example, architecture dominates the hexagon symmetry C_4 in buildings. Threefold symmetry with group C_3 is frequently associated with magic symbols of Celts and trinity symbols of Christian religion.

Geometrically, C_n symmetry is related with regular polygon, and this polygon can be generated by reflections in two mirrors, representing the wedge with angle $\frac{\pi}{n}$.

This become origin of a kaleidoscope (Fig 1.1), which is an optical instrument with two mirrors as a wedge domain. Then, one or more objects in this wedge domain are seen as a regular symmetrical pattern (Fig 1.2) due to repeated reflections. The name is invented by David Brewster in 1817 from the ancient greek $\kappa\alpha\lambda\acute{o}\varsigma$ (kalos), $\varepsilon\tilde{\imath}\delta o\varsigma$ (eidos),

" form, shape" and $\sigma\kappa o\pi\epsilon\omega$ (skopeo), "to look to, to examine". This means "observation of beautiful forms"(Online Etymology Dictionary).

Figure 1.1. Kaleidoscope

Figure 1.2. Symmetrical pattern

In electrostatics and hydrodynamics, the kaleidoscope principle relates with so called method of images. The method of images is a mathematical tool for solving problems, in which the domain of a function can be extended to its mirror image. As a result, specific boundary conditions allows one to solve original problem easily for hydrodynamics of incompressible and irrotational fluid flow in the wedge domain. This method, formulated for an arbitrary flow is known as the wedge theorem (Pashaev, 2015). In particular, this wedge theorem allows to construct from the given vortex in plane, the vortex kaleidoscope, corresponding to the wedge domain.

Analytic description of hydrodynamics flow by complex analytic function can be related with description of quantum states. This representation is called the Fock-Bargmann representation (Peremolov, 1997). In this representation, an arbitrary state in Fock space, projected to the coherent states can be described by complex analytic function. This implies that the idea of method of images and the kaleidoscope of images can be applied also to quantum states. The main tool for this construction is based on coherent states. The coherent states were introduced by Schröndinger in 1926 (Schrödinger, 1926) to describe non-spreading wave packet of quantum oscillator. These states, as superposition of Gaussian wave functions, minimize Heinsenberg uncertainity relations and represent most classical quantum states. The states are related with Heinsenberg-Weyl group, constructed from coordinate and momentum operators. In 1963, Glauber has applied coherent states for description of photons and the states are called "Glauber coherent states" (Glauber, 1963). The Glauber coherent states have wide applications in quantum optics (Klauder and Skagerstain, 1985), (Wolfgang, 2001). Especially, in recent research related with construction of entangled photon states for quantum information and computation processing (Sanders, 1992), (Munro, Milburn and Sanders, 1974), (Cochrane, Milburn and Munro, 1999). Therefore, fundamental question is how to create units of quantum information from quantum states of photons.

The coherent states of photons become important tool to study quantum information, but the problem is that the set of coherent states is not orthogonal. The inner product of coherent states

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2+\bar{\alpha}\beta} \tag{1.1}$$

is never zero. Therefore, the coherent states can not be directly used to describe quantum information units. These states are orthogonal approximately; for $\alpha \geq 2$, $|\langle\alpha| - \alpha\rangle| \leq 1.1x10^{-7}$ (Ralph, Gilchrist, Milburn, Munro and Glancy , 2003). However, it is still important from theoratical point of view to have exactly orthogonal states from coherent states of photons. The simplest idea is to use superposition of coherent states and impose some orthogonality conditions on it. There is well known Gram-Schmidtt orthogonalization procedure to create orthogonal state to the given one. But it is not constructive and symmetric way for an arbitrary number of coherent states. The first example of two orthogonal states, as superposition of two coherent states was derived in 1974 (Dodonov, Malkin and Manko, 1974) in the form

$$|cat_e\rangle \equiv |\alpha_+\rangle = N_+ (|\alpha\rangle + | - \alpha\rangle) \quad , \quad |cat_o\rangle \equiv |\alpha_-\rangle = N_- (|\alpha\rangle - | - \alpha\rangle) \ . \tag{1.2}$$

These states are called the Schrödinger cat states. They are orthogonal and normalized. Several researchers use them for description of a single qubit, as a unit of quantum information for photons (Sanders, 1992). The tensor product of these states describe two qubit entangled photon state (Munro, Milburn and Sanders, 1974), (Cochrane, Milburn and Munro, 1999).

It is known that qubit, as a unit of quantum information represents quantum version of a bit, as a unit of classical information. It is characterized by two orthogonal states $|0\rangle$ and $|1\rangle$. A bit and qubit are based on binary representation of numbers with base 2. However, for quantum information processing, ternary and quaternary number systems could be more efficient with base 3 and 4 as basis for position notation. Such quantum states for $n = 3$ are called the qutrit. Then, the problem appears how to construct these states from coherent states since qutrit requires three orthogonal basis states.

In present thesis, we propose a unique approach to construct superposition of coherent states, based on kaleidoscope symmetry of regular $n-$polygon and the roots of unity. This superposition gives kaleidoscope of quantum coherent states. The construction has form of the quantum Fourier transform. For description of kaleidoscope of quantum states, we introduce superposition of exponential functions with mod n symmetry. This type of functions has been discussed as generalized hyperbolic functions in mathematical research (Ungar, 1982). Our study shows that they can be applied to represent physical characteristics of kaleidoscope states and units of quantum information.

The thesis is organized as follows.

In Chapter 2, we give short introduction to consepts of qubit, Bloch sphere, one qubit gates and generic unit of quantum information as qudit.

Definition and main properties of Glauber coherent states are subject of Chapter 3. In Section 3.1, the Heinsenberg-Weyl algebra is introduced. In Section 3.2, relations between coherent states and complex plane are given. Also, matrix representation, inner product, completeness relation and Heinsenberg uncertainty relation are discussed. Average number of photons in coherent states is calculated in Section 3.3. We identify coordinate representation of coherent states in terms of Hermite polynomials in Section 3.4.

Chapter 4 is devoted to mod n exponential functions, which are associated with primitive roots of unity. In Section 4.1, we start with definition of scale and phase invariance. Section 4.2. introduces even and odd exponential functions as mod 2 exponentials. Factorization of mod 2 exponential functions, with operator argument is constructed in Section 4.3. Then, we derive mod 3 functions and mod 3 exponential functions in Section 4.4. Generalizations to mod 4 functions (Section 4.6) and mod n functions, associated to arbitrary roots of unity (Section 4.8) are studied. In Section 4.9, the ordinary differential equation for mod n exponential functions is derived. Applications to the Fock-Bargmann representation and kaleidoscope of quantum coherent states are discussed in Section 4.10. Mod n displacement operator and mod n Hermite polynomials are studied in Sections 4.11 and 4.12, respectively.

In Chapter 5, we introduce the Schrödinger cat states by mod 2 exponential function with operator argument (Section 5.1). Then, eigenvalue problem for cat states are obtained in Section 5.2. Number of photons in these states are derived in Section 5.3 in terms of mod 2 exponential functions. Then, fermionic representation of cat states is subject of Section 5.4. The Heinsenberg uncertainty relations for these states are constructed in Section 5.5. Coordinate representation of cat states is described in Section 5.6.

In Chapter 6, trinity states are introduced and main properties of these states are studied (Section 6.1,6.2). In Section 6.3, mod 3 form of trinity states is defined. In Section 6.4, the eigenvalue problem for cubic power of annihilation operator is derived. Then, number of photons in trinity states is calculated in Section 6.5. Matrix representation of operators in trinity basis and Heinsenberg uncertainty relation for trinity states are derived in Section 6.6 and 6.7.

In Chapter 7, we give definition of quartet states, and relate them with cat states (Section 7.1). We represent quartet states in mod 4 form in Section 7.2. Eigenvalue problem (Section 7.3) for quartet states is constructed. Then, number of photons in these states are derived in terms of mod 4 exponential function in Section 7.4.

In Chapter 8, we generalize the procedure to the n orthonormal states as kaleidoscope of quantum coherent states (Section 8.1). The form of these states, related with mod n exponential functions is given in Section 8.2. Then, we construct eigenvalue problem of kaleidoscope states and derive qudit coherent states in Section 8.3. Matrix representation of operators in this basis is found in Section 8.5. We identify Heinsenberg uncertainty relation for kaleidoscope states by mod n exponential functions in Section 8.6. Coordinate representation of kaleidoscope quantum states is derived in Section 8.7 by using mod n generating function for Hermite polynomials.

Chapter 9 is devoted to quantum group symmetry, which is related with our construction of kaleidoscope of quantum coherent states.

In Conclusion, we summarize our results. Some calculations are given in appendices A,B,C and D.

CHAPTER 2

INTRODUCTION TO QUANTUM INFORMATION

This chapter introduces the basic notions and principles of quantum information based on (Benenti, Casati and Strini, 2004).

2.1. Qubit as a Unit of Quantum Information

A classical bit is a system, which can exist in two different states that are represented by 0 and 1. An arbitrary integer a in binary representation, positional notation with base 2, is

$$\begin{aligned} a &\equiv a_n 2^n + a_{n-1} 2^{n-1} + \cdots + a_1 2^1 + a_0 2^0 \,, && (2.1) \\ &\equiv a_n a_{n-1} \cdots a_1 a_0 && (2.2) \end{aligned}$$

where $a_k = 0, 1$ and $k = 0, 1, ..., n$. In quantum theory , this number is represented by vector

$$|a_n a_{n-1} \cdots a_1 a_0\rangle = |a_n\rangle \otimes |a_{n-1}\rangle \otimes \cdots \otimes |a_1\rangle \otimes |a_0\rangle \qquad (2.3)$$

in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \equiv \mathbb{C}^{2(n+1)}$ dimensional Hilbert space. Every vector $|a_k\rangle \in \mathbb{C}^2$ in this tensor product represent state $|0\rangle$ or $|1\rangle$. Superposition of these states determines the qubit state. A quantum bit or qubit is a two-level quantum system, described by vector in a two-dimensional complex Hilbert space $\mathbb{C}^2$. In this space, one can introduce a pair of normalized and orthogonal quantum states,

$$|0\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |1\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad (2.4)$$

corresponding to values 0 and 1 of classical bit. These two states form a computational basis. From superposition principle, any state of the qubit can be written as

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}, \tag{2.5}$$

where the amplitudes α and β are complex numbers with normalization condition

$$|\alpha_0|^2 + |\alpha_1|^2 = 1. \tag{2.6}$$

In two level system, $|0\rangle$ is called as the ground state and $|1\rangle$ as the excited state. Another realization of qubit is by using particles with two spin states: the "down" state $|\downarrow\rangle$ and the "up" state $|\uparrow\rangle$ corresponding to $|1\rangle$ and $|0\rangle$ states respectively. Due to that, any such system can be mapped onto an effective spin$\frac{1}{2}$ system.

2.1.1. The Bloch Sphere

By parametrizing qubit state (2.5) with $\alpha = \cos\frac{\theta}{2}$ and $\beta = e^{i\phi}\sin\frac{\theta}{2}$, satisfying (2.6) up to the global phase factor, we get the Bloch sphere representation with latitude and longitude (θ, ϕ), where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$.

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle = \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix} \tag{2.7}$$

It provides a geometric picture of the qubit states as points on the unit sphere $\mathbb{S}^2$. This representation helps also in visualizing of unitary transformations acting on qubits as rotations and reflections of Bloch sphere.

2.1.2. Measuring States of Qubit

The qubit state can be measured in the computational basis. When the qubit state $|\psi\rangle$ is measured , there are two possible outcomes:

- $|0\rangle$ state with probability $p_0 = |\langle 0|\psi\rangle|^2$,
- $|1\rangle$ state with probability $p_1 = |\langle 1|\psi\rangle|^2$.

This follows from the completeness of states $|0\rangle$ and $|1\rangle$ according to proposition.

Proposition 2.1 *Computational basis is complete:*

$$\hat{I} = |0\rangle\langle 0| + |1\rangle\langle 1|. \tag{2.8}$$

From completeness relation (2.8), we can write the qubit state as

$$|\psi\rangle = \hat{I}|\psi\rangle = \Big(|0\rangle\langle 0| + |1\rangle\langle 1|\Big)|\psi\rangle = (\langle 0|\psi\rangle)|0\rangle + (\langle 1|\psi\rangle)|1\rangle \tag{2.9}$$

$$= \alpha|0\rangle + \beta|1\rangle \tag{2.10}$$

where $\alpha = \langle 0|\psi\rangle$, $\beta = \langle 1|\psi\rangle$. This gives probabilities for these states as $p_0 = |\langle 0|\psi\rangle|^2 = |\alpha|^2 = \cos^2\frac{\theta}{2}$ and $p_1 = |\langle 1|\psi\rangle|^2 = |\beta|^2 = \sin^2\frac{\theta}{2}$. Then, normalization condition means that in the measurement of qubit , the total probability should be one , $p_0 + p_1 = 1$.

2.2. One Qubit Gates

The linear transformations on a qubit, preserving normalization condition (2.6) are described by 2x2 unitary matrices, satisfying

$$UU^\dagger = U^\dagger U = \hat{I}. \tag{2.11}$$

These unitary transformations move a qubit from one point to the another point on Bloch sphere, which corresponds to rotation or reflection of the Bloch sphere.

2.2.1. Hadamard Gate and Phase-Shift Gate

In the following, we introduce basic unitary transformations acting on one qubit, as the Hadamard gate and the phase-shift gate.

Definition 2.1 *The Hadamard gate is defined as*

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{2.12}$$

This gate transforms the computational basis $\{|0\rangle, |1\rangle\}$ to the new basis $\{|+\rangle, |-\rangle\}$, which is called the Hadamard basis, as superposition of computational states:

$$\hat{H}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \equiv |+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{2.13}$$

$$\hat{H}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \equiv |-\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{2.14}$$

From this, completeness relation is valid also for the Hadamard basis:

$$\hat{I} = |+\rangle\langle+| + |-\rangle\langle-| \,. \tag{2.15}$$

The Hadamard gate (2.12) is both Hermitian and unitary. It satisfies following properties

i) Hermitian: $\hat{H} = \hat{H}^{\dagger}$,

ii) Unitarity: $\hat{H} = \hat{H}^{-1} \;\Rightarrow\; \hat{H}^2 = \hat{I}$.

Definition 2.2 *The phase-shift gate is defined as*

$$\hat{R}_z(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}. \tag{2.16}$$

This gate $\hat{R}_z(\theta)$ acts on the computational basis $|0\rangle, |1\rangle$ in the following way,

$$\hat{R}_z(\theta)|0\rangle = |0\rangle, \tag{2.17}$$

$$\hat{R}_z(\theta)|1\rangle = e^{i\theta}|1\rangle. \tag{2.18}$$

and for a single-qubit state $|\psi\rangle$, it gives

$$\hat{R}_z(\theta)|\psi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix} = \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i(\phi+\theta)}\sin\frac{\theta}{2} \end{bmatrix}. \tag{2.19}$$

This shows the counter-clockwise rotation on the Bloch sphere with angle θ about z-axis.

2.2.2. Pauli Gates

Definition 2.3 *Pauli matrices $\hat{\sigma}_x, \hat{\sigma}_y$ and $\hat{\sigma}_z$,*

$$\hat{\sigma}_x = \hat{\sigma}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \hat{\sigma}_y = \hat{\sigma}_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \hat{\sigma}_z = \hat{\sigma}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

define the Pauli gates or $\hat{X}, \hat{Y}$ and $\hat{Z}$ gates: $\hat{X} = \hat{\sigma}_1$, $\hat{Y} = \hat{\sigma}_2$ and $\hat{Z} = \hat{\sigma}_3$.

These matrices have the following properties:

i) $\hat{\sigma}_1^2 = \hat{\sigma}_2^2 = \hat{\sigma}_3^2 = \hat{I}$

ii) $\hat{\sigma}_k\hat{\sigma}_l + \hat{\sigma}_l\hat{\sigma}_k = 2\delta_{kl}$, $k, l = 1, 2, 3$

iii) $\hat{\sigma}_1\hat{\sigma}_2 = i\hat{\sigma}_3, \qquad \hat{\sigma}_2\hat{\sigma}_3 = i\hat{\sigma}_1, \qquad \hat{\sigma}_3\hat{\sigma}_1 = i\hat{\sigma}_2$

From this, the commutation relations hold:

$$[\hat{\sigma}_1, \hat{\sigma}_2] = 2i\hat{\sigma}_3, \qquad [\hat{\sigma}_2, \hat{\sigma}_3] = 2i\hat{\sigma}_1, \qquad [\hat{\sigma}_3, \hat{\sigma}_1] = 2i\hat{\sigma}_2 \tag{2.20}$$

On computational basis, these Pauli gates act as

i) $\hat{\sigma}_1|0\rangle = |1\rangle, \qquad \hat{\sigma}_1|1\rangle = |0\rangle$ (The flipping gate)

ii) $\hat{\sigma}_2|0\rangle = +i|1\rangle, \qquad \hat{\sigma}_2|1\rangle = -i|0\rangle$ (The phase-flipping gate)

iii) $\hat{\sigma}_3|0\rangle = |0\rangle, \qquad \hat{\sigma}_3|1\rangle = -1|1\rangle$ (The phase-shift gate with $\theta = \pi$)

Pauli matrices can be written as follows:

$$\hat{\sigma}_1 = |0\rangle\langle 1| + |1\rangle\langle 0| \tag{2.21}$$

$$\hat{\sigma}_2 = -i|0\rangle\langle 1| + i|1\rangle\langle 0| \tag{2.22}$$

$$\hat{\sigma}_3 = |0\rangle\langle 0| - |1\rangle\langle 1| \tag{2.23}$$

Especially, Pauli matrices $\hat{\sigma}_1$ and $\hat{\sigma}_3$ represent an example of so called the clock and shift matrices in two dimensions. These matrices are connected by the Hadamard gate $\hat{H}$:

$$\hat{\sigma}_1 = \hat{H}\hat{\sigma}_3\hat{H}^\dagger \,. \tag{2.24}$$

Generalization of this relation for qutrit, ququad and qudit, we will discuss in Chapter 9.

2.3. Universality of One Qubit Computations

For quantum computation, important is principle of universality, allowing to rewrite arbitrary quantum gate as implementation of finite number of fundamental gates. The following theorem establishes universality for one qubit gates;

Theorem 2.1 *Any unitary operation acting on a single qubit can be constructed using only Hadamard $\hat{H}$ and phase-shift $\hat{R}_z(\theta)$ gates. This is why, these gates are called universal one-qubit gates.*

The proof of this statement consists from two steps;
i)First, we show that $\forall|\psi\rangle$ can be generated from $|0\rangle$ by $\hat{H}$ and $\hat{R}_z(\theta)$ gates as

$$|\psi\rangle \equiv e^{i\frac{\theta}{2}}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right) = R_z\left(\frac{\pi}{2}+\phi\right)\hat{H}R_z(\theta)\hat{H}|0\rangle \tag{2.25}$$

Indeed, by using matrix form of gates, we have

$$
\begin{aligned}
R_z\left(\frac{\pi}{2}+\phi\right)\hat{H}R_z(\theta)\hat{H}|0\rangle &= \begin{bmatrix} 1 & 0 \\ 0 & e^{i\left(\frac{\pi}{2}+\phi\right)} \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & ie^{i\phi} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & ie^{i\phi} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ e^{i\theta} \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & ie^{i\phi} \end{bmatrix}\begin{bmatrix} 1+e^{i\theta} \\ 1-e^{i\theta} \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 1+e^{i\theta} \\ ie^{i\phi}\left(1-e^{i\theta}\right) \end{bmatrix} = e^{i\frac{\theta}{2}}\begin{bmatrix} \frac{e^{-i\frac{\theta}{2}}+e^{i\frac{\theta}{2}}}{2} \\ e^{i\phi}\frac{e^{-i\frac{\theta}{2}}-e^{i\frac{\theta}{2}}}{2} \end{bmatrix}. \qquad (2.26)
\end{aligned}
$$

It gives us one qubit on the Bloch sphere

$$
R_z\left(\frac{\pi}{2}+\phi\right)\hat{H}R_z(\theta)\hat{H}|0\rangle = e^{i\frac{\theta}{2}}\begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix} = e^{i\frac{\theta}{2}}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right) \equiv |\psi\rangle \qquad (2.27)
$$

ii) Then, $\forall$ state $|\psi_1\rangle$ parametrized by (θ_1,ϕ_1) can be transformed to $\forall$ state $|\psi_2\rangle$ parametrized by (θ_2,ϕ_2). It is evident from the following revertable chain

$$
|\psi_1\rangle \quad \to \quad |0\rangle \quad \to \quad |\psi_2\rangle. \qquad (2.28)
$$

For first part of this chain,we have

$$
|0\rangle = \hat{H}^\dagger R_z(-\theta_1)\hat{H}^\dagger R_z\left(-\frac{\pi}{2}-\phi_1\right)|\psi_1\rangle \qquad (2.29)
$$

Then, the state $|\psi_2\rangle$ can be obtained by

$$
\begin{aligned}
|\psi_2\rangle &\overset{(2.25)}{=} R_z\left(\frac{\pi}{2}+\phi_2\right)\hat{H}R_z(\theta_2)\hat{H}|0\rangle \qquad (2.30) \\
&= R_z\left(\frac{\pi}{2}+\phi_2\right)\hat{H}R_z(\theta_2)\hat{H}\hat{H}^\dagger R_z(-\theta_1)\hat{H}^\dagger R_z\left(-\frac{\pi}{2}-\phi_1\right)|\psi_1\rangle. \qquad (2.31)
\end{aligned}
$$

Due to $\hat{H}^{\dagger} = \hat{H}^{-1} = \hat{I}$, this unitary operator transforms the state $|\psi_1\rangle$ to the state $|\psi_2\rangle$

$$|\psi_2\rangle = R_z\left(\frac{\pi}{2} + \phi_2\right)\hat{H}R_z(\theta_2 - \theta_1)\hat{H}R_z\left(-\frac{\pi}{2} - \phi_1\right)|\psi_1\rangle. \quad (2.32)$$

2.4. Qutrits, Ququads and Qudits

In previous section, we have introduced qubit as a superposition of two orthogonal states, corresponding to binary representation of numbers. If instead of binary system, one considers ternary, quaternary, decimal or other number systems, then for corresponding units of quantum information we need to extend the Hilbert space. These extended units of quantum information are known as qutrit, ququad and in general qudit, and they could provide more storage for quantum information and quantum computation.

2.4.1. Qutrit

The qutrit is analog of the classical trit, which corresponds to ternary position system. Ternary representation of number b with base 3 is

$$\begin{aligned} b &= b_n 3^n + b_{n-1}3^{n-1} + \cdots + b_1 3^1 + b_0 3^0 \quad (2.33) \\ &\equiv b_n b_{n-1}\cdots b_1 b_0 \,, \quad (2.34) \end{aligned}$$

where b_k is trit which takes values $0, 1, 2$ and $k = 0, 1, ..., n$. In quantum theory, this number corresponds to vector

$$|b_n b_{n-1}\cdots b_1 b_0\rangle = |b_n\rangle \otimes |b_{n-1}\rangle \otimes \cdots \otimes |b_1\rangle \otimes |b_0\rangle \quad (2.35)$$

in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \cdots \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \equiv \mathbb{C}^{3(n+1)}$ dimensional Hilbert space. Every vector $|b_k\rangle \in \mathbb{C}^3$ in this tensor product represents states $|0\rangle, |1\rangle\, |2\rangle$. The qutrit is a unit of quantum information that exists as a superposition of three orthogonal quantum states $|0\rangle, |1\rangle$ or $|2\rangle$;

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle \quad (2.36)$$

$$= \alpha_0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad (2.37)$$

where the coefficients are probability amplitudes, satisfying

$$|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 = 1. \quad (2.38)$$

Measurement of one qutrit gives one of the states $|k\rangle$ with probability

$$p_k = |\langle k|\psi\rangle|^2 = |\alpha_k|^2, \quad (2.39)$$

where $k = 0, 1, 2$. Comparing with qubits, qutrits require a Hilbert space of higher dimension, namely $\mathbb{C}^3$. Physically, they can be realized by 3−level quantum system. In particular case, spin $s = 1$ particle has 3 orthogonal projections $|-1\rangle, |0\rangle$ and $|1\rangle$ and can be considered as a qutrit.

2.4.2. Ququat

Quaternary is the base−4 numeral system. It uses four digits 0, 1, 2 and 3 for c_k , $k = 0, 1, ..., n-1$, to represent any real number c

$$c = c_n 4^n + c_{n-1} 4^{n-1} + \cdots + c_1 4^1 + c_0 4^0 , \quad (2.40)$$

$$\equiv c_n c_{n-1} \cdots c_1 c_0 . \quad (2.41)$$

In quantum theory, this number corresponds to vector

$$|c_n c_{n-1} \cdots c_1 c_0\rangle = |c_n\rangle \otimes |c_{n-1}\rangle \otimes \cdots \otimes |c_1\rangle \otimes |c_0\rangle \quad (2.42)$$

in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \cdots \otimes \mathbb{C}^4 \otimes \mathbb{C}^4 \equiv \mathbb{C}^{4(n+1)}$ dimensional Hilbert space. Every vector $|c_k\rangle \in \mathbb{C}^4$ in this tensor product, represents states $|0\rangle, |1\rangle, |2\rangle$ and $|3\rangle$. The ququat is a unit of quantum information as superposition of four orthogonal quantum states $|0\rangle, |1\rangle, |2\rangle$ and $|3\rangle$;

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle + \alpha_3|3\rangle\,, \tag{2.43}$$

where $|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 = 1$. Physically, it can be realized by 4−level quantum system.

2.4.3. Qudit

If one uses representation of positive integer number by finite set of numbers $0, 1, 2, ..., n-1$, the quantum analogue of this number is given by tensor product of quantum states $|0\rangle, |1\rangle, |2\rangle, ..., |n-1\rangle$ from $\mathbb{C}^n$. Superposition of these states give qudit unit of quantum information. Qudit, as a generic state is the higher-dimensional analogue of qubit.

$$|\psi\rangle = \sum_{k=0}^{n-1} \alpha_k |k\rangle, \tag{2.44}$$

where $\sum_{k=0}^{n-1} |\alpha_k|^2 = 1$. It corresponds to states of $n-$level quantum system and all previous cases can be considered as particular case of this general qubit state. In the next chapters, we are going to introduce quantum coherent states in quantum optics and superposition of these states as units of quantum information.

CHAPTER 3

COHERENT STATES

In this chapter,we review definition and main properties of coherent states. More detalized expositions can be found in (Peremolov, 1997) and (Wolfgang, 2001).

3.1. The Heinsenberg Weyl Algebra

In classical mechanics, the coordinate operator $\hat{x}$ and momentum operator $\hat{p}$ are just real numbers but these simplest operators are used in describing a quantum system by Hermitian operators . They act in the Hilbert space $\mathcal{H}$ and satisfy Heinsenberg following commutation relations:

$$[\hat{x}, \hat{p}] = i\hbar\hat{\mathrm{I}} \ , \ \left[\hat{x}, \hat{\mathrm{I}}\right] = \left[\hat{p}, \hat{\mathrm{I}}\right] = 0 \, . \tag{3.1}$$

Here $\hat{\mathrm{I}}$ is the identity operator and $\hbar$ is Planck's constant , and the bracket means the commutator $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$.

Instead of operators $\hat{x}$ and $\hat{p}$, another pair of operators as the annihilation operator $\hat{a}$ and creation operator $\hat{a}^{\dagger}$ is defined

$$\hat{a}^{\dagger} = \frac{\hat{x} - i\hat{p}}{\sqrt{2\hbar}} \qquad , \qquad \hat{a} = \frac{\hat{x} + i\hat{p}}{\sqrt{2\hbar}} \, . \tag{3.2}$$

The commutation relation for these operators follows from (3.1) and (3.2) :

$$\left[\hat{a}, \hat{a}^{\dagger}\right] = \hat{a}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a} = \hat{\mathrm{I}} \, . \tag{3.3}$$

The vectors belonging to the Hilbert space are denoted by Dirac's symbol $|\Psi\rangle$.

There is a vacuum vector $|0\rangle \in \mathcal{H}$ defined as

$$\hat{a}|0\rangle = 0 \ , \quad \text{where } \langle 0|0\rangle = 1 \ . \tag{3.4}$$

Application of creation operator n-times to the vacuum state, gives us n-particle state

$$|n\rangle = \frac{(\hat{a}^{\dagger})^n}{\sqrt{n!}}|0\rangle \ , \qquad n = 0, 1, 2, ... \tag{3.5}$$

The vectors $|n\rangle$ form a basis in $\mathcal{H}$. The action of the operators are given by

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad \& \quad \hat{a}^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle \ . \tag{3.6}$$

There is a number operator $\widehat{N}$ which is Hermitian and defined by multiplication of operators $\hat{a}, \hat{a}^{\dagger}$, with eigenvalues $\widehat{N}$:

$$\widehat{N} = \hat{a}^{\dagger}\hat{a} \quad \Rightarrow \quad \widehat{N}|n\rangle = n|n\rangle \quad , n \geq 0 \ . \tag{3.7}$$

3.2. Coherent States and Complex Plane

In this section, we introduce main properties of coherent states (Peremolov, 1997). A coherent state is the specific quantum state introdued by Schrödinger for the quantum harmonic oscillator, which has dynamics most close to the behaviour of classical harmonic oscillator.

Definition 3.1 *A Glauber coherent state $|\alpha\rangle$ is defined as eigenstate of the annihilation operator $\hat{a}$, with eigenvalue $\alpha \in \mathbb{C}$*

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{3.8}$$

Now, we want to define displacement operator which generates the coherent states from vacuum state.

Definition 3.2 *The displacement operator $D(\alpha)$, where $\alpha \in \mathbb{C}$, is defined by*

$$D(\alpha) = e^{\alpha \hat{a}^\dagger - \bar{\alpha}\hat{a}}. \tag{3.9}$$

Properties of Displacement operator:

- $D^\dagger(\alpha) = D^{-1}(\alpha) = D(-\alpha) \qquad ''unitarity''$ (3.10)
- $D^\dagger(\alpha)\hat{a}D(\alpha) = \hat{a} + \alpha$ (3.11)
- $D^\dagger(\alpha)\hat{a}^\dagger D(\alpha) = \hat{a}^\dagger + \bar{\alpha}$ (3.12)
- $D(\alpha + \beta) = D(\alpha)D(\beta)e^{-i\,Im(\alpha\bar{\beta})}$ (3.13)

Proposition 3.1 *Coherent states are obtained by applying displacement operator $D(\alpha)$ to the vacuum state:*

$$|\alpha\rangle \;\; = \;\; D(\alpha)|0\rangle \tag{3.14}$$

Proposition 3.2 *The equation $|\alpha\rangle = D(\alpha)|0\rangle$ satisfies the eigenvalue problem for coherent state.*

Proof Starting with

$$D(\alpha)|0\rangle = |\alpha\rangle \Rightarrow \hat{a}D(\alpha)|0\rangle = \hat{a}|\alpha\rangle\,, \tag{3.15}$$

where

$$\hat{a}D(\alpha) = \hat{I}\hat{a}D(\alpha) = D(\alpha)D^\dagger(\alpha)\hat{a}D(\alpha) \overset{(3.11)}{=} D(\alpha)\,(\hat{a} + \alpha) = D(\alpha)\hat{a} + D(\alpha)\alpha\,, \tag{3.16}$$

then, we have

$$\hat{a}|\alpha\rangle = \hat{a}D(\alpha)|0\rangle = D(\alpha)\hat{a}|0\rangle + D(\alpha)\alpha|0\rangle = \alpha D(\alpha)|0\rangle = \alpha|\alpha\rangle \tag{3.17}$$

□

Proposition 3.3 *The displacement operator $D(\alpha)$, where $\alpha \in \mathbb{C}$, can be written in the form*

$$D(\alpha) = e^{\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}} = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^\dagger} e^{-\bar{\alpha}\hat{a}} \tag{3.18}$$

Proof We will prove this equation by using Baker−Campbell−Hausdorff formula, which says that operators $\hat{A}$ and $\hat{B}$ with ommutator $[\hat{A}, \hat{B}] = c$, where $[c, \hat{A}] = [c, \hat{B}] = 0$, give us

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}} e^{[\hat{A},\hat{B}]/2} . \tag{3.19}$$

The commutator of $\hat{A} = \alpha\hat{a}^\dagger$ and $\hat{B} = -\bar{\alpha}\hat{a}$ can be calculated as

$$[\hat{A}, \hat{B}] = [\alpha\hat{a}^\dagger, -\bar{\alpha}\hat{a}] = [\alpha, -\bar{\alpha}\hat{a}]\hat{a}^\dagger + \alpha[\hat{a}^\dagger, -\bar{\alpha}\hat{a}] \tag{3.20}$$

$$= \alpha\left(-\hat{a}^\dagger\bar{\alpha}\hat{a} + \bar{\alpha}\hat{a}\hat{a}^\dagger)\right) \tag{3.21}$$

$$= -\alpha\bar{\alpha}\left(\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger\right) = -|\alpha|^2[\hat{a}^\dagger, \hat{a}] = |\alpha|^2 . \tag{3.22}$$

□

By substituting this result into (3.19) , we get

$$e^{\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}} = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^\dagger} e^{-\bar{\alpha}\hat{a}} \quad . \tag{3.23}$$

The coherent state $|\alpha\rangle$ can be written in terms of $|0\rangle$ in a compact form,

$$|\alpha\rangle = \frac{e^{\alpha\hat{a}^\dagger}}{\sqrt{e^{|\alpha|^2}}}|0\rangle \,. \tag{3.24}$$

Definition 3.3 *Representation of coherent states in the Fock basis is*

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \,, \tag{3.25}$$

where $|n\rangle$ is eigenstate of number operator (3.7).

Proof To prove equation (3.25) , we use above results

$$\begin{aligned}
|\alpha\rangle &= D(\alpha)|0\rangle = e^{\alpha\hat{a}^\dagger-\bar{\alpha}\hat{a}}|0\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a}^\dagger}e^{-\bar{\alpha}\hat{a}}|0\rangle & (3.26)\\
&= e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a}^\dagger}\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}\,(\hat{a})^n\,|0\rangle & (3.27)\\
&= e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a}^\dagger}\left(|0\rangle-\alpha\,\underbrace{\hat{a}|0\rangle}_{0}+\frac{\alpha^2}{2}\underbrace{\hat{a}^2|0\rangle}_{0}+...\right) & (3.28)\\
&= e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a}^\dagger}|0\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{(\alpha)^n}{n!}\left(\hat{a}^\dagger\right)^n|0\rangle & (3.29)\\
&= e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{(\alpha)^n}{\sqrt{n!}}\underbrace{\frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}|0\rangle}_{|n\rangle} & (3.30)\\
&= e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle & (3.31)
\end{aligned}$$

□

3.2.1. Matrix Representation of Coherent States

Using matrix representation for n-particle states $|n\rangle$, we can find matrix representation of coherent states. In the Fock basis, $|n\rangle$ has an infinite column matrix form with unit element at $(n+1)^{th}$ place,

$$\begin{aligned}
|0\rangle^T &= (1,0,0,...), \\
|1\rangle^T &= (0,1,0,...), && (3.32)\\
|2\rangle^T &= (0,0,1,...), && (3.33)\\
&\vdots && (3.34)\\
|n\rangle^T &= (0,0,...,1,...)\,. && (3.35)
\end{aligned}$$

The coherent state $|\alpha\rangle$ has also infinite matrix form with powers of α ;

$$\begin{aligned}
|\alpha\rangle &= e^{-\frac{1}{2}|\alpha|^2}\left(1|0\rangle + \alpha|1\rangle + \frac{\alpha^2}{\sqrt{2!}}|2\rangle + ... + \frac{\alpha^n}{\sqrt{n!}}|n\rangle + ...\right) \\
&= e^{-\frac{1}{2}|\alpha|^2}\left(\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix} + \alpha\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix} + \frac{\alpha^2}{\sqrt{2!}}\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix} + ... + \frac{\alpha^n}{\sqrt{n!}}\begin{pmatrix}0\\0\\\vdots\\1\\\vdots\end{pmatrix} + ...\right) = e^{\frac{-|\alpha|^2}{2}}\begin{pmatrix}1\\\alpha\\\frac{\alpha^2}{\sqrt{2!}}\\\vdots\\\frac{\alpha^n}{\sqrt{n!}}\\\vdots\end{pmatrix}
\end{aligned}$$

3.2.2. Inner Product of Coherent States

Proposition 3.4 *Inner product of two coherent states is given by*

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2+\bar{\alpha}\beta} \tag{3.36}$$

Proof For any two coherent states $|\alpha\rangle$ and $|\beta\rangle$, we have

$$\begin{aligned}
\langle\alpha|\beta\rangle &= e^{\frac{-|\alpha|^2-|\beta|^2}{2}}\sum_{n=0}\sum_{m=0}^{\infty}\frac{(\bar{\alpha})^n\beta^m}{\sqrt{n!m!}}\underbrace{\langle n|m\rangle}_{\delta_{n,m}} \\
&= e^{\frac{-|\alpha|^2-|\beta|^2}{2}}\sum_{n=0}^{\infty}\frac{(\bar{\alpha}\beta)^n}{n!} = e^{\frac{-|\alpha|^2-|\beta|^2}{2}}e^{\bar{\alpha}\beta} = e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2+\bar{\alpha}\beta}
\end{aligned}$$

□

This result implies following corollary;

Corollary 3.1 *Coherent states are not orthogonal.*

$$|\langle\alpha|\beta\rangle|^2 = \langle\alpha|\beta\rangle\langle\beta|\alpha\rangle = e^{-(|\alpha|^2+|\beta|^2-\overline{\alpha}\beta-\overline{\beta}\alpha)} = e^{-|\alpha-\beta|^2} \neq 0$$

Since exponential of complex number is never zero, coherent states are not orthogonal.

3.2.3. Completeness of Coherent States

Proposition 3.5 *The collection of coherent states $|\alpha\rangle$, where $\alpha \in \mathbb{C}$, forms an overcomplete set:*

$$\frac{1}{\pi}\int_{\mathbb{C}} |\alpha\rangle\langle\alpha|\; d^2\alpha = \hat{I}\,. \tag{3.37}$$

Proof

$$\int_{\mathbb{C}} |\alpha\rangle\langle\alpha|\; d^2\alpha = \int_{\mathbb{C}} e^{-|\alpha|^2} \sum_{n,m=0}^{\infty} \frac{(\overline{\alpha})^n\alpha^m}{\sqrt{n!m!}}|m\rangle\langle n|\; d^2\alpha \tag{3.38}$$

$$= \sum_{n,m=0}^{\infty} \frac{|m\rangle\langle n|}{\sqrt{n!m!}} \int_{\mathbb{C}} e^{-|\alpha|^2}(\overline{\alpha})^n\alpha^m d^2\alpha\,, \tag{3.39}$$

where the measure $d^2\alpha = i d\alpha_1 d\alpha_2$ means integrating over the whole complex plane. The integral in the right hand side of equation (3.39) , say $I_{\mathbb{C}}$, can be evaluated by choosing α in polar form, such that $\alpha = re^{i\phi}$ and $d^2\alpha = rd\phi dr$,

$$I_{\mathbb{C}} = \int_0^{\infty} dr\, r\, e^{-r^2} r^{m+n} \underbrace{\int_0^{2\pi} d\phi\, e^{i(m-n)\phi}}_{2\pi\delta_{n,m}} = 2\pi \int_0^{\infty} dr\, r\,(r^2)^n e^{-r^2}\,. \tag{3.40}$$

Substitution $t = r^2$ to rewrite the integral, by using definition of gamma function

$$2\pi \int_0^{\infty} \frac{dt}{2}\, t^n e^{-t} = \pi\,\Gamma(n+1) = \pi\, n! \tag{3.41}$$

gives the completeness relation

$$\int_{\mathbb{C}} |\alpha\rangle\langle\alpha| \, d^2\alpha = \sum_{n=0}^{\infty} \frac{|n\rangle \otimes \langle n|}{n!} \pi n! = \pi \hat{I} \quad .$$

□

It is clear that coherent states are not orthogonal, but we say that the set $|\alpha\rangle$ is overcomplete, since there is the completeness relation.

3.2.4. Heinsenberg Uncertainty Relation

Coherent states are satisfying minimum uncertainty relation, thus we say that they are "most classical states".

$$(\Delta\hat{x})_\alpha \, (\Delta\hat{p})_\alpha = \frac{\hbar}{2} \tag{3.42}$$

In the proof of Heinsenberg uncertainty relation for the coherent state $|\alpha\rangle$, we will use definitions of $\hat{x}$ and $\hat{p}$ operators in terms of $\hat{a}$ and $\hat{a}^\dagger$:

$$\hat{x} = \sqrt{\frac{\hbar}{2}} \left(\hat{a} + \hat{a}^\dagger\right) \qquad , \qquad \hat{p} = -i\sqrt{\frac{\hbar}{2}} \left(\hat{a} - \hat{a}^\dagger\right) \tag{3.43}$$

Since Heinsenberg uncertainty relation includes expectation value and variance of $\hat{x}$ and $\hat{p}$ operators, these will be calculated respectively. First, we will calculate expectation value of $\hat{x}$ and $\hat{p}$ operators in the coherent state $|\alpha\rangle$,

$$\begin{aligned}
\langle\hat{x}\rangle_\alpha = \langle\alpha|\hat{x}|\alpha\rangle &= \langle\alpha|\sqrt{\frac{\hbar}{2}}(\hat{a} + \hat{a}^\dagger)|\alpha\rangle \\
&= \sqrt{\frac{\hbar}{2}}\langle\alpha|(\hat{a} + \hat{a}^\dagger)|\alpha\rangle \\
&= \sqrt{\frac{\hbar}{2}}\left(\langle\alpha|\hat{a}|\alpha\rangle + \langle\alpha|\hat{a}^\dagger|\alpha\rangle\right) = \sqrt{\frac{\hbar}{2}}(\alpha + \bar{\alpha}) \ ,
\end{aligned}$$

$$
\begin{aligned}
\langle\hat{p}\rangle_\alpha = \langle\alpha|\hat{p}|\alpha\rangle &= \langle\alpha| - i\sqrt{\frac{\hbar}{2}}(\hat{a} - \hat{a}^\dagger)|\alpha\rangle \\
&= -i\sqrt{\frac{\hbar}{2}}\langle\alpha|(\hat{a} - \hat{a}^\dagger)|\alpha\rangle \\
&= -i\sqrt{\frac{\hbar}{2}}\left(\langle\alpha|\hat{a}|\alpha\rangle - \langle\alpha|\hat{a}^\dagger|\alpha\rangle\right) = -i\sqrt{\frac{\hbar}{2}}\,(\alpha - \bar{\alpha}) \ .
\end{aligned}
$$

Then, variance for coordinate and momentum operators can be calculated as,

$$
\begin{aligned}
\langle\hat{x}^2\rangle_\alpha = \langle\alpha|\hat{x}^2|\alpha\rangle &= \langle\alpha|\left(\sqrt{\frac{\hbar}{2}}(\hat{a} + \hat{a}^\dagger)\right)^2|\alpha\rangle \\
&= \frac{\hbar}{2}\langle\alpha|\left(\hat{a} + \hat{a}^\dagger\right)^2|\alpha\rangle \\
&= \frac{\hbar}{2}\langle\alpha|\left(\hat{a}\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger\right)|\alpha\rangle \\
&\overset{(3.3)}{=} \langle\alpha|\left(\hat{a}^2 + 2\hat{a}^\dagger\hat{a} + \hat{\mathrm{I}} + (\hat{a}^\dagger)^2\right)|\alpha\rangle \\
&= \frac{\hbar}{2}\left(\langle\alpha|\hat{a}^2|\alpha\rangle + 2\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle + \langle\alpha|\alpha\rangle + \langle\alpha|(\hat{a}^\dagger)^2|\alpha\rangle\right) \\
&= \frac{\hbar}{2}\left(\alpha^2 + 2\alpha\overline{\alpha} + 1 + \overline{\alpha}^2\right) = \frac{\hbar}{2}\left((\alpha + \overline{\alpha})^2 + 1\right) ,
\end{aligned}
\tag{3.44}
$$

$$
\begin{aligned}
\langle\hat{p}^2\rangle_\alpha = \langle\alpha|\hat{p}^2|\alpha\rangle &= \langle\alpha|\left(-i\sqrt{\frac{\hbar}{2}}(\hat{a} - \hat{a}^\dagger)\right)^2|\alpha\rangle \\
&= -\frac{\hbar}{2}\langle\alpha|\left(\hat{a} - \hat{a}^\dagger\right)^2|\alpha\rangle \\
&= -\frac{\hbar}{2}\langle\alpha|\left(\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger\right)|\alpha\rangle \\
&\overset{(3.3)}{=} -\frac{\hbar}{2}\langle\alpha|\left(\hat{a}^2 - (\hat{a}^\dagger\hat{a} + \hat{\mathrm{I}}) - \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2\right)|\alpha\rangle \\
&= -\frac{\hbar}{2}\langle\alpha|\left(\hat{a}^2 - 2\hat{a}^\dagger\hat{a} - \hat{\mathrm{I}} + (\hat{a}^\dagger)^2\right)|\alpha\rangle \\
&= -\frac{\hbar}{2}\left(\langle\alpha|\hat{a}^2|\alpha\rangle - 2\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle - \langle\alpha|\alpha\rangle + \langle\alpha|(\hat{a}^\dagger)^2|\alpha\rangle\right) \\
&= -\frac{\hbar}{2}\left(\alpha^2 - 2\alpha\overline{\alpha} - 1 + \overline{\alpha}^2\right) = -\frac{\hbar}{2}\left((\alpha - \overline{\alpha})^2 - 1\right) .
\end{aligned}
\tag{3.45}
$$

Finally, there are relations between expectation value and variance of coordinate and momentum operators:

$$
\langle\hat{x}^2\rangle_\alpha = \langle\hat{x}\rangle_\alpha^2 + \frac{\hbar}{2} \qquad , \qquad \langle\hat{p}^2\rangle_\alpha = \langle\hat{p}\rangle_\alpha^2 + \frac{\hbar}{2}
\tag{3.46}
$$

Then, by using (A.2), we can calculate uncertainty in coherent states

$$\sqrt{(\Delta\hat{x})^2_\alpha} \equiv (\Delta\hat{x})_\alpha = \sqrt{\langle\hat{x}^2\rangle_\alpha - \langle\hat{x}\rangle^2_\alpha} = \sqrt{\frac{\hbar}{2}}\,, \quad (3.47)$$

$$\sqrt{(\Delta\hat{p})^2_\alpha} \equiv (\Delta\hat{p})_\alpha = \sqrt{\langle\hat{p}^2\rangle_\alpha - \langle\hat{p}\rangle^2_\alpha} = \sqrt{\frac{\hbar}{2}}\,. \quad (3.48)$$

From this, coherent state $|\alpha\rangle$ minimize the uncertainty relations in (3.42). When we calculate deviation of coordinate and momentum operators in state $|\alpha\rangle$, we observe that it does not depend on α and results for arbitrary α correspond to the case $\alpha = 0$ or to vacuum state:

$$(\Delta\hat{x})^2_\alpha = (\Delta\hat{x})^2_0 = \frac{\hbar}{2} \quad , \quad (\Delta\hat{p})^2_\alpha = (\Delta\hat{p})^2_0 = \frac{\hbar}{2}\,. \quad (3.49)$$

3.3. Average Number of Photons in Coherent States

Proposition 3.6 *Matrix representation of number operator $\hat{N}$ in coherent state basis is*

$$\langle\alpha|\hat{N}|\beta\rangle = \overline{\alpha}\beta\, e^{-\left(\frac{|\alpha|^2+|\alpha|^2}{2} - \bar{\alpha}\beta\right)}. \quad (3.50)$$

Proof We introduce expansion (3.25) for coherent states

$$\langle\alpha|\widehat{N}|\beta\rangle = \left(e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\overline{\alpha}^{\,n}}{\sqrt{n!}}\langle n|\right)\widehat{N}\left(e^{-\frac{1}{2}|\beta|^2}\sum_{m=0}^{\infty}\frac{\beta^m}{\sqrt{m!}}|m\rangle\right) \quad (3.51)$$

$$= e^{-\frac{1}{2}|\alpha|^2}e^{-\frac{1}{2}|\beta|^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\overline{\alpha}^{\,n}}{\sqrt{n!}}\frac{\beta^m}{\sqrt{m!}}\langle n|\widehat{N}|m\rangle\,. \quad (3.52)$$

Since $\widehat{N}$ is the number operator, we have relation $\widehat{N}|m\rangle = m|m\rangle$, so that

$$e^{-\frac{1}{2}|\alpha|^2}e^{-\frac{1}{2}|\beta|^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\bar{\alpha}^{\,n}}{\sqrt{n!}}\frac{\beta^{\,m}}{\sqrt{m!}}m\underbrace{\langle n|m\rangle}_{\delta_{nm}} = e^{-\left(\frac{|\alpha|^2+|\beta|^2}{2}\right)}\sum_{m=0}^{\infty}m\frac{\overline{\alpha}^{\,m}\beta^{\,m}}{m!}$$

$$= e^{-\left(\frac{|\alpha|^2+|\beta|^2}{2}\right)}\overline{\alpha}\beta\sum_{m=1}^{\infty}\frac{\overline{\alpha}^{\,(m-1)}\beta^{\,(m-1)}}{(m-1)!}\,. \quad (3.53)$$

Introducing new summation index $n = m - 1$ gives

$$\langle\alpha|\hat{N}|\beta\rangle = \overline{\alpha}\beta\, e^{-\left(\frac{|\alpha|^2+|\alpha|^2}{2} - \bar{\alpha}\beta\right)}. \tag{3.54}$$

□

By choosing $\beta = \alpha$ in (3.50) , average number of photons in coherent state can be calculated as following;

$$\langle\alpha|\hat{N}|\alpha\rangle = \overline{\alpha}\alpha\, e^{-\left(\frac{|\alpha|^2+|\alpha|^2}{2} - \overline{\alpha}\alpha\right)} = |\alpha|^2\, e^{-\left(|\alpha|^2 - |\alpha|^2\right)} = |\alpha|^2\,. \tag{3.55}$$

From the eigenvalue equation (3.8), we can also calculate average number of photons in coherent state :

$$\langle\alpha|\hat{N}|\alpha\rangle = \langle\alpha|\hat{a}^{\dagger}\hat{a}|\alpha\rangle = \overline{\alpha}\alpha\langle\alpha|\alpha\rangle = |\alpha|^2\,, \tag{3.56}$$

where we have used the formulas $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ and $\langle\alpha|\hat{a}^{\dagger} = \langle\alpha|\overline{\alpha}$. According to this result, modulus square $|\alpha|^2$ of complex number α has simple physical meaning as average number of photons in coherent state $|\alpha\rangle$.

3.4. Coordinate Representation of Coherent States

Coherent state in coordinate representation give non-stationary wave function of Gaussian form, which is the generating function of Hermite Polynomials. It was shown by Schrödinger (Schrödinger, 1926) that it provides solution of quantum harmonic oscillator, where position of Gaussian function oscillates according to equation of classical harmonic oscillator. In order to find this representation, we consider the wave function

$$\psi_{\alpha}(x) = \langle x|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\langle x|n\rangle \tag{3.57}$$

where

$$\langle x|n\rangle = \frac{1}{\pi^{1/4}} \frac{e^{-\frac{x^2}{2}}}{2^{n/2}\sqrt{n!}} H_n(x)\,. \tag{3.58}$$

After substituting and by using equation generating function for Hermite polynomials $H_n(x)$;

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = e^{-t^2+2tx}\,, \tag{3.59}$$

we get the wave function in Gaussian form

$$\langle x|\alpha\rangle = \frac{1}{\pi^{1/4}} e^{-\frac{|\alpha|^2}{2}-\frac{x^2}{2}} e^{-\frac{\alpha^2}{2}+\sqrt{2}x\alpha} \tag{3.60}$$

The probability density

$$|\psi_\alpha(x)|^2 = \frac{1}{\pi^{1/2}} e^{-\left(x-\frac{\alpha_1}{\sqrt{2}}\right)^2} e^{-\frac{3}{2}\alpha_1^2} \tag{3.61}$$

is the Gaussian distribution with centered at $\frac{\alpha_1}{\sqrt{2}}$, where $\alpha = \alpha_1 + i\alpha_2$.

CHAPTER 4

MOD *N* EXPONENTIAL FUNCTIONS

For description of kaleidoscope of coherent states, it is convenient to use new set of functions. In this Chapter, we introduce the set of functions with mod n symmetry, related with discrete symmetry of regular polygon and the root of unity $q^{2n} = 1$. These type of functions have origin from the wedge theorem, describing hydrodynamic flow in wedge domain with angle $\frac{\pi}{n}$, (Pashaev, 2015). Exponential form of these functions was derived as generalized hyperbolic functions in (Ungar, 1982), and some applications to superposition of coherent states in (Spiridonov V V. , 1995). Here we describe properties of these functions, their relations with discrete Fourier transformation and applications to non-commutative operator argument.

4.1. Scale and Phase Invariance

Definition 4.1 *A function is said to be scale-invariant if it satisfies following property;*

$$f(\lambda z) = \lambda^d f(z)\,, \tag{4.1}$$

for some choice of exponent $d \in \mathbb{R}$ and fixed scale factor $\lambda > 0$, which can be taken as a length or size of re-scaling.

If $\lambda = e^{i\varphi}$ and as follows $|\lambda| = 1$, then this formula gives

$$f(e^{i\varphi} z) = e^{i\varphi d} f(z) \tag{4.2}$$

In this case, rotation of argument z to angle φ implies rotation of function f to angle φd, and scale invariance becomes rotational or phase(gauge) invariance. If $\lambda = q^2$ is the primitive root of unity

$$q^{2n} = 1 \tag{4.3}$$

so that $q^2 = e^{i\frac{2\pi}{n}}$, then

$$f(e^{i\frac{2\pi}{n}} z) = e^{i\frac{2\pi}{n}d} f(z)\,. \tag{4.4}$$

This means that rotation of argument z to angle $\frac{2\pi}{n}$ of $n-$sided polygon, leads to rotation of f on $d-$times of this angle. We call this as discrete phase gauge invariant function, with order d. Simplest example of phase invariant functions is given by even and odd functions with $q^4 = 1$;

$$f_{even}(q^2 x) = f_{even}(x)\ ,\ \ f_{odd}(q^2 x) = q^2 f_{odd}(x) \tag{4.5}$$

where $\lambda = q^2 = -1$ and $d = 0$, $d = 1$ respectively.

4.2. Even and Odd Exponential Functions

Proposition 4.1 *Arbitrary function $f(x)$ is a superposition of even and odd functions.*

$$f(x) = f_{even}(x) + f_{odd}(x)\,, \tag{4.6}$$

where $f_{even}(-x) = f_{even}(x) = \dfrac{f(x) + f(-x)}{2}\ ,\ \ f_{odd}(-x) = -f_{odd}(x) = \dfrac{f(x) - f(-x)}{2}.$

Proof It is evident from following identity

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = f_{even}(x) + f_{odd}(x)\,. \tag{4.7}$$

□

In particular case of standard exponential function we have

$$e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \cosh x + \sinh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \tag{4.8}$$

It is instructive to rewrite this expression in terms of primitive roots of unity $q^4 = 1$. If $q^4 = 1$, so that $q^2 = \bar{q}^2 = -1 = e^{i\pi}$, then

$$e^x = \frac{e^x + e^{q^2 x}}{2} + \frac{e^x + \bar{q}^2 e^{q^2 x}}{2} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x^n}{n!} + \frac{(q^2 x)^n}{n!}\right) + \frac{1}{2}\sum_{m=0}^{\infty}\left(\frac{x^m}{m!} + \bar{q}^2\frac{(q^2 x)^m}{m!}\right)$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{n!}\underbrace{(1+q^{2n})}_{2\delta_{n,0(mod2)}} + \frac{1}{2}\sum_{m=0}^{\infty}\frac{x^m}{m!}\underbrace{(1+q^{2(m-1)})}_{2\delta_{m,1(mod2)}}$$
$$= \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!} + \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!} = \cosh x + \sinh x \qquad (4.9)$$

Definition 4.2 *mod 2 exponential functions* ${}_0e^x$ *and* ${}_1e^x$ *are even and odd combinations of exponential function* e^x*, which are known as with hyperbolic functions:*

$$(mod\, 2) \quad {}_0e^x \equiv \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!} = \frac{e^x + e^{q^2 x}}{2} = \cosh x, \qquad (4.10)$$
$$(mod\, 2) \quad {}_1e^x \equiv \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!} = \frac{e^x + \bar{q}^2 e^{q^2 x}}{2} = \sinh x. \qquad (4.11)$$

Here indices 0 and 1 correspond to values of remainder in $(mod\, 2)$ for even and odd numbers $2k \equiv 0\,(mod\, 2)$ and $2k+1 \equiv 1\,(mod\, 2)$. We combine these functions to the matrix form

$$\begin{bmatrix} {}_0e^x \\ {}_1e^x \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & \bar{q}^2 \end{bmatrix}\begin{bmatrix} e^x \\ e^{q^2 x} \end{bmatrix} = \frac{\hat{H}}{\sqrt{2}}\begin{bmatrix} e^x \\ e^{q^2 x} \end{bmatrix}. \qquad (4.12)$$

where $\hat{H}$ is the Hadamard gate (2.12). Due to ${}_0e^{-x} = {}_0e^x$ and ${}_1e^{-x} = -{}_1e^x$, the standard exponential can be written as superposition of mod 2 exponential functions:

$$e^x = {}_0e^x + {}_1e^x \quad (mod\, 2), \qquad (4.13)$$
$$e^{-x} = {}_0e^x - {}_1e^x \quad (mod\, 2). \qquad (4.14)$$

For arbitrary function $f(x)$ from proposition 4.1 , we have

$$\begin{bmatrix} f_{even}(x) \\ f_{odd}(x) \end{bmatrix} = \begin{bmatrix} f_0(x) \\ f_1(x) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & \overline{q}^2 \end{bmatrix}\begin{bmatrix} f(x) \\ f(q^2x) \end{bmatrix}. \tag{4.15}$$

It implies the inverse transformation

$$f(x) = f_0(x) + f_1(x), \tag{4.16}$$

$$f(q^2x) = f_0(x) + q^2 f_1(x) = f_0(x) - f_1(x) \tag{4.17}$$

We call functions $f_0(x)$ and $f_1(x)$ as mod 2 functions. These functions exist for an arbitrary analytic function. If $f(z)$ is an analytic function in disk D around the origin, it can be represented by power series

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(0) z^n . \tag{4.18}$$

Then, mod 2 functions $f_0(x)$ and $f_1(x)$ would include even or odd powers of z correspondingly

$$f_0(z) = \sum_{k=0}^{\infty} \frac{1}{2k!} f^{2k}(0) z^{2k} \quad , \quad f_1(z) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} f^{2k+1}(0) z^{2k+1} . \tag{4.19}$$

4.3. Factorization of Mod 2 Exponential Functions with Operator Argument

It is well known that exponential function with operator argument can be factorized in the form

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]} , \tag{4.20}$$

where $\hat{A}$ and $\hat{B}$ are c-commutative: $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$. Here we are going to derive similar factorization formulas for mod 2 exponential functions.

Proposition 4.2 *Let $\hat{A}$ and $\hat{B}$ are two c-commutative operators. Then*

$$
\begin{aligned}
{}_0e^{\hat{A}+\hat{B}} &= \left({}_0e^{\hat{A}}\,{}_0e^{\hat{B}} + {}_1e^{\hat{A}}\,{}_1e^{\hat{B}}\right)e^{-\frac{1}{2}[\hat{A},\hat{B}]} && (4.21)\\
{}_1e^{\hat{A}+\hat{B}} &= \left({}_0e^{\hat{A}}\,{}_1e^{\hat{B}} + {}_1e^{\hat{A}}\,{}_0e^{\hat{B}}\right)e^{-\frac{1}{2}[\hat{A},\hat{B}]}, && (4.22)
\end{aligned}
$$

where $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$.

Proof We will use definition 4.2 to prove equations (4.21) and (4.22). Then,

$$
\begin{aligned}
{}_0e^{\hat{A}+\hat{B}} &= \frac{e^{\hat{A}+\hat{B}} + e^{-(\hat{A}+\hat{B})}}{2} \overset{(4.20)}{=} \frac{e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]} + e^{-\hat{A}}e^{-\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}}{2}\\
&= \left[\left({}_0e^{\hat{A}} + {}_1e^{\hat{A}}\right)\left({}_0e^{\hat{B}} + {}_1e^{\hat{B}}\right) + \left({}_0e^{-\hat{A}} + {}_1e^{-\hat{A}}\right)\left({}_0e^{-\hat{B}} + {}_1e^{-\hat{B}}\right)\right]\frac{e^{-\frac{1}{2}[\hat{A},\hat{B}]}}{2}\\
&= \left[\left({}_0e^{\hat{A}} + {}_1e^{\hat{A}}\right)\left({}_0e^{\hat{B}} + {}_1e^{\hat{B}}\right) + \left({}_0e^{\hat{A}} - {}_1e^{\hat{A}}\right)\left({}_0e^{\hat{B}} - {}_1e^{\hat{B}}\right)\right]\frac{e^{-\frac{1}{2}[\hat{A},\hat{B}]}}{2}\\
&= \left[{}_0e^{\hat{A}}\,{}_0e^{\hat{B}} + {}_1e^{\hat{A}}\,{}_1e^{\hat{B}}\right]e^{-\frac{1}{2}[\hat{A},\hat{B}]} \qquad (4.23)
\end{aligned}
$$

and

$$
\begin{aligned}
{}_1e^{\hat{A}+\hat{B}} &= \frac{e^{\hat{A}+\hat{B}} - e^{-(\hat{A}+\hat{B})}}{2} \overset{(4.20)}{=} \frac{e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]} - e^{-\hat{A}}e^{-\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}}{2}\\
&= \left[\left({}_0e^{\hat{A}} + {}_1e^{\hat{A}}\right)\left({}_0e^{\hat{B}} + {}_1e^{\hat{B}}\right) - \left({}_0e^{-\hat{A}} + {}_1e^{-\hat{A}}\right)\left({}_0e^{-\hat{B}} + {}_1e^{-\hat{B}}\right)\right]\frac{e^{-\frac{1}{2}[\hat{A},\hat{B}]}}{2}\\
&= \left[\left({}_0e^{\hat{A}} + {}_1e^{\hat{A}}\right)\left({}_0e^{\hat{B}} + {}_1e^{\hat{B}}\right) - \left({}_0e^{\hat{A}} - {}_1e^{\hat{A}}\right)\left({}_0e^{\hat{B}} - {}_1e^{\hat{B}}\right)\right]\frac{e^{-\frac{1}{2}[\hat{A},\hat{B}]}}{2}\\
&= \left[{}_0e^{\hat{A}}\,{}_1e^{\hat{B}} + {}_1e^{\hat{A}}\,{}_0e^{\hat{B}}\right]e^{-\frac{1}{2}[\hat{A},\hat{B}]} \qquad (4.24)
\end{aligned}
$$

□

Factorization formula (4.20) gives $q-$commutative relation between operators $e^{\hat{A}}$ and $e^{\hat{B}}$,

$$e^{\hat{A}}e^{\hat{B}} = e^{[\hat{A},\hat{B}]}e^{\hat{B}}e^{\hat{A}} = qe^{\hat{B}}e^{\hat{A}}\,. \qquad (4.25)$$

We have analogue of this formula for mod 2 exponential functions due to following proposition.

Proposition 4.3 *For operators $\hat{A}$ and $\hat{B}$ such that $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$ following identities hold*

$$
\begin{aligned}
{}_0e^{\hat{A}}\,{}_0e^{\hat{B}} &= {}_0e^{\hat{B}}\,{}_0e^{\hat{A}}\,{}_0e^{[\hat{A},\hat{B}]} + {}_1e^{\hat{B}}\,{}_1e^{\hat{A}}\,{}_1e^{[\hat{A},\hat{B}]} && (4.26)\\
{}_1e^{\hat{A}}\,{}_1e^{\hat{B}} &= {}_1e^{\hat{B}}\,{}_1e^{\hat{A}}\,{}_0e^{[\hat{A},\hat{B}]} + {}_0e^{\hat{B}}\,{}_0e^{\hat{A}}\,{}_1e^{[\hat{A},\hat{B}]} && (4.27)\\
{}_0e^{\hat{A}}\,{}_1e^{\hat{B}} &= {}_1e^{\hat{B}}\,{}_0e^{\hat{A}}\,{}_0e^{[\hat{A},\hat{B}]} + {}_0e^{\hat{B}}\,{}_1e^{\hat{A}}\,{}_1e^{[\hat{A},\hat{B}]} && (4.28)\\
{}_1e^{\hat{A}}\,{}_0e^{\hat{B}} &= {}_0e^{\hat{B}}\,{}_1e^{\hat{A}}\,{}_0e^{[\hat{A},\hat{B}]} + {}_1e^{\hat{B}}\,{}_0e^{\hat{A}}\,{}_1e^{[\hat{A},\hat{B}]} && (4.29)
\end{aligned}
$$

Proof By explicit calculation,

$$
\begin{aligned}
{}_0e^{\hat{A}}\,{}_0e^{\hat{B}} &= \left(\frac{e^{\hat{A}}+e^{-\hat{A}}}{2}\right)\left(\frac{e^{\hat{B}}+e^{-\hat{B}}}{2}\right)\\
&= \frac{1}{4}\left[e^{\hat{A}}e^{\hat{B}} + e^{-\hat{A}}e^{-\hat{B}} + e^{\hat{A}}e^{-\hat{B}} + e^{-\hat{A}}e^{\hat{B}}\right]\\
&\overset{(4.25)}{=} \frac{1}{4}\left[\left(e^{\hat{B}}e^{\hat{A}} + e^{-\hat{B}}e^{-\hat{A}}\right)e^{[\hat{A},\hat{B}]} + \left(e^{-\hat{B}}e^{\hat{A}} + e^{\hat{B}}e^{-\hat{A}}\right)e^{-[\hat{A},\hat{B}]}\right]\\
&= \frac{1}{4}\Big[\Big(\left({}_0e^{\hat{B}} + {}_1e^{\hat{B}}\right)\left({}_0e^{\hat{A}} + {}_1e^{\hat{A}}\right) + \left({}_0e^{\hat{B}} - {}_1e^{\hat{B}}\right)\left({}_0e^{\hat{A}} - {}_1e^{\hat{A}}\right)\Big)e^{[\hat{A},\hat{B}]}\\
&\quad + \Big(\left({}_0e^{\hat{B}} - {}_1e^{\hat{B}}\right)\left({}_0e^{\hat{A}} + {}_1e^{\hat{A}}\right) + \left({}_0e^{\hat{B}} + {}_1e^{\hat{B}}\right)\left({}_0e^{\hat{A}} - {}_1e^{\hat{A}}\right)\Big)e^{-[\hat{A},\hat{B}]}\Big]\\
&= \frac{1}{4}\left[2\left({}_0e^{\hat{B}}\,{}_0e^{\hat{A}} + {}_1e^{\hat{B}}\,{}_1e^{\hat{A}}\right)e^{[\hat{A},\hat{B}]} + 2\left({}_0e^{\hat{B}}\,{}_0e^{\hat{A}} - {}_1e^{\hat{B}}\,{}_1e^{\hat{A}}\right)e^{-[\hat{A},\hat{B}]}\right]\\
&= \frac{1}{2}\left[2\,{}_0e^{\hat{B}}\,{}_0e^{\hat{A}}\,{}_0e^{[\hat{A},\hat{B}]} + 2\,{}_1e^{\hat{B}}\,{}_1e^{\hat{A}}\,{}_1e^{[\hat{A},\hat{B}]}\right] && (4.30)
\end{aligned}
$$

we find

$$
{}_0e^{\hat{A}}\,{}_0e^{\hat{B}} = {}_0e^{\hat{B}}\,{}_0e^{\hat{A}}\,{}_0e^{[\hat{A},\hat{B}]} + {}_1e^{\hat{B}}\,{}_1e^{\hat{A}}\,{}_1e^{[\hat{A},\hat{B}]}. \tag{4.31}
$$

Similar calculations give

$$
\begin{aligned}
{}_1e^{\hat{A}}\,{}_1e^{\hat{B}} &= \left(\frac{e^{\hat{A}} - e^{-\hat{A}}}{2}\right)\left(\frac{e^{\hat{B}} - e^{-\hat{B}}}{2}\right) \\
&= \frac{1}{4}\left[e^{\hat{A}}e^{\hat{B}} + e^{-\hat{A}}e^{-\hat{B}} - e^{\hat{A}}e^{-\hat{B}} - e^{-\hat{A}}e^{\hat{B}}\right] \\
&\overset{(4.25)}{=} \frac{1}{4}\left[\left(e^{\hat{B}}e^{\hat{A}} + e^{-\hat{B}}e^{-\hat{A}}\right)e^{[\hat{A},\hat{B}]} - \left(e^{-\hat{B}}e^{\hat{A}} + e^{\hat{B}}e^{-\hat{A}}\right)e^{-[\hat{A},\hat{B}]}\right] \\
&= \frac{1}{4}\Big[\Big(\left({}_0e^{\hat{B}} + {}_1e^{\hat{B}}\right)\left({}_0e^{\hat{A}} + {}_1e^{\hat{A}}\right) + \left({}_0e^{\hat{B}} - {}_1e^{\hat{B}}\right)\left({}_0e^{\hat{A}} - {}_1e^{\hat{A}}\right)\Big)e^{[\hat{A},\hat{B}]} \\
&\quad - \Big(\left({}_0e^{\hat{B}} - {}_1e^{\hat{B}}\right)\left({}_0e^{\hat{A}} + {}_1e^{\hat{A}}\right) + \left({}_0e^{\hat{B}} + {}_1e^{\hat{B}}\right)\left({}_0e^{\hat{A}} - {}_1e^{\hat{A}}\right)\Big)e^{-[\hat{A},\hat{B}]}\Big] \\
&= \frac{1}{4}\left[2\left({}_0e^{\hat{B}}\,{}_0e^{\hat{A}} + {}_1e^{\hat{B}}\,{}_1e^{\hat{A}}\right)e^{[\hat{A},\hat{B}]} - 2\left({}_0e^{\hat{B}}\,{}_0e^{\hat{A}} - {}_1e^{\hat{B}}\,{}_1e^{\hat{A}}\right)e^{-[\hat{A},\hat{B}]}\right] \\
&= \frac{1}{2}\left[2\,{}_0e^{\hat{B}}\,{}_0e^{\hat{A}}\,{}_1e^{[\hat{A},\hat{B}]} + 2\,{}_1e^{\hat{B}}\,{}_1e^{\hat{A}}\,{}_0e^{[\hat{A},\hat{B}]}\right] \qquad (4.32)
\end{aligned}
$$

so that

$$
{}_1e^{\hat{A}}\,{}_1e^{\hat{B}} = {}_1e^{\hat{B}}\,{}_1e^{\hat{A}}\,{}_0e^{[\hat{A},\hat{B}]} + {}_0e^{\hat{B}}\,{}_0e^{\hat{A}}\,{}_1e^{[\hat{A},\hat{B}]}. \qquad (4.33)
$$

In a similar way, mixed product of mod 2 exponentials can be calculated. □

The mod 2 identities from proposition 4.3 can be rewritten as commutativity relations for hyperbolic functions of operator argument.

Corollary 4.1 *Let $\hat{A}$ and $\hat{B}$ are two $c-$commutative operators , then hyperbolic functions with operator argument satisfy following identities*

$$
\begin{aligned}
\cosh\hat{A}\cosh\hat{B} &= \cosh\hat{B}\cosh\hat{A}\cosh[\hat{A},\hat{B}] + \sinh\hat{B}\sinh\hat{A}\sinh[\hat{A},\hat{B}], \qquad (4.34) \\
\sinh\hat{A}\sinh\hat{B} &= \sinh\hat{B}\sinh\hat{A}\cosh[\hat{A},\hat{B}] + \cosh\hat{B}\cosh\hat{A}\sinh[\hat{A},\hat{B}], \qquad (4.35) \\
\cosh\hat{A}\sinh\hat{B} &= \sinh\hat{B}\cosh\hat{A}\cosh[\hat{A},\hat{B}] + \cosh\hat{B}\sinh\hat{A}\sinh[\hat{A},\hat{B}], \qquad (4.36) \\
\sinh\hat{A}\cosh\hat{B} &= \cosh\hat{B}\sinh\hat{A}\cosh[\hat{A},\hat{B}] + \sinh\hat{B}\cosh\hat{A}\sinh[\hat{A},\hat{B}], \qquad (4.37)
\end{aligned}
$$

where $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0.$

Formulas (4.21) – (4.22), implies also addition formulas for hyperbolic functions of operator argument:

$$
\begin{aligned}
\cosh\left(\hat{A}+\hat{B}\right) &= \left(\cosh\hat{A}\cosh\hat{B}+\sinh\hat{A}\sinh\hat{B}\right)e^{-\frac{1}{2}[\hat{A},\hat{B}]} && (4.38)\\
\cosh\left(\hat{A}-\hat{B}\right) &= \left(\cosh\hat{A}\cosh\hat{B}-\sinh\hat{A}\sinh\hat{B}\right)e^{\frac{1}{2}[\hat{A},\hat{B}]} && (4.39)\\
\sinh\left(\hat{A}+\hat{B}\right) &= \left(\sinh\hat{A}\cosh\hat{B}+\sinh\hat{B}\cosh\hat{A}\right)e^{-\frac{1}{2}[\hat{A},\hat{B}]} && (4.40)\\
\sinh\left(\hat{A}-\hat{B}\right) &= \left(\sinh\hat{A}\cosh\hat{B}-\sinh\hat{B}\cosh\hat{A}\right)e^{\frac{1}{2}[\hat{A},\hat{B}]} && (4.41)
\end{aligned}
$$

For special case, when $[\hat{A},\hat{B}]=0$, these addition formulas reduce to usual formulas for hyperbolic functions.

4.4. Mod 3 Functions

Here we are going to generalize previous results to the case of mod 3 functions.

Definition 4.3 *For arbitrary function $f(x)$, mod 3 functions are defined by formulas*

$$
\begin{aligned}
f_0(x) &\equiv \frac{f(x)+f(q^2x)+f(q^4x)}{3}, && (4.42)\\
f_1(x) &\equiv \frac{f(x)+\bar{q}^{\,2}f(q^2x)+\bar{q}^{\,4}f(q^4x)}{3}, && (4.43)\\
f_2(x) &\equiv \frac{f(x)+\bar{q}^{\,4}f(q^2x)+\bar{q}^{\,2}f(q^4x)}{3}, && (4.44)
\end{aligned}
$$

where q^2 is primitive root of unity, $q^6=1$.

Proposition 4.4 *An arbitrary function $f(x)$ is a superposition of mod 3 functions*

$$f(x)=f_0(x)+f_1(x)+f_2(x)\,. \tag{4.45}$$

Proof By adding these functions

$$
\begin{aligned}
f_0(x)+f_1(x)+f_2(x) &= \frac{f(x)+f(q^2x)+f(q^4x)}{3}+\frac{f(x)+\bar{q}^{\,2}f(q^2x)+\bar{q}^{\,4}f(q^4x)}{3}\\
&+ \frac{f(x)+\bar{q}^{\,4}f(q^2x)+\bar{q}^{\,2}f(q^4x)}{3}\\
&= \frac{1}{3}\Big(3f(x)+\left(1+\bar{q}^{\,2}+\bar{q}^{\,4}\right)f(q^2x)+\left(1+\bar{q}^{\,4}+\bar{q}^{\,2}\right)f(q^4x)\Big)
\end{aligned}
$$

and using equation $(A.8)$ with $q^6 = (e^{i\frac{2\pi}{3}})^3 = 1$,

$$q^{6n} = 1 \quad \Rightarrow \quad (q^{2n} - 1)(1 + q^{2n} + q^{4n}) = 0 \tag{4.46}$$

$$\Rightarrow \quad 1 + q^{2n} + q^{4n} = 3\delta_{n,0\,(mod3)} \tag{4.47}$$

$$\Rightarrow \quad 1 + \overline{q}^{\,2n} + \overline{q}^{\,4n} = 3\delta_{n,0\,(mod3)} \tag{4.48}$$

we have

$$f_0(x) + f_1(x) + f_2(x) = f(x). \tag{4.49}$$

□

As we can notice, structure of functions $f_0(x)\,, f_1(x)$ and $f_2(x)$ in definition (4.3) is determined by superposition of function f(x) at three points x, q^2x and q^4x, so that sum of coefficients for $f(q^2x)$ and $f(q^4x)$ are zero due to $1 + q^2 + q^4 = 0$. Combining mod 3 functions as the column matrix, we can rewrite definition 4.3 in matrix form;

$$\begin{bmatrix} f_0(x) \\ f_1(x) \\ f_2(x) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \overline{q}^{\,2} & \overline{q}^{\,4} \\ 1 & \overline{q}^{\,4} & \overline{q}^{\,2} \end{bmatrix} \begin{bmatrix} f(x) \\ f(q^2x) \\ f(q^4x) \end{bmatrix}. \tag{4.50}$$

This transformation can be considered as a discrete Fourier transformation(see Appendix C). Moreover, this matrix represents 3×3 analogue of the Hadamard gate matrix

$$\hat{H} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \overline{q}^{\,2} & \overline{q}^{\,4} \\ 1 & \overline{q}^{\,4} & \overline{q}^{\,2} \end{bmatrix}, \tag{4.51}$$

which is unitary $\hat{H}\hat{H}^{\dagger} = \hat{I}$. Then,

$$\begin{bmatrix} f_0(x) \\ f_1(x) \\ f_2(x) \end{bmatrix} = \frac{1}{\sqrt{3}}\hat{H} \begin{bmatrix} f(x) \\ f(q^2x) \\ f(q^4x) \end{bmatrix}. \tag{4.52}$$

Mod 3 functions $f_0(x)$, $f_1(x)$ and $f_2(x)$ are phase(gauge) invariant functions with following transformation rules;

$$f_0(q^2 x) = f_0(x)\,, \tag{4.53}$$

$$f_1(q^2 x) = q^2 f_1(x)\,, \tag{4.54}$$

$$f_2(q^2 x) = q^4 f_2(x)\,. \tag{4.55}$$

where $q^2 = e^{i\frac{2\pi}{3}}$. If we have complex argument z and $f(z)$ is an analytic in disk D, so that

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(0) z^n\,, \tag{4.56}$$

then mod 3 functions are $f_0(z)$, $f_1(z)$ and $f_2(z)$ are

$$\begin{aligned} f_0(z) &= \sum_{k=0}^{\infty} \frac{1}{3k!} f^{3k}(0) z^{3k}\,, \\ f_1(z) &= \sum_{k=0}^{\infty} \frac{1}{(3k+1)!} f^{3k+1}(0) z^{3k+1}\,, \\ f_2(z) &= \sum_{k=0}^{\infty} \frac{1}{(3k+2)!} f^{3k+2}(0) z^{3k+2}\,. \end{aligned}$$

4.4.1. Mod 3 Exponential Functions

Here, we apply mod 3 splitting to the standard exponential function

$$\begin{aligned} e^x &= \frac{e^x + e^{q^2 x} + e^{q^4 x}}{3} + \frac{e^x + \overline{q}^2 e^{q^2 x} + \overline{q}^2 e^{q^4 x}}{3} + \frac{e^x + \overline{q}^4 e^{q^2 x} + \overline{q}^2 e^{q^4 x}}{3} \\ &= \frac{1}{3}\left(\sum_{t=0}^{\infty} \frac{x^t}{t!}(1 + q^{2t} + q^{4t}) + \sum_{l=0}^{\infty} \frac{x^l}{l!}(1 + q^{2(l-1)} + q^{4(l-1)}) \right. \\ &\quad \left. + \sum_{r=0}^{\infty} \frac{x^r}{r!}(1 + q^{2(r-2)} + q^{4(r-2)}) \right) \\ &\overset{(4.47)}{=} \sum_{k=0}^{\infty} \frac{x^{3k}}{3k!} + \sum_{k=0}^{\infty} \frac{x^{3k+1}}{(3k+1)!} + \sum_{k=0}^{\infty} \frac{x^{3k+2}}{(3k+2)!}\,. \end{aligned}$$

Definition 4.4 *mod* 3 *exponential functions* ${}_0e^x$, ${}_1e^x$ *and* ${}_2e^x$ *are defined by power series expansions*

$$
\begin{aligned}
{}_0e^x &\equiv \frac{e^x + e^{q^2x} + e^{q^4x}}{3} = \sum_{k=0}^{\infty} \frac{x^{3k}}{3k!}, \qquad (4.57)\\
{}_1e^x &\equiv \frac{e^x + \bar{q}^2 e^{q^2x} + \bar{q}^2 e^{q^4x}}{3} = \sum_{k=0}^{\infty} \frac{x^{3k+1}}{(3k+1)!}, \qquad (4.58)\\
{}_2e^x &\equiv \frac{e^x + \bar{q}^4 e^{q^2x} + \bar{q}^2 e^{q^4x}}{3} = \sum_{k=0}^{\infty} \frac{x^{3k+2}}{(3k+2)!}. \qquad (4.59)
\end{aligned}
$$

The standard exponential function can be written as superposition of these three functions:

$$e^x = {}_0e^x + {}_1e^x + {}_2e^x \quad (mod\, 3)\,. \qquad (4.60)$$

Matrix form of these functions in (4.57) – (4.59) gives discrete fourier transformation

$$\begin{bmatrix} {}_0e^x \\ {}_1e^x \\ {}_2e^x \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \bar{q}^2 & \bar{q}^4 \\ 1 & \bar{q}^4 & \bar{q}^2 \end{bmatrix} \begin{bmatrix} e^x \\ e^{q^2x} \\ e^{q^4x} \end{bmatrix}. \qquad (4.61)$$

Mod 3 exponential functions can be considered as generalized Hyperbolic functions(see Appendix B) and these functions can be represented in terms of exponential e^x and trigonometric functions,

$$
\begin{aligned}
(mod\, 3) \quad {}_0e^x &\equiv \frac{1}{3}\left(e^x + 2e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right)\right), \qquad (4.62)\\
(mod\, 3) \quad {}_1e^x &\equiv \frac{1}{3}\left(e^x + 2e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x - \frac{2\pi}{3}\right)\right), \qquad (4.63)\\
(mod\, 3) \quad {}_2e^x &\equiv \frac{1}{3}\left(e^x + 2e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x + \frac{2\pi}{3}\right)\right). \qquad (4.64)
\end{aligned}
$$

Due to equations in (4.53) – (4.55), these funtions admit mod 3 symmetry but in complex domain:

$$ {}_0e^{q^2x} = {}_0e^x \quad (mod\, 3)\,, \tag{4.65} $$
$$ {}_1e^{q^2x} = q^2\, {}_1e^x \quad (mod\, 3)\,, \tag{4.66} $$
$$ {}_2e^{q^2x} = q^4\, {}_2e^x \quad (mod\, 3)\,. \tag{4.67} $$

In (Fig 4.1), we plot these functions and they are not showing even or odd symmetry.

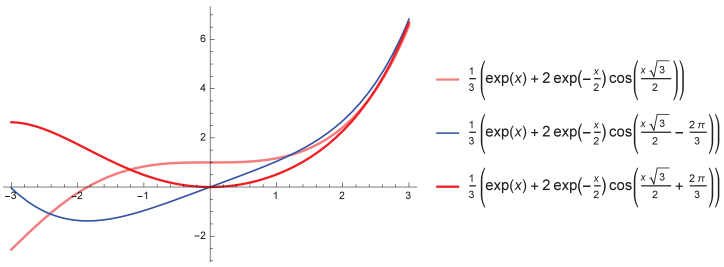

Figure 4.1. Mod 3 Exponential functions

4.5. Euler Type Formula for Mod 3 Exponential Functions

Replacing $x \to ix$ in equations (4.62) – (4.64) we get analogue of Euler formula for mod 3 exponential functions

Proposition 4.5 *Analogue of Euler formula for mod* 3 *exponential functions is defined as*

$$ (mod\, 3)\, {}_0e^{ix} = \frac{1}{3}\left[\cos(x) + 2\cosh\left(\frac{\sqrt{3}}{2}x\right)\cos\left(\frac{x}{2}\right)\right] + \frac{i}{3}\left[\sin(x) - 2\cosh\left(\frac{\sqrt{3}}{2}x\right)\sin\left(\frac{x}{2}\right)\right], $$

$$ \begin{aligned} (mod\, 3)\, {}_1e^{ix} &= \frac{1}{3}\left[\cos(x) - \cos\left(\frac{x}{2}\right)\cosh\left(\frac{\sqrt{3}}{2}x\right) + \sqrt{3}\sin\left(\frac{x}{2}\right)\sinh\left(\frac{\sqrt{3}}{2}x\right)\right] \\ &+ \frac{i}{3}\left[\sin(x) + \sin\left(\frac{x}{2}\right)\cosh\left(\frac{\sqrt{3}}{2}x\right) + \sqrt{3}\cos\left(\frac{x}{2}\right)\sinh\left(\frac{\sqrt{3}}{2}x\right)\right] \\ (mod\, 3)\, {}_2e^{ix} &= \frac{1}{3}\left[\cos(x) - \cos\left(\frac{x}{2}\right)\cosh\left(\frac{\sqrt{3}}{2}x\right) - \sqrt{3}\sin\left(\frac{x}{2}\right)\sinh\left(\frac{\sqrt{3}}{2}x\right)\right] \\ &+ \frac{i}{3}\left[\sin(x) + \sin\left(\frac{x}{2}\right)\cosh\left(\frac{\sqrt{3}}{2}x\right) - \sqrt{3}\cos\left(\frac{x}{2}\right)\sinh\left(\frac{\sqrt{3}}{2}x\right)\right]. \end{aligned} $$

4.6. Mod 4 Functions

Here, we study the case of mod 4 functions, associated with $q^8 = 1$.

Definition 4.5 *Mod* 4 *functions are defined as*

$$f_0(x) \equiv \frac{f(x) + f(q^2 x) + f(q^4 x) + f(q^6 x)}{4}, \tag{4.68}$$

$$f_1(x) \equiv \frac{f(x) + \overline{q}^{\,2} f(q^2 x) + \overline{q}^{\,4} f(q^4 x) + \overline{q}^{\,6} f(q^6 x)}{4}, \tag{4.69}$$

$$f_2(x) \equiv \frac{f(x) + \overline{q}^{\,4} f(q^2 x) + \overline{q}^{\,8} f(q^4 x) + \overline{q}^{\,4} f(q^6 x)}{4}, \tag{4.70}$$

$$f_3(x) \equiv \frac{f(x) + \overline{q}^{\,6} f(q^2 x) + \overline{q}^{\,4} f(q^4 x) + \overline{q}^{\,2} f(q^6 x)}{4}. \tag{4.71}$$

Proposition 4.6 *Arbitrary function $f(x)$ can be written as superposition of mod* 4 *functions,*

$$f(x) = f_0(x) + f_1(x) + f_2(x) + f_3(x).$$

Proof By addition

$$\begin{aligned} f_0(x) + f_1(x) + f_2(x) + f_3(x) &= \frac{f(x) + f(q^2 x) + f(q^4 x) + f(q^6 x)}{4} \\ &+ \frac{f(x) + \overline{q}^{\,2} f(q^2 x) + \overline{q}^{\,4} f(q^4 x) + \overline{q}^{\,6} f(q^6 x)}{4} \\ &+ \frac{f(x) + \overline{q}^{\,4} f(q^2 x) + \overline{q}^{\,8} f(q^4 x) + \overline{q}^{\,4} f(q^6 x)}{4} \\ &+ \frac{f(x) + \overline{q}^{\,6} f(q^2 x) + \overline{q}^{\,4} f(q^4 x) + \overline{q}^{\,2} f(q^6 x)}{4} \\ &= \frac{1}{4}\Big(4f(x) + \left(1 + \overline{q}^{\,2} + \overline{q}^{\,4} + \overline{q}^{\,6}\right) f(q^2 x) \\ &+ \left(1 + \overline{q}^{\,4} + \overline{q}^{\,8} + \overline{q}^{\,4}\right) f(q^4 x) \\ &+ \left(1 + \overline{q}^{\,6} + \overline{q}^{\,4} + \overline{q}^{\,2}\right) f(q^6 x)\Big) \end{aligned} \tag{4.72}$$

Since $q^8 = (e^{i\frac{\pi}{2}})^4 = 1$, then we have

$$1 + \overline{q}^{\,2n} + \overline{q}^{\,4n} + \overline{q}^{\,6n} = 4\delta_{n,0(mod4)}. \tag{4.73}$$

As a result,

$$f(x) = f_0(x) + f_1(x) + f_2(x) + f_3(x). \tag{4.74}$$

□

We notice that structure of functions $f_0(x)$, $f_1(x)$, $f_2(x)$ and $f_3(x)$ is given by superposition of function f(x) at four points of argument x, q^2x, q^4x and q^6x so that sum of coefficients for $f(q^2x), f(q^4x)$ and $f(q^6x)$ is zero due to equation (4.73).

4.6.1. Mod 4 Exponential Functions

For the standard exponential function, we have expansion

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = \sum_{k=0}^{\infty} \frac{x^{4k}}{4k!} + \sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!} + \sum_{k=0}^{\infty} \frac{x^{4k+2}}{(4k+2)!} + \sum_{k=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!} \tag{4.75}$$

Definition 4.6 *$mod\,4$ exponential functions ${}_0e^x$, ${}_1e^x$, ${}_2e^x$ and ${}_3e^x$ are defined by power series*

$$ {}_0e^x \equiv \frac{e^x + e^{q^2x} + e^{q^4x} + e^{q^6x}}{4} = \sum_{k=0}^{\infty} \frac{x^{4k}}{4k!}, \tag{4.76}$$

$$ {}_1e^x \equiv \frac{e^x + \overline{q}^2 e^{q^2x} + \overline{q}^4 e^{q^4x} + \overline{q}^6 e^{q^6x}}{4} = \sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!}, \tag{4.77}$$

$$ {}_2e^x \equiv \frac{e^x + \overline{q}^4 e^{q^2x} + \overline{q}^8 e^{q^2x} e^{q^4x} + \overline{q}^4 e^{q^6x}}{4} = \sum_{k=0}^{\infty} \frac{x^{4k+2}}{(4k+2)!}, \tag{4.78}$$

$$ {}_3e^x \equiv \frac{e^x + \overline{q}^6 e^{q^2x} + \overline{q}^4 e^{q^4x} + \overline{q}^2 e^{q^6x}}{4} = \sum_{k=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!}. \tag{4.79}$$

The indices $0, 1, 2$ and 3 are related with remainer in mod 4. From this definition, the standard exponential function can be written as superposition of four functions:

$$e^x = {}_0e^x + {}_1e^x + {}_2e^x + {}_3e^x \quad (mod\,4)\,. \tag{4.80}$$

We can combine them in matrix form

$$\begin{bmatrix} {}_0e^x \\ {}_1e^x \\ {}_2e^x \\ {}_3e^x \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \overline{q}^2 & \overline{q}^4 & \overline{q}^6 \\ 1 & \overline{q}^4 & \overline{q}^8 & \overline{q}^4 \\ 1 & \overline{q}^6 & \overline{q}^4 & \overline{q}^2 \end{bmatrix}\begin{bmatrix} e^x \\ e^{q^2x} \\ e^{q^4x} \\ e^{q^6x} \end{bmatrix} = \frac{\hat{H}}{\sqrt{4}}\begin{bmatrix} e^x \\ e^{q^2x} \\ e^{q^4x} \\ e^{q^6x} \end{bmatrix} \tag{4.81}$$

as mod 4 discrete Fourier transformation, with unitary Hadamard gate matrix

$$\hat{H} = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \overline{q}^2 & \overline{q}^4 & \overline{q}^6 \\ 1 & \overline{q}^4 & \overline{q}^8 & \overline{q}^4 \\ 1 & \overline{q}^6 & \overline{q}^4 & \overline{q}^2 \end{bmatrix}. \tag{4.82}$$

Mod 4 exponential functions in definition (4.6) are superpositions of hyperbolic and trigonometric functions,

$$(mod\,4) \quad {}_0e^x \equiv \frac{1}{2}(\cosh x + \cos x)\,, \tag{4.83}$$

$$(mod\,4) \quad {}_1e^x \equiv \frac{1}{2}(\sinh x + \sin x)\,, \tag{4.84}$$

$$(mod\,4) \quad {}_2e^x \equiv \frac{1}{2}(\cosh x - \cos x)\,, \tag{4.85}$$

$$(mod\,4) \quad {}_3e^x \equiv \frac{1}{2}(\sinh x - \sin x)\,. \tag{4.86}$$

It is easy to see that ${}_0e^x$, ${}_2e^x$ are even functions and ${}_1e^x$, ${}_3e^x$ are odd functions such that

$${}_0e^{-x} = {}_0e^x \quad , \quad {}_2e^{-x} = {}_2e^x\,,, \tag{4.87}$$

$${}_1e^{-x} = -\,{}_1e^x \quad , \quad {}_3e^{-x} = -\,{}_3e^x\,. \tag{4.88}$$

We plot these exponential functions in Figure (4.2) according to variable x in real domain.

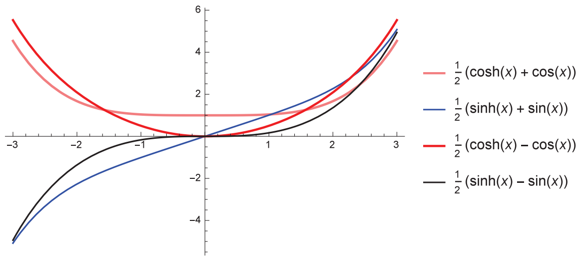

Figure 4.2. Mod 4 Exponential functions

4.7. Euler Type Formula for Mod 4 Exponential Functions

Replacing $x \to ix$ in equations (4.83) – (4.86) we get analogue of Euler formula for mod 4 exponential functions

Proposition 4.7 *Analogue of Euler formula for mod* 3 *exponential functions is defined as*

$$(mod\, 4) \quad {}_0e^{ix} \equiv \frac{1}{2}(\cosh x + \cos x) = {}_1e^{x}, \tag{4.89}$$

$$(mod\, 4) \quad {}_1e^{ix} \equiv i\frac{1}{2}(\sinh x + \sin x) = i\,{}_1e^{x} = q^{\,2}\,{}_1e^{x}, \tag{4.90}$$

$$(mod\, 4) \quad {}_2e^{ix} \equiv -\frac{1}{2}(\cosh x - \cos x) = -\,{}_2e^{x} = q^{\,4}\,{}_2e^{x}, \tag{4.91}$$

$$(mod\, 4) \quad {}_3e^{ix} \equiv -i\frac{1}{2}(\sinh x - \sin x) == -i\,{}_3e^{x} = q^{\,6}\,{}_3e^{x}. \tag{4.92}$$

4.8. Mod n Functions

Previous considerations can be generalized to arbitrary mod n functions. These functions would appear in the study of normalization constants and calculations of average number of photons in kaleidoscope of quantum Coherent States. For $q^{2n} = 1$ root of unity, we consider n values of argument $x, q^2x, q^4x, ..., q^{2(n-1)}x$. Every mod n function is superposition of functions with these arguments such that addition of coefficients for $f(q^2x), f(q^4x), ..., f(q^{2(n-1)}x)$ are equal to zero, due to

$$1 + \bar{q}^{\,2k} + \bar{q}^{\,4k} + ... + \bar{q}^{\,2(n-1)k} = 0\,, 1 \leq k \leq n-1\,. \tag{4.93}$$

Definition 4.7 *Mod n functions are defined by relation*

$$\begin{bmatrix} f_0(x) \\ f_1(x) \\ f_2(x) \\ \vdots \\ f_{n-1}(x) \end{bmatrix} = \frac{1}{n} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \overline{q}^{\,2} & \overline{q}^{\,4} & \dots & \overline{q}^{\,2(n-1)} \\ 1 & \overline{q}^{\,4} & \overline{q}^{\,8} & \dots & \overline{q}^{\,4(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \overline{q}^{\,2(n-1)} & \overline{q}^{\,4(n-1)} & \dots & \overline{q}^{\,2(n-1)^2} \end{bmatrix} \begin{bmatrix} f(x) \\ f(q^2x) \\ f(q^4x) \\ \vdots \\ f(q^{2(n-1)}x) \end{bmatrix}, \tag{4.94}$$

where transformation matrix is discrete Fourier transformation, related with unitary Hadamard gate matrix $\hat{H}$,

$$\hat{H} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \overline{q}^{\,2} & \overline{q}^{\,4} & \dots & \overline{q}^{\,2(n-1)} \\ 1 & \overline{q}^{\,4} & \overline{q}^{\,8} & \dots & \overline{q}^{\,4(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \overline{q}^{\,2(n-1)} & \overline{q}^{\,4(n-1)} & \dots & \overline{q}^{\,2(n-1)^2} \end{bmatrix}, \tag{4.95}$$

$$\textit{such that} \quad \begin{bmatrix} f_0(x) \\ f_1(x) \\ f_2(x) \\ \vdots \\ f_{n-1}(x) \end{bmatrix} = \frac{1}{\sqrt{n}} \hat{H} \begin{bmatrix} f(x) \\ f(q^2x) \\ f(q^4x) \\ \vdots \\ f(q^{2(n-1)}x) \end{bmatrix}. \tag{4.96}$$

Proposition 4.8 *Arbitrary function $f(x)$ can be written as a superposition of n functions.*

$$f(x) = \sum_{k=0}^{n-1} f_k(x)\ , \tag{4.97}$$

where $f_k(x) = \dfrac{1}{n} \sum_{s=0}^{n-1} \overline{q}^{2sk} f(q^{2s})\ , \quad k = 0, 1, 2, ..., n-1.$

Proof The proof follows easily if we add these functions

$$\sum_{k=0}^{n-1} f_k(x) = f_0(x) + f_1(x) + f_2(x) + \cdots + f_{n-1}(x)\,. \tag{4.98}$$

Combining terms with the same arguments

$$\left(1+\overline{q}^{\,2}+\overline{q}^{\,4}+\ldots+\overline{q}^{\,2(n-1)}\right)f(q^2x)=0$$
$$\left(1+\overline{q}^{\,4}+\overline{q}^{\,8}+\ldots+\overline{q}^{\,4(n-1)}\right)f(q^4x)=0$$
$$\left(1+\overline{q}^{\,6}+\overline{q}^{\,12}+\ldots+\overline{q}^{\,6(n-1)}\right)f(q^6x)=0$$
$$\vdots$$
$$\left(1+\overline{q}^{\,2(n-1)}+\overline{q}^{\,4(n-1)}+\ldots+\overline{q}^{\,2(n-1)^2}\right)f(q^{2(n-1)}x)=0 \tag{4.99}$$

shows that all terms vanish except one with argument x and the term with this argument gives $\frac{1}{n}nf(x)=f(x)$. □

4.8.1. Mod n Exponential Functions

Instead of arbitray function, we can expand the standard exponential in the following form;

$$e^x=\sum_{m=0}^{\infty}\frac{x^m}{m!}=\sum_{k=0}^{\infty}\frac{x^{nk}}{nk!}+\sum_{k=0}^{\infty}\frac{x^{nk+1}}{(nk+1)!}+\sum_{k=0}^{\infty}\frac{x^{nk+2}}{(nk+2)!}+\ldots+\sum_{k=0}^{\infty}\frac{x^{nk+(n-1)}}{(nk+(n-1))!}.$$

Each function in this addition represents mod n exponential function with following definition;

Definition 4.8 *mod n exponential functions ${}_se^x(mod\,n)$, where $0\le s\le n-1$ as a remainder in mod n, are defined by*

$${}_se^x(mod\,n)=f_s(x)=\sum_{k=0}^{\infty}\frac{x^{nk+s}}{(nk+s)!}, \tag{4.100}$$

and they can be expressed as superposition of standard exponentials;

$${}_se^x(mod\,n)=\frac{1}{n}\sum_{k=0}^{n-1}\overline{q}^{2sk}e^{q^{2k}x}\qquad ,0\le s\le n-1\,. \tag{4.101}$$

These functions can be combined as discrete Fourier transformation,

$$\begin{bmatrix} {}_0e^x \\ {}_1e^x \\ {}_2e^x \\ {}_3e^x \\ \vdots \\ {}_{n-1}e^x \end{bmatrix} = \frac{1}{n} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \overline{q}^{\,2} & \overline{q}^{\,4} & \dots & \overline{q}^{\,2(n-1)} \\ 1 & \overline{q}^{\,4} & \overline{q}^{\,8} & \dots & \overline{q}^{\,4(n-1)} \\ 1 & \overline{q}^{\,6} & \overline{q}^{\,12} & \dots & \overline{q}^{\,6(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \overline{q}^{\,2(n-1)} & \overline{q}^{\,4(n-1)} & \dots & \overline{q}^{\,2(n-1)^2} \end{bmatrix} \begin{bmatrix} e^x \\ e^{q^2 x} \\ e^{q^4 x} \\ e^{q^6 x} \\ \vdots \\ e^{q^2(n-1)x} \end{bmatrix} . \tag{4.102}$$

4.9. Differential Equation and Initial Value Problem for Mod n Exponential Functions

The standard exponential e^x is a solution of first order differential equation

$$\frac{d}{dx} f(x) = f(x) , \tag{4.103}$$

with inital value $f(0) = 1$. Applying derivative operator $n-$times shows that e^x is also a solution of $n-$th order equation

$$\frac{d^n}{dx^n} f(x) = f(x) , \tag{4.104}$$

with initial value $f(0) = 1$ and $f'(0) = f''(0) = \dots = f^{n-1}(0) = 0$. In order to find other $n-1$ solutions of the last equation, we write $f(x) = e^{\lambda x}$ and get $\lambda^n - 1 = 0$, which implies that every root of unity $\lambda^n = 1$ determines particular solution of equation (4.104). If we parametrize non-trivial roots as

$$\lambda_0 = 1 , \lambda_1 = q^2 , \lambda_2 = q^4 , \dots , \lambda_{n-1} = q^{2(n-1)} , \tag{4.105}$$

where $q^{2n} = 1$. Then we get the set of solutions

$$e^x , e^{q^2 x} , e^{q^4 x} , \dots , e^{q^{2(n-1)} x} , \tag{4.106}$$

and the general solution of (4.104) is linear combination

$$f(x) = c_0 e^x + c_1 e^{q^2 x} + c_2 e^{q^4 x} + ... + c_{n-1} e^{q^{2(n-1)} x} . \tag{4.107}$$

The set of mod n exponential functions appears by choosing constants $c_0, c_1, c_2, ..., c_{n-1}$ in a proper way.

4.9.1. Mod 2 Exponential Functions

Proposition 4.9 *mod 2 exponential functions ${}_0e^x, {}_1e^x$ (mod 2) are solutions of second order differential equation*

$$\frac{d^2}{dx^2} f(x) = f(x), \quad where \quad f(x) = {}_0e^x,\ {}_1e^x \ (mod\, 2) \tag{4.108}$$

with initial values: $f(0) = 1\,, f'(0) = 0$ for ${}_0e^x$ and $f(0) = 0\,, f'(0) = 1$ for ${}_1e^x$.

These two functions are related by derivative

$$\frac{d}{dx}\, {}_0e^x = \frac{d}{dx}\left(\frac{e^x + e^{q^2 x}}{2}\right) = {}_1e^x \tag{4.109}$$

$$\frac{d}{dx}\, {}_1e^x = \frac{d}{dx}\left(\frac{e^x + \overline{q}^{\,2} e^{q^2 x}}{2}\right) = {}_0e^x \tag{4.110}$$

4.9.2. Mod 3 Exponential Functions

Proposition 4.10 *mod 3 exponential functions ${}_0e^x$, ${}_1e^x$ and ${}_2e^x$ are solutions of third order differential equation*

$$\frac{d^3}{dx^3}\, {}_se^x = {}_se^x \ (mod\, 3)\,, \tag{4.111}$$

with initial values,

$$\frac{d^k f_s}{dx^k}(0) = \begin{cases} 1, & s = k \ ; \\ 0, & s \neq k \ , \end{cases} \tag{4.112}$$

where $f_s(x) = \ _se^x \ (mod\, 3)$, $0 \leq s, k \leq 2$.

Proposition 4.11 *Derivative relation between mod* 3 *exponential functions is following*

$$\frac{d}{dx}\,{}_0e^x = {}_2e^x, \qquad \frac{d}{dx}\,{}_2e^x = {}_1e^x, \qquad \frac{d}{dx}\,{}_1e^x = {}_0e^x . \tag{4.113}$$

Proof We will prove by using definitions in (4.4) and $q^6 = 1$,

$$\begin{aligned}
\frac{d}{dx}\,{}_0e^x &= \frac{d}{dx}\left(\frac{e^x + e^{q^2 x} + e^{q^4 x}}{3}\right) = \frac{1}{3}\left(e^x + \overline{q}^4 e^{q^2 x} + \overline{q}^2 e^{q^4 x}\right) = {}_2e^x ,\\
\frac{d}{dx}\,{}_2e^x &= \frac{d}{dx}\left(\frac{e^x + \overline{q}^4 e^{q^2 x} + \overline{q}^2 e^{q^4 x}}{3}\right) = \frac{1}{3}\left(e^x + \overline{q}^2 e^{q^2 x} + \overline{q}^4 e^{q^4 x}\right) = {}_1e^x ,\\
\frac{d}{dx}\,{}_1e^x &= \frac{d}{dx}\left(\frac{e^x + \overline{q}^2 e^{q^2 x} + \overline{q}^4 e^{q^4 x}}{3}\right) = \frac{1}{3}\left(e^x + e^{q^2 x} + e^{q^4 x}\right) = {}_0e^x .
\end{aligned}$$

□

4.9.3. Mod 4 Exponential Functions

Proposition 4.12 *mod* 4 *exponential functions* ${}_0e^x$, ${}_1e^x$, ${}_2e^x$ *and* ${}_3e^x$ *are solution of fourth order differential equation*

$$\frac{d^4}{dx^4}\,{}_se^x = {}_se^x \ (mod\, 4) , \tag{4.114}$$

with initial values,

$$\frac{d^k f_s}{dx^k}(0) = \begin{cases} 1, & s = k \ ; \\ 0, & s \neq k \ , \end{cases} \tag{4.115}$$

where $f_s(x) = \ _se^x \ (mod\, 4)$, $0 \leq s, k \leq 3$.

Proposition 4.13 *Derivative relation of mod 4 exponential functions is following*

$$\frac{d}{dx}\,{}_0e^x = {}_3e^x, \qquad \frac{d}{dx}\,{}_3e^x = {}_2e^x, \qquad \frac{d}{dx}\,{}_2e^x = {}_1e^x, \qquad \frac{d}{dx}\,{}_1e^x = {}_0e^x. \qquad (4.116)$$

Proof We prove it by using definition (4.6) and $q^6 = 1$,

$$\begin{aligned}
\frac{d}{dx}\,{}_0e^x &= \frac{d}{dx}\left(\frac{e^x + e^{q^2x} + e^{q^4x} + e^{q^6x}}{4}\right) = \frac{1}{4}\left(e^x + \overline{q}^{\,6}e^{q^2x} + \overline{q}^{\,4}e^{q^4x} + \overline{q}^{\,2}e^{q^6x}\right) = {}_3e^x\,,\\
\frac{d}{dx}\,{}_3e^x &= \frac{d}{dx}\left(\frac{e^x + \overline{q}^{\,6}e^{q^2x} + \overline{q}^{\,4}e^{q^4x} + \overline{q}^{\,2}e^{q^6x}}{4}\right) = \frac{1}{4}\left(e^x + \overline{q}^{\,4}e^{q^2x} + \overline{q}^{\,8}e^{q^4x} + \overline{q}^{\,4}e^{q^6x}\right) = {}_2e^x\,,\\
\frac{d}{dx}\,{}_2e^x &= \frac{d}{dx}\left(\frac{e^x + \overline{q}^{\,4}e^{q^2x} + \overline{q}^{\,8}e^{q^4x} + \overline{q}^{\,4}e^{q^6x}}{4}\right) = \frac{1}{4}\left(e^x + \overline{q}^{\,2}e^{q^2x} + \overline{q}^{\,4}e^{q^4x} + \overline{q}^{\,6}e^{q^6x}\right) = {}_1e^x\,,\\
\frac{d}{dx}\,{}_1e^x &= \frac{d}{dx}\left(\frac{e^x + \overline{q}^{\,2}e^{q^2x} + \overline{q}^{\,4}e^{q^4x} + \overline{q}^{\,6}e^{q^6x}}{4}\right) = \frac{1}{4}\left(e^x + e^{q^2x} + e^{q^4x} + e^{q^6x}\right) = {}_0e^x\,,
\end{aligned}$$

□

4.9.4. Mod n Exponential Functions

Proposition 4.14 *mod n exponential functions ${}_se^x$ satisfy nth order differential equation*

$$\frac{d^n}{dx^n} f_s = f_s\,, \qquad (4.117)$$

where $f_s = {}_se^x$, $\quad 0 \le s \le n-1$, *with initial values:* $f_s^{(s)}(0) = 1$ *and* $f_s(0) = f_s'(0) = \ldots = f_s^{(s-1)}(0) = f_s^{(s+1)}(0) = \ldots = f_s^{(n-1)}(0) = 0.$

Proposition 4.15 *mod n exponential functions have following derivative relations*

$$\frac{d}{dx}\,{}_se^x = {}_{s-1}e^x \quad , \quad \frac{d}{dx}\,{}_0e^x = {}_{n-1}e^x, \quad 1 \le s \le n-1. \qquad (4.118)$$

Proof In order to prove the first relation, we use expansion (4.100),

$$\frac{d}{dx}\,{}_se^x = \frac{d}{dx}\left(\sum_{k=0}^{\infty}\frac{x^{nk+s}}{(nk+s)!}\right) = \sum_{k=0}^{\infty}\frac{d}{dx}\left(\frac{x^{nk+s}}{(nk+s)!}\right) \qquad (4.119)$$

$$= \sum_{k=0}^{\infty}\frac{(nk+s)x^{nk+s-1}}{(nk+s)!} \qquad (4.120)$$

$$= \sum_{k=0}^{\infty}\frac{x^{nk+(s-1)}}{(nk+(s-1))!} = {}_{s-1}e^x\,. \qquad (4.121)$$

For the second relation, setting s=0 in (4.100)

$$\frac{d}{dx}\,{}_0e^x = \frac{d}{dx}\left(\sum_{k=0}^{\infty}\frac{x^{nk}}{(nk)!}\right) = \sum_{k=0}^{\infty}\frac{d}{dx}\left(\frac{x^{nk}}{(nk)!}\right) = \sum_{k=0}^{\infty}\frac{(nk)x^{nk-1}}{(nk)!} = \sum_{k=1}^{\infty}\frac{x^{nk-1}}{(nk-1)!}\,, \quad (4.122)$$

and changing index gives us

$$\sum_{k=1}^{\infty}\frac{x^{nk-1}}{(nk-1)!} = \sum_{k-1=0}^{\infty}\frac{x^{n(k-1)+n-1}}{(n(k-1)+n-1)!} = \sum_{m=0}^{\infty}\frac{x^{nm+n-1}}{(nm+n-1)!} = {}_{n-1}e^x\,. \quad (4.123)$$

□

4.10. Application of Mod n Exponential Functions

Mod n exponential functions have several applications in quantum theory.

4.10.1. Fock-Bargmann Representation

In Fock-Bargmann representation, every quantum state corresponds to entire analytic function and vice versa. If

$$|\psi\rangle = \sum_{n=0}^{\infty} C_n|n\rangle, \quad \langle\psi|\psi\rangle = 1\,, \quad (4.124)$$

, where $\sum_{n=0}^{\infty}|C_n|^2 = 1.$ is an arbitrary vector in Fock space, then the wave function in coherent state basis $|\alpha\rangle$ is

$$\langle\alpha|\psi\rangle = \sum_{n=0}^{\infty} C_n\langle\alpha|n\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n,m=0}^{\infty}\frac{(\overline{\alpha})^n}{\sqrt{n!}}\underbrace{\langle m|n\rangle}_{\delta_{n,m}} = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}C_n\frac{(\overline{\alpha})^n}{\sqrt{n!}} \quad (4.125)$$

or

$$\langle\alpha|\psi\rangle = e^{-\frac{1}{2}|\alpha|^2}\psi(\overline{\alpha})\,, \quad (4.126)$$

where

$$\psi(z) = \sum_{n=0}^{\infty} C_n U_n(z) \quad , \quad U_n(z) = \frac{z^n}{\sqrt{n!}} \tag{4.127}$$

is entire function of argument z. In this representation, the annihilation operator $\hat{a}$ is represented as a complex derivative and the creation operator $\hat{a}^\dagger$ is defined as a multiplication operator

$$\hat{a} \leftrightarrow \frac{d}{dz} \quad , \quad \hat{a}^\dagger \leftrightarrow z \tag{4.128}$$

so that

$$\left[\hat{a}, \hat{a}^\dagger\right] = \hat{\mathrm{I}} \quad \leftrightarrow \quad \left[\frac{d}{dz}, z\right] = 1 \tag{4.129}$$

and state $|\alpha\rangle$ is represented by $e^{\alpha z}$. From this, we have following relation for coherent states

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \quad \leftrightarrow \quad \frac{d}{dz} e^{\alpha z} = \alpha e^{\alpha z} \,. \tag{4.130}$$

Then, the eigenvalue problem $\hat{a}^n|\alpha\rangle = \alpha^n|\alpha\rangle$ for operator $\hat{a}^n$ in Fock−Bargmann representation becomes

$$\frac{d^n}{dz^n} e^{\alpha z} = \alpha^n e^{\alpha z} \,. \tag{4.131}$$

This n-th order ordinary differential equation admits n independent solutions

$$e^{\alpha z}\,, e^{q^2 \alpha z}\,, e^{q^4 \alpha z}\,, \ldots\,, e^{q^{2(n-1)} \alpha z}\,, \tag{4.132}$$

corresponding to rotated coherent states, respectively

$$|\alpha\rangle\,,|q^2\alpha\rangle\,,|q^4\alpha\rangle\,,\ldots,|q^{2(n-1)}\alpha\rangle\,. \tag{4.133}$$

Then, mod n exponential functions

$${}_0e^{\alpha z}\,,\;{}_1e^{\alpha z}\,,\;{}_2e^{\alpha z}\,,\ldots,\;{}_{n-1}e^{\alpha z}\,, \tag{4.134}$$

are related to the superposition of rotated coherent states in the form of kaleidoscope, associated with roots of unity $q^{2n} = 1$(see Chapter 8).

4.10.2. Kaleidoscope of Quantum Coherent States

In Chapter 8, we derive kaleidoscope of coherent states $|k\rangle_\alpha$ by using mod n exponential functions of operator argument

$$|k\rangle_\alpha = \frac{{}_ke^{\alpha\hat{a}^\dagger}}{\sqrt{{}_ke^{|\alpha|^2}}}|0\rangle \quad (mod\,n), \quad k = 0,1,2,...,n-1. \tag{4.135}$$

Furthermore, calculation of average number of photons in these states is given by ratio of two consecutive mod n exponential functions,

$${}_\alpha\langle k|\widehat{N}|k\rangle_\alpha = |\alpha|^2\left[\frac{{}_{k-1}e^{|\alpha|^2}}{{}_ke^{|\alpha|^2}}\right] \quad , \quad {}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = |\alpha|^2\left[\frac{{}_{n-1}e^{|\alpha|^2}}{{}_0e^{|\alpha|^2}}\right].$$

4.11. Mod n Form of Displacement Operator

The coherent states in (3.14) are generated by the displacement operator $D(\alpha)$ as exponential function of operator argument. For kaleidoscope of coherent states, the displacement operator can be represented as mod n operator valued exponential functions.

For coherent states $|\mp\alpha\rangle$;

$$|\mp\alpha\rangle = D(\mp\alpha)|0\rangle\,, \tag{4.136}$$

the displacement operators $D(\mp\alpha)$ are

$$D(\mp\alpha) = e^{\mp\alpha\hat{a}^{\dagger}\pm\bar{\alpha}\hat{a}} = e^{-\frac{1}{2}|\alpha|^2}e^{\mp\alpha\hat{a}^{\dagger}}e^{\pm\bar{\alpha}\hat{a}}\,. \tag{4.137}$$

Then, superpositions of these states

$$|\tilde{0}\rangle_\alpha = \frac{|\alpha\rangle + |-\alpha\rangle}{2} = \left(\frac{D(\alpha)+D(-\alpha)}{2}\right)|0\rangle = {}_0D(\alpha)|0\rangle\,, \tag{4.138}$$

$$|\tilde{1}\rangle_\alpha = \frac{|\alpha\rangle - |-\alpha\rangle}{2} = \left(\frac{D(\alpha)-D(-\alpha)}{2}\right)|0\rangle = {}_1D(\alpha)|0\rangle\,, \tag{4.139}$$

which are the Schrödinger cat states, is generated by mod 2 displacement operators ${}_0D(\alpha)$ and ${}_1D(\alpha)$. These operators can be written as

$${}_0D(\alpha) = e^{-\frac{1}{2}|\alpha|^2}(\cosh\alpha\hat{a}^{\dagger}\cosh\alpha\hat{a} - \sinh\alpha\hat{a}^{\dagger}\sinh\alpha\hat{a})\,, \tag{4.140}$$

$${}_1D(\alpha) = e^{-\frac{1}{2}|\alpha|^2}(\sinh\alpha\hat{a}^{\dagger}\cosh\alpha\hat{a} + \cosh\alpha\hat{a}^{\dagger}\sinh\alpha\hat{a})\,. \tag{4.141}$$

Then, the Schrödinger cat states become

$$|\tilde{0}\rangle_\alpha = {}_0D(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2}\cosh\alpha\hat{a}^{\dagger}|0\rangle\,, \tag{4.142}$$

$$|\tilde{1}\rangle_\alpha = {}_1D(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2}\sinh\alpha\hat{a}^{\dagger}|0\rangle\,. \tag{4.143}$$

For trinity states, we define displacement operators as mod 3 operator valued functions;

$${}_0D(\alpha) = \frac{D(\alpha)+D(q^2\alpha)+D(q^4\alpha)}{3}\,, \tag{4.144}$$

$${}_1D(\alpha) = \frac{D(\alpha)+\bar{q}^2D(q^2\alpha)+\bar{q}^4D(q^4\alpha)}{3}\,, \tag{4.145}$$

$${}_2D(\alpha) = \frac{D(\alpha)+\bar{q}^4D(q^2\alpha)+\bar{q}^2D(q^4\alpha)}{3}\,. \tag{4.146}$$

and corresponding trinity states are

$$
\begin{aligned}
|\tilde{0}\rangle_\alpha &= {}_0D(\alpha)|0\rangle\,, && (4.147)\\
|\tilde{1}\rangle_\alpha &= {}_1D(\alpha)|0\rangle\,, && (4.148)\\
|\tilde{2}\rangle_\alpha &= {}_2D(\alpha)|0\rangle\,. && (4.149)
\end{aligned}
$$

Generalization to the kaleidoscope states $|k\rangle_\alpha$ for $q^{2n} = 1$, is generated by mod n displacement operator

$$
{}_kD(\alpha) = \frac{1}{n}\sum_{j=0}^{n-1}\overline{q}^{2jk}D(q^{2j}\alpha)|0\rangle\ , 0 \leq k \leq n-1\,. \tag{4.150}
$$

Acting to vacuum states, it produces $|\tilde{k}\rangle_\alpha$ state as

$$
|\tilde{k}\rangle_\alpha = {}_kD(\alpha)|0\rangle\,. \tag{4.151}
$$

We note that the above states are not normalized this why notation is $\tilde{k}$.

4.12. Generating Function for Mod *n* Hermite Polynomials

For standard hermite polynomials, we have generating function (3.59). In coordinate representation of Kaleidoscope states, we have mod *n* Hermite polynomials.

4.12.1. Mod 2 Hermite Polynomials

Coordinate representation of Cat states requires generating function for $H_{2k}(x)$ and $H_{2k+1}(x)$ as following;

$$
\begin{aligned}
\sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!}H_{2k}(x) &= e^{-z^2}\cosh(2zx)\,, && (4.152)\\
\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!}H_{2k+1}(x) &= e^{-z^2}\sinh(2zx)\,. && (4.153)
\end{aligned}
$$

4.12.2. Mod 3 Hermite Polynomials

Coordinate representation of trinity states related with mod 3 exponential functions which are generating function for $H_{3k}(x), H_{3k+1}(x)$ and $H_{3k+2}(x)$. Generating functions for mod 3 Hermite polynomials defined as

$$\sum_{k=0}^{\infty} \frac{z^{3k}}{(3k)!} H_{3k}(x) = {}_0e^{-z^2+2zx}, \tag{4.154}$$

$$\sum_{k=0}^{\infty} \frac{z^{3k+1}}{(3k+1)!} H_{3k+1}(x) = {}_1e^{-z^2+2zx} \tag{4.155}$$

$$\sum_{k=0}^{\infty} \frac{z^{3k+2}}{(3k+2)!} H_{3k+2}(x) = {}_2e^{-z^2+2zx} \tag{4.156}$$

CHAPTER 5

SCHRÖDINGER'S CAT STATES

As we have seen, the coherent states are not orthogonal, but we can take superposition of these states to get orthogonal ones. In the description of Schrödinger cat states two orthogonal states are introduced, which are called even and odd cat states in (Dodonov, Malkin and Manko, 1974). These are even and odd superposition of $|\alpha\rangle$ and $|-\alpha\rangle$ states:

$$|cat_e\rangle \equiv |\alpha_+\rangle = N_+\left(|\alpha\rangle + |-\alpha\rangle\right) \qquad |cat_o\rangle \equiv |\alpha_-\rangle = N_-\left(|\alpha\rangle - |-\alpha\rangle\right) . \tag{5.1}$$

The states can be considered as a superposition of two coherent states related by rotation to angle π, which corresponds to primitive root of unity $q^2 = \bar{q}^2 = -1$, so that $q^4 = 1$. It is convenient to use notations $|\alpha_+\rangle \equiv |0\rangle_\alpha$ and $|\alpha_-\rangle \equiv |1\rangle_\alpha$. The normalized states are calculated as following:

$$|0\rangle_\alpha = \frac{e^{\frac{|\alpha|^2}{2}}}{2\sqrt{\cosh|\alpha|^2}}\left(\,|\alpha\rangle + |q^2\alpha\rangle\,\right) \quad , \quad |1\rangle_\alpha = \frac{e^{\frac{|\alpha|^2}{2}}}{2\sqrt{\sinh|\alpha|^2}}\left(\,|\alpha\rangle + \bar{q}^2|q^2\alpha\rangle\,\right). \tag{5.2}$$

By acting with Hadamard gate, these states can be represent in the matrix form:

$$\begin{bmatrix} |0\rangle_\alpha \\ |1\rangle_\alpha \end{bmatrix} = \mathbf{N}\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & \bar{q}^2 \end{bmatrix}}_{Hadamard\ gate}\begin{bmatrix} |\alpha\rangle \\ |q^2\alpha\rangle \end{bmatrix}, \tag{5.3}$$

where

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & \bar{q}^2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{5.4}$$

is Hadamard gate. The normalization matrix

$$\mathbf{N} = \begin{bmatrix} N_0 & 0 \\ 0 & N_1 \end{bmatrix} = \frac{e^{\frac{|\alpha|^2}{2}}}{\sqrt{2}} \begin{bmatrix} {}_0e^{|\alpha|^2} & 0 \\ 0 & {}_1e^{|\alpha|^2} \end{bmatrix}^{-1/2} \quad (mod\, 2) \tag{5.5}$$

is defined by even ($0\, mod\, 2$) and odd ($1\, mod\, 2$) exponential functions (4.2);

$$(mod\, 2) \quad {}_0e^{|\alpha|^2} \equiv \sum_{k=0}^{\infty} \frac{\left(|\alpha|^2\right)^{2k}}{(2k)!} = \frac{e^{|\alpha|^2} + e^{q^2|\alpha|^2}}{2} = \cosh|\alpha|^2, \tag{5.6}$$

$$(mod\, 2) \quad {}_1e^{|\alpha|^2} \equiv \sum_{k=0}^{\infty} \frac{\left(|\alpha|^2\right)^{2k+1}}{(2k+1)!} = \frac{e^{|\alpha|^2} + \bar{q}^2 e^{q^2|\alpha|^2}}{2} = \sinh|\alpha|^2. \tag{5.7}$$

5.1. Mod 2 Cat States

In terms of these exponential functions, we can rewrite the Schrödinger cat states in a compact form(see Appendix D.2):

$$|0\rangle_\alpha = \frac{{}_0e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_0e^{|\alpha|^2}}}|0\rangle \quad (mod\, 2) = \frac{\cosh\alpha\hat{a}^\dagger}{\sqrt{\cosh|\alpha|^2}}|0\rangle\,, \tag{5.8}$$

$$|1\rangle_\alpha = \frac{{}_1e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_1e^{|\alpha|^2}}}|0\rangle \quad (mod\, 2) = \frac{\sinh\alpha\hat{a}^\dagger}{\sqrt{\sinh|\alpha|^2}}|0\rangle. \tag{5.9}$$

This form can be written in the following form ;

$$|0\rangle_\alpha = \frac{1}{\sqrt{{}_0e^{|\alpha|^2}}} \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{\sqrt{2k!}}|2k\rangle \quad (mod\, 2)\,, \tag{5.10}$$

$$|1\rangle_\alpha = \frac{1}{\sqrt{{}_1e^{|\alpha|^2}}} \sum_{k=0}^{\infty} \frac{\alpha^{2k+1}}{\sqrt{(2k+!)!}}|2k+1\rangle \quad (mod\, 2)\,. \tag{5.11}$$

(5.12)

This representation will be useful for our generalization and some calculations.

5.2. Eigenvalue Relation for Cat States

Since $|\alpha\rangle$ is an eigenstate of annihilation operator $\hat{a}$, such that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, it is also the eigenstate of operator $\hat{a}^2$:

$$\hat{a}^2|\alpha\rangle = \alpha^2|\alpha\rangle. \tag{5.13}$$

However, the last equation admits one more eigenstate $|-\alpha\rangle$ with the same eigenvalue α^2, so that

$$\hat{a}^2|\mp\alpha\rangle = \alpha^2|\mp\alpha\rangle. \tag{5.14}$$

Hence, any superposition of states $\{|+\alpha\rangle, |-\alpha\rangle\}$ is also an eigenstate of operator $\hat{a}^2$, with the same eigenvalue.

Proposition 5.1 *Schrödinger cat states are eigenstates of operator* $\hat{a}^2$

$$\hat{a}^2|0\rangle_\alpha = \alpha^2|0\rangle_\alpha, \qquad \hat{a}^2|1\rangle_\alpha = \alpha^2|1\rangle_\alpha, \tag{5.15}$$

constituting orthonormal basis $\{|0\rangle_\alpha, |1\rangle_\alpha\}$.

Proof In order to prove (5.15), first, we check application of annihilation operator to the cat states, respectively:

$$\begin{aligned}
\hat{a}|0\rangle_\alpha = \frac{N_0}{\sqrt{2}}\left(\hat{a}|\alpha\rangle + \hat{a}|q^2\alpha\rangle\right) &= \frac{N_0}{\sqrt{2}}\left(\alpha|\alpha\rangle + q^2\alpha|q^2\alpha\rangle\right) \\
&= \frac{N_0}{\sqrt{2}}\alpha\left(|\alpha\rangle + \bar{q}^2|q^2\alpha\rangle\right) \\
&= \frac{N_0}{\sqrt{2}}\alpha\left(\frac{\sqrt{2}}{N_1}|1\rangle_\alpha\right) \\
&= \alpha\frac{N_0}{N_1}|1\rangle_\alpha\,,
\end{aligned}$$

and

$$\begin{aligned}\hat{a}|1\rangle_\alpha = \frac{N_1}{\sqrt{2}}\left(\hat{a}|\alpha\rangle + \overline{q}^2\hat{a}|q^2\alpha\rangle\right) &= \frac{N_1}{\sqrt{2}}\left(\alpha|\alpha\rangle + \overline{q}^2 q^2\alpha|q^2\alpha\rangle\right)\\ &= \frac{N_1}{\sqrt{2}}\alpha\left(|\alpha\rangle + |q^2\alpha\rangle\right)\\ &= \frac{N_1}{\sqrt{2}}\alpha\left(\frac{\sqrt{2}}{N_0}|0\rangle_\alpha\right)\\ &= \alpha\frac{N_1}{N_0}|0\rangle_\alpha\,.\end{aligned}$$

Thus, annihilation operator $\hat{a}$ gives flipping between cat states $|0\rangle_\alpha$ and $|1\rangle_\alpha$

$$\hat{a}|0\rangle_\alpha = \alpha\frac{N_0}{N_1}|1\rangle_\alpha\,, \quad \hat{a}|1\rangle_\alpha = \alpha\frac{N_1}{N_0}|0\rangle_\alpha\,. \tag{5.16}$$

Now, we will annihilation operator $\hat{a}$ to (5.16):

$$\begin{aligned}\hat{a}^2|0\rangle_\alpha = \alpha\frac{N_0}{N_1}\hat{a}|1\rangle_\alpha &= \alpha\frac{N_0}{N_1}\frac{N_1}{\sqrt{2}}\left(\hat{a}|\alpha\rangle + \overline{q}^2\hat{a}|q^2\alpha\rangle\right)\\ &= \alpha\frac{N_0}{\sqrt{2}}\left(\alpha|\alpha\rangle + \overline{q}^2 q^2\alpha|q^2\alpha\rangle\right)\\ &= \alpha^2\frac{N_0}{\sqrt{2}}\left(|\alpha\rangle + |q^2\alpha\rangle\right)\\ &= \alpha^2\frac{N_0}{\sqrt{2}}\left(\frac{\sqrt{2}}{N_0}|0\rangle_\alpha)\right) = \alpha^2|0\rangle_\alpha\,,\end{aligned}$$

and

$$\begin{aligned}\hat{a}^2|1\rangle_\alpha = \alpha\frac{N_1}{N_0}\hat{a}|0\rangle_\alpha &= \alpha\frac{N_1}{N_0}\frac{N_0}{\sqrt{2}}\left(\hat{a}|\alpha\rangle + \hat{a}|q^2\alpha\rangle\right)\\ &= \alpha\frac{N_1}{\sqrt{2}}\left(\alpha|\alpha\rangle + q^2\alpha|q^2\alpha\rangle\right)\\ &= \alpha^2\frac{N_1}{\sqrt{2}}\left(|\alpha\rangle + \overline{q}^2|q^2\alpha\rangle\right)\\ &= \alpha^2\frac{N_1}{\sqrt{2}}\left(\frac{\sqrt{2}}{N_1}|1\rangle_\alpha)\right) = \alpha^2|1\rangle_\alpha\,.\end{aligned}$$

□

These results imply that the states $|0\rangle_\alpha$ and $|1\rangle_\alpha$ can be used to define the qubit coherent state:

$$|\psi\rangle_\alpha = c_0|0\rangle_\alpha + c_1|1\rangle_\alpha, \tag{5.17}$$

where $|c_0|^2+|c_1|^2 = 1$, representing a unit of quantum information in quantum optics. This qubit state is an eigenstate of operator $\hat{a}^2$ as well:

$$\hat{a}^2|\psi\rangle_\alpha = \alpha^2|\psi\rangle_\alpha. \tag{5.18}$$

5.2.1. Representation of Annihilation Operator in Cat States

Here, we find matrix representation of annihilation operator $\hat{a}$ in cat states basis by using (5.16);

$$\begin{aligned}\hat{a} = \begin{bmatrix} \hat{a}_{00} & \hat{a}_{01} \\ \hat{a}_{10} & \hat{a}_{11} \end{bmatrix} = \begin{bmatrix} {}_\alpha\langle 0|\hat{a}|0\rangle_\alpha & {}_\alpha\langle 0|\hat{a}|1\rangle_\alpha \\ {}_\alpha\langle 1|\hat{a}|0\rangle_\alpha & {}_\alpha\langle 1|\hat{a}|1\rangle_\alpha \end{bmatrix} &= \begin{bmatrix} {}_\alpha\langle 0|\alpha\frac{N_0}{N_1}|1\rangle_\alpha & {}_\alpha\langle 0|\alpha\frac{N_1}{N_0}|0\rangle_\alpha \\ {}_\alpha\langle 1|\alpha\frac{N_0}{N_1}|1\rangle_\alpha & {}_\alpha\langle 1|\alpha\frac{N_1}{N_0}|0\rangle_\alpha \end{bmatrix} \\ &= \alpha\begin{bmatrix} \frac{N_0}{N_1}{}_\alpha\langle 0|1\rangle_\alpha & \frac{N_1}{N_0}{}_\alpha\langle 0|0\rangle_\alpha \\ \frac{N_0}{N_1}{}_\alpha\langle 1|1\rangle_\alpha & \frac{N_1}{N_0}{}_\alpha\langle 1|0\rangle_\alpha \end{bmatrix}\end{aligned}$$

From ${}_\alpha\langle n|m\rangle_\alpha = \delta_{nm}$, where $n, m = 0, 1$, we get

$$\hat{a} = \alpha\begin{bmatrix} 0 & \frac{N_1}{N_0} \\ \frac{N_0}{N_1} & 0 \end{bmatrix}. \tag{5.19}$$

Since we have normalization matrix in diagonal form, its inverse will be reciprocal of diagonal elements

$$N = \begin{bmatrix} N_0 & 0 \\ 0 & N_1 \end{bmatrix} \quad\Rightarrow\quad N^{-1} = \begin{bmatrix} \frac{1}{N_0} & 0 \\ 0 & \frac{1}{N_1} \end{bmatrix}. \tag{5.20}$$

Representation of annihilation operator(5.19) can be written with Pauli spin matrix σ_x as below;

$$\hat{a} = \alpha N^{-1}\sigma_x N = \alpha \begin{bmatrix} \frac{1}{N_0} & 0 \\ 0 & \frac{1}{N_1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} N_0 & 0 \\ 0 & N_1 \end{bmatrix} \tag{5.21}$$

$$= \alpha \begin{bmatrix} \frac{1}{N_0} & 0 \\ 0 & \frac{1}{N_1} \end{bmatrix} \begin{bmatrix} 0 & N_1 \\ N_0 & 0 \end{bmatrix} = \alpha \begin{bmatrix} 0 & \frac{N_1}{N_0} \\ \frac{N_0}{N_1} & 0 \end{bmatrix} \tag{5.22}$$

5.3. Number of Photons in Cat States

Proposition 5.2 *Average number of photons in cat states is calculated as ratio of normalization constants multiple $|\alpha|^2$;*

$${}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = |\alpha|^2 \frac{N_0^2}{N_1^2} = |\alpha|^2 \frac{{}_1e^{|\alpha|^2}}{{}_0e^{|\alpha|^2}} = |\alpha|^2 \tanh|\alpha|^2 , \tag{5.23}$$

$${}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha = |\alpha|^2 \frac{N_1^2}{N_0^2} = |\alpha|^2 \frac{{}_0e^{|\alpha|^2}}{{}_1e^{|\alpha|^2}} = |\alpha|^2 \coth|\alpha|^2 . \tag{5.24}$$

Proof Annihilation operator $\hat{a}$ gives flipping between cat states $|0\rangle_\alpha$ and $|1\rangle_\alpha$,

$$\hat{a}|0\rangle_\alpha = \alpha \frac{N_0}{N_1}|1\rangle_\alpha , \quad \hat{a}|1\rangle_\alpha = \alpha \frac{N_1}{N_0}|0\rangle_\alpha , \tag{5.25}$$

where

$$N_0 = \frac{1}{\sqrt{2\,{}_0e^{|\alpha|^2}}} = \frac{1}{\sqrt{2\cosh|\alpha|^2}} \quad , \quad N_1 = \frac{1}{\sqrt{2\,{}_1e^{|\alpha|^2}}} = \frac{1}{\sqrt{2\sinh|\alpha|^2}} \quad (mod\, 2) . \tag{5.26}$$

By using these equations, we calculate

$${}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = {}_\alpha\langle 0|\hat{a}^\dagger\hat{a}|0\rangle_\alpha = \left({}_\alpha\langle 1|\frac{N_0}{N_1}\overline{\alpha}\right)\left(\alpha\frac{N_0}{N_1}|1\rangle_\alpha\right) = \overline{\alpha}\alpha\frac{N_0^2}{N_1^2}{}_\alpha\langle 1|1\rangle_\alpha = |\alpha|^2\frac{N_0^2}{N_1^2} . \tag{5.27}$$

Substituting (5.26) gives explicitly

$$\boxed{{}_{\alpha}\langle 0|\widehat{N}|0\rangle_{\alpha} = |\alpha|^2 \frac{N_0^2}{N_1^2} = |\alpha|^2 \frac{{}_1e^{|\alpha|^2}}{{}_0e^{|\alpha|^2}} = |\alpha|^2 \tanh|\alpha|^2} \tag{5.28}$$

Then,

$${}_{\alpha}\langle 1|\widehat{N}|1\rangle_{\alpha} = {}_{\alpha}\langle 1|\hat{a}^{\dagger}\hat{a}|1\rangle_{\alpha} = \left({}_{\alpha}\langle 1|\frac{N_1}{N_0}\overline{\alpha}\right)\left(\alpha\frac{N_1}{N_0}|1\rangle_{\alpha}\right) = \overline{\alpha}\alpha\frac{N_1^2}{N_0^2}{}_{\alpha}\langle 0|0\rangle_{\alpha} = |\alpha|^2\frac{N_1^2}{N_0^2}\,. \tag{5.29}$$

and in explicit form, we have

$$\boxed{{}_{\alpha}\langle 1|\widehat{N}|1\rangle_{\alpha} = |\alpha|^2 \frac{N_1^2}{N_0^2} = |\alpha|^2 \frac{{}_0e^{|\alpha|^2}}{{}_1e^{|\alpha|^2}} = |\alpha|^2 \coth|\alpha|^2} \tag{5.30}$$

□

The average number of photons in cat states are is shown in Fig 5.1. As easy to evaluate, asymptotically these numbers are approaching the usual coherent states number $|\alpha|^2$:

$$\lim_{|\alpha|\to\infty} {}_{\alpha}\langle 0|\widehat{N}|0\rangle_{\alpha} = \lim_{|\alpha|\to\infty} {}_{\alpha}\langle 1|\widehat{N}|1\rangle_{\alpha} \approx |\alpha|^2 = \langle \pm\alpha|\widehat{N}| \pm \alpha\rangle\,. \tag{5.31}$$

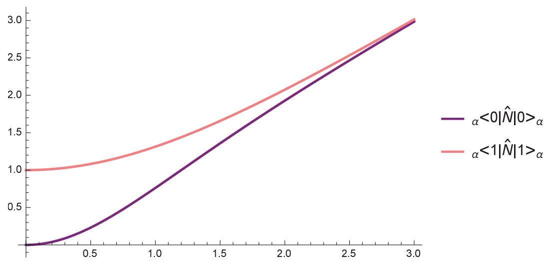

Figure 5.1. Photon numbers in Schrödinger's cat states

5.3.1. Schrödinger's Kitten States

Coherent state with small number of photons corresponds to the limit $|\alpha| \to 0$. For the Schrödinger's cat states, we get number of photons 0 and 1, in the so called Schrödinger's kitten states:

$$\lim_{|\alpha|\to 0} {}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = 0, \quad \lim_{|\alpha|\to 0} {}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha = 1. \tag{5.32}$$

By using (5.23), we calculate the limit as $|\alpha| \to 0$

$$\lim_{|\alpha|\to 0} {}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = \lim_{|\alpha|\to 0} |\alpha|^2 \tanh|\alpha|^2 = 0 \tanh 0 = 0\,. \tag{5.33}$$

By using (5.24), we calculate the limits as $|\alpha| \to 0$

$$\lim_{|\alpha|\to 0} {}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha = \lim_{|\alpha|\to 0} |\alpha|^2 \coth|\alpha|^2 = \lim_{|\alpha|\to 0} \frac{|\alpha|^2 \cosh|\alpha|^2}{\sinh|\alpha|^2} \tag{5.34}$$

$$= \lim_{|\alpha|\to 0} \cosh|\alpha|^2 \lim_{|\alpha|\to 0} \frac{|\alpha|^2}{\sinh|\alpha|^2} \tag{5.35}$$

$$= \lim_{|\alpha|\to 0} \frac{|\alpha|^2}{\sinh|\alpha|^2} \tag{5.36}$$

$$\stackrel{L.H}{=} \lim_{|\alpha|\to 0} \frac{1}{\cosh|\alpha|^2} = \frac{1}{\cosh 0} = 1\,. \tag{5.37}$$

5.4. Fermionic Representation of Cat States

Cat states are intrinsically related with fermion oscillator representation. These states are expansions to even and odd powers of α, which are distinguished by parity operator. This parity operator is the projection operator to even and odd numbers.

5.4.1. Fermionic Oscillator

Here, we give definition and main properties of fermionic oscillator based on (Louisell, 1964). In Chapter 2, we have worked with the annihilation operator $\hat{a}$ and creation operator $\hat{a}^\dagger$, which satisfy bosonic commutation relation $\left[\hat{a}, \hat{a}^\dagger\right] = \hat{I}$. Therefore, particles that obey this relation are bosons and they may occupy the same quantum state. Light quanta(photons) and phonons are examples of bosons. There is another class of physical particles in nature, called fermions, which have property that no two of them can occupy the same quantum state. This is called the Pauli exclusion principle. Electrons, protons and neutrons are examples of fermions.

For description of fermions there are two operators $\hat{b}$ and $\hat{b}^\dagger$, which are interpreted as fermion annihilation operator and creation operator, respectively. Operators $\hat{b}$ and $\hat{b}^\dagger$ obey the anticommutation relation

$$\{\hat{b}, \hat{b}^\dagger\}_+ = \hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b} = I, \qquad \hat{b}^2 = 0, \qquad (\hat{b}^\dagger)^2 = 0\,, \tag{5.38}$$

where the anticommutator of $\hat{A}$ and $\hat{B}$ is defined by

$$\{\hat{A}, \hat{B}\}_+ \equiv \hat{A}\hat{B} + \hat{B}\hat{A}\,. \tag{5.39}$$

The fermion number operator $\widehat{N}_F = \hat{b}^\dagger\hat{b} = \widehat{N}_F^\dagger$ is diagonal in states $|0\rangle$ and $|1\rangle$, defined by

$$\hat{b}|0\rangle = 0 \quad , \quad \hat{b}^\dagger|0\rangle = |1\rangle\,. \tag{5.40}$$

Then, matrix representation of this operator is

$$\widehat{N}_F = \begin{bmatrix} \langle 0|\widehat{N}_F|0\rangle & \langle 0|\widehat{N}_F|1\rangle \\ \langle 1|\widehat{N}_F|0\rangle & \langle 1|\widehat{N}_F|1\rangle \end{bmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.41}$$

where

$$\widehat{N}_F|0\rangle = 0\,, \tag{5.42}$$

$$\widehat{N}_F|1\rangle = |1\rangle\,, \tag{5.43}$$

and the matrix elements of $\hat{b}$ and $\hat{b}^\dagger$ in the $\widehat{N}$ representation are

$$\hat{b} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \hat{b}^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{5.44}$$

Now, we are going to show that dilatation operator $q^{2\widehat{N}}$ determines q^2-number operator, which in cat state basis is fermionic number operator.

Proposition 5.3 *The dilatation operator $q^{2\widehat{N}} = e^{i\pi\widehat{N}} = (-1)^{\widehat{N}}$ is the parity operator for cat states, so that $|0\rangle_\alpha$ and $|1\rangle_\alpha$ states are eigenstates of this operator.*

$$q^{2\widehat{N}}|0\rangle_\alpha = |0\rangle_\alpha\,, \quad q^{2\widehat{N}}|1\rangle_\alpha = q^2|1\rangle_\alpha.$$

The first state is the q^2-periodic state and the second one is q^2-self-similar state,

Proof The proof will be done by using equation(5.8);

$$\begin{aligned} q^{2\widehat{N}}|0\rangle_\alpha = q^{2\widehat{N}} \frac{{}_0e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_0e^{|\alpha|^2}}}|0\rangle &= \frac{q^{2\widehat{N}}}{\sqrt{{}_0e^{|\alpha|^2}}} \sum_{k=0}^{\infty} \frac{\left(\alpha\hat{a}^\dagger\right)^{2k}}{2k!}|0\rangle \\ &= \frac{q^{2\widehat{N}}}{\sqrt{{}_0e^{|\alpha|^2}}} \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{\sqrt{2k!}} \underbrace{\frac{(\hat{a}^\dagger)^{2k}}{\sqrt{2k!}}|0\rangle}_{|2k\rangle} \\ &= \frac{1}{\sqrt{{}_0e^{|\alpha|^2}}} \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{\sqrt{2k!}} q^{2\widehat{N}}|2k\rangle \quad (mod\,2)\,. \end{aligned}$$

The number operator $\widehat{N}$ gives $q^{2\widehat{N}}|2k\rangle = q^{2(2k)}|2k\rangle = q^{4k}|2k\rangle = |2k\rangle$ so that

$$\begin{aligned} q^{2\widehat{N}}|0\rangle_\alpha &= \frac{1}{\sqrt{{}_0e^{|\alpha|^2}}} \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{\sqrt{2k!}}|2k\rangle \\ &= \frac{1}{\sqrt{{}_0e^{|\alpha|^2}}} \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{\sqrt{2k!}} \frac{(\hat{a}^\dagger)^{2k}}{\sqrt{2k!}}|0\rangle \\ &= \frac{1}{\sqrt{{}_0e^{|\alpha|^2}}} \sum_{k=0}^{\infty} \frac{\left(\alpha\hat{a}^\dagger\right)^{2k}}{2k!}|0\rangle = \frac{{}_0e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_0e^{|\alpha|^2}}}|0\rangle = |0\rangle_\alpha \quad (mod\,2)\,, \end{aligned}$$

For the state $|1\rangle_\alpha$, we follow the same steps

$$\begin{aligned}
q^{2\widehat{N}}|1\rangle_\alpha = q^{2\widehat{N}}\frac{{}_1e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_1e^{|\alpha|^2}}}|0\rangle &= \frac{q^{2\widehat{N}}}{\sqrt{{}_1e^{|\alpha|^2}}}\sum_{k=0}^{\infty}\frac{\left(\alpha\hat{a}^\dagger\right)^{2k+1}}{2k+1!}|0\rangle \\
&= \frac{q^{2\widehat{N}}}{\sqrt{{}_1e^{|\alpha|^2}}}\sum_{k=0}^{\infty}\frac{\alpha^{2k+1}}{\sqrt{2k+1!}}\underbrace{\frac{(\hat{a}^\dagger)^{2k+1}}{\sqrt{2k+1!}}|0\rangle}_{|2k+1\rangle} \\
&= \frac{1}{\sqrt{{}_1e^{|\alpha|^2}}}\sum_{k=0}^{\infty}\frac{\alpha^{2k+1}}{\sqrt{2k+1!}}q^{2\widehat{N}}|2k+1\rangle \quad (mod\, 2)\,,
\end{aligned}$$

and

$$q^{2\widehat{N}}|2k+1\rangle = q^{2(2k+1)}|2k\rangle = q^2q^{4k}|2k\rangle = q^2|2k\rangle\,. \tag{5.45}$$

It gives that

$$\begin{aligned}
q^{2\widehat{N}}|1\rangle_\alpha &= \frac{1}{\sqrt{{}_1e^{|\alpha|^2}}}\sum_{k=0}^{\infty}\frac{\alpha^{2k}}{\sqrt{2k!}}q^2|0\rangle \\
&= q^2\frac{1}{\sqrt{{}_1e^{|\alpha|^2}}}\sum_{k=0}^{\infty}\frac{\alpha^{2k+1}}{\sqrt{2k+1!}}\frac{(\hat{a}^\dagger)^{2k+1}}{\sqrt{2k+1!}}|0\rangle \\
&= q^2\frac{1}{\sqrt{{}_1e^{|\alpha|^2}}}\sum_{k=0}^{\infty}\frac{\left(\alpha\hat{a}^\dagger\right)^{2k+1}}{2k+1!}|0\rangle = \frac{{}_0e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_1e^{|\alpha|^2}}}|1\rangle = q^2|1\rangle_\alpha \quad (mod\, 2)\,.
\end{aligned}$$

□

The $|0\rangle_\alpha$ and $|1\rangle_\alpha$ states can be rewritten in terms of parity operator

$$|0\rangle_\alpha = \frac{N_0}{\sqrt{2}}[2]_{q^{2\widehat{N}}}|\alpha\rangle = \frac{N_0}{\sqrt{2}}(I + q^{2\widehat{N}})|\alpha\rangle, \tag{5.46}$$

$$|1\rangle_\alpha = \frac{N_1}{\sqrt{2}}[2]_{q^{2\widehat{N}+2}}|\alpha\rangle = \frac{N_1}{\sqrt{2}}(I + q^2q^{2\widehat{N}})|\alpha\rangle, \tag{5.47}$$

or

$$|0\rangle_\alpha = \frac{N_0}{\sqrt{2}}(I + (-1)^{\widehat{N}})|\alpha\rangle\,, \tag{5.48}$$

$$|1\rangle_\alpha = \frac{N_1}{\sqrt{2}}(I - (-1)^{\widehat{N}})|\alpha\rangle\,. \tag{5.49}$$

It is noticed that the cat states are eigenstates also of q^2- non-symmetric number operator

$$[\widehat{N}]_{q^2} = \frac{q^{2\widehat{N}} - 1}{q^2 - 1},$$

where $q^2 = -1$,

$$[\widehat{N}]_{q^2}|0\rangle_\alpha = [0]_{q^2}|0\rangle_\alpha, \qquad [\widehat{N}]_{q^2}|1\rangle_\alpha = [1]_{q^2}|1\rangle_\alpha, \tag{5.50}$$

with eigenvalues $[0]_{q^2} = 0$ and $[1]_{q^2} = 1$. In the Fock basis $|n\rangle$, $n = 0, 1, 2...$, these number operator is diagonal, with eigenvalues 0 for even numbers $n = 2k$, and 1 for odd numbers $n = 2k + 1$. This number operator in the cat state basis is 2×2 matrix of the fermion number operator

$$[\widehat{N}]_{q^2} = \begin{bmatrix} {}_\alpha\langle 0|[N]_{q^2}|0\rangle_\alpha & {}_\alpha\langle 0|[N]_{q^2}|1\rangle_\alpha \\ {}_\alpha\langle 1|[N]_{q^2}|0\rangle_\alpha & {}_\alpha\langle 1|[N]_{q^2}|1\rangle_\alpha \end{bmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \widehat{N}_F$$

factorized by fermionic creation and annihilation operators $\widehat{N}_F = \hat{b}^\dagger \hat{b}$.

5.5. Heinsenberg Uncertainity Relation for Cat States

As we have seen in Section 3.2, the coherent states are satisfying minimum uncertainty relation. Here, we calculate uncertainty relation for the cat states as superposition of coherent states. First, expectation value of $\hat{q}$ and $\hat{p}$ operators for state $|0\rangle_\alpha$ are

$$\langle \hat{x} \rangle_{|0\rangle_\alpha} = {}_\alpha\langle 0|\hat{x}|0\rangle_\alpha = {}_\alpha\langle 0|\sqrt{\frac{\hbar}{2}}(\hat{a} + \hat{a}^\dagger)|0\rangle_\alpha = \sqrt{\frac{\hbar}{2}}\left({}_\alpha\langle 0|\hat{a}|0\rangle_\alpha + {}_\alpha\langle 0|\hat{a}^\dagger|0\rangle_\alpha\right) = 0\,,$$

and

$$\langle \hat{p} \rangle_{|0\rangle_\alpha} = {}_\alpha\langle 0|\hat{p}|0\rangle_\alpha = {}_\alpha\langle 0| - i\sqrt{\frac{\hbar}{2}}(\hat{a} - \hat{a}^\dagger)|0\rangle_\alpha = -i\sqrt{\frac{\hbar}{2}}\left({}_\alpha\langle 0|\hat{a}|0\rangle_\alpha + {}_\alpha\langle 0|\hat{a}^\dagger|0\rangle_\alpha\right) = 0\,.$$

It happens due to that the application of $\hat{a}$ to the cat state produce the orthogonal state as in (5.16). Similar calculations for state $|1\rangle_\alpha$ gives

$$\begin{aligned}
\langle\hat{x}\rangle_{|1\rangle_\alpha} &= {}_\alpha\langle 1|\hat{x}|1\rangle_\alpha = 0\,, && (5.51)\\
\langle\hat{p}\rangle_{|1\rangle_\alpha} &= {}_\alpha\langle 1|\hat{p}|1\rangle_\alpha = 0\,. && (5.52)
\end{aligned}$$

Then, for state $|0\rangle_\alpha$ the variance in coordinate operator is

$$\begin{aligned}
\langle\hat{x}^2\rangle_{|0\rangle_\alpha} = {}_\alpha\langle 0|\hat{x}^2|0\rangle_\alpha &= \frac{\hbar}{2}{}_\alpha\langle 0|\left(\hat{a}+\hat{a}^\dagger\right)^2|0\rangle_\alpha\\
&\overset{(3.3)}{=} \frac{\hbar}{2}{}_\alpha\langle 0|\left(\hat{a}^2+2\hat{a}^\dagger\hat{a}+\hat{\mathrm{I}}+(\hat{a}^\dagger)^2\right)|0\rangle_\alpha\\
&= \frac{\hbar}{2}\left({}_\alpha\langle 0|\hat{a}^2|0\rangle_\alpha+2{}_\alpha\langle 0|\hat{a}^\dagger\hat{a}|0\rangle_\alpha+{}_\alpha\langle 0|0\rangle_\alpha+{}_\alpha\langle 0|(\hat{a}^\dagger)^2|0\rangle_\alpha\right).
\end{aligned}$$

and variance in momentum operator is

$$\begin{aligned}
\langle\hat{p}^2\rangle_{|0\rangle_\alpha} = {}_\alpha\langle 0|\hat{p}^2|0\rangle_\alpha &= -\frac{\hbar}{2}{}_\alpha\langle 0|\left(\hat{a}-\hat{a}^\dagger\right)^2|0\rangle_\alpha\\
&\overset{(3.3)}{=} -\frac{\hbar}{2}{}_\alpha\langle 0|\left(\hat{a}^2-2\hat{a}^\dagger\hat{a}-\hat{\mathrm{I}}+(\hat{a}^\dagger)^2\right)|0\rangle_\alpha\\
&= -\frac{\hbar}{2}\left({}_\alpha\langle 0|\hat{a}^2|0\rangle_\alpha-2{}_\alpha\langle 0|\hat{a}^\dagger\hat{a}|0\rangle_\alpha-{}_\alpha\langle 0|0\rangle_\alpha+{}_\alpha\langle 0|(\hat{a}^\dagger)^2|0\rangle_\alpha\right).
\end{aligned}$$

By substituting ${}_\alpha\langle 0|\hat{a}^2|0\rangle_\alpha = \alpha^2$ and ${}_\alpha\langle 0|(\hat{a}^\dagger)^2|0\rangle_\alpha = \overline{\alpha}^2$, it gives

$$\begin{aligned}
\langle\hat{x}^2\rangle_{|0\rangle_\alpha} &= {}_\alpha\langle 0|\hat{x}^2|0\rangle_\alpha = \frac{\hbar}{2}\left(2{}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha+1+\alpha^2+\overline{\alpha}^2\right), && (5.53)\\
\langle\hat{p}^2\rangle_{|0\rangle_\alpha} &= {}_\alpha\langle 0|\hat{p}^2|0\rangle_\alpha = \frac{\hbar}{2}\left(2{}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha+1-\alpha^2-\overline{\alpha}^2\right). && (5.54)
\end{aligned}$$

By following the same steps for state $|1\rangle_\alpha$,we have variance in coordinate and momentum operator as

$$\begin{aligned}
\langle\hat{x}^2\rangle_{|1\rangle_\alpha} &= {}_\alpha\langle 1|\hat{x}^2|1\rangle_\alpha = \frac{\hbar}{2}\left(2{}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha+1+\alpha^2+\overline{\alpha}^2\right), && (5.55)\\
\langle\hat{p}^2\rangle_{|1\rangle_\alpha} &= {}_\alpha\langle 1|\hat{p}^2|1\rangle_\alpha = \frac{\hbar}{2}\left(2{}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha+1-\alpha^2-\overline{\alpha}^2\right). && (5.56)
\end{aligned}$$

These relations give uncertainty relation for cat states (see appendix (A.2))

$$
\begin{aligned}
(\Delta\hat{q})_{|0\rangle_\alpha}(\Delta\hat{p})_{|0\rangle_\alpha} &= \frac{\hbar}{2}\sqrt{\left(1+2\,{}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha\right)-\left(\alpha^2+\overline{\alpha}^2\right)^2}\,, & (5.57)\\
(\Delta\hat{q})_{|1\rangle_\alpha}(\Delta\hat{p})_{|1\rangle_\alpha} &= \frac{\hbar}{2}\sqrt{\left(1+2\,{}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha\right)-\left(\alpha^2+\overline{\alpha}^2\right)^2}\,, & (5.58)
\end{aligned}
$$

where ${}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = |\alpha|^2\tanh|\alpha|^2$ and ${}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha = |\alpha|^2\coth|\alpha|^2$. These formulas show that in contrast to coherent states, the cat states are not minimal uncertainty states. Only in the case $\alpha = 0$, uncertainty for the cat states coincide with coherent states minimal case.

5.6. Coordinate Representation of Cat States

Here, we construct coordinate representation of cat states as

$$
\langle x|0\rangle_\alpha = \frac{e^{\frac{|\alpha|^2}{2}}}{2\sqrt{\cosh|\alpha|^2}}\left(\langle x|\alpha\rangle + \langle x|q^2\alpha\rangle\right). \qquad (5.59)
$$

By substituting (3.60) it gives

$$
\begin{aligned}
\langle x|0\rangle_\alpha &= \frac{e^{\frac{|\alpha|^2}{2}}}{2\pi^{1/4}\sqrt{\cosh|\alpha|^2}} e^{-\frac{|\alpha|^2}{2}-\frac{x^2}{2}} e^{-\frac{\alpha^2}{2}}\left(e^{\sqrt{2}x\alpha}+e^{\sqrt{2}xq^2\alpha}\right) & (5.60)\\
&= \frac{e^{-\frac{\alpha^2}{2}-\frac{x^2}{2}}}{\pi^{1/4}\sqrt{\cosh|\alpha|^2}}\cosh\left(\sqrt{2}x\alpha\right). & (5.61)
\end{aligned}
$$

Following the same steps, we get

$$
\langle x|1\rangle_\alpha = \frac{e^{-\frac{\alpha^2}{2}-\frac{x^2}{2}}}{\pi^{1/4}\sqrt{\sinh|\alpha|^2}}\sinh\left(\sqrt{2}x\alpha\right). \qquad (5.62)
$$

Also,we obtain coordinate representation of cat states in terms of Hermite polynomials;

$$\langle x|0\rangle_\alpha = \frac{e^{\frac{|\alpha|^2}{2}}}{2\sqrt{\cosh|\alpha|^2}}\left(\langle x|\alpha\rangle + \langle x|q^2\alpha\rangle\right) \tag{5.63}$$

$$= \frac{e^{\frac{|\alpha|^2}{2}}}{2\pi^{1/4}\sqrt{\cosh|\alpha|^2}} e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{1+(-1)^n}{n!}\left(\frac{\alpha}{\sqrt{2}}\right)^n H_n(x) e^{-\frac{x^2}{2}}\,. \tag{5.64}$$

Terms with $n = 2k$ terms survive in summation, then

$$\langle x|0\rangle_\alpha = \frac{e^{-\frac{x^2}{2}}}{\pi^{1/4}\sqrt{\cosh|\alpha|^2}} \sum_{n=0}^{\infty} \frac{H_{2k}(x)}{(2k)!}\left(\frac{\alpha}{\sqrt{2}}\right)^{2k}\,. \tag{5.65}$$

And similar calculations give us for state $|1\rangle_\alpha$;

$$\langle x|1\rangle_\alpha = \frac{e^{-\frac{x^2}{2}}}{\pi^{1/4}\sqrt{\sinh|\alpha|^2}} \sum_{n=0}^{\infty} \frac{H_{2k+1}(x)}{(2k+1)!}\left(\frac{\alpha}{\sqrt{2}}\right)^{2k+1} \tag{5.66}$$

Probability density in coordinate representation of cat states $|\langle x|0\rangle_\alpha|^2$ and $|\langle x|1\rangle_\alpha|^2$ for $\alpha = 1 + i$ is shown in the Fig.(5.2) and Fig.(5.3) correspondingly.

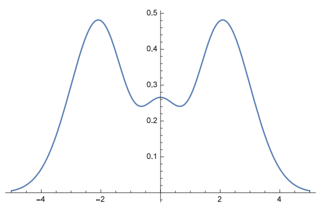

Figure 5.2. Coordinate representation of $|0\rangle_\alpha$

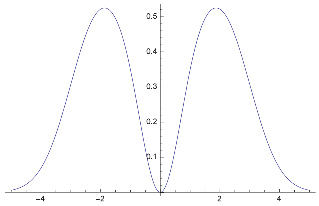

Figure 5.3. Coordinate representation of $|1\rangle_\alpha$

CHAPTER 6

TRINITY STATES

As a first generalization of Schrödinger cat states, we introduce the trinity states. If the cat states are associated with $q^4 = 1$, the trinity states are related with $q^6 = 1$ so that $q^2 = e^{i\frac{2\pi}{3}}$. We start this generalization from the set of coherent states, rotated by angle $\frac{2\pi}{3}$. The states $|\alpha\rangle, |q^2\alpha\rangle$ and $|q^4\alpha\rangle$ correspond to vertices of equilateral triangle in $Fig.$(6.1) such that

$$\langle\alpha|\alpha\rangle = \langle q^2\alpha|q^2\alpha\rangle = \langle q^4\alpha|q^4\alpha\rangle = 1\,. \tag{6.1}$$

Then, the inner product of these states can be calculated as following

- $\langle\alpha|q^2\alpha\rangle = \langle q^2\alpha|q^4\alpha\rangle = \langle q^4\alpha|\alpha\rangle = e^{|\alpha|^2(q^2-1)}$,
- $\langle\alpha|q^4\alpha\rangle = \langle q^4\alpha|q^2\alpha\rangle = \langle q^2\alpha|\alpha\rangle = e^{|\alpha|^2(q^4-1)}$.

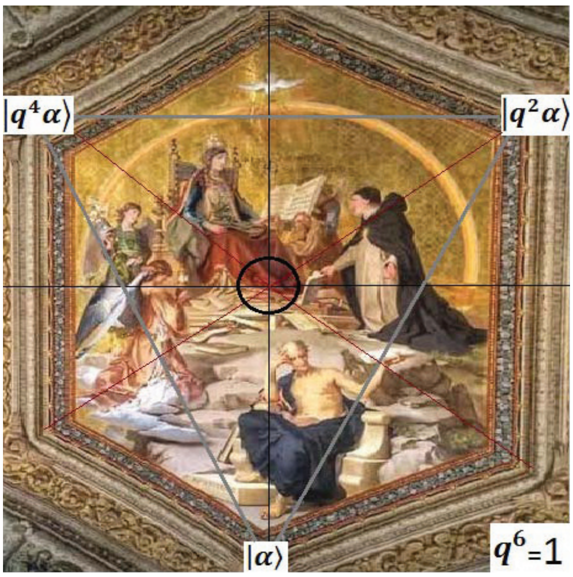

Figure 6.1. Trinity States

First,we define $|0\rangle_\alpha$ state with normalization constant N_0 in matrix representation

$$
\begin{aligned}
|0\rangle_\alpha &= N_0\left(|\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle\right) = N_0 e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}(|n\rangle + q^{2n}|n\rangle + q^{4n}|n\rangle) \\
&= N_0 e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left[\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ q^{2n} \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ q^{4n} \\ \vdots \end{bmatrix}\right].
\end{aligned}
$$

The summation of terms $1+q^{2n}+q^{4n}$ is not zero, when n is divisible by 3, or $n \equiv 0\,(mod\,3)$, due to $q^6 = 1$ such that

$$
1 + q^{2n} + q^{4n} = 3\delta_{n,0(mod3)}\,, \tag{6.2}
$$

where $\delta_{n,0\,(mod\,3)} = \begin{cases} 1, & n = 0\,(mod\,3); \\ 0, & n \neq 0\,(mod\,3). \end{cases}$

This is why only $n = 3k$ terms will survive, corresponding to $|3k\rangle$ state with unit element at position $(3k + 1)$th row as

$$
|0\rangle_\alpha = 3N_0 e^{\frac{-|\alpha|^2}{2}} \sum_{k=0}^{\infty} \frac{\alpha^{3k}}{\sqrt{(3k)!}} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} = 3N_0 e^{\frac{-|\alpha|^2}{2}} \sum_{k=0}^{\infty} \frac{\alpha^{3k}}{\sqrt{(3k)!}} |3k\rangle. \tag{6.3}
$$

Normalization of state $|0\rangle_\alpha$

$$
{}_\alpha\langle 0|0\rangle_\alpha = 1 = 9|N_0|^2 e^{-|\alpha|^2} \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\bar{\alpha})^{3m}}{\sqrt{(3m)!}} \frac{\alpha^{3k}}{\sqrt{(3k)!}} \underbrace{\langle 3m|3k\rangle}_{\delta_{m,k}} \tag{6.4}
$$

$$
= 9|N_0|^2 e^{-|\alpha|^2} \sum_{k=0}^{\infty} \frac{\left(|\alpha|^2\right)^{3k}}{(3k)!} \overset{(4.57)}{=} 9|N_0|^2 e^{-|\alpha|^2}\, {}_0e^{|\alpha|^2}\ (mod\,3) \tag{6.5}
$$

gives

$$|0\rangle_\alpha = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle}{\sqrt{3}\sqrt{e^{|\alpha|^2} + e^{q^2|\alpha|^2} + e^{q^4|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle}{3\sqrt{{}_0e^{|\alpha|^2}(mod\,3)}} \tag{6.6}$$

As a next step, to find states $|1\rangle_\alpha$ and $|2\rangle_\alpha$ orthogonal to state $|0\rangle_\alpha$, we define

$$|1\rangle_\alpha = a_1|\alpha\rangle + b_1|q^2\alpha\rangle + c_1|q^4\alpha\rangle \tag{6.7}$$

$$|2\rangle_\alpha = a_2|\alpha\rangle + b_2|q^2\alpha\rangle + c_2|q^4\alpha\rangle. \tag{6.8}$$

Due to orthogonality condition,

$$\begin{aligned}
{}_\alpha\langle 0|1\rangle_\alpha = 0 &= \Big(\langle\alpha| + \langle q^2\alpha| + \langle q^4\alpha|\Big)\Big(a_1|\alpha\rangle + b_1|q^2\alpha\rangle + c_1|q^4\alpha\rangle\Big) \\
&= a_1\langle\alpha|\alpha\rangle + b_1\langle\alpha|q^2\alpha\rangle + c_1\langle\alpha|q^4\alpha\rangle + a_1\langle q^2\alpha|\alpha\rangle + b_1\langle q^2\alpha|q^2\alpha\rangle + c_1\langle q^2\alpha|q^4\alpha\rangle \\
&+ a_1\langle q^4\alpha|\alpha\rangle + b_1\langle q^4\alpha|q^2\alpha\rangle + c_1\langle q^4\alpha|q^4\alpha\rangle \\
&= (a_1 + b_1 + c_1)\Big(1 + \langle\alpha|q^2\alpha\rangle + \langle\alpha|q^4\alpha\rangle\Big) \\
&= (a_1 + b_1 + c_1)\underbrace{\Big(1 + e^{|\alpha|^2(q^2-1)} + e^{|\alpha|^2(q^4-1)}\Big)}_{\neq 0} \quad \Rightarrow \quad a_1 + b_1 + c_1 = 0
\end{aligned}$$

Similar calculation for ${}_\alpha\langle 0|2\rangle_\alpha = 0$ gives $a_2+b_2+c_2 = 0$. Orthogonality of states ${}_\alpha\langle 2|1\rangle_\alpha = 0$ is described by system of equations in matrix form;

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ b_1 & c_1 & a_1 \\ c_1 & a_1 & b_1 \end{bmatrix}\begin{bmatrix} \overline{a_2} \\ \overline{b_2} \\ \overline{c_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{6.9}$$

From orthogonality relations, due to (6.2) coefficients can be choosen up to constant as

$$(a_1, b_1, c_1) = (1, \bar{q}^2, \bar{q}^4), \tag{6.10}$$

$$(a_2, b_2, c_2) = (1, \bar{q}^4, \bar{q}^2). \tag{6.11}$$

Thus, we can construct orthonormal states $|1\rangle_\alpha$ and $|2\rangle_\alpha$ with normalization constants N_1, N_2:

$$|1\rangle_\alpha = N_1\left(|\alpha\rangle + \overline{q}^2|q^2\alpha\rangle + \overline{q}^4|q^4\alpha\rangle\right), \tag{6.12}$$

$$|2\rangle_\alpha = N_2\left(|\alpha\rangle + \overline{q}^4|q^2\alpha\rangle + \overline{q}^2|q^4\alpha\rangle\right). \tag{6.13}$$

The matrix form of state $|1\rangle_\alpha$ is

$$\begin{aligned}
|1\rangle_\alpha = N_1\left(|\alpha\rangle + \overline{q}^2|q^2\alpha\rangle + \overline{q}^4|q^4\alpha\rangle\right) &= N_1 e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} (|n\rangle + \overline{q}^2 q^{2n}|n\rangle + \overline{q}^4 q^{4n}|n\rangle) \\
&= N_1 e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left[\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} + \overline{q}^2 \begin{bmatrix} 0 \\ 0 \\ \vdots \\ q^{2n} \\ \vdots \end{bmatrix} + \overline{q}^4 \begin{bmatrix} 0 \\ 0 \\ \vdots \\ q^{4n} \\ \vdots \end{bmatrix}\right],
\end{aligned}$$

where addition of the terms is not zero if $n = 3k + 1(n \equiv 1\ (mod\,3))$, This means

$$1 + q^{2(n-1)} + q^{4(n-1)} = 3\delta_{n,1(mod3)}, \tag{6.14}$$

and the state $|3k + 1\rangle$ has element 1 at position of $(3k + 2)-$th row

$$|1\rangle_\alpha = 3N_1 e^{\frac{-|\alpha|^2}{2}} \sum_{k=0}^{\infty} \frac{\alpha^{3k+1}}{\sqrt{(3k+1)!}} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} = 3N_1 e^{\frac{-|\alpha|^2}{2}} \sum_{k=0}^{\infty} \frac{\alpha^{3k+1}}{\sqrt{(3k+1)!}} |3k+1\rangle. \tag{6.15}$$

Then, we normalize state $|1\rangle_\alpha$

$$\begin{aligned}
{}_\alpha\langle 1|1\rangle_\alpha = 1 &= 9|N_1|^2 e^{-|\alpha|^2} \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\bar{\alpha})^{3m+1}}{\sqrt{(3m+1)!}} \frac{\alpha^{3k+1}}{\sqrt{(3k+1)!}} \underbrace{\langle 3m|3k\rangle}_{\delta_{m,k}} && (6.16) \\
&= 9|N_1|^2 e^{-|\alpha|^2} \sum_{k=0}^{\infty} \frac{\left(|\alpha|^2\right)^{3k+1}}{(3k+1)!} \overset{(4.58)}{=} 9|N_1|^2 e^{-|\alpha|^2}\ {}_1e^{|\alpha|^2}\ (mod\,3) && (6.17)
\end{aligned}$$

so that

$$|1\rangle_\alpha = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \overline{q}^2|q^2\alpha\rangle + \overline{q}^4|q^4\alpha\rangle}{\sqrt{3}\sqrt{e^{|\alpha|^2} + \overline{q}^2 e^{q^2|\alpha|^2} + \overline{q}^4 e^{q^4|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \overline{q}^2|q^2\alpha\rangle + \overline{q}^4|q^4\alpha\rangle}{3\sqrt{{}_1e^{|\alpha|^2}(mod\,3)}}. \quad (6.18)$$

Following the same steps, we find normalized state $|2\rangle_\alpha$ with mod 3 exponential function:

$$|2\rangle_\alpha = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \overline{q}^4|q^2\alpha\rangle + \overline{q}^2|q^4\alpha\rangle}{\sqrt{3}\sqrt{e^{|\alpha|^2} + \overline{q}^4 e^{q^2|\alpha|^2} + \overline{q}^2 e^{q^4|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \overline{q}^4|q^2\alpha\rangle + \overline{q}^2|q^4\alpha\rangle}{3\sqrt{{}_2e^{|\alpha|^2}(mod\,3)}}. \quad (6.19)$$

Finally , we have derived the set of three orthonormal states $|0\rangle_\alpha, |1\rangle_\alpha$ and $|2\rangle_\alpha$ and call them as the trinity states

$$\begin{aligned}
|0\rangle_\alpha &= e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle}{\sqrt{3}\sqrt{e^{|\alpha|^2} + e^{q^2|\alpha|^2} + e^{q^4|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle}{3\sqrt{{}_0e^{|\alpha|^2}(mod\,3)}}, \\
|1\rangle_\alpha &= e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \overline{q}^2|q^2\alpha\rangle + \overline{q}^4|q^4\alpha\rangle}{\sqrt{3}\sqrt{e^{|\alpha|^2} + \overline{q}^2 e^{q^2|\alpha|^2} + \overline{q}^4 e^{q^4|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \overline{q}^2|q^2\alpha\rangle + \overline{q}^4|q^4\alpha\rangle}{3\sqrt{{}_1e^{|\alpha|^2}(mod\,3)}}, \\
|2\rangle_\alpha &= e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \overline{q}^4|q^2\alpha\rangle + \overline{q}^2|q^4\alpha\rangle}{\sqrt{3}\sqrt{e^{|\alpha|^2} + \overline{q}^4 e^{q^2|\alpha|^2} + \overline{q}^2 e^{q^4|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \overline{q}^4|q^2\alpha\rangle + \overline{q}^2|q^4\alpha\rangle}{3\sqrt{{}_2e^{|\alpha|^2}(mod\,3)}}.
\end{aligned}$$

6.1. Matrix Form of Trinity States

The trinity states appear by action of the trinity gate , which is three dimensional analogue of the Hadamard gate

$$\begin{bmatrix} |0\rangle_\alpha \\ |1\rangle_\alpha \\ |2\rangle_\alpha \end{bmatrix} = \mathbf{N} \underbrace{\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \overline{q}^2 & \left(\overline{q}^2\right)^2 \\ 1 & \overline{q}^4 & \left(\overline{q}^4\right)^2 \end{bmatrix}}_{Trinity\ gate} \begin{bmatrix} |\alpha\rangle \\ |q^2\alpha\rangle \\ |q^4\alpha\rangle \end{bmatrix}.$$

In this representation, first diagonal matrix determines normalization factor

$$\mathbf{N} = \begin{bmatrix} N_0 & 0 & 0 \\ 0 & N_1 & 0 \\ 0 & 0 & N_2 \end{bmatrix} = \frac{e^{\frac{|\alpha|^2}{2}}}{\sqrt{3}} \begin{bmatrix} {}_0e^{|\alpha|^2} & 0 & 0 \\ 0 & {}_1e^{|\alpha|^2} & 0 \\ 0 & 0 & {}_2e^{|\alpha|^2} \end{bmatrix}^{-\frac{1}{2}} \quad (mod\, 3)\,, \tag{6.20}$$

where

$$N_0 = \frac{e^{\frac{|\alpha|^2}{2}}}{\sqrt{3}} \left(e^{|\alpha|^2} + e^{q^2|\alpha|^2} + e^{q^4|\alpha|^2}\right)^{-\frac{1}{2}} = \frac{e^{\frac{|\alpha|^2}{2}}}{3} \left({}_0e^{|\alpha|^2}\right)^{-\frac{1}{2}} \quad (mod\, 3)\,, \tag{6.21}$$

$$N_1 = \frac{e^{\frac{|\alpha|^2}{2}}}{\sqrt{3}} \left(e^{|\alpha|^2} + \bar{q}^2 e^{q^2|\alpha|^2} + \bar{q}^4 e^{q^4|\alpha|^2}\right)^{-\frac{1}{2}} = \frac{e^{\frac{|\alpha|^2}{2}}}{3} \left({}_1e^{|\alpha|^2}\right)^{-\frac{1}{2}} \quad (mod\, 3)\,, \tag{6.22}$$

$$N_2 = \frac{e^{\frac{|\alpha|^2}{2}}}{\sqrt{3}} \left(e^{|\alpha|^2} + \bar{q}^4 e^{q^2|\alpha|^2} + \bar{q}^2 e^{q^4|\alpha|^2}\right)^{-\frac{1}{2}} = \frac{e^{\frac{|\alpha|^2}{2}}}{3} \left({}_2e^{|\alpha|^2}\right)^{-\frac{1}{2}} \quad (mod\, 3)\,. \tag{6.23}$$

and the second matrix produces orthogonal states from $|\alpha\rangle$, $|q^2\alpha\rangle$ and $|q^4\alpha\rangle$. For normalization , we have identity

$$1 + \bar{q}^{2(n-k)} + \bar{q}^{4(n-k)} = 3\,\delta_{n,k(mod\ 3)} \quad ,0 \le k \le 2\,, \tag{6.24}$$

where $\delta_{n,k\,(mod\,3)} = \begin{cases} 1, & n = k\,(mod\,3); \\ 0, & n \neq k\,(mod\,3). \end{cases}$

6.2. Phase Structure of Trinity States

Trinity states as superposition of coherent states with explicit phase shift are following

$$\begin{aligned} |0\rangle_\alpha &= N_0\left(|\alpha\rangle + \left|e^{i\frac{2\Pi}{3}}\alpha\right\rangle + \left|e^{-i\frac{2\Pi}{3}}\alpha\right\rangle\right)\,, \\ |1\rangle_\alpha &= N_1\left(|\alpha\rangle + e^{-i\frac{2\Pi}{3}}\left|e^{i\frac{2\Pi}{3}}\alpha\right\rangle + e^{i\frac{2\Pi}{3}}\left|e^{-i\frac{2\Pi}{3}}\alpha\right\rangle\right)\,, \\ |2\rangle_\alpha &= N_2\left(|\alpha\rangle + e^{i\frac{2\Pi}{3}}\left|e^{i\frac{2\Pi}{3}}\alpha\right\rangle + e^{-i\frac{2\Pi}{3}}\left|e^{-i\frac{2\Pi}{3}}\alpha\right\rangle\right)\,. \end{aligned}$$

where $q^2 = e^{i\frac{2\pi}{3}}$ and $\bar{q}^2 = e^{-i\frac{2\pi}{3}}$.

6.3. Mod 3 Form of Trinity States

As we have seen, the coherent state can be derived by application of exponential function of creation operator to the vacuum state (3.24). Then, the cat states are written by application of hyperbolic functions of creation operator to the vacuum state. Here, we use definition of mod 3 exponential functions to obtain trinity states from $|0\rangle$.

Proposition 6.1 *Trinity states are written in a compact form*

$$|0\rangle_\alpha = \frac{{}_0e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_0e^{|\alpha|^2}}}|0\rangle\,, \quad |1\rangle_\alpha = \frac{{}_1e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_1e^{|\alpha|^2}}}|0\rangle\,, \quad |2\rangle_\alpha = \frac{{}_2e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_2e^{|\alpha|^2}}}|0\rangle \qquad (mod\ 3).$$

Proof The proof will be done by using definition of coherent state(3.24) and *mod* 3 exponential functions. First, for $|0\rangle_\alpha$ state we have

$$\begin{aligned}
|0\rangle_\alpha &= e^{\frac{1}{2}|\alpha|^2}\frac{|\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle}{3\sqrt{{}_0e^{|\alpha|^2}}}\,(mod\ 3) \\
&= \frac{e^{\frac{1}{2}|\alpha|^2}}{3\sqrt{{}_0e^{|\alpha|^2}}}\left(e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a}^\dagger} + e^{-\frac{1}{2}|q^2\alpha|^2}e^{q^2\alpha\hat{a}^\dagger} + e^{-\frac{1}{2}|q^4\alpha|^2}e^{q^4\alpha\hat{a}^\dagger}\right)|0\rangle\,(mod\ 3) \\
&= \frac{e^{\frac{1}{2}|\alpha|^2}}{3\sqrt{{}_0e^{|\alpha|^2}}}e^{-\frac{1}{2}|\alpha|^2}\left(e^{\alpha\hat{a}^\dagger} + e^{q^2\alpha\hat{a}^\dagger} + e^{q^4\alpha\hat{a}^\dagger}\right)|0\rangle\,(mod\ 3) \\
&\overset{(4.57)}{=} \frac{1}{3\sqrt{{}_0e^{|\alpha|^2}}}3\,{}_0e^{\alpha\hat{a}^\dagger}|0\rangle = \frac{{}_0e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_0e^{|\alpha|^2}}}|0\rangle\,(mod\ 3)\,.
\end{aligned} \tag{6.25}$$

Then, similar calculations give

$$\begin{aligned}
|1\rangle_\alpha &= e^{\frac{1}{2}|\alpha|^2}\frac{|\alpha\rangle + \bar{q}^2|q^2\alpha\rangle + \bar{q}^4|q^4\alpha\rangle}{3\sqrt{{}_1e^{|\alpha|^2}}}\,(mod\ 3) \\
&= \frac{e^{\frac{1}{2}|\alpha|^2}}{3\sqrt{{}_1e^{|\alpha|^2}}}\left(e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a}^\dagger} + \bar{q}^2e^{-\frac{1}{2}|q^2\alpha|^2}e^{q^2\alpha\hat{a}^\dagger} + \bar{q}^4e^{-\frac{1}{2}|q^4\alpha|^2}e^{q^4\alpha\hat{a}^\dagger}\right)|0\rangle \\
&= \frac{e^{\frac{1}{2}|\alpha|^2}}{3\sqrt{{}_1e^{|\alpha|^2}}}e^{-\frac{1}{2}|\alpha|^2}\left(e^{\alpha\hat{a}^\dagger} + \bar{q}^2e^{q^2\alpha\hat{a}^\dagger} + \bar{q}^4e^{q^4\alpha\hat{a}^\dagger}\right)|0\rangle \\
&\overset{(4.58)}{=} \frac{1}{3\sqrt{{}_1e^{|\alpha|^2}}}3\,{}_1e^{\alpha\hat{a}^\dagger}|0\rangle = \frac{{}_1e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_1e^{|\alpha|^2}}}|0\rangle\,(mod\ 3)\,,
\end{aligned} \tag{6.26}$$

and

$$
\begin{aligned}
|2\rangle_\alpha &= e^{\frac{1}{2}|\alpha|^2} \frac{|\alpha\rangle + \bar{q}^4 |q^2\alpha\rangle + \bar{q}^2 |q^4\alpha\rangle}{3\sqrt{{}_2e^{|\alpha|^2}}} \,(mod\ 3) \\
&= \frac{e^{\frac{1}{2}|\alpha|^2}}{3\sqrt{{}_2e^{|\alpha|^2}}} \left(e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^\dagger} + \bar{q}^4 e^{-\frac{1}{2}|q^2\alpha|^2} e^{q^2\alpha\hat{a}^\dagger} + \bar{q}^2 e^{-\frac{1}{2}|q^4\alpha|^2} e^{q^4\alpha\hat{a}^\dagger} \right) |0\rangle \\
&= \frac{e^{\frac{1}{2}|\alpha|^2}}{3\sqrt{{}_2e^{|\alpha|^2}}} e^{-\frac{1}{2}|\alpha|^2} \left(e^{\alpha\hat{a}^\dagger} + \bar{q}^4 e^{q^2\alpha\hat{a}^\dagger} + \bar{q}^2 e^{q^4\alpha\hat{a}^\dagger} \right) |0\rangle \\
&\overset{(4.59)}{=} \frac{1}{3\sqrt{{}_2e^{|\alpha|^2}}} 3\,{}_2e^{\alpha\hat{a}^\dagger} |0\rangle = \frac{{}_2e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_2e^{|\alpha|^2}}} |0\rangle \,(mod\ 3)\,.
\end{aligned} \tag{6.27}
$$

□

6.4. Eigenvalue Problem for Trinity States

Coherent states $\{|\alpha\rangle, |q^2\alpha\rangle, |q^4\alpha\rangle\}$ are eigenstates of operator $\hat{a}$ with different eigenvalues $\alpha, q^2\alpha, q^4\alpha$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle\,, \tag{6.28}$$

$$\hat{a}|q^2\alpha\rangle = q^2\alpha|q^2\alpha\rangle\,, \tag{6.29}$$

$$\hat{a}|q^4\alpha\rangle = q^4\alpha|q^4\alpha\rangle\,, \tag{6.30}$$

and the eigenstates of operator $\hat{a}^3$ with the same eigenvalue α^3 ,

$$\hat{a}^3|q^{2k}\alpha\rangle = \alpha^3|q^{2k}\alpha\rangle \quad k = 0, 1, 2. \tag{6.31}$$

Due to the equation (6.31), any superposition of these states is also eigenstate of $\hat{a}^3$ with eigenvalue α^3. We have following theorem

Proposition 6.2 *Trinity states* $\{|0\rangle_\alpha, |1\rangle_\alpha, |2\rangle_\alpha\}$ *are eigenstates of operator* $\hat{a}^3$

$$\hat{a}^3|k\rangle_\alpha = \alpha^3|k\rangle_\alpha \quad k = 0, 1, 2. \tag{6.32}$$

From trinity states, we can construct the qutrit coherent state

$$|\psi\rangle_\alpha = c_0|0\rangle_\alpha + c_1|1\rangle_\alpha + c_2|2\rangle_\alpha \,, \tag{6.33}$$

where $|c_0|^2 + |c_1|^2 + |c_2|^2 = 1$, as a unit of quantum information with base 3 (2.35). It turns out that this state is an eigenstate of operator $\hat{a}^3$ as well

$$\hat{a}^3|\psi\rangle_\alpha = \alpha^3|\psi\rangle_\alpha \,. \tag{6.34}$$

6.5. Number of Photons in Trinity States

For calculating number of photons in trinity states, it is convenient to apply annihilation operator $\hat{a}$ to the states $|0\rangle_\alpha, |1\rangle_\alpha$ and $|2\rangle_\alpha$.

Proposition 6.3 *The annihilation operator $\hat{a}$ acts on $|0\rangle_\alpha, |1\rangle_\alpha$ and $|2\rangle_\alpha$ states as cyclic permutation*

$$\hat{a}|0\rangle_\alpha = \alpha\frac{N_0}{N_2}|2\rangle_\alpha \,, \;\; \hat{a}|1\rangle_\alpha = \alpha\frac{N_1}{N_0}|0\rangle_\alpha \,, \;\; \hat{a}|2\rangle_\alpha = \alpha\frac{N_2}{N_1}|1\rangle_\alpha \,. \tag{6.35}$$

Proof We apply $\hat{a}$ to the trinity states, respectively;

$$\begin{aligned} \hat{a}|0\rangle_\alpha &= \frac{N_0}{\sqrt{3}}\left(\hat{a}|\alpha\rangle + \hat{a}|q^2\alpha\rangle + \hat{a}|q^4\alpha\rangle\right) \\ &= \frac{N_0}{\sqrt{3}}\left(\alpha|\alpha\rangle + q^2\alpha|q^2\alpha\rangle + q^4\alpha|q^4\alpha\rangle\right) \\ &= \frac{N_0}{\sqrt{3}}\alpha\left(|\alpha\rangle + \bar{q}^4|q^2\alpha\rangle + \bar{q}^2|q^4\alpha\rangle\right) \\ &= \alpha\frac{N_0}{\sqrt{3}}\frac{\sqrt{3}}{N_2}|2\rangle_\alpha = \alpha\frac{N_0}{N_2}|2\rangle_\alpha \,, \end{aligned} \tag{6.36}$$

Then,

$$\begin{aligned}
\hat{a}|1\rangle_\alpha &= \frac{N_1}{\sqrt{3}}\left(\hat{a}|\alpha\rangle + \overline{q}^2\hat{a}|q^2\alpha\rangle + \overline{q}^4\hat{a}|q^4\alpha\rangle\right) \\
&= \frac{N_1}{\sqrt{3}}\left(\alpha|\alpha\rangle + \overline{q}^2 q^2\alpha|q^2\alpha\rangle + \overline{q}^4 q^4\alpha|q^4\alpha\rangle\right) \\
&= \frac{N_1}{\sqrt{3}}\alpha\left(|\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle\right) \\
&= \alpha\frac{N_1}{\sqrt{3}}\frac{\sqrt{3}}{N_0}|0\rangle_\alpha = \alpha\frac{N_1}{N_0}|0\rangle_\alpha\,, \qquad (6.37)
\end{aligned}$$

and

$$\begin{aligned}
\hat{a}|2\rangle_\alpha &= \frac{N_2}{\sqrt{3}}\left(\hat{a}|\alpha\rangle + \overline{q}^4\hat{a}|q^2\alpha\rangle + \overline{q}^2\hat{a}|q^4\alpha\rangle\right) \\
&= \frac{N_2}{\sqrt{3}}\left(\alpha|\alpha\rangle + \overline{q}^4 q^2\alpha|q^2\alpha\rangle + \overline{q}^4 q^4\alpha|q^4\alpha\rangle\right) \\
&= \frac{N_2}{\sqrt{3}}\alpha\left(|\alpha\rangle + \overline{q}^2|q^2\alpha\rangle + \overline{q}^4|q^4\alpha\rangle\right) \\
&= \alpha\frac{N_2}{\sqrt{3}}\frac{\sqrt{3}}{N_1}|1\rangle_\alpha = \alpha\frac{N_2}{N_1}|1\rangle_\alpha\,. \qquad (6.38)
\end{aligned}$$

□

Proposition 6.4 *Number of photons in trinity states is defined by ratio of two consecutive mod* 3 *exponential functions, multiplied by* $|\alpha|^2$:

$$\begin{aligned}
{}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha &= |\alpha|^2\left(\frac{N_0}{N_2}\right)^2 = |\alpha|^2\left(\frac{{}_2e^{|\alpha|^2}}{{}_0e^{|\alpha|^2}}\right), \qquad (6.39) \\
{}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha &= |\alpha|^2\left(\frac{N_1}{N_0}\right)^2 = |\alpha|^2\left(\frac{{}_0e^{|\alpha|^2}}{{}_1e^{|\alpha|^2}}\right), \qquad (6.40) \\
{}_\alpha\langle 2|\widehat{N}|2\rangle_\alpha &= |\alpha|^2\left(\frac{N_2}{N_1}\right)^2 = |\alpha|^2\left(\frac{{}_1e^{|\alpha|^2}}{{}_2e^{|\alpha|^2}}\right) \qquad (6.41)
\end{aligned}$$

Proof To show this, first we use equation (6.35) so that

$${}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = {}_\alpha\langle 0|\hat{a}^\dagger\hat{a}|0\rangle_\alpha = \left({}_\alpha\langle 2|\frac{N_0}{N_2}\overline{\alpha}\right)\left(\alpha\frac{N_0}{N_2}|2\rangle_\alpha\right) = \overline{\alpha}\alpha\left(\frac{N_0}{N_2}\right)^2 {}_\alpha\langle 1|1\rangle_\alpha = |\alpha|^2\left(\frac{N_0}{N_2}\right)^2 .$$

Then, in a similar way we get

$$ {}_{\alpha}\langle 1|\widehat{N}|1\rangle_{\alpha} = |\alpha|^2 \left(\frac{N_1}{N_0}\right)^2 \quad , \quad {}_{\alpha}\langle 2|\widehat{N}|2\rangle_{\alpha} = |\alpha|^2 \left(\frac{N_2}{N_1}\right)^2 , $$

where normalization constants are related with mod 3 exponential functions

$$ N_0 = \frac{e^{\frac{|\alpha|^2}{2}}}{3}\left({}_0e^{|\alpha|^2}\right)^{-\frac{1}{2}} , N_1 = \frac{e^{\frac{|\alpha|^2}{2}}}{3}\left({}_1e^{|\alpha|^2}\right)^{-\frac{1}{2}} , N_2 = \frac{e^{\frac{|\alpha|^2}{2}}}{3}\left({}_2e^{|\alpha|^2}\right)^{-\frac{1}{2}} \quad (mod\, 3) . \tag{6.42} $$

□

If we use explicit form of mod 3 exponential functions from equations (4.62) – (4.64), then number of photons can be written in terms of standard trigonometric functions ;

$$ \begin{aligned} {}_{\alpha}\langle 0|\widehat{N}|0\rangle_{\alpha} &= |\alpha|^2 \left[\frac{1 + 2e^{\frac{-3|\alpha|^2}{2}}\cos\left(\frac{\sqrt{3}}{2}|\alpha|^2 + \frac{2\pi}{3}\right)}{1 + 2e^{\frac{-3|\alpha|^2}{2}}\cos\left(\frac{\sqrt{3}}{2}|\alpha|^2\right)}\right] , \\ {}_{\alpha}\langle 1|\widehat{N}|1\rangle_{\alpha} &= |\alpha|^2 \left[\frac{1 + 2e^{\frac{-3|\alpha|^2}{2}}\cos\left(\frac{\sqrt{3}}{2}|\alpha|^2\right)}{1 + 2e^{\frac{-3|\alpha|^2}{2}}\cos\left(\frac{\sqrt{3}}{2}|\alpha|^2 - \frac{2\pi}{3}\right)}\right] , \\ {}_{\alpha}\langle 2|\widehat{N}|2\rangle_{\alpha} &= |\alpha|^2 \left[\frac{1 + 2e^{\frac{-3|\alpha|^2}{2}}\cos\left(\frac{\sqrt{3}}{2}|\alpha|^2 - \frac{2\pi}{3}\right)}{1 + 2e^{\frac{-3|\alpha|^2}{2}}\cos\left(\frac{\sqrt{3}}{2}|\alpha|^2 + \frac{2\pi}{3}\right)}\right] . \end{aligned} $$

For small number of photons, We can find these number of photons corresponding to the limit as $|\alpha|^2 \to 0$. For simplicity of calculations, we denote $|\alpha|^2 \equiv x$ and have

$$ \begin{aligned} \lim_{x\to 0} {}_{\alpha}\langle 0|\widehat{N}|0\rangle_{\alpha} &= \lim_{x\to\infty} x \left[\frac{{}_2e^x}{{}_0e^x}\right] (mod\, 3) = \lim_{x\to 0} x \frac{\sum_{k=0}^{\infty} \frac{x^{3k+2}}{(3k+2)!}}{\sum_{k=0}^{\infty} \frac{x^{3k}}{(3k)!}} \\ &= \lim_{x\to 0} x^3 \frac{\frac{1}{2!} + \frac{x^3}{5!} + \frac{x^6}{8!} + \dots}{1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \dots} = \lim_{x\to 0} x^3 \frac{1}{2} = 0 . \end{aligned} \tag{6.43} $$

By the same steps, we obtain following equations

$$
\begin{aligned}
\lim_{x\to 0} {}_{\alpha}\langle 1|\widehat{N}|1\rangle_{\alpha} = \lim_{x\to\infty} x \left[\frac{{}_0e^x}{{}_1e^x}\right] (mod\,3) \;&=\; \lim_{x\to\infty} x \frac{\displaystyle\sum_{k=0}^{\infty}\frac{x^{3k}}{(3k)!}}{\displaystyle\sum_{k=0}^{\infty}\frac{x^{3k+1}}{(3k+1)!}} \\
&=\; \lim_{x\to 0} x \frac{1+\dfrac{x^3}{3!}+\dfrac{x^6}{6!}+\ldots}{x+\dfrac{x^4}{4!}+\dfrac{x^7}{7!}+\ldots} = 1\,, \qquad (6.44)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{x\to 0} {}_{\alpha}\langle 2|\widehat{N}|2\rangle_{\alpha} \;&=\; \lim_{x\to\infty} x \left[\frac{{}_1e^x}{{}_2e^x}\right] (mod\,3) = \lim_{x\to 0} x \frac{\displaystyle\sum_{k=0}^{\infty}\frac{x^{3k+1}}{(3k+1)!}}{\displaystyle\sum_{k=0}^{\infty}\frac{x^{3k+2}}{(3k+2)!}} \\
&=\; \lim_{x\to 0} x \frac{x+\dfrac{x^4}{4!}+\dfrac{x^7}{7!}+\ldots}{\dfrac{x^2}{2!}+\dfrac{x^5}{5!}+\dfrac{x^8}{8!}+\ldots} = \lim_{x\to 0}\frac{1+\dfrac{x^3}{4!}+\dfrac{x^6}{7!}+\ldots}{\dfrac{1}{2!}+\dfrac{x^3}{5!}+\dfrac{x^6}{8!}+\ldots} = 2\,. \quad (6.45)
\end{aligned}
$$

As a result, we get

$$
\lim_{|\alpha|^2\to 0\to 0} {}_{\alpha}\langle 0|\widehat{N}|0\rangle_{\alpha} = 0\,, \quad \lim_{|\alpha|^2\to 0\to 0} {}_{\alpha}\langle 1|\widehat{N}|1\rangle_{\alpha} = 1\,, \quad \lim_{|\alpha|^2\to 0\to 0} {}_{\alpha}\langle 2|\widehat{N}|2\rangle_{\alpha} = 2\,, \qquad (6.46)
$$

shown in Fig.(6.2) :

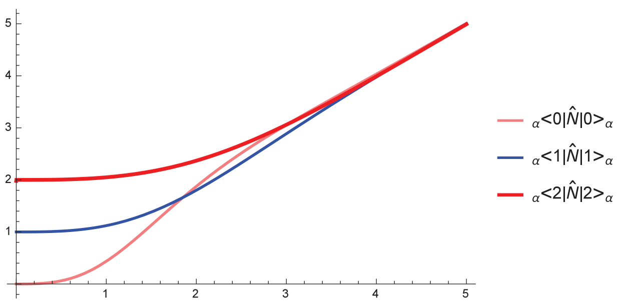

Figure 6.2. Photon numbers in trinity states

6.6. Matrix Representation of Operators in Trinity Basis

Main operators in our construction act in the Fock space, which becomes direct product of 3−dimensional subspaces. Operators acting in these subspaces are described by 3×3 matrices in trinity basis. Firstly, we find matrix representation of annihilation operator $\hat{a}$ as following;

$$\hat{a} = \begin{bmatrix} {}_\alpha\langle 0|\hat{a}|0\rangle_\alpha & {}_\alpha\langle 0|\hat{a}|1\rangle_\alpha & {}_\alpha\langle 0|\hat{a}|2\rangle_\alpha \\ {}_\alpha\langle 1|\hat{a}|0\rangle_\alpha & {}_\alpha\langle 1|\hat{a}|1\rangle_\alpha & {}_\alpha\langle 1|\hat{a}|2\rangle_\alpha \\ {}_\alpha\langle 2|\hat{a}|0\rangle_\alpha & {}_\alpha\langle 2|\hat{a}|1\rangle_\alpha & {}_\alpha\langle 2|\hat{a}|2\rangle_\alpha \end{bmatrix} \overset{(6.35)}{=} \alpha \begin{bmatrix} 0 & \frac{N_1}{N_0} & 0 \\ 0 & 0 & \frac{N_2}{N_1} \\ \frac{N_0}{N_2} & 0 & 0 \end{bmatrix}$$

As a result, $\hat{a}$ can be written in terms of normalization matrix and the 3×3 shift matrix Σ_1 as

$$\hat{a} = \alpha \begin{bmatrix} \frac{1}{N_0} & 0 & 0 \\ 0 & \frac{1}{N_1} & 0 \\ 0 & 0 & \frac{1}{N_2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} N_0 & 0 & 0 \\ 0 & N_1 & 0 \\ 0 & 0 & N_2 \end{bmatrix} = \alpha N^{-1} \Sigma_1^\dagger N \tag{6.47}$$

Since $\widehat{N}|n\rangle = n|n\rangle, n \geq 0$, the operator $q^{2\widehat{N}} = \left(e^{i\frac{2\pi}{3}}\right)^{\widehat{N}}$ acts on the trinity states as following

$$q^{2\widehat{N}}|0\rangle_\alpha = |0\rangle_\alpha, \quad q^{2\widehat{N}}|1\rangle_\alpha = q^2|1\rangle_\alpha, \quad q^{2\widehat{N}}|2\rangle_\alpha = q^4|2\rangle_\alpha\,. \tag{6.48}$$

Thus, operator $q^{2\widehat{N}}$ in trinity basis is represented by the clock matrix (Appendix C.3)

$$q^{2\widehat{N}} = \begin{bmatrix} {}_\alpha\langle 0|q^{2\widehat{N}}|0\rangle_\alpha & {}_\alpha\langle 0|q^{2\widehat{N}}|1\rangle_\alpha & {}_\alpha\langle 0|q^{2\widehat{N}}|2\rangle_\alpha \\ {}_\alpha\langle 1|q^{2\widehat{N}}|0\rangle_\alpha & {}_\alpha\langle 1|q^{2\widehat{N}}|1\rangle_\alpha & {}_\alpha\langle 1|q^{2\widehat{N}}|2\rangle_\alpha \\ {}_\alpha\langle 2|q^{2\widehat{N}}|0\rangle_\alpha & {}_\alpha\langle 2|q^{2\widehat{N}}|1\rangle_\alpha & {}_\alpha\langle 2|q^{2\widehat{N}}|2\rangle_\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & q^4 \end{bmatrix}.$$

Equation (6.48) gives the eigenvalue problem for the q^2 number operator $[\widehat{N}]_{q^2}$

$$\frac{q^{2\widehat{N}}-1}{q^2-1}|0\rangle_\alpha = \frac{1-1}{q^2-1}|0\rangle_\alpha \;\;\Rightarrow\;\; [\widehat{N}]_{q^2}|0\rangle_\alpha = [0]_{q^2}|0\rangle_\alpha\,, \tag{6.49}$$

$$\frac{q^{2\widehat{N}}-1}{q^2-1}|1\rangle_\alpha = \frac{q^2-1}{q^2-1}|1\rangle_\alpha \;\;\Rightarrow\;\; [\widehat{N}]_{q^2}|1\rangle_\alpha = [1]_{q^2}|1\rangle_\alpha\,, \tag{6.50}$$

$$\frac{q^{2\widehat{N}}-1}{q^2-1}|2\rangle_\alpha = \frac{q^4-1}{q^2-1}|2\rangle_\alpha \;\;\Rightarrow\;\; [\widehat{N}]_{q^2}|2\rangle_\alpha = [2]_{q^2}|2\rangle_\alpha\,. \tag{6.51}$$

and for the diagonal matrix elements

$$_\alpha\langle 0|[\widehat{N}]_{q^2}|0\rangle_\alpha = [0]_{q^2}, \quad _\alpha\langle 1|[\widehat{N}]_{q^2}|1\rangle_\alpha = [1]_{q^2}, \quad _\alpha\langle 2|[\widehat{N}]_{q^2}|2\rangle_\alpha = [2]_{q^2}\,. \tag{6.52}$$

Therefore, matrix representation of $[\widehat{N}]_{q^2}$ operator in trinity basis is

$$[\widehat{N}]_{q^2} = \begin{bmatrix} [0]_{q^2} & 0 & 0 \\ 0 & [1]_{q^2} & 0 \\ 0 & 0 & [2]_{q^2} \end{bmatrix},$$

where the eigenvalues are q^2 numbers $[0]_{q^2} = 0$, $[1]_{q^2} = 1$ and $[2]_{q^2} = \frac{1+i\sqrt{3}}{2}$.

6.7. Heinsenberg Uncertainty Relation for Trinity States

Here, we construct uncertainty relations for trinity states. By using direct calculations, we get

$$(\Delta\hat{q})_{|0\rangle_\alpha}(\Delta\hat{p})_{|0\rangle_\alpha} = \frac{\hbar}{2}\left(1 + 2\,{_\alpha\langle 0|\widehat{N}|0\rangle_\alpha}\right), \tag{6.53}$$

$$(\Delta\hat{q})_{|1\rangle_\alpha}(\Delta\hat{p})_{|1\rangle_\alpha} = \frac{\hbar}{2}\left(1 + 2\,{_\alpha\langle 1|\widehat{N}|1\rangle_\alpha}\right), \tag{6.54}$$

$$(\Delta\hat{q})_{|2\rangle_\alpha}(\Delta\hat{p})_{|2\rangle_\alpha} = \frac{\hbar}{2}\left(1 + 2\,{_\alpha\langle 2|\widehat{N}|2\rangle_\alpha}\right), \tag{6.55}$$

where ${}_{\alpha}\langle k|\widehat{N}|k\rangle_{\alpha}\,, 0 \leq k \leq 2$ are given by (6.39)-(6.41). In the limiting case, $|\alpha| \to 0$ follows from AS (6.46)

$$\lim_{|\alpha|^2\to 0} (\Delta\hat{q})_{|0\rangle_\alpha} (\Delta\hat{p})_{|0\rangle_\alpha} = \frac{\hbar}{2}\,, \tag{6.56}$$

$$\lim_{|\alpha|^2\to 0} (\Delta\hat{q})_{|1\rangle_\alpha} (\Delta\hat{p})_{|1\rangle_\alpha} = \frac{3\hbar}{2}\,, \tag{6.57}$$

$$\lim_{|\alpha|^2\to 0} (\Delta\hat{q})_{|2\rangle_\alpha} (\Delta\hat{p})_{|2\rangle_\alpha} = \frac{5\hbar}{2}\,. \tag{6.58}$$

This shows that uncertainty is growing with states number.

CHAPTER 7

QUARTET STATES

In this Chapter, the quartet states associated with $q^8 = 1$ roots of unity are treated in details. We define four states, rotated by angle $\frac{\pi}{2}$ and related with vertices of the square shown in Fig.(7.1). Superposition of these states, with choosen by proper coefficients, give us quartet of orthonormal basis states

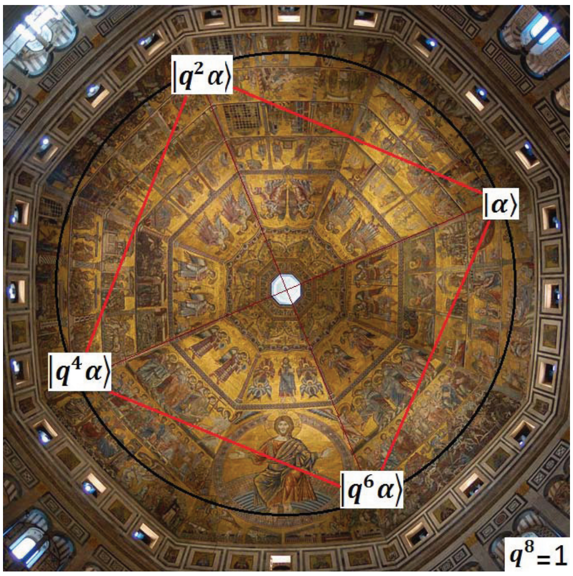

Figure 7.1. Quartet states

$$
\begin{aligned}
|0\rangle_\alpha &= e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle + |q^6\alpha\rangle}{\sqrt{4}\sqrt{e^{|\alpha|^2} + e^{q^2|\alpha|^2} + e^{q^4|\alpha|^2} + e^{q^6|\alpha|^2}}} \\
&= e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle + |q^6\alpha\rangle}{4\sqrt{{}_0e^{|\alpha|^2}(mod\,4)}},
\end{aligned} \tag{7.1}
$$

$$
\begin{aligned}
|1\rangle_\alpha &= e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \bar{q}^2|q^2\alpha\rangle + \bar{q}^4|q^4\alpha\rangle + \bar{q}^6|q^6\alpha\rangle}{\sqrt{4}\sqrt{e^{|\alpha|^2} + \bar{q}^2 e^{q^2|\alpha|^2} + \bar{q}^4 e^{q^4|\alpha|^2} + \bar{q}^6 e^{q^6|\alpha|^2}}} \\
&= e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \bar{q}^2|q^2\alpha\rangle + \bar{q}^4|q^4\alpha\rangle + \bar{q}^6|q^6\alpha\rangle}{4\sqrt{{}_1e^{|\alpha|^2}(mod\,4)}},
\end{aligned} \tag{7.2}
$$

$$
\begin{aligned}
|2\rangle_\alpha &= e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \bar{q}^4|q^2\alpha\rangle + |q^4\alpha\rangle + \bar{q}^4|q^6\alpha\rangle}{\sqrt{4}\sqrt{e^{|\alpha|^2} + \bar{q}^4 e^{q^2|\alpha|^2} + e^{q^4|\alpha|^2} + \bar{q}^4 e^{q^6|\alpha|^2}}} \qquad (7.3)\\
&= e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \bar{q}^4|q^2\alpha\rangle + |q^4\alpha\rangle + \bar{q}^4|q^6\alpha\rangle}{4\sqrt{{}_2e^{|\alpha|^2}(mod\,4)}},\\
|3\rangle_\alpha &= e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \bar{q}^6|q^2\alpha\rangle + \bar{q}^2|q^4\alpha\rangle + \bar{q}^4|q^6\alpha\rangle}{\sqrt{4}\sqrt{e^{|\alpha|^2} + \bar{q}^6 e^{q^2|\alpha|^2} + \bar{q}^4 e^{q^4|\alpha|^2} + \bar{q}^4 e^{q^6|\alpha|^2}}} \qquad (7.4)\\
&= e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \bar{q}^6|q^2\alpha\rangle + \bar{q}^2|q^4\alpha\rangle + \bar{q}^4|q^6\alpha\rangle}{4\sqrt{{}_3e^{|\alpha|^2}(mod\,4)}},
\end{aligned}
$$

We combine them in matrix form

$$
\begin{bmatrix} |0\rangle_\alpha \\ |1\rangle_\alpha \\ |2\rangle_\alpha \\ |3\rangle_\alpha \end{bmatrix} = \mathbf{N}\, \underbrace{\frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{q}^2 & \left(\bar{q}^2\right)^2 & \left(\bar{q}^2\right)^3 \\ 1 & \bar{q}^4 & \left(\bar{q}^4\right)^2 & \left(\bar{q}^4\right)^3 \\ 1 & \bar{q}^6 & \left(\bar{q}^6\right)^2 & \left(\bar{q}^6\right)^3 \end{bmatrix}}_{Quartet\ gate} \begin{bmatrix} |\alpha\rangle \\ |q^2\alpha\rangle \\ |q^4\alpha\rangle \\ |q^6\alpha\rangle \end{bmatrix},
$$

with normalization matrix, defined as

$$
\mathbf{N} = \begin{bmatrix} N_0 & 0 & 0 & 0 \\ 0 & N_1 & 0 & 0 \\ 0 & 0 & N_2 & 0 \\ 0 & 0 & 0 & N_3 \end{bmatrix} = \frac{e^{\frac{|\alpha|^2}{2}}}{\sqrt{4}} \begin{bmatrix} {}_0e^{|\alpha|^2} & 0 & 0 & 0 \\ 0 & {}_1e^{|\alpha|^2} & 0 & 0 \\ 0 & 0 & {}_2e^{|\alpha|^2} & 0 \\ 0 & 0 & 0 & {}_3e^{|\alpha|^2} \end{bmatrix}^{-1/2} (mod\,4), \qquad (7.5)
$$

obtained by using identity

$$
1 + \bar{q}^{2(n-k)} + \bar{q}^{4(n-k)} + \bar{q}^{6(n-k)} = 4\,\delta_{n,k(mod\,4)} \quad ,0 \leq k \leq 3 \qquad (7.6)
$$

with

$$
\delta_{n,k\,(mod\,4)} = \begin{cases} 1, & n = k\,(mod\,4); \\ 0, & n \neq k\,(mod\,4). \end{cases} \qquad (7.7)
$$

7.1. Quartet States in Terms of Cat States

The quartet states are superpositions of cat states with explicit form of phase shift as

$$\begin{aligned}
|0\rangle_\alpha &= \frac{N_0}{\sqrt{4}}\left[(|\alpha\rangle + |-\alpha\rangle) + (|i\alpha\rangle + |-i\alpha\rangle)\right], \\
|1\rangle_\alpha &= \frac{N_1}{\sqrt{4}}\left[(|\alpha\rangle - |-\alpha\rangle) - i(|i\alpha\rangle - |-i\alpha\rangle)\right], \\
|2\rangle_\alpha &= \frac{N_2}{\sqrt{4}}\left[(|\alpha\rangle + |-\alpha\rangle) - (|i\alpha\rangle + |-i\alpha\rangle)\right], \\
|3\rangle_\alpha &= \frac{N_3}{\sqrt{4}}\left[(|\alpha\rangle - |-\alpha\rangle) + i(|i\alpha\rangle - |-i\alpha\rangle)\right].
\end{aligned}$$

with $q^2 = e^{i\frac{\pi}{2}} = i$ and $q^4 = e^{i\pi} = -1$.

7.2. Mod 4 Form of Quartet States

By using $(mod\,4)$ exponential functions we get representation of these states in a compact form:

$$|0\rangle_\alpha = \frac{{}_0e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_0e^{|\alpha|^2}}}|0\rangle, \quad |1\rangle_\alpha = \frac{{}_1e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_1e^{|\alpha|^2}}}|0\rangle, \quad |2\rangle_\alpha = \frac{{}_2e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_2e^{|\alpha|^2}}}|0\rangle, \quad |3\rangle_\alpha = \frac{{}_3e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_3e^{|\alpha|^2}}}|0\rangle. \quad (7.8)$$

7.3. Eigenvalue Problem for Quartet States

As easy to see, the quartet states are eigenstates of operator $\hat{a}^4$ with eigenvalue α^4

$$\hat{a}^4|q^{2k}\alpha\rangle = \alpha^4|q^{2k}\alpha\rangle \quad \Rightarrow \quad \hat{a}^4|k\rangle_\alpha = \alpha^4|k\rangle_\alpha \quad k = 0, 1, 2, 3. \quad (7.9)$$

In terms of these states the ququat state is defined as

$$|\psi\rangle_\alpha = c_0|0\rangle_\alpha + c_1|1\rangle_\alpha + c_2|2\rangle_\alpha + c_3|3\rangle_\alpha, \tag{7.10}$$

where $|c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 = 1$. It describes a unit of quantum information with base 4, and it is an eigenstate of operator $\hat{a}^4$:

$$\hat{a}^4|\psi\rangle_\alpha = \alpha^4|\psi\rangle_\alpha. \tag{7.11}$$

7.4. Number of Photons in Quartet States

The annihilation operator $\hat{a}$ implements cyclic permutation of quartet states as following

$$\hat{a}|0\rangle_\alpha = \alpha\frac{N_0}{N_3}|3\rangle_\alpha\,,\ \hat{a}|1\rangle_\alpha = \alpha\frac{N_1}{N_0}|0\rangle_\alpha\,,\ \hat{a}|2\rangle_\alpha = \alpha\frac{N_2}{N_1}|1\rangle_\alpha\,,\ \hat{a}|3\rangle_\alpha = \alpha\frac{N_3}{N_2}|2\rangle_\alpha\,, \tag{7.12}$$

These equations are allowing us to calculate number of photons easily.

Proposition 7.1 *Number of photons in quartet states are defined as*

$$\begin{aligned}
{}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha &= |\alpha|^2\left[\frac{{}_3e^{|\alpha|^2}}{{}_0e^{|\alpha|^2}}\right] = |\alpha|^2\left[\frac{\sinh|\alpha|^2 - \sin|\alpha|^2}{\cosh|\alpha|^2 + \cos|\alpha|^2}\right],\\
{}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha &= |\alpha|^2\left[\frac{{}_0e^{|\alpha|^2}}{{}_1e^{|\alpha|^2}}\right] = |\alpha|^2\left[\frac{\cosh|\alpha|^2 + \cos|\alpha|^2}{\sinh|\alpha|^2 + \sin|\alpha|^2}\right],\\
{}_\alpha\langle 2|\widehat{N}|2\rangle_\alpha &= |\alpha|^2\left[\frac{{}_1e^{|\alpha|^2}}{{}_2e^{|\alpha|^2}}\right] = |\alpha|^2\left[\frac{\sinh|\alpha|^2 + \sin|\alpha|^2}{\cosh|\alpha|^2 - \cos|\alpha|^2}\right],\\
{}_\alpha\langle 3|\widehat{N}|3\rangle_\alpha &= |\alpha|^2\left[\frac{{}_2e^{|\alpha|^2}}{{}_3e^{|\alpha|^2}}\right] = |\alpha|^2\left[\frac{\cosh|\alpha|^2 - \cos|\alpha|^2}{\sinh|\alpha|^2 - \sin|\alpha|^2}\right].
\end{aligned}$$

As one can see, if we use definition of mod 4 exponential function, we get number of photons in terms of trigonometric and hyperbolic functions.

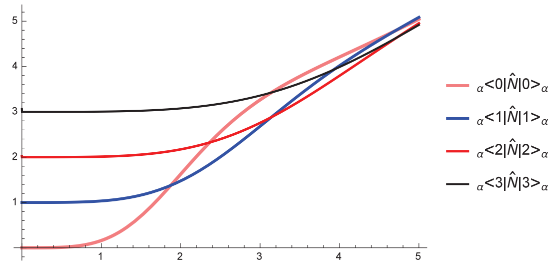

Figure 7.2. Photon numbers in quartet states

From the Fig.(7.2), it is clear that for small number of photons we have following limits as $|\alpha|^2 \rightarrow 0$,

$$\lim_{|\alpha|^2\rightarrow 0} {}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = 0\,, \qquad \lim_{|\alpha|^2\rightarrow 0} {}_\alpha\langle 1|\widehat{N}|1\rangle_\alpha = 1\,, \tag{7.13}$$

$$\lim_{|\alpha|^2\rightarrow 0} {}_\alpha\langle 2|\widehat{N}|2\rangle_\alpha = 2\,, \qquad \lim_{|\alpha|^2\rightarrow 0} {}_\alpha\langle 3|\widehat{N}|3\rangle_\alpha = 3\,. \tag{7.14}$$

7.5. Matrix Representation of Operators in Quartet Basis

Here, operators in the quartet basis as 4×4 matrices are constructed. We start with annihilation operator by using equation in (7.12);

$$\hat{a} = \alpha \begin{bmatrix} 0 & \frac{N_1}{N_0} & 0 & 0 \\ 0 & 0 & \frac{N_2}{N_1} & 0 \\ 0 & 0 & 0 & \frac{N_3}{N_2} \\ \frac{N_0}{N_3} & 0 & 0 & 0 \end{bmatrix}.$$

Then, this form can be written in terms of normalization matrix and 4 dimensional shift matrix;

$$\hat{a} = \alpha \begin{bmatrix} \frac{1}{N_0} & 0 & 0 & 0 \\ 0 & \frac{1}{N_1} & 0 & 0 \\ 0 & 0 & \frac{1}{N_2} & 0 \\ 0 & 0 & 0 & \frac{1}{N_3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} N_0 & 0 & 0 & 0 \\ 0 & N_1 & 0 & 0 \\ 0 & 0 & N_2 & 0 \\ 0 & 0 & 0 & N_3 \end{bmatrix} = \alpha N^{-1} \Sigma_1^\dagger N\,.$$

The matrix representation of q^2-number operator in quartet basis is derived from the following relations,

$$q^{2\widehat{N}}|k\rangle_\alpha = q^{2k}|k\rangle_\alpha\,, \tag{7.15}$$

where $k = 0, 1, 2, 3$. It is the four dimensional clock matrix

$$q^{2\widehat{N}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q^2 & 0 & 0 \\ 0 & 0 & q^4 & 0 \\ 0 & 0 & 0 & q^6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}.$$

Due to (7.15) , we have eigenvalue problem for operator $[\widehat{N}]_{q^2}$ in quartet basis

$$\begin{aligned} [\widehat{N}]_{q^2}|0\rangle_\alpha &= [0]_{q^2}|0\rangle_\alpha, \\ [\widehat{N}]_{q^2}|1\rangle_\alpha &= [1]_{q^2}|1\rangle_\alpha, \\ [\widehat{N}]_{q^2}|2\rangle_\alpha &= [2]_{q^2}|2\rangle_\alpha, \\ [\widehat{N}]_{q^2}|3\rangle_\alpha &= [3]_{q^2}|3\rangle_\alpha. \end{aligned}$$

with the eigenvalues as q^2-numbers $[0]_{q^2} = 0$, $[1]_{q^2} = 1$, $[2]_{q^2} = i + 1$ and $[3]_{q^2} = i$. Thus, representation of $[\widehat{N}]_{q^2}$ operator in quartet basis is the diagonal matrix.

$$[\widehat{N}]_{q^2} = \begin{bmatrix} [0]_{q^2} & 0 & 0 & 0 \\ 0 & [1]_{q^2} & 0 & 0 \\ 0 & 0 & [2]_{q^2} & 0 \\ 0 & 0 & 0 & [3]_{q^2} \end{bmatrix}$$

CHAPTER 8

KALEIDOSCOPE OF QUANTUM COHERENT STATES

As a generalization of previous results, here we consider superposition of n coherent states, which are belonging to vertices of regular n-polygon and are rotated by angle $\frac{\pi}{n}$ Fig.(8.1). It is related with primitive roots of unity $q^{2n} = e^{i\frac{\pi}{n}} = 1$.

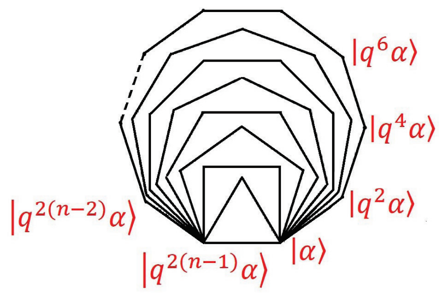

Figure 8.1. The regular n-polygon states

First, we start to define $|0\rangle_\alpha$ with normalization constant N_0, as a superposition of rotated states in Fig.(8.1);

$$\begin{aligned} |0\rangle_\alpha &= N_0\left(|\alpha\rangle + |q^2\alpha\rangle + |q^4\alpha\rangle + ... + |q^{2(n-1)}\alpha\rangle\right) \\ &= N_0 e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} (|n\rangle + q^{2n}|n\rangle + q^{4n}|n\rangle + ... + q^{2(n-1)}|n\rangle) \\ &= N_0 e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left(1 + q^{2n} + q^{4n} + ... + q^{2(n-1)}\right)|n\rangle) \,. \end{aligned} \tag{8.1}$$

By using identity, which is proven in Appendix A.3.1, for $q^{2n} = 1$,

$$1 + q^{2m} + q^{4m} + ... + q^{2m(n-1)} = n\delta_{m,0(mod\,n)} \tag{8.2}$$

where

$$\delta_{m,0\,(mod\,n)} = \begin{cases} 1, & m = 0\,(mod\,n); \\ 0, & m \neq 0\,(mod\,n)\,, \end{cases} \tag{8.3}$$

it gives

$$|0\rangle_\alpha = nN_0 e^{\frac{-|\alpha|^2}{2}} \sum_{k=0}^{\infty} \frac{\alpha^{nk}}{\sqrt{(nk)!}} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} = nN_0 e^{\frac{-|\alpha|^2}{2}} \sum_{k=0}^{\infty} \frac{\alpha^{nk}}{\sqrt{(nk)!}} |nk\rangle\,. \tag{8.4}$$

where the state $|nk\rangle$ has the unit element at $(nk+1)-$th row. Then, normalized state $|0\rangle_\alpha$ becomes

$$|0\rangle_\alpha = \frac{e^{\frac{|\alpha|^2}{2}}}{\sqrt{n \sum_{k=0}^{n-1} e^{q^{2k}|\alpha|^2}}} \sum_{k=0}^{n-1} |q^{2k}\alpha\rangle = \frac{e^{\frac{|\alpha|^2}{2}}}{n\sqrt{{}_0 e^{|\alpha|^2}(mod\,n)}} \sum_{k=0}^{n-1} |q^{2k}\alpha\rangle \tag{8.5}$$

The other $n-1$ states, which are orthogonal to $|0\rangle_\alpha$ and to each other, are denoted by $\widetilde{|1\rangle}_\alpha\,, \widetilde{|2\rangle}_\alpha\,, ..., \widetilde{|n-1\rangle}_\alpha$, so that ${}_\alpha\widetilde{\langle k|l\rangle}_\alpha = 0, 0 \leq k, l \leq n-1$. We can build these states by using orthogonality relations and following inner products of q^{2k} rotated coherent states;

$$\langle q^{2k}\alpha|q^{2k}\alpha\rangle = 1 \quad , \quad \langle q^{2k}\alpha|q^{2l}\alpha\rangle = e^{|\alpha|^2(q^{2(l-k)}-1)},\ 0 \leq k, l \leq n-1. \tag{8.6}$$

Our construction shows that this set of orthogonal states can be described by the $n \times n$ matrix

$$\begin{bmatrix} \widetilde{|0\rangle}_\alpha \\ \widetilde{|1\rangle}_\alpha \\ \widetilde{|2\rangle}_\alpha \\ \widetilde{|3\rangle}_\alpha \\ \vdots \\ \widetilde{|n-1\rangle}_\alpha \end{bmatrix} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \bar{q}^2 & \bar{q}^4 & \dots & \bar{q}^{2(n-1)} \\ 1 & \bar{q}^4 & \bar{q}^8 & \dots & \bar{q}^{4(n-1)} \\ 1 & \bar{q}^6 & \bar{q}^{12} & \dots & \bar{q}^{6(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \bar{q}^{2(n-1)} & \bar{q}^{4(n-1)} & \dots & \bar{q}^{2(n-1)^2} \end{bmatrix} \begin{bmatrix} |\alpha\rangle \\ |q^2\alpha\rangle \\ |q^4\alpha\rangle \\ |q^6\alpha\rangle \\ \vdots \\ |q^{2(n-1)}\alpha\rangle \end{bmatrix}, \tag{8.7}$$

where $\bar{q}^2 = e^{\frac{-2\pi i}{n}}$ is the n-th root of unity, so that corresponding transformation is

$$|\widetilde{k}\rangle_\alpha = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \bar{q}^{2jk} |q^{2j}\alpha\rangle = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} Q_{jk} |q^{2j}\alpha\rangle \quad 0 \le k \le n-1, , \tag{8.8}$$

This is the Quantum Fourier transformation Q, which represents the unitary gate.

8.1. Construction of Kaleidoscope States

Here, from the states $|\widetilde{k}\rangle_\alpha$ in (8.7), we obtain normalized states $|k\rangle_\alpha$

$$\begin{bmatrix} |0\rangle_\alpha \\ |1\rangle_\alpha \\ |2\rangle_\alpha \\ \vdots \\ |n-2\rangle_\alpha \\ |n-1\rangle_\alpha \end{bmatrix} = \mathbf{N} \begin{bmatrix} |\widetilde{0}\rangle_\alpha \\ |\widetilde{1}\rangle_\alpha \\ |\widetilde{2}\rangle_\alpha \\ |\widetilde{3}\rangle_\alpha \\ \vdots \\ |\widetilde{n-1}\rangle_\alpha \end{bmatrix} = \mathbf{NQ} \begin{bmatrix} |\alpha\rangle \\ |q^2\alpha\rangle \\ |q^4\alpha\rangle \\ \vdots \\ |q^{2(n-2)}\alpha\rangle \\ |q^{2(n-1)}\alpha\rangle \end{bmatrix},$$

which we called as "Kaleidoscope of Quantum coherent states." Here, **N** is normalization matrix and **Q** is the Quantum Fourier transformation.
Explicitly it gives

$$\begin{bmatrix} |0\rangle_\alpha \\ |1\rangle_\alpha \\ |2\rangle_\alpha \\ \vdots \\ |n-1\rangle_\alpha \end{bmatrix} = \frac{e^{\frac{|\alpha|^2}{2}}}{\sqrt{n}} \begin{bmatrix} {}_0e^{|\alpha|^2} & 0 & 0 & \dots & 0 \\ 0 & {}_1e^{|\alpha|^2} & 0 & \dots & 0 \\ 0 & 0 & {}_2e^{|\alpha|^2} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & {}_{(n-1)}e^{|\alpha|^2} \end{bmatrix}^{-\frac{1}{2}} \begin{bmatrix} |\widetilde{0}\rangle_\alpha \\ |\widetilde{1}\rangle_\alpha \\ |\widetilde{2}\rangle_\alpha \\ \vdots \\ |\widetilde{n-1}\rangle_\alpha \end{bmatrix} \tag{8.9}$$

or

$$|k\rangle_\alpha = \frac{e^{\frac{|\alpha|^2}{2}}}{n\sqrt{{}_ke^{|\alpha|^2}}} \sum_{j=0}^{n-1} \bar{q}^{2jk} |q^{2j}\alpha\rangle \quad 0 \le k \le n-1, . \tag{8.10}$$

In terms of ($mod\ n$) exponential functions(see Section 4.6)

$$f_k(|\alpha|^2) = {}_ke^{|\alpha|^2}(mod\, n) \equiv \sum_{s=0}^{\infty} \frac{(|\alpha|^2)^{ns+k}}{(ns+k)!}\ , \quad 0 \le k \le n-1\,, \tag{8.11}$$

These functions have been introduced in Chapter 4 and they represent superposition of standard exponentials

$${}_ke^{|\alpha|^2}(mod\, n) = \frac{1}{n}\sum_{s=0}^{n-1} \bar{q}^{2sk} e^{q^{2k}|\alpha|^2}\ , \quad 0 \le k \le n-1\,, \tag{8.12}$$

related to each other by derivatives

$$\frac{\partial}{\partial|\alpha|^2}\left[{}_ke^{|\alpha|^2}\right] = {}_{k-1}e^{|\alpha|^2}\ , \quad \frac{\partial}{\partial|\alpha|^2}\left[{}_0e^{|\alpha|^2}\right] ={}_{n-1}\ e^{|\alpha|^2}\ . \tag{8.13}$$

This reflects the cyclic permutation of states by annihilation operator in Fock-Bargmann representation.

8.2. Mod n Form of Kaleidoscope States

By using mod n functions introduced in Chapter 4, we derive compact expression for the kaleidoscope states.

Proposition 8.1 *Kaleidoscope of Quantum coherent states has the form*

$$|k\rangle_\alpha = \frac{{}_ke^{\alpha\hat{a}^\dagger}}{\sqrt{{}_ke^{|\alpha|^2}}}|0\rangle \quad (mod\, n),\ \ 0 \le k \le n-1. \tag{8.14}$$

Proof Since

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^\dagger}|0\rangle \ \Rightarrow\ |q^{2j}\alpha\rangle \ = \ e^{-\frac{1}{2}|q^{2j}\alpha|^2} e^{q^{2j}\alpha\hat{a}^\dagger}|0\rangle \tag{8.15}$$

$$= \ e^{-\frac{1}{2}|\alpha|^2} e^{q^{2j}\alpha\hat{a}^\dagger}|0\rangle\,,\ 0 \le j \le n-1\,, \tag{8.16}$$

then kaleidoscope states can be derived from $|0\rangle$ state as

$$|k\rangle_\alpha = \frac{e^{\frac{|\alpha|^2}{2}}}{n\sqrt{{}_k e^{|\alpha|^2}}}\sum_{j=0}^{n-1}\bar{q}^{2jk}|q^{2j}\alpha\rangle \overset{(8.16)}{=} \frac{1}{\sqrt{{}_k e^{|\alpha|^2}}}\frac{1}{n}\sum_{j=0}^{n-1}\bar{q}^{2jk}e^{q^{2j}\alpha\hat{a}^\dagger}|0\rangle \tag{8.17}$$

The summation coincides with definiton of mod n function in equation (8.12), so that

$$|k\rangle_\alpha = \frac{{}_k e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_k e^{|\alpha|^2}}}|0\rangle \quad (mod\, n), \quad 0 \le k \le n-1. \tag{8.18}$$

□

8.3. Eigenvalue Problem for Kaleidoscope States

The kaleidoscope quantum coherent states are eigenstates of operator $\hat{a}^n$, with eigenvalue α^n and

$$\hat{a}^n|q^{2k}\alpha\rangle = \alpha^n|q^{2k}\alpha\rangle \quad \Rightarrow \quad \hat{a}^n|k\rangle_\alpha = \alpha^n|k\rangle_\alpha \quad k = 0, ..., n-1. \tag{8.19}$$

Taking superposition of kaleidoscope states

$$|\psi\rangle_\alpha = \sum_{k=0}^{n-1} C_k|k\rangle_\alpha, \tag{8.20}$$

with normalization $\sum_{k=0}^{n}|C_k|^2 = 1$, the qudit state is defined. The qudit describes a unit of quantum information with base n, and it is an eigenstate of operator $\hat{a}^n$:

$$\hat{a}^n|\psi\rangle_\alpha = \alpha^n|\psi\rangle_\alpha. \tag{8.21}$$

8.4. Number of Photons in Kaleidoscope States

Due to the compact form of kaleidoscope states in equation (8.14), we can use application of annihilation operator to calculate number of photons easily. For calculation we need following lemma.

Lemma 8.1 *If $f(z)$ is an analytic function, then for the operator argument $\hat{a}^\dagger$*

$$f(\hat{a}^\dagger) = \sum_{n=0}^{\infty} C_n (\hat{a}^\dagger)^n \,, \tag{8.22}$$

the following commutation relation valid

$$\left[\hat{a}, f(\hat{a}^\dagger)\right] = \frac{d}{d\hat{a}^\dagger} f(\hat{a}^\dagger) \,. \tag{8.23}$$

Proof The commutator of $\hat{a}$ and $f(\hat{a}^\dagger)$ is

$$\left[\hat{a}, f(\hat{a}^\dagger)\right] = \hat{a} f(\hat{a}^\dagger) - f(\hat{a}^\dagger)\hat{a} = \sum_{n=0}^{\infty} C_n \left[\hat{a}, (\hat{a}^\dagger)^n\right] = \sum_{n=0}^{\infty} C_n n (\hat{a}^\dagger)^{n+1} = \frac{d}{d\hat{a}^\dagger} f(\hat{a}^\dagger) \,. \tag{8.24}$$

□

Thus it follows that

$$\hat{a} f(\hat{a}^\dagger) = \frac{d}{d\hat{a}^\dagger} f(\hat{a}^\dagger) + f(\hat{a}^\dagger)\hat{a}. \tag{8.25}$$

Proposition 8.2 *The kaleidoscope of quantum states is generated by annihilation operator $\hat{a}$ as cyclic permutation of states;*

$$\hat{a}|0\rangle_\alpha = \alpha \frac{N_0}{N_{n-1}} |n-1\rangle_\alpha \,, \tag{8.26}$$

$$\hat{a}|k\rangle_\alpha = \alpha \frac{N_k}{N_{k-1}} |k-1\rangle_\alpha \,, 1 \le k \le n-1 \,, \tag{8.27}$$

Proof The proof is splitted into two parts for states $|0\rangle_\alpha$ and $|k\rangle_\alpha$, $1 \le k \le n-1$. Firstly, application of annihilation operator $\hat{a}$ to $|0\rangle_\alpha$ state by using equation (8.14)

$$\hat{a}|0\rangle_\alpha = \frac{\hat{a}\,{}_0e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_0e^{|\alpha|^2}}}|0\rangle \overset{(8.25)}{=} \frac{1}{\sqrt{{}_0e^{|\alpha|^2}}}\left(\frac{d}{d\hat{a}^\dagger}\,{}_0e^{\alpha\hat{a}^\dagger} + {}_0e^{\alpha\hat{a}^\dagger}\hat{a}\right)|0\rangle \tag{8.28}$$

$$= \alpha\frac{1}{\sqrt{{}_0e^{|\alpha|^2}}}\left({}_{n-1}e^{\alpha\hat{a}^\dagger}|0\rangle + {}_0e^{\alpha\hat{a}^\dagger}\hat{a}|0\rangle\right) \tag{8.29}$$

gives

$$\hat{a}|0\rangle_\alpha = \frac{\alpha}{\sqrt{{}_0e^{|\alpha|^2}}}\,{}_{n-1}e^{\alpha\hat{a}^\dagger}\frac{\sqrt{{}_{n-1}e^{|\alpha|^2}}}{\sqrt{{}_{n-1}e^{|\alpha|^2}}}|0\rangle = \alpha\frac{\sqrt{{}_{n-1}e^{|\alpha|^2}}}{\sqrt{{}_0e^{|\alpha|^2}}}\frac{{}_{n-1}e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_{n-1}e^{|\alpha|^2}}}|0\rangle \tag{8.30}$$

$$= \alpha\frac{\sqrt{{}_{n-1}e^{|\alpha|^2}}}{\sqrt{{}_0e^{|\alpha|^2}}}|n-1\rangle_\alpha \tag{8.31}$$

$$= \alpha\frac{N_0}{N_{n-1}}|n-1\rangle_\alpha \quad (mod\, n)\,. \tag{8.32}$$

Following the same steps for states $|k\rangle_\alpha$, $1 \le k \le n-1$, we have

$$\hat{a}|k\rangle_\alpha = \frac{\hat{a}\,{}_ke^{\alpha\hat{a}^\dagger}}{\sqrt{{}_ke^{|\alpha|^2}}}|0\rangle \overset{(8.25)}{=} \frac{1}{\sqrt{{}_ke^{|\alpha|^2}}}\left(\frac{d}{d\hat{a}^\dagger}\,{}_ke^{\alpha\hat{a}^\dagger} + {}_ke^{\alpha\hat{a}^\dagger}\hat{a}\right)|0\rangle \tag{8.33}$$

$$= \frac{1}{\sqrt{{}_ke^{|\alpha|^2}}}\left(\alpha\,{}_{k-1}e^{\alpha\hat{a}^\dagger}|0\rangle + {}_ke^{\alpha\hat{a}^\dagger}\hat{a}|0\rangle\right) \tag{8.34}$$

$$= \frac{\alpha}{\sqrt{{}_ke^{|\alpha|^2}}}\,{}_{k-1}e^{\alpha\hat{a}^\dagger}\frac{\sqrt{{}_{k-1}e^{|\alpha|^2}}}{\sqrt{{}_{k-1}e^{|\alpha|^2}}}|0\rangle \tag{8.35}$$

$$= \alpha\frac{\sqrt{{}_{k-1}e^{|\alpha|^2}}}{\sqrt{{}_ke^{|\alpha|^2}}}\frac{{}_{k-1}e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_{k-1}e^{|\alpha|^2}}}|0\rangle \tag{8.36}$$

$$= \alpha\frac{\sqrt{{}_{k-1}e^{|\alpha|^2}}}{\sqrt{{}_ke^{|\alpha|^2}}}|k-1\rangle_\alpha = \frac{N_k}{N_{k-1}}|k-1\rangle_\alpha \quad (mod\, n)\,. \tag{8.37}$$

□

Now, we are ready to calculate number of photons in Kaleidoscope of quantum coherent states.

Proposition 8.3 *Average number of photons in kaleidoscope of coherent states is given as*

$$
\begin{aligned}
{}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha &= |\alpha|^2 \left(\frac{{}_{n-1}e^{|\alpha|^2}}{{}_0e^{|\alpha|^2}} \right) , &(8.38)\\
{}_\alpha\langle k|\widehat{N}|k\rangle_\alpha &= |\alpha|^2 \left(\frac{{}_{k-1}e^{|\alpha|^2}}{{}_ke^{|\alpha|^2}} \right) , 1 \leq k \leq n-1 . &(8.39)
\end{aligned}
$$

Proof To derive number of photons in kaleidoscope states, first we use equation (8.26) so that

$$
\begin{aligned}
{}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = {}_\alpha\langle 0|\hat{a}^\dagger\hat{a}|0\rangle_\alpha &= \left({}_\alpha\langle n-1|\frac{N_0}{N_{n-1}}\overline{\alpha}\right)\left(\alpha\frac{N_0}{N_{n-1}}|n-1\rangle_\alpha\right)\\
&= \overline{\alpha}\alpha\left(\frac{N_0}{N_n-1}\right)^2 {}_\alpha\langle n-1|n-1\rangle_\alpha &(8.40)\\
&= |\alpha|^2\left(\frac{N_0}{N_n-1}\right)^2 = |\alpha|^2 \left(\frac{{}_{n-1}e^{|\alpha|^2}}{{}_0e^{|\alpha|^2}} \right) . &(8.41)
\end{aligned}
$$

Similar calculations give (see (8.27))

$$
\begin{aligned}
{}_\alpha\langle k|\widehat{N}|k\rangle_\alpha = {}_\alpha\langle k|\hat{a}^\dagger\hat{a}|k\rangle_\alpha &= \left({}_\alpha\langle k-1|\frac{N_k}{N_{k-1}}\overline{\alpha}\right)\left(\alpha\frac{N_k}{N_{k-1}}|k-1\rangle_\alpha\right)\\
&= \overline{\alpha}\alpha\left(\frac{N_k}{N_k-1}\right)^2 {}_\alpha\langle k-1|k-1\rangle_\alpha &(8.42)\\
&= |\alpha|^2\left(\frac{N_k}{N_k-1}\right)^2 = |\alpha|^2 \left(\frac{{}_{k-1}e^{|\alpha|^2}}{{}_ke^{|\alpha|^2}} \right) , 1 \leq k \leq n-1 . &(8.43)
\end{aligned}
$$

□

In the limiting case $|\alpha|^2 \equiv x \rightarrow 0$, we have following results. Number of photons in kaleidoscope state have the limit for state $|0\rangle_\alpha$:

$$\lim_{x\to 0} {}_\alpha\langle 0|\widehat{N}|0\rangle_\alpha = \lim_{x\to\infty} x \left[\frac{{}_{n-1}e^x}{{}_0e^x}\right] (mod\, n) \tag{8.44}$$

$$= \lim_{x\to 0} x \frac{\sum_{s=0}^{\infty} \frac{x^{ns+(n-1)}}{(ns+(n-1))!}}{\sum_{s=0}^{\infty} \frac{x^{ns}}{(ns)!}} \tag{8.45}$$

$$= \lim_{x\to 0} x x^{n-1} \left(\frac{\frac{1}{(n-1)!} + \frac{x^n}{(2n-1)!} + \frac{x^2n}{(3n-1)!} + ...}{1 + \frac{x^n}{(n)!} + \frac{x^2n}{(2n)!} + ...} \right) \tag{8.46}$$

$$= \lim_{x\to 0} \frac{x^n}{(n-1)!} = 0 \tag{8.47}$$

and for states $|k\rangle_\alpha$, $1 \le k \le n-1$;

$$\lim_{x\to 0} {}_\alpha\langle k|\widehat{N}|k\rangle_\alpha = \lim_{x\to\infty} x \left[\frac{{}_{k-1}e^x}{{}_ke^x}\right] (mod\, n) = \lim_{x\to 0} x \frac{\sum_{s=0}^{\infty} \frac{x^{ns+(k-1)}}{(ns+(k-1))!}}{\sum_{s=0}^{\infty} \frac{x^{ns+k}}{(ns+k)!}} \tag{8.48}$$

$$= \lim_{x\to 0} x \frac{x^{k-1}}{x^k} \left(\frac{\frac{1}{(k-1)!} + \frac{x^n}{((n+k)-1)!} + \frac{x^2n}{(2n+k)-1)!} + ...}{\frac{1}{k!} + \frac{x^n}{(n+k)!} + \frac{x^2n}{(2n+k)!} + ...} \right) \tag{8.49}$$

$$= \lim_{x\to 0} \frac{k!}{(k-1)!} = k \tag{8.50}$$

Asymptotically, for small occupation numbers they approach the integers k values

$$\lim_{|\alpha|\to 0} {}_\alpha\langle k|\widehat{N}|k\rangle_\alpha = k\,, 0 \le k \le n-1\,. \tag{8.51}$$

8.5. Matrix Representation of Operators in Kaleidoscope Basis

The operators in kaleidoscope basis are described by $n \times n$ matrix. We start with construction of annihilation operator $\hat{a}$ with matrix elements

$$ {}_\alpha\langle k|\hat{a}|l\rangle_\alpha = \hat{a}_{kl}\,, 0 \le k, l \le n-1\,. \tag{8.52}$$

In the kaleidoscope basis, since annihilation operator $\hat{a}$ applied to a state produces the orthogonal one, only $l = 0, k = n-1$

$$ {}_\alpha\langle n-1|\hat{a}|0\rangle_\alpha \overset{(8.26)}{=} {}_\alpha\langle n-1|\left(\alpha\frac{N_0}{N_{n-1}}|n-1\rangle_\alpha\right) = \alpha\frac{N_0}{N_{n-1}}\,{}_\alpha\langle n-1|n-1\rangle_\alpha = \alpha\frac{N_0}{N_{n-1}} \tag{8.53}$$

and $l = k+1\,, 0 \le k \le n-1$

$$ {}_\alpha\langle k|\hat{a}|k+1\rangle_\alpha \overset{(8.27)}{=} {}_\alpha\langle k|\left(\alpha\frac{N_{k+1}}{N_k}|k\rangle_\alpha\right) = \alpha\frac{N_{k+1}}{N_k}\,{}_\alpha\langle k|k\rangle_\alpha = \alpha\frac{N_{k+1}}{N_k} \tag{8.54}$$

terms survive. Then, the matrix representation of annihilation operator $\hat{a}$ is

$$\hat{a} = \alpha\begin{bmatrix} 0 & \frac{N_1}{N_0} & 0 & 0 & ... & 0 \\ 0 & 0 & \frac{N_2}{N_1} & 0 & ... & 0 \\ 0 & 0 & 0 & \frac{N_3}{N_2} & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & ... & \frac{N_{n-1}}{N_{n-2}} \\ \frac{N_0}{N_{n-1}} & 0 & 0 & 0 & ... & 0 \end{bmatrix} = \alpha N^{-1}\Sigma_1^\dagger N\,, \tag{8.55}$$

where N is normalization matrix and Σ_1 is the $n \times n$ shift matrix. To construct matrix representation of the operator $q^{2\widehat{N}} = \left(e^{i\frac{2\pi}{n}}\right)^{\widehat{N}}$, we have following proposition

Proposition 8.4 *Let $q^{2n} = 1$, the operator $q^{2\widehat{N}}$ acts on the kaleidoscope states as*

$$ q^{2\widehat{N}}|k\rangle_\alpha = q^{2k}|k\rangle_\alpha\,, 0 \le k \le n-1\,. \tag{8.56}$$

Proof We prove the statement by using compact form of kaleidoscope states in mod n representation

$$
\begin{aligned}
q^{2\widehat{N}}|k\rangle_{\alpha} &= q^{2\widehat{N}} \frac{{}_k e^{\alpha\hat{a}^{\dagger}}}{\sqrt{{}_k e^{|\alpha|^2}}}|0\rangle \quad (mod\, n) = \frac{q^{2\widehat{N}}}{\sqrt{{}_k e^{|\alpha|^2}}} \sum_{s=0}^{\infty} \frac{\left(\alpha\hat{a}^{\dagger}\right)^{ns+k}}{(ns+k)!}|0\rangle \\
&= \frac{q^{2\widehat{N}}}{\sqrt{{}_k e^{|\alpha|^2}}} \sum_{s=0}^{\infty} \frac{(\alpha)^{ns+k}}{\sqrt{(ns+k!)}} \frac{(\hat{a}^{\dagger})^{ns+k}}{\sqrt{(ns+k!)}} .
\end{aligned}
$$

From (3.5) , we have

$$
\begin{aligned}
\frac{q^{2\widehat{N}}}{\sqrt{{}_k e^{|\alpha|^2}}} \sum_{s=0}^{\infty} \frac{(\alpha)^{ns+k}}{\sqrt{(ns+k)!}}|ns+k\rangle &= \frac{1}{\sqrt{{}_k e^{|\alpha|^2}}} \sum_{s=0}^{\infty} \frac{(\alpha)^{ns+k}}{\sqrt{(ns+k)!}} q^{2\widehat{N}}|ns+k\rangle \\
&= \frac{1}{\sqrt{{}_k e^{|\alpha|^2}}} \sum_{s=0}^{\infty} \frac{(\alpha)^{ns+k}}{\sqrt{(ns+k)!}} q^{2k}|ns+k\rangle ,
\end{aligned}
$$

due to $\widehat{N}|n\rangle = n|n\rangle \Rightarrow q^{2\widehat{N}}|ns+k\rangle = q^{2(ns+k)}|ns+k\rangle = q^{2k}|ns+k\rangle$, then

$$
q^{2\widehat{N}}|k\rangle_{\alpha} = \frac{q^{2k}}{\sqrt{{}_k e^{|\alpha|^2}}} \sum_{s=0}^{\infty} \frac{\left(\alpha\hat{a}^{\dagger}\right)^{ns+k}}{(ns+k)!}|0\rangle = q^{2k} \frac{{}_k e^{\alpha\hat{a}^{\dagger}}}{\sqrt{{}_k e^{|\alpha|^2}}}|0\rangle = q^{2k}|k\rangle_{\alpha}. \tag{8.57}
$$

□

Therefore, operator $q^{2\widehat{N}}$ has diagonal form in the kaleidoscope basis, which is represented by the clock matrix Σ_3;

$$
q^{2\widehat{N}} = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & q^2 & 0 & ... & 0 \\ 0 & 0 & q^4 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & q^{2(n-1)} \end{bmatrix} , \tag{8.58}
$$

with diagonal matrix elements ${}_{\alpha}\langle k|q^{2\widehat{N}}|k\rangle_{\alpha} = q^{2k}$, $0 \le k \le n-1$. Equation (8.56) becomes the eigenvalue problem for the q^2 number operator $[\widehat{N}]_{q^2}$ as

$$[\widehat{N}]_{q^2}|k\rangle_{\alpha} = [k]_{q^2}|k\rangle_{\alpha} \,, 0 \le k \le n-1 \,. \tag{8.59}$$

Therefore, matrix representation of $[\widehat{N}]_{q^2}$ operator in kaleidoscope basis is

$$[\widehat{N}]_{q^2} = \begin{bmatrix} [0]_{q^2} & 0 & 0 & \dots & 0 \\ 0 & [1]_{q^2} & 0 & \dots & 0 \\ 0 & 0 & [2]_{q^2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & [n-1]_{q^2} \end{bmatrix} , \tag{8.60}$$

where the diagonal elements are q^2 numbers $[k]_{q^2} = \frac{q^{2k-1}}{q^2-1}$, $0 \le k \le n-1$.

8.6. Heinsenberg Uncertainty Relation for Kaleidoscope States

Here, we calculate uncertainty relation for Kaleidoscope states. For this, we need to find expectation value of $\hat{q}$ and $\hat{p}$ operators for state $|k\rangle_{\alpha}, 0 \le k \le n-1$, as following

$$\begin{aligned} \langle \hat{x} \rangle_{|k\rangle_{\alpha}} = {}_{\alpha}\langle k|\hat{x}|k\rangle_{\alpha} &= {}_{\alpha}\langle k| \sqrt{\frac{\hbar}{2}}(\hat{a} + \hat{a}^{\dagger})|k\rangle_{\alpha} \\ &= \sqrt{\frac{\hbar}{2}}\left({}_{\alpha}\langle k|\hat{a}|k\rangle_{\alpha} + {}_{\alpha}\langle k|\hat{a}^{\dagger}|k\rangle_{\alpha}\right) = 0 \,, \end{aligned} \tag{8.61}$$

and

$$\begin{aligned} \langle \hat{p} \rangle_{|k\rangle_{\alpha}} = {}_{\alpha}\langle k|\hat{p}|k\rangle_{\alpha} &= {}_{\alpha}\langle k| - i\sqrt{\frac{\hbar}{2}}(\hat{a} - \hat{a}^{\dagger})|k\rangle_{\alpha} \\ &= -i\sqrt{\frac{\hbar}{2}}\left({}_{\alpha}\langle k|\hat{a}|k\rangle_{\alpha} + {}_{\alpha}\langle k|\hat{a}^{\dagger}|k\rangle_{\alpha}\right) = 0 \,. \end{aligned} \tag{8.62}$$

It happens due to that application of $\hat{a}$ to state $|k\rangle_\alpha$, creating the orthogonal states to $|k\rangle_\alpha$ according to (8.26) – (8.27). Then, for state $|k\rangle_\alpha$ the variance in coordinate operator is

$$\begin{aligned}
\langle \hat{x}^2 \rangle_{|k\rangle_\alpha} = {}_\alpha\langle k|\hat{x}^2|k\rangle_\alpha &= \frac{\hbar}{2}{}_\alpha\langle k|\left(\hat{a}+\hat{a}^\dagger\right)^2|k\rangle_\alpha \\
&\overset{(3.3)}{=} \frac{\hbar}{2}{}_\alpha\langle k|\left(\hat{a}^2+2\hat{a}^\dagger\hat{a}+\hat{\mathrm{I}}+(\hat{a}^\dagger)^2\right)|k\rangle_\alpha \\
&= \frac{\hbar}{2}\left({}_\alpha\langle k|\hat{a}^2|k\rangle_\alpha + 2{}_\alpha\langle k|\hat{a}^\dagger\hat{a}|k\rangle_\alpha + {}_\alpha\langle k|k\rangle_\alpha + {}_\alpha\langle k|(\hat{a}^\dagger)^2|k\rangle_\alpha\right) \\
&= \frac{\hbar}{2}\left(1+2{}_\alpha\langle k|\widehat{N}|k\rangle_\alpha\right) = \frac{\hbar}{2}\left(1+2|\alpha|^2\frac{N_k^2}{N_{k-1}^2}\right), \qquad (8.63)
\end{aligned}$$

and variance in momentum operator is

$$\begin{aligned}
\langle \hat{p}^2 \rangle_{|k\rangle_\alpha} = {}_\alpha\langle k|\hat{p}^2|k\rangle_\alpha &= -\frac{\hbar}{2}{}_\alpha\langle k|\left(\hat{a}-\hat{a}^\dagger\right)^2|k\rangle_\alpha \\
&\overset{(3.3)}{=} -\frac{\hbar}{2}{}_\alpha\langle k|\left(\hat{a}^2-2\hat{a}^\dagger\hat{a}-\hat{\mathrm{I}}+(\hat{a}^\dagger)^2\right)|k\rangle_\alpha \\
&= -\frac{\hbar}{2}\left({}_\alpha\langle k|\hat{a}^2|k\rangle_\alpha - 2{}_\alpha\langle k|\hat{a}^\dagger\hat{a}|k\rangle_\alpha - {}_\alpha\langle k|k\rangle_\alpha + {}_\alpha\langle k|(\hat{a}^\dagger)^2|k\rangle_\alpha\right) \\
&= \frac{\hbar}{2}\left(1+2{}_\alpha\langle k|\widehat{N}|k\rangle_\alpha\right) = \frac{\hbar}{2}\left(1+2|\alpha|^2\frac{N_k^2}{N_{k-1}^2}\right), \qquad (8.64)
\end{aligned}$$

where we have used

$${}_\alpha\langle k|\hat{a}^2|k\rangle_\alpha = 0 \quad \& \quad {}_\alpha\langle k|(\hat{a}^\dagger)^2|k\rangle_\alpha = 0 \ 0 \le k \le n-1\,. \qquad (8.65)$$

As a result, we have

$$\begin{aligned}
\langle \hat{x}^2 \rangle_{|k\rangle_\alpha} &= \frac{\hbar}{2}\left(1+2|\alpha|^2\frac{N_k^2}{N_{k-1}^2}\right), \qquad (8.66) \\
\langle \hat{q}^2 \rangle_{|k\rangle_\alpha} &= \frac{\hbar}{2}\left(1+2|\alpha|^2\frac{N_k^2}{N_{k-1}^2}\right). \qquad (8.67)
\end{aligned}$$

The form of variance is different from the cat states $n = 2$, due to that the cat states are eigenstates of operator $\hat{a}^2$, so that the expression (8.65) is not zero. We have derived uncertainty relation for cat states in Chapter 4. The following uncertainty relations are for $n \ge 3$;

$$(\Delta\hat{q})_{|k\rangle_\alpha}\,(\Delta\hat{p})_{|k\rangle_\alpha} = \frac{\hbar}{2}\left(1 + 2|\alpha|^2 \frac{N_k^2}{N_{k-1}^2}\right) = \frac{\hbar}{2}\left(1 + 2|\alpha|^2 \frac{{}_{k-1}e^{|\alpha|^2}}{{}_{k}e^{|\alpha|^2}}\right), \tag{8.68}$$

where

$$(\Delta\hat{q})_{|k\rangle_\alpha} \equiv (\Delta\hat{p})_{|k\rangle_\alpha} = \sqrt{\frac{\hbar}{2}\left(1 + 2|\alpha|^2 \frac{N_k^2}{N_{k-1}^2}\right)}. \tag{8.69}$$

This uncertainty relation for kaleidoscope states $|k\rangle_\alpha$ have the following limit;

$$\lim_{|\alpha|^2 \to 0} (\Delta\hat{q})_{|k\rangle_\alpha}\,(\Delta\hat{p})_{|k\rangle_\alpha} \overset{(8.51)}{=} \frac{\hbar}{2}(2k+1) \quad , 0 \le k \le n-1\,. \tag{8.70}$$

It is noted that uncertainty relation (8.68) includes mod n exponential funtions, and in the limit $|\alpha|^2 \to 0$ coincides with spectrum of harmonic oscillator with finite number of energy levels

$$E_k = \hbar\left(k + \frac{1}{2}\right) \quad , 0 \le k \le n-1\,. \tag{8.71}$$

8.7. Coordinate Representation of Kaleidoscope States

The wave function for kaleidoscope of quantum coherent states $|k\rangle_\alpha\,, 0 \le k \le n-1$,in coordinate representation is given by

$$\langle x|k\rangle_\alpha = \frac{e^{-\frac{x^2}{2}}}{\pi^{1/4}\sqrt{{}_{k}e^{|\alpha|^2}}} \sum_{s=0}^{\infty} \frac{H_{ns+k}(x)}{(ns+k)!}\left(\frac{\alpha}{\sqrt{2}}\right)^{ns+k}, \tag{8.72}$$

By using (mod n) generating function for mod n Hermite polynomials

$$\sum_{s=0}^{\infty} \frac{z^{ns+k}}{(ns+k)!} H_{ns+k}(x) = {}_{k}e^{-z^2+2zx}, \tag{8.73}$$

where

$$ {}_{k}e^{-z^2+2zx} = \frac{1}{n}\left(e^{-z^2+2zx} + e^{-(q^2z)^2+2q^2zx} + ... + e^{-(q^{2(n-1)}z)^2+2q^{2(n-1)}zx}\right), \tag{8.74} $$

this wave function appears as superposition of Gaussian wave functions:

$$ \langle x|k\rangle_{\alpha} \quad = \quad \frac{e^{-\frac{x^2}{2}}}{\pi^{1/4}\sqrt{{}_{k}e^{|\alpha|^2}}}\ {}_{k}e^{-\frac{\alpha^2}{2}+\sqrt{2}\alpha x}\,. \tag{8.75} $$

CHAPTER 9

QUANTUM GROUP SYMMETRY

In the Fock space the operator $q^{2\widehat{N}}$ is an infinite matrix of the form

$$\Sigma_3 \equiv q^{2\hat{N}} = I \otimes \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & q^2 & 0 & \dots & 0 \\ 0 & 0 & q^4 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & q^{2(n-1)} \end{pmatrix}, \quad \Sigma_1 = I \otimes \begin{pmatrix} 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{9.1}$$

Here the $n \times n$ matrices are called the Sylvester clock and shift matrices correspondingly. They are q-commutative

$$\Sigma_1 \Sigma_3 = q^2 \Sigma_3 \Sigma_1, \tag{9.2}$$

satisfy relations

$$\Sigma_1^n = I, \quad \Sigma_3^n = I \tag{9.3}$$

and are connected by the unitary transformation:

$$\Sigma_1 = (I \otimes Q) q^{2\hat{N}} (I \otimes Q^+). \tag{9.4}$$

Hermann Weyl in book (Weyl, 1931) proposed them for description of quantum mechanics of finite dimensional systems. By dilatation operator $q^{2\hat{N}}$ we define q^2-number operator

$$[\hat{N}]_{q^2} = \frac{q^{2\hat{N}} - 1}{q^2 - 1}$$

for non-symmetrical q-calculus, and

$$[\hat{N}]_{\tilde{q}^2} = \frac{q^{2\hat{N}} - q^{-2\hat{N}}}{q^2 - q^{-2}}\,, \tag{9.5}$$

for the symmetrical one. In our kaleidoscope basis, these number operators are diagonal with eigenvalues given by q-numbers:

$$[\hat{N}]_{q^2} = diag([0]_{q^2}, [1]_{q^2}, ..., [n-1]_{q^2})\,, \tag{9.6}$$

with $[n]_{q^2} = \frac{q^{2n}-1}{q^2-1}$ for non-symmetric case, and

$$[\hat{N}]_{\tilde{q}^2} = diag([0]_{\tilde{q}^2}, [1]_{\tilde{q}^2}, ..., [n-1]_{\tilde{q}^2})\,, \tag{9.7}$$

with $[n]_{\tilde{q}^2} = \frac{q^{2n}-q^{-2n}}{q^2-q^{-2}}$ for the symmetrical one. For symmetric case the q-number operator is Hermitian and can be factorized as

$$[\hat{N}] = \hat{B}^{+}\hat{B}, \quad [\hat{N}+1] = \hat{B}\hat{B}^{+}, \tag{9.8}$$

where

$$\hat{B} = \hat{a}\sqrt{\frac{[N]_{\tilde{q}^2}}{N}}. \tag{9.9}$$

Explicitly in matrix form it is

$$\hat{B} = I \otimes \begin{pmatrix} 0 & \sqrt{[1]} & 0 & ... & 0 \\ 0 & 0 & \sqrt{[2]} & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}, \quad \hat{B}^{+} = I \otimes \begin{pmatrix} 0 & 0 & 0 & ... & 0 \\ \sqrt{[1]} & 0 & 0 & ... & 0 \\ 0 & \sqrt{[2]} & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}, \tag{9.10}$$

and $\hat{B}^n = 0$, $(\hat{B}^{+})^n = 0$. In non-symmetric case the number operator is not Hermitian.

9.1. Symmetric Case

For symmetric case we have the quantum algebra

$$\hat{B}\hat{B}^{+} - q^{2}\hat{B}^{+}\hat{B} = q^{-2\hat{N}}, \tag{9.11}$$
$$\hat{B}\hat{B}^{+} - q^{-2}\hat{B}^{+}\hat{B} = q^{2\hat{N}}, \tag{9.12}$$

and quantum q^2-oscillator with Hamiltonian

$$\hat{H} = \frac{\hbar\omega}{2}\left([\hat{N}]_{\tilde{q}^2} + [\hat{N} + I]_{\tilde{q}^2}\right). \tag{9.13}$$

On the kaleidoscope states as the eigenstates, the spectrum of this Hamiltonian is

$$E_k = \frac{\hbar\omega}{2}\frac{\sin\frac{2\pi}{n}(k+\frac{1}{2})}{\sin\frac{\pi}{n}}.$$

The same spectrum was obtained in (Floratos and Tomaras, 1990) for description of physical system of two anyons. Appearance of quantum algebraic structure in two different physical systems, as optical coherent states and the anyons problem is instructive.

9.2. Non-symmetric Case

In this case the quantum algebra of operators is q^2-deformed

$$\hat{B}\hat{B}^{+} - q^{2}\hat{B}^{+}\hat{B} = I, \tag{9.14}$$
$$\hat{B}\hat{B}^{+} - \hat{B}^{+}\hat{B} = q^{2\hat{N}},$$

with periodic (mod n) ($[k+n]_{q^2} = [k]_{q^2}$) q^2-numbers

$$[k]_{q^2} = e^{i\frac{\pi}{n}(k-1)}\frac{\sin\frac{\pi}{n}k}{\sin\frac{\pi}{n}}.$$

CHAPTER 10

CONCLUSION

In the present thesis the Glauber coherent states, as most classical quantum states, were used for description of units of quantum information in terms of quantum kaleidoscope of states, associated with regular polygon symmetry and roots of unity.

For description of this kaleidoscope, a new type of functions with mod n symmetry was introduced by discrete Fourier transform. These functions include generalized hyperbolic functions, satisfying ordinary differential equations with proper initial conditions. For mod 2 functions with operator argument, non-commutative addition formulas were derived. Mod n representation of displacement operator for quantum states was obtained. Mod n Hermite polynomials and corresponding generating functions were constructed by mod n exponential functions. In terms of these functions, normalization constants, average number of photons, uncertainty relations, and coordinate representation of kaleidoscope states were derived.

The Schrödinger Cat States generated by Hadamard gate as superposition of coherent states with opposite phases, were related to primitive roots of unity $q^4 = 1$. These states have been used for description of qubit unit of quantum information. The trinity and the quartet states, related with $q^6 = 1$ and $q^8 = 1$ primitive roots of unity, were described in details. These states provide computational basis for quantum units of information as the qutrit and the ququat, correspondingly.

Generalization of this construction to the kaleidoscope of quantum coherent states, related with regular n-polygon and the roots of unity $q^{2n} = 1$ was derived. This construction was generated by quantum Fourier transform of Glauber coherent states. Calculation of uncertainty relations, show that kaleidoscope states are not minimal uncertainty states. In coordinate representation these states were described by superposition of Gaussian states with mod n symmetry. A Superposition of these states represents the qudit unit of quantum information, corresponding to arbitrary base in position notation of numbers. The kaleidoscope states, as well as the qudit state are eigenstates of the n-th power of annihilation operator. Relations of these states with q-number operators and quantum group symmetry, in terms of the clock and shift matrices were established. The kaleidoscope coherent states, derived in this thesis can be used for creating two and multiple qubit entangled states of photons.

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APPENDIX A

PRELIMINARIES

A.1. Expectation Value

In quantum mechanics , the expectation value is the probabilistic expected value of the result of an measurement.

Definition A.1 *Let $\hat{A}$ be an operator on a Hilbert space and $|\varphi\rangle$ is a normalized state, then the expectation value of $\hat{A}$ in the state $|\varphi\rangle$ is defined as*

$$\langle\hat{A}\rangle = \langle\hat{A}\rangle_{\varphi} = \langle\varphi|\hat{A}|\varphi\rangle \tag{A.1}$$

A.2. Uncertainty(Deviation)

Definition A.2 *The uncertainty of the observable A is a measure of the spread of results around the mean $\langle\hat{A}\rangle$. It is defined in the usual way, that is the difference between each measured result and the mean is calculated.*

$$(\Delta A)^2_{\varphi} = \langle\hat{A}^2\rangle_{\varphi} - \langle\hat{A}\rangle^2_{\varphi} \tag{A.2}$$

A.3. Root of Unity

Definition A.3 *Given a positive number n, a complex number w is called an nth root of unity if*

$$w^n = 1\,.$$

Define w_n as following;

$$w_n \equiv w = e^{\frac{2\pi}{n}i} \equiv \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right) \tag{A.3}$$

From $w^n = 1$, the complex numbers

$$1, \, , w \, , w^2, \, \ldots, \, w^{n-1}, \tag{A.4}$$

considered as points in the complex plane which are vertices of $n-$sided regular polygon, inscribed to unit circle. For example, when n=6, they are corresponding to vertices of hexagon.

A.3.1. Main Properties of Roots of Unity

Any integer power of an nth root of unity is also $n-$th root of unity:

$$(w^k)^n = w^{kn} = (w^n)^k = 1^k = 1. \tag{A.5}$$

Here k may be negative. In particular, the reciprocal of an n-th root of unity is its complex conjugate, and is also an nth root of unity:

$$\frac{1}{w} = w^{-1} = w^{n-1} = \overline{w}. \tag{A.6}$$

Definition A.4 *A primitive n-th root of unity is a n-th root of unity that is not a k-th root of unity for any positive integer $k < n$. That is, w is a primitive root of unity if and only if*

$$w^n = 1 \, , \, w^k \neq 1 \, . \tag{A.7}$$

Proposition A.1 *For primitive roots of unity $w^n = 1 \, , 0 < s, k \leq n-1$ following identities* *hold;*

$$1)\qquad 1 + w^k + w^{2k} + ... + w^{k(n-1)} = n\delta_{k,0(modn)} \tag{A.8}$$

$$2)\qquad 1 + w^{(k-s)} + w^{2(k-s)} + ... + w^{(k-s)(n-1)} = n\delta_{k,s(modn)} \tag{A.9}$$

where

$$\delta_{k,0\,(modn)} = \begin{cases} 1, & k = 0\,(mod\,n); \\ 0, & k \neq 0\,(mod\,n). \end{cases} \qquad \& \qquad \delta_{k,s\,(modn)} = \begin{cases} 1, & k = s\,(mod\,n); \\ 0, & k \neq s\,(mod\,n). \end{cases} \tag{A.10}$$

Proof For the first statement, due to $w^{nk} = 1\,, \forall\, k = nl\,, l \in \mathbb{Z}$;

$$1 - (w^k)^n = (1 - w^k)(1 + w^k + w^{2k} + ... + q^{k(n-1)}) \tag{A.11}$$

$$= (1 - (w^n)^l)(1 + w^{nl} + w^{2nl} + ... + q^{nl(n-1)}) = 0 \tag{A.12}$$

Since $1 - (w^n)^l = 0$, then

$$1 + w^{nl} + w^{2nl} + ... + w^{nl(n-1)} = 1 + (w^n)^l + (w^{2n})^l + ... + (w^{n(n-1)})^l = n \tag{A.13}$$

When $k \neq nl\,, l \in \mathbb{Z}$, it gives us $w^k \neq 1$ and

$$1 + w^k + w^{2k} + ... + w^{k(n-1)} = 0\,. \tag{A.14}$$

Therefore, the following result holds

$$1 + w^k + w^{2k} + ... + w^{k(n-1)} = n\delta_{k,0\,(modn)} = n\begin{cases} 1, & k = 0\,(mod\,n); \\ 0, & k \neq 0\,(mod\,n). \end{cases} \tag{A.15}$$

For the second statement,it is enough to choose $k \equiv k - s$ and substitute to equation $(A.8)$;

$$
\begin{aligned}
1 + w^k + w^{2k} + ... + w^{k(n-1)} &= 1 + w^{(k-s)} + w^{2(k-s)} + ... + w^{(k-s)(n-1)} && \text{(A.16)} \\
&= n\delta_{k,s\,(mod n)} && \text{(A.17)}
\end{aligned}
$$

□

A.4. The Baker−Campbell−Hausdorff formula

Definition A.5 *The Baker−Campbell−Hausdorff formula for the product of the exponentials of two operators $\hat{A}$ and $\hat{B}$ is*

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+[\hat{A},\hat{B}]/2+...} \tag{A.18}$$

which involves nested commutators of $\hat{A}$ and $\hat{B}$.

APPENDIX B

GENERALIZED HYPERBOLIC FUNCTIONS

This chapter introduces the basic notions and properties of Generalized hyperbolic functions based on (Ungar, 1982).

Definition B.1 *For any pair of integers* (n, r)*,* $n \geq 2$*,* $0 \leq r \leq n-1$ *, let the entire function in the complex plane C,*

$$F_{n,r}(z) = \sum_{k=0}^{\infty} \frac{z^{nk+r}}{(nk+r)!} \tag{B.1}$$

be referred to as the hyperbolic function of order n and rth kind.

The two hyperbolic functions of order $n = 2$ are thus $F_{2,0}(z) = \cosh z$ and $F_{2,1}(z) = \sinh z$. Since

$$\frac{d}{dz} F_{n,r}(z) = F_{n,r-1}(z) \, , \tag{B.2}$$

where we define

$$F_{n,-1}(z) = F_{n,r-1}(z) \, , \tag{B.3}$$

the hyperbolic functions of order n,

$$F_{n,r}(z), \qquad r = 0, 1, 2, ..., n-1 \, , \tag{B.4}$$

form a set of n linearly independent solutions of differential equation

$$\frac{d^n}{dz^n} \Phi(z) = \Phi(z) \tag{B.5}$$

Furthermore, we can extend the definition of $F_{n,r}(z)$ to any integer r,

$$F_{n,r}(z) = F_{n,r(mod\, n)}(z)\,, \tag{B.6}$$

The hyperbolic functions of order $n \geq 2$, satisfy two basic properties. The first property is that they form a set of n linearly independent solutions of an ordinary differential equation that are obtained from one another by differentiations. The second property is that they form a continuous commutative group, represented by an nxn matrix of a single complex variable, the determinant of which is unity.
As an application to the theory of ordinary differential equations, let us consider the particular case of $n = 3$. The three hyperbolic functions of order 3, $F_{3,0}(x)$, $F_{3,1}(x)$, and $F_{3,2}(x), -\infty < x < \infty$ may be written in terms of the exponential function,

$$\begin{aligned} F_{3,0}(x) &= \frac{1}{3}\,(e^x + e^{q_1 x} + e^{q_2 x})\,, && \text{(B.7)}\\ F_{3,1}(x) &= \frac{1}{3}\,(e^x + q_2 e^{q_1 x} + q_1 e^{q_2 x})\,, && \text{(B.8)}\\ F_{3,2}(x) &= \frac{1}{3}\,(e^x + q_1 e^{q_1 x} + q_2 e^{q_2 x})\,. && \text{(B.9)} \end{aligned}$$

where $q_1 = \frac{-1}{\sqrt{2}} + i\frac{\sqrt{3}}{2}$ and $q_2 = \frac{-1}{\sqrt{2}} - i\frac{\sqrt{3}}{2}$ are two non-real cube roots of unity that is $(q_1)^6 = (q_2)^6 = 1$. Equivalently, the hyperbolic functions of order 3 can be written as

$$\begin{aligned} F_{3,0}(x) &\equiv \frac{1}{3}\left(e^x + 2e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)\right), && \text{(B.10)}\\ F_{3,1}(x) &\equiv \frac{1}{3}\left(e^x + 2e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x - \frac{2\pi}{3}\right)\right), && \text{(B.11)}\\ F_{3,2}(x) &\equiv \frac{1}{3}\left(e^x + 2e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x + \frac{2\pi}{3}\right)\right). && \text{(B.12)} \end{aligned}$$

thus indicating that the ordinary differential equation

$$\frac{d^3}{dx^3}\Phi(x) = \Phi(x) \tag{B.13}$$

has three linearly independent solutions, $F_{3,r}(x)$, $r = 0, 1, 2$. The functions are approaching $\frac{e^x}{3}$ for $x \to \infty$, and rapidly oscillating near $x \to -\infty$.

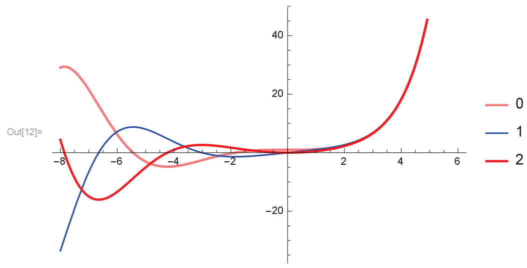

Figure B.1. Hyperbolic functions with order 3

APPENDIX C

QUANTUM FOURIER TRANSFORMATION

C.1. Discrete Fourier Transformation

Definition C.1 *The discrete fourier transform of a vector with complex components* $f(0)$, $f(1), ..., f(N-1)$ *is a new complex vector* $F(0), F(1), ..., F(N-1)$, *defined as*

$$F(K) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} w^{jk} f(j) \tag{C.1}$$

C.2. Quantum Fourier Transformation

In quantum computing, the quantum Fourier transform (for short: QFT) is a linear transformation on quantum bits, and is the quantum analogue of the discrete Fourier transform.Using a simple decomposition, the discrete Fourier transform on 2^n amplitudes can be implemented as a quantum circuit consisting of only $O(n^2)$Hadamard gates and controlled phase shift gates, where n is the number of qubits. This can be compared with the classical discrete Fourier transform, which takes $O(n2^n)$ gates, where n is the number of bits, which is exponentially more than $O(n^2)$. However, the quantum Fourier transform acts on a quantum state, whereas the classical Fourier transform acts on a vector. The quantum Fourier transform can be performed efficiently on a quantum computer, with a particular decomposition into a product of simpler unitary matrices.

$$|\psi\rangle \longmapsto \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \overline{q}^{\,2jk} |\phi\rangle \quad 0 \leq k \leq n-1 \tag{C.2}$$

C.3. The Clock and Shift Matrices

Let $q = e^{\frac{\pi}{n}i}$ be a primitive root of unity. Since $q^{2n} = 1$ and $q \neq 1$, the sum of all roots satisfies

$$1 + q^2 + q^4 + \ldots + q^{2(n-1)} = 0\,. \tag{C.3}$$

We define the clock and shift matrices, correspondingly;

$$\Sigma_3 = \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & q^2 & 0 & ... & 0 \\ 0 & 0 & q^4 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & q^{2(n-1)} \end{pmatrix}, \qquad \Sigma_1 = \begin{pmatrix} 0 & 0 & 0 & ... & 1 \\ 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}.$$

The clock and shift matrices are generalizations of Pauli matrices σ_3 and σ_1, respectively. Since we are in higher dimension than 2, these matrices are unitary and traceless, but not Hermitian.The group generated by the clock and shift matrices is sometimes called Weyl-Heinsenberg group.

$$\Sigma_1^n = \hat{I} \quad , \quad \Sigma_3^n = \hat{I} \tag{C.4}$$

Furthermore, they satisfy

$$\Sigma_3\Sigma_1 = q^2\Sigma_1\Sigma_3 = e^{\frac{2\pi}{n}i}\Sigma_1\Sigma_3 \tag{C.5}$$

which is quantum canonical commutation relation for finite dimensional spaces.

APPENDIX D

SCHRÖDINGER'S CAT STATES

D.1. Normalization of Schrödinger's Cat States

In this part, we will use this identification where $q^2 = \bar{q}^{\,2} = -1 = e^{i\pi}$.

$$|0\rangle_\alpha = |\alpha_+\rangle = N_+\left(|\alpha\rangle + |q^2\alpha\rangle\right) \quad , \quad |1\rangle_\alpha = |\alpha_-\rangle = N_-\left(|\alpha\rangle + \bar{q}^{\,2}|q^2\alpha\rangle\right) \tag{D.1}$$

We will show orthogonality of cat states and normalization constants N_+, N_- of these in $(D.1)$ will be calculated by using inner product of coherent states:

- $\langle\alpha|\alpha\rangle = \langle q^2\alpha|q^2\alpha\rangle = 1$
- $\langle\alpha|q^2\alpha\rangle = \langle q^2\alpha|\alpha\rangle = e^{-2|\alpha|^2}$

First, we will show that these states are orthogonal :

$$\begin{aligned}
{}_\alpha\langle 1|0\rangle_\alpha &= N_+N_-(\langle\alpha| + \bar{q}^{\,2}\langle q^{\,2}\alpha|)(|\alpha\rangle + |q^{\,2}\alpha\rangle) \qquad (D.2)\\
&= N_+N_-(\langle\alpha|\alpha\rangle + \langle\alpha|q^{\,2}\alpha\rangle + \bar{q}^{\,2}\langle q^{\,2}\alpha|\alpha\rangle + \bar{q}^{\,2}\langle q^{\,2}\alpha|q^{\,2}\alpha\rangle)\\
&= N_+N_-(1 + e^{-2|\alpha|^2} - e^{-2|\alpha|^2} - 1) = 0
\end{aligned}$$

Then, normalization factors will be calculates respectively:

$$\begin{aligned}
{}_\alpha\langle 0|0\rangle_\alpha = 1 \Rightarrow 1 &= |N_+|^2\left(\langle\alpha| + \langle q^{\,2}\alpha|)(|\alpha\rangle + |q^{\,2}\alpha\rangle\right) \qquad (D.3)\\
&= |N_+|^2\left(\langle\alpha|\alpha\rangle + \langle\alpha|q^{\,2}\alpha\rangle + \langle\alpha|\alpha\rangle + \langle q^{\,2}\alpha|q^{\,2}\alpha\rangle\right)\\
&= |N_+|^2\left(2 + 2e^{-2|\alpha|^2}\right) = 2|N_+|^2 e^{-|\alpha|^2}\left(e^{|\alpha|^2} + e^{q^2|\alpha|^2}\right)
\end{aligned}$$

Thus,we have normalized state $|0\rangle_c$:

$$\boxed{|0\rangle_c = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + |q^2\alpha\rangle}{\sqrt{2}\sqrt{e^{|\alpha|^2} + e^{q^2|\alpha|^2}}}} \tag{D.4}$$

Normalization factor can be rewritten as:

$$\begin{aligned} e^{|\alpha|^2} + e^{q^2|\alpha|^2} &= \sum_{n=0}^{\infty}\left(\frac{\left(|\alpha|^2\right)^n}{n!} + \frac{\left(q^2|\alpha|^2\right)^n}{n!}\right) = \sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}(\underbrace{1+q^{2n}}_{2\delta_{n=0(mod2)}}) \\ &= 2\sum_{k=0}^{\infty}\frac{\left(|\alpha|^2\right)^{2k}}{(2k)!} = 2\,{}_0e^{|\alpha|^2} = 2cosh(|\alpha|^2) \end{aligned}$$

Similarly, we can apply same process to $|1\rangle_\alpha$

$$\begin{aligned} {}_\alpha\langle 1|1\rangle_\alpha = 1 \Rightarrow 1 &= |N_-|^2\left(\langle\alpha| + \overline{q}^2\langle q^2\alpha|)(|\alpha\rangle + \overline{q}^2|q^2\alpha\rangle\right) \qquad \text{(D.5)} \\ &= |N_-|^2\left(\langle\alpha|\alpha\rangle + \overline{q}^2\langle\alpha|q^2\alpha\rangle + \overline{q}^2\langle\alpha|\alpha\rangle + \langle q^2\alpha|q^2\alpha\rangle\right) \\ &= |N_-|^2\left(2 - 2e^{-2|\alpha|^2}\right) = 2|N_-|^2e^{-|\alpha|^2}\left(e^{|\alpha|^2} + \overline{q}^2e^{q^2|\alpha|^2}\right) \end{aligned}$$

Thus,we have normalized state $|1\rangle_\alpha$:

$$\boxed{|1\rangle_\alpha = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + |q^2\alpha\rangle}{\sqrt{2}\sqrt{e^{|\alpha|^2} + \overline{q}^2e^{q^2|\alpha|^2}}}} \tag{D.6}$$

Normalization factor can be rewritten as:

$$\begin{aligned} e^{|\alpha|^2} + \overline{q}^2e^{q^2|\alpha|^2} &= \sum_{n=0}^{\infty}\left(\frac{\left(|\alpha|^2\right)^n}{n!} + \overline{q}^2\frac{\left(q^2|\alpha|^2\right)^n}{n!}\right) = \sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}(\underbrace{1+q^{2(n-1)}}_{2\delta_{n=1(mod2)}}) \\ &= 2\sum_{k=0}^{\infty}\frac{\left(|\alpha|^2\right)^{2k+1}}{(2k+1)!} = 2\,{}_1e^{|\alpha|^2} = 2\sinh\left(|\alpha|^2\right) \end{aligned}$$

So that,we have normalized cat states $|0\rangle_\alpha$ and $|1\rangle_\alpha$:

$$\boxed{|0\rangle_\alpha = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + |q^2\alpha\rangle}{\sqrt{2}\sqrt{e^{|\alpha|^2} + e^{q^2|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + |q^2\alpha\rangle}{2\sqrt{{}_0e^{|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + |q^2\alpha\rangle}{2\sqrt{\cosh|\alpha|^2}}} \quad \text{(D.7)}$$

$$\boxed{|1\rangle_\alpha = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \bar{q}^2|q^2\alpha\rangle}{\sqrt{2}\sqrt{e^{|\alpha|^2} + \bar{q}^2 e^{q^2|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \bar{q}^2|q^2\alpha\rangle}{2\sqrt{{}_1e^{|\alpha|^2}}} = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \bar{q}^2|q^2\alpha\rangle}{2\sqrt{\sinh|\alpha|^2}}} \quad \text{(D.8)}$$

D.2. Mod 2 Cat States

Since $|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger}|0\rangle$ and $|q^2\alpha\rangle = e^{-\frac{|q^2\alpha|^2}{2}} e^{q^2\alpha\hat{a}^\dagger}|0\rangle = e^{-\frac{|\alpha|^2}{2}} e^{q^2\alpha\hat{a}^\dagger}|0\rangle$ where $q^4 = 1$, Cat states can be written as follows

$$|0\rangle_\alpha = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + |q^2\alpha\rangle}{2\sqrt{\cosh(|\alpha|^2)}} = e^{\frac{|\alpha|^2}{2}} \frac{e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger} + e^{-\frac{|\alpha|^2}{2}} e^{q^2\alpha\hat{a}^\dagger}}{2\sqrt{\cosh(|\alpha|^2)}}|0\rangle \quad \text{(D.9)}$$

$$= \frac{e^{\alpha\hat{a}^\dagger} + e^{q^2\alpha\hat{a}^\dagger}}{2\sqrt{\cosh(|\alpha|^2)}}|0\rangle \quad \text{(D.10)}$$

$$= \frac{\cosh\alpha\hat{a}^\dagger}{\sqrt{\cosh|\alpha|^2}}|0\rangle = \frac{{}_0e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_0e^{|\alpha|^2}}}|0\rangle \quad (mod\, 2) \quad \text{(D.11)}$$

and

$$|1\rangle_\alpha = e^{\frac{|\alpha|^2}{2}} \frac{|\alpha\rangle + \bar{q}^2|q^2\alpha\rangle}{2\sqrt{\sinh(|\alpha|^2)}} = e^{\frac{|\alpha|^2}{2}} \frac{e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger} + \bar{q}^2 e^{-\frac{|\alpha|^2}{2}} e^{q^2\alpha\hat{a}^\dagger}}{2\sqrt{\sinh(|\alpha|^2)}}|0\rangle \quad \text{(D.12)}$$

$$= \frac{e^{\alpha\hat{a}^\dagger} - e^{q^2\alpha\hat{a}^\dagger}}{2\sqrt{\sinh(|\alpha|^2)}}|0\rangle \quad \text{(D.13)}$$

$$= \frac{\sinh\alpha\hat{a}^\dagger}{\sqrt{\sinh|\alpha|^2}}|0\rangle = \frac{{}_1e^{\alpha\hat{a}^\dagger}}{\sqrt{{}_1e^{|\alpha|^2}}}|0\rangle \quad (mod\, 2) \quad \text{(D.14)}$$