ON THE STRUCTURE OF MODULES CHARACTERIZED BY OPPOSITES OF INJECTIVITY

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Ferhat ALTINAY

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We approve the thesis of **Ferhat ALTINAY**

Examining Committee Members:

Prof. Dr. Engin BÜYÜKAŞIK
Department of Mathematics, İzmir Institute of Technology

Assoc. Prof. Dr. Başak AY SAYLAM
Department of Mathematics, İzmir Institute of Technology

Assoc. Prof. Dr. Salahattin ÖZDEMİR
Department of Mathematics, Dokuz Eylül University

6 July 2018

Prof. Dr. Engin BÜYÜKAŞIK
Supervisor, Department of Mathematics
İzmir Institute of Technology

Prof. Dr. Engin BÜYÜKAŞIK
Head of the Department of
Mathematics

Prof. Dr. Aysun SOFUOĜLU
Dean of the Graduate School of
Engineering and Sciences

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ABSTRACT

ON THE STRUCTURE OF MODULES CHARACTERIZED BY OPPOSITES OF INJECTIVITY

In this thesis we consider some problems and also generalize some results related to indigent modules and subinjectivity domains. We prove that subinjectivity domain of any right module is closed under factor modules if and only if the ring is right hereditary. Indigent modules are the modules whose subinjectivity domain is as small as possible, namely the modules whose subinjectivity domain is exactly the class of injective modules. We give a complete characterization of indigent modules over commutative hereditary Noetherian rings. The commutative rings whose simple modules are injective or indigent are fully determined. The rings whose cyclic right modules are indigent are shown to be semisimple Artinian. We also give a characterization of t.i.b.s. modules over Dedekind domains.

ÖZET

İNJEKTİFLİĞİN TERSİ İLE KARAKTERİZE EDİLEN MODÜLLERİN YAPISI ÜZERİNE

Bu tezde yoksul modüller ile ilgili bazı problemler ele alınmakta ve aynı zamanda mevcut bazı sonuçlar genelleştirilmektedir. Her sağ modülün altinjektiflik bölgesinin faktör modüllere göre kapalı olması için gerek ve yeter koşulun halkanın sağ kalıtsal halka olduğu kanıtlanmıştır. Yoksul modüller mümkün olan en küçük altinjektiflik bölgesine sahip olan modüllerdir, yani altinjektiflik bölgesi tam olarak injektif moduller olan modüllerdir. Yoksul modüller değişmeli kalıtsal Noether halkalar üzerinde tam olarak karakterize edilmiştir. Basit modülleri yoksul veya injektif olan değişmeli halkalar tam olarak belirlenmiştir. Devirli sağ modülleri yoksul olan halkaların yarı basit Artin olduğu gösterilmiştir. Aynı zamanda, t.i.b.s. modüller Dedekind tamlık bölgeleri üzerinde karakterize edilmiştir.

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LIST OF ABBREVIATIONS

R	an associative ring with unit unless otherwise stated
$\mathbb{Z}, \mathbb{Z}^+$	the ring of integers, the set of all positive integers
$\mathbb{Q}$	the field of rational numbers
$R - MOD$	the category of *left* R-modules
$MOD - R$	the category of *right* R-modules
$Hom_R(M, N)$	all R-module homomorphisms from M to N
$M \otimes_R N$	the tensor product of the *right* R-module M and the *left* R-module N
$\oplus_{i \in I} M_i$	direct sum of R- modules M_i
$\prod_{i \in I} M_i$	direct product of R- modules M_i
$Kerf$	the kernel of the map f
imf	the image of the map f
$SocM$	the socle of the R-module M
$RadM$	the radical of the R-module M
$E(M)$	the injective envelope (hull) of a module M
$T(M)$	the torsion submodule of a module M
$Z(M)$	the singular submodule of a module M
$\ll$	small (or superfluous) submodule
$\trianglelefteq$	essential submodule
$In^{-1}(M)$	the injectivity domain of a module M
$Jn^{-1}(M)$	the subinjectivity domain of a module M
$Annl_R(X)$	$= \{r \in R \mid rX = 0\}$ = the *left* annihilator of a subset X of a *left* R-module M
$Annr_R(X)$	$= \{r \in R \mid Xr = 0\}$ = the *right* annihilator of a subset X of a *right* R-module M
$\cong$	isomorphic
$\leq$	submodule

CHAPTER 1

INTRODUCTION

The notions of injectivity and relative injectivity have been studied extensively in the literature. Given two right R-modules M and N, the module M is said to be relative injective to N or N-injective if for each submodule K of N any homomorphism from K to M can be extended to a homomorphism from N to M. The injectivity domain of M is the collection of all right R-modules N such that M is N-injective. The injectivity domain of any right module is closed under submodules, factor modules and finite direct sums. It is evident that any right module is S-injective for each semisimple right module S. In other word, semisimple modules are contained in injectivity domain of any right module.

A right module M is called *poor* if its injectivity domain consists of exactly the class of semisimple right R-modules (Alahmadi, Alkan and López-Permouth, 2010). Every ring has a poor right module (Er, Lòpez-Permouth and Sökmez, 2011). The main results related to poor modules can be found in (Alahmadi, Alkan and López-Permouth, 2010), (Er, Lòpez-Permouth and Sökmez, 2011), (Alizade and Büyükaşık, 2017) and (Alizade, Büyükaşık, López-Permouth and Yang, 2018).

Recently, an opposite notions of poor modules and relative injectivity introduced in (Aydoğdu and López-Permouth, 2011). A right module M is said to be N-subinjective for some right module N, if every homomorphism from N to M can be extended to a homomorphism from the injective hull $E(N)$ of N to M. The subinjectivity domain of M is defined as the collection of all right modules N such that M is N-subinjective. Injective right modules are contained in the subinjectivity domain of any right module. In contrast to injectivity domains, subinjectivity domains need not be closed under submodules and factor modules. We prove that the subinjectivity domain of any right module is closed under factor modules if and only if the ring is right hereditary.

A right module M is called *indigent* if its subinjectivity domain is exactly the class of injective right modules. Existence of indigent modules is not known over arbitrary rings. In (Aydoğdu and López-Permouth, 2011), the authors ask whether the direct sum of non-injective uniform right modules is indigent. We give an example to show that this module is not indigent in general. Namely we show that, over a right semiartinian right V-ring the direct sum of non-injective uniform right modules is not indigent. On the other hand, it is indigent over right PCI-domains.

The structure of indigent abelian groups determined in (Alizade and Büyükaşık, 2017). We give a complete characterization of indigent modules over commutative hereditary Noetherian rings. We prove that, over a commutative hereditary Noetherian, a module M is indigent if and only if $Z(M)$ is indigent if and only if $Hom(S, M') \neq 0$ for every singular simple module S, where $Z(M)$ is the singular submodule and M' is the reduced part of M.

The commutative rings whose simple modules are injective or indigent are fully determined. Over a commutative ring R, every simple module is injective or indigent if and only if R is a V-ring, or $R = A \times B$, where B is semisimple, and A is either zero or, A is a DVR, or, A is local QF-ring. The rings whose cyclic right modules are indigent are shown to be semisimple Artinian.

A right module M is said to be a test module for injectivity by subinjectivity (t.i.b.s., for short) if whenever M is N-subinjective for some right module N, then N is injective (Alizade, Büyükaşık and Er, 2014). We prove that, a commutative domain R is Dedekind if and only if every nonzero ideal of R is indigent if and only if a nonzero R-module M is t.i.b.s. exactly when $Hom(M, R) \neq 0$.

CHAPTER 2

PRELIMINARIES

In this chapter we give the basic definitions and results that are used in the sequel.

2.1. Rings and Their Homomorphisms

Definition 2.1 *A ring is defined as a non-empty set R with two compositions $+, \cdot : R \times R \to R$ with the properties :*

(i) $(R, +)$ *is an abelian group (zero element* 0*);*

(ii) $(R, \cdot)$ *is a semigroup;*

(iii) *for all $a, b, c \in R$ the distributivity laws are valid:*

$(a + b)c = ac + bc,\ a(b + c) = ab + ac.$

Definition 2.2 *A subset S of a ring R is called a subring if it is a ring with the operations of R, and $1_R = 1_S$ in case R has identity.*

Proposition 2.1 *(The Subring Criterion) Let R be a ring and S be a subset of R. Then S is a subring of R if and only if for every $a, b \in S$:*

(i) $a - b \in S$;

(ii) $ab \in S$.

Definition 2.3 *Let R, S be rings. The mapping $f : R \to S$ is called a ring homomorphism if it satisfies the following:*

(i) $f(a + b) = f(a) + f(b)$ *for all $a, b \in R$;*

(ii) $f(ab) = f(a)f(b)$ *for all $a, b \in R$;*

(iii) $f(1_R) = 1_S$.

2.2. Ideals and Factor Rings

Definition 2.4 *Let R be a ring. We say that the subset I of R is a left ideal of R if the following are satisfied:*

(i) $I \neq \emptyset$*;*

(ii) *whenever $a, b, \in I$, then $a + b \in I$;*

(iii) *whenever $a \in I$ and $r \in R$, then $ra \in I$, also.*

Similarly a right ideal of a ring can be defined by changing the left multiplication in the definition with right multiplication. If I is both left and right ideal, we say that I is a two sided ideal. Clearly, for a commutative ring, left and right ideals coincide. By an ideal we will always mean a two sided ideal.

The kernel of a homomorphism $f : R \to S$ is the set

$$Ker f = \{r \in R : f(r) = 0\}.$$

Suppose that I is a proper ideal of a ring R. The relation defined by

$$a \equiv b (mod\ I) \Leftrightarrow a - b \in I$$

determines an equivalence relation on R. The congruence class of an element a is defined by $a + I = \{a + x : x \in I\}$ and is called a coset of the element a, and the set R/I of all cosets of I is a ring with operations defined by

$$(a + I) + (b + I) = (a + b) + I \ \ and \ \ (a + I)(b + I) = ab + I.$$

Additive and multiplicative identities are $0 + I$ and $1 + I$.

The ring R/I is called the factor ring of R modulo I. Further, the map $\sigma : R \to R/I$ defined by $r \mapsto r + I$ is an epimorphism with kernel I, is called the natural or canonical epimorphism.

Definition 2.5 *We say that an ideal M of a ring R is a maximal ideal, if*

(i) *$M \subsetneq R$, and*

(ii) *$M \subsetneq I \subseteq R$ implies that $I = R$ for every ideal I of R.*

2.3. Modules, Submodules and Module Homomorphisms

Although modules are in fact considered as a pair (M, λ), where M is an additive abelian group and λ is a map from R to the set of endomorphisms of M, we find the following definition more common and simple:

Definition 2.6 *Let R be a ring (with unity 1). A right R-module is an additive abelian group M together with a mapping $M \times R \to M$, which we call a scalar multiplication, denoted by*

$$(m, r) \mapsto mr$$

such that the following properties hold: for all $m, n \in M$ and $r, s \in R$;

(1) *$(m + n)r = mr + nr$,*

(2) *$m(r + s) = mr + ms$,*

(3) *$m(rs) = (mr)s$.*

If, in addition, for every $m \in M$ we have $m1 = m$, then M is called a unitary right R-module. If M is a right R-module, we denote it by M_R.

2.3.1. Submodules

Let M be a left module over R. A subgroup N of $(M, +)$ is called a *submodule* of M if N is closed under multiplication with elements in R, i.e. $rn \in N$ for all $r \in R$, $n \in N$. Then N is also an R-module by the operations induced from M:

$$R \times N \to N, (r, n) \mapsto rn, r \in R, n \in N.$$

M is called *simple* if $M \neq 0$ and it has no submodules except 0 and M. The submodules of ${}_R R$ (resp. ${}_R R_R$) are just the left (resp. two-sided) ideals.

For non-empty subsets $N_1, N_2, N \subset M$, $A \subset R$ we define:

$$\begin{aligned} N_1 + N_2 &= \{n_1 + n_2 \mid n_1 \in N_1, n_2 \in N_2\} \subset M, \\ AN &= \{\Sigma_{i=1}^{k} a_i n_i \mid a_i \in A, n_i \in N, k \in \mathbb{N}\} \subset M. \end{aligned}$$

If N_1, N_2 are submodules, then $N_1 + N_2$ is also a submodule of M. For a left ideal $A \subset R$, the product AN is always a submodule of M.

For any infinite family $\{N_i\}_{i\in\Lambda}$ of submodules of M, a sum is defined by

$$\Sigma_{\lambda\in\Lambda} N_\lambda = \{\Sigma_{k=1}^{r} n_{\lambda_k} \mid r \in \mathbb{N}, \lambda_k \in \Lambda, n_{\lambda_k} \in N_{\lambda_k}\} \subset M.$$

This is a submodule in M. Also the intersection $\bigcap_{\lambda\in\Lambda} N_\lambda$ is also a submodule of M. $\sum_{\lambda\in\Lambda} N_\lambda$ is the smallest submodule of M which contains all N_λ, $\bigcap_{\lambda\in\Lambda} N_\lambda$is the largest submodule of M which is contained in all N_λ.

Proposition 2.2 *((Anderson and Fuller, 1992), Proposition 2.3)*
Let M be a left R module and let N be a non-empty subset of M. Then the following are equivalent:

(a) *N is a submodule of M ;*

(b) *$RN = N$;*

(c) *For all $a, b \in R$ and all $x, y \in N$*

$$ax + ay \in N.$$

Proposition 2.3 ***Modularity condition*** *(Wisbauer, 1991)*
If H, K, L are submodules of an R-module M and $K \subset H$, then

$$H \cap (K + L) = K + (H \cap L).$$

Definition 2.7 *If N is a submodule of a left R-module M, then the **quotient module** is the quotient group M/N (M is an abelian group and N is a subgroup) equipped with the scalar multiplication*

$$r(m + N) = rm + N.$$

*The **natural map** $\pi : M \to M/N$, given by $m \mapsto m + N$, is easily seen to be an R-map.*

Scalar multiplication in the definition of quotient module is well-defined: if $m + N = m' + N$, then $m - m' \in N$. Hence, $r(m - m') \in N$ (because N is a submodule), $rm - rm' \in N$, and $rm + N = rm' + N$.

Definition 2.8 *Let M and N be left modules over the ring R. A map $f : M \to N$ is called an (R-module) homomorphism (also R-linear map) if*

$$\begin{aligned} f(m_1 + m_2) &= f(m_1) + f(m_2) \ \ for\,all\,m_1, m_2 \in M, \\ f(mr) &= r[f(m)] \quad for\,all\,m \in M,\ r \in R. \end{aligned}$$

Proposition 2.4 *((Rotman, 2009), Proposition 2.4) Let R be a ring, and let A, B, B' be left R-modules.*

(i) *$Hom_R(A, \square)$ is an additive functor ${}_R\textbf{Mod} \to \textbf{Ab}$.*

(ii) *If A is a left R-module, then $Hom_R(A, B)$ is a $Z(R)$-module, where $Z(R)$ is the center of R, if we define*

$$rf : a \mapsto f(ra)$$

for $r \in Z(R)$ and $f : A \to B$. If $q : B \to B'$ is an R-map, then the induced map $q_* : Hom_R(A, B) \to Hom_R(A, B')$ is a $Z(R)$-map, and $Hom_R(A, \square)$ takes values in ${}_{Z(R)}\textbf{Mod}$. In particular, if R is commutative, then $Hom_R(A, \square)$ is a functor ${}_R\textbf{Mod} \to {}_R\textbf{Mod}$.

Theorem 2.1 ***The Factor Theorem**. ((Anderson and Fuller, 1992), Theorem 3.6)*
Let M, M', N and N' be left R-modules and let $f : M \to N$ be an R-homomorphism.

(1) *If $g : M \to M^{'}$ is an epimorphism with $Ker(g) \subseteq Ker(f)$, then there exists a unique homomorphism $h : M^{'} \to N$ such that*

$$f = hg.$$

Moreover, $Kerh = g(Ker(f))$ and $Im(h) = Im(f)$, so that h is monic if and only if $Ker(g) = Ker(f)$ and h is epic if and only if f is epic.

(2) *If $g : N' \to N$ is a monomorphism with $Im(f) \subseteq Im(g)$, then there exists a unique homomorphism $h : M \to N'$ such that*

$$f = gh.$$

Moreover, $Ker(h) = Ker(f)$ and $Im(f) = g^{\leftarrow}(Im(f))$, so that h is monic if and only if f is monic and h is epic if and only if $Im(g) = Im(f)$.

Corollary 2.1 ***Isomorphism Theorems***. *((Anderson and Fuller, 1992), Corollary 3.7) Let M and N be left R-modules.*

(1) *If $f : M \to N$ is an epimorphism with $Kerf = K$, then there is a unique isomorphism*

$$\eta : M/K \to N \text{ such that } \eta(m + K) = f(m)$$

for all $m \in M$.

(2) *If $K \leq L \leq M$, then*

$$(M/K)/(L/K) \cong M/L.$$

(3) *If $H \leq M$ and $K \leq M$, then*

$$(H + K)/K \cong H/(H \cap K).$$

Definition 2.9 *If $f : M \to N$ is an R-map between left R-modules, then*

$$\begin{aligned} \textbf{\textit{kernel f}} &= ker\, f = \{m \in M : f(m) = 0\}, \\ \textbf{\textit{image f}} &= im\, f = \{n \in N : there\, exist\, m \in M\, with\, n = f(m)\}. \end{aligned}$$

It is routine to check that ker f is a submodule of M and that im f is a submodule of N.

2.4. Exact Sequences

Definition 2.10 *A finite or infinite sequence of R-maps and left R-modules*

$$\cdots \longrightarrow M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \longrightarrow \cdots$$

is called an exact sequence if $Im(f_{n+1}) = ker(f_n)$ for all n.

Proposition 2.5 *((Anderson and Fuller, 1992), Proposition 2.18)*

(i) *A sequence* $0 \longrightarrow A \xrightarrow{f} B$ *is exact if and only if f is injective.*

(ii) *A sequence* $B \xrightarrow{g} C \longrightarrow 0$ *is exact if and only if g is surjective.*

(iii) *A sequence* $0 \longrightarrow A \xrightarrow{h} B \longrightarrow 0$ *is exact if and only if h is an isomorphism.*

2.5. Adjoint Isomorphisms

Theorem 2.2 *(Adjoint Isomorphism, First Version) ((Rotman, 2009), Theorem 2.75) Given modules A_R, ${}_RB_S$, and C_S, where R and S are rings, there is a natural isomorphism:*

$$\tau_{A,B,C} : Hom_S(A \otimes_R B, C) \to Hom_R(A, Hom_S(B, C)),$$

namely, for $f : A \otimes_R B \to C$, $a \in A$, *and* $b \in B$.

$$\tau_{A,B,C} : f \mapsto \tau(f), \text{ where } \tau(f)_a : b \mapsto f(a \otimes b).$$

Theorem 2.3 *(Adjoint Isomorphism, Second Version) ((Rotman, 2009), Theorem 2.76) Given modules* ${}_RA$, ${}_SB_R$, *and* ${}_SC$, *where* R *and* S *are rings, there is a natural isomorphism:*

$$\tau'_{A,B,C} : Hom_S(B \otimes_R A, C) \to Hom_R(A, Hom_S(B, C)),$$

namely, for $f : B \otimes_R A \to C$, $a \in A$, *and* $b \in B$.

$$\tau'_{A,B,C} : f \mapsto \tau'(f), \text{ where } \tau'(f)_a : b \mapsto f(a \otimes b).$$

2.6. Definitions

Definition 2.11 *A submodule* $N \subset M$ *is called* ***maximal*** *if* $N \neq M$ *and it is not properly contained in any proper submodule of* M.

In a finitely generated R-module, every proper submodule is contained in a maximal submodule.

Definition 2.12 *A submodule* K *of an R-module* M *is called* ***essential*** *or* ***large in*** M *if, for every nonzero submodule* $L \subset M$, *we have* $K \cap L \neq 0$.

Then M *is called an essential extension of* K *and we write* $K \trianglelefteq M$. *A monomorphism* $f : L \to M$ *is said to be essential if* Imf *is an essential submodule of* M.

Hence a submodule $K \subset M$ *is essential if and only if the inclusion map* $K \to M$ *is an essential monomorphism. For example, in* $\mathbb{Z}$ *every non-zero submodule (=ideal) is essential.*

Definition 2.13 *A submodule* K *of an R- module* M *is called* ***superfluous*** *or* ***small in*** M, *written* $K \ll M$, *if for every submodule* $L \subset M$,*the equality* $K + L = M$ *implies* $L = M$.

An epimorphism $f : M \to N$ *is called superfluous if* $Kerf \ll M$.

Obviously $K \ll M$ if and only if the canonical projection $M \to M/K$ is a superfluous epimorphism.

It is easy to see that e.g. in $\mathbb{Z}$ there are no non-zero superfluous submodules.

Definition 2.14 *Let M be an R-module. As socle of M ($= Soc(M), SocM$) we denote the sum of all simple (minimal) submodules of M. If there no minimal submodules in M we put $Soc(M) = 0$.*

$Soc(M)$ is a semisimple submodule of M. Clearly, M is semisimple if and only if $M = Soc(M)$. An important multiple characterization of the socle is

Proposition 2.6 *If M is a left R-module, then*

$$\begin{aligned} Soc\,(M) &= \Sigma\{K \leq M \mid K \text{ is minimal in } M\} \\ &= \bigcap\{L \leq M \mid L \text{ is essential in } M\}. \end{aligned}$$

Properties of the Socle ((Wisbauer, 1991), 21.2)

Let M be an R-module.

(1) For any morphism $f : M \to N$, we have $f(Soc(M)) \subset Soc(N)$.

(2) For any submodule $K \subset M$, we have $Soc(K) = K \cap Soc(M)$.

(3) $Soc(M) \trianglelefteq M$ if and only $Soc(K) \neq 0$ for every non-zero submodule $K \subset M$.

(4) $Soc(M)$ is an $\mathrm{End}_R(M)$-submodule, i.e. $Soc(M)$ is fully invariant in M.

(5) $Soc(\bigoplus_\Lambda M_\lambda) = \bigoplus_\Lambda Soc(M_\lambda)$.

Definition 2.15 *Dual to the socle we define as* ***radical of an R-module*** *M ($= Rad(M), RadM$) the intersection of all maximal submodules of M. If M has no maximal submodule we set $Rad(M) = M$.*

The characterization of the radical

Proposition 2.7 *Let M be a left R-module. Then*

$$\begin{aligned} Rad\,(M) &= \bigcap\{K \leq M \mid K \text{ is maximal in } M\} \\ &= \Sigma\{L \leq M \mid L \text{ is superfluous in } M\}. \end{aligned}$$

Properties of the radical ((Wisbauer, 1991), 21.6)
Let M be an R-module.

(1) For a morphism $f : M \to N$ we have
(i) $f(RadM) \subset RadN$,
(ii) $Rad(M/RadM) = 0$, and
(iii) $f(RadM) = Rad(f(M))$, if $Kerf \subset RadM$.

(2) $RadM$ is an $\mathrm{End}_R(M)$-submodule of M (fully invariant).

(3) If every proper submodule of M is contained in a maximal submodule, then $RadM \ll M$ (e.g. if M is finitely generated).

(4) M is finitely generated if and only if $RadM \ll M$ and $M/RadM$ is finitely generated.

(5) If $M = \bigoplus_\Lambda M_\lambda$, then $RadM = \bigoplus_\Lambda RadM_\lambda$ and $M/RadM \simeq \bigoplus_\Lambda M_\lambda/RadM_\lambda$.

(6) If M is finitely cogenerated and $RadM = 0$, then M is semisimple and finitely generated.

(7) If $\overline{M} = M/RadM$ is semisimple and $RadM \ll M$, then every proper submodule of M is contained in a maximal submodule.

Definition 2.16 *The radical of* $_RR$ *is called the **Jacabson radical** of R, i.e.*

$$Jac(R) = Rad(_RR)$$

As a fully invariant submodule of the ring, $Jac(R)$ *is two-sided ideal in R.*

Definition 2.17 *An element* $x \in R$ *is left quasi-regular in case* $1 - x$ *has a left inverse in R. Similarly* $x \in R$ *is right quasi-regular (quasi-regular) in case* $1 - x$ *has a right (two-sided) inverse in R.*

Proposition 2.8 ***Characterization of the Jacobson radical***
In a ring R with unit, $Jac(R)$ *can be described as the*

(a) *intersection of the maximal left ideals in R (= definition);*

(b) *sum of all superfluous left ideals in R;*

(c) *sum of all left quasi-regular left ideals;*

(d) *largest (left) quasi-regular ideal;*

(e) $\{r \in R \mid 1 - ar$ *is invertible for any* $a \in R\}$*;*

(f) *intersection of the annihilators of the simple left R-modules;*

(a*) *intersection of the maximal right ideals.*
*Replacing 'left' by 'right' further characterizations (**b***) - (**f***) are possible.*

2.7. Singular Submodule

Given any right module M, the *singular submodule of M* is the set

$$Z(M) = \{m \in M : mI = 0 \ for\ some\ essential\ right\ ideal\ I\ of\ R\}.$$

Equivalently, $Z(M)$ is the set of those $m \in M$ for which the right ideal $ann_R(m)$ is essential in R. An R-module M is called singular if $Z(M) = M$, and it is called a nonsingular module if $Z(M) = 0$. A ring R is called a right nonsingular ring if R is nonsingular as a right R-module. $Z_r(R)$ will be used for $Z(R_R)$. Similarly, we say that R is *left nonsingular ring* if $Z_l(R) = 0$.

Proposition 2.9 *(Goodearl, 1976) The following hold for any ring R.*

(1) A module N is nonsingular if and only if $Hom(M, N) = 0$ *for all singular modules* M.

(2) If R is a right semihereditary ring, then $Z_r(R) = 0$.

(3) If $Z_r(R) = 0$*, then* $Z(M/Z(M)) = 0$ *for all right R-modules* M.

(4) If $N \leq M$*, then* $Z(N) = N \cap Z(M)$.

(5) Suppose that $Z_r(R) = 0$*. A right module M is singular if and only if* $Hom(M, N) = 0$ *for all nonsingular right modules* N.

Let M be an R-module and $N \leq M$. If N is an essential submodule of M, then M/N is singular. Converse is not true in general. For example, let $M = \mathbb{Z}/2\mathbb{Z}$ and $N = 0$. M/N is singular but N is not an essential submodule of M. The following Proposition shows that when the converse true.

Proposition 2.10 *((Goodearl, 1976), Proposition 1.21) Let M be a nonsingular module and $N \leq M$. Then M/N is singular if and only if N is an essential submodule of M.*

The class of all singular right modules is closed under submodules, factor modules and direct sums. On the other hand, the class of all nonsingular right modules is closed under submodules, direct products, essential extensions, and module extensions.

Proposition 2.11 *((Goodearl, 1976), Proposition 1.24) If M is any simple right R-module, then M is either singular or projective, but not both.*

A ring R is called a right *SI-ring* if every singular right R-module is injective. A ring R is called a right *PCI-ring* if each proper cyclic right R-module is injective. Right PCI-rings are right Noetherian and right hereditary. The right SI-ring and right PCI-ring conditions are equivalent for domains.

2.8. Small Rings and Small Modules

Definition 2.18 *A right R-module M is called a* ***small module*** *if it is a small submodule in its injective hull $E(M)$, i.e $M \ll E(M)$.*

The following characterization of small module is well-known

Proposition 2.12 *For a right R-module M, the followings are equivalent:*

(i) *M is small.*

(ii) *$M \ll E(M)$.*

(iii) *$M \ll E$ for some injective right R-module E.*

(iv) *$M \ll L$ for some right R-module L containing M.*

Proposition 2.13 *If M is small then M/N is small for every $N \leq M$.*

Proof Suppose M is small i.e. $M \ll E(M)$. Let $N \leq M$, then $M/N \leq E(M)/N$. Let $L/N \leq E(M)/N$ such that $M/N + L/N = E(M)/N$, then $M + L = E(M)$. Since $M \ll E(M)$, $L = E(M)$. Hence $L/N = E(M)/N$ and M/L is small. □

Definition 2.19 *A ring R is called **left small** if $_RR$ is a small module; e.g. $\mathbb{Z}$ is a small ring as it is small in $_{\mathbb{Z}}\mathbb{Q}$.*

Proposition 2.14 *((Ramamurthi, 1982), 3.3), ((Pareigis, 1966), 4.8) Let R be a ring and let $E(R)$ be the injective hull of $_RR$. Then the followings conditions are equivalent:*

(i) *R is a left small ring.*

(ii) *$Rad(M) = M$ for every injective left R-module M.*

(iii) *$Rad(E(R)) = E(R)$.*

CHAPTER 3

PROJECTIVE, INJECTIVE AND FLAT MODULES

In this chapter we give the definitions and main properties and characterization of projective, injective and flat modules.

3.1. Projective Modules

Definition 3.1 *A left R-module P is* ***projective*** *if, whenever p is surjective and h is any map, there exists a lifting g; that is, there exists a map g making the following diagram commute:*

$$\begin{tikzcd} & P \arrow[dl, dashed, "g"'] \arrow[d, "h"] & \\ C \arrow[r, "p"'] & A \arrow[r] & 0 \end{tikzcd}$$

Proposition 3.1 *((Rotman, 2009), Proposition 3.2) A left R-module P is projective if and only if $Hom_R(P, \square)$ is an exact functor.*

Proposition 3.2 *((Rotman, 2009), Proposition 3.3) A left R-module P is projective if and only if every short exact squence* $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} P \longrightarrow 0$ *splits.*

Definition 3.2 *A ring R is* ***left hereditary*** *if every left ideal is projective; a ring R is* ***right hereditary*** *if every right ideal is projective. A* ***Dedekind ring*** *is a hereditary domain.*

Theorem 3.1 ***(Cartan-Eilenberg)*** *The following statements are equivalent for a ring R.*

(i) *R is left hereditary.*

(ii) *Every submodule of a projective module is projective.*

(iii) *Every quotient of an injective module is injective.*

3.2. Injective Modules

Definition 3.3 *A left R-module E is* ***injective*** *if, whenever i is an injection, a dashed arrow exists making the following diagram commute.*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[dl, dashed, "g"] \\ & E & \end{tikzcd}$$

Proposition 3.3 *((Rotman, 2009), Proposition 3.28)*

(i) *If $(E_k)_{k\in K}$ is a family of injective left R-modules, then $\prod_{k\in K} E_k$ is also an injective left R-module.*

(ii) *Every direct summand of an injective left R-module E is injective.*

Proof

(i) Consider the diagram in which $E = \prod E_k$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[dl, dashed, "g"] \\ & E & \end{tikzcd}$$

Let $p_k : E \to E_k$ be the kth projection, so that $p_k f : A \to E_k$. Since E_k is an injective module, there is $g_k : B \to E_k$ with $g_k i = p_k f$. Now define $g : B \to E$ by $g : b \mapsto (g_k(b))$. The map g does extend f, for if $b = ia$, then

$$g(ia) = (g_k(ia)) = (p_k fa) = fa,$$

because $x = (p_k x)$ for every x in the product.

(ii) Assume that $E = E_1 \oplus E_2$, let $i : E_1 \to E$ be the inclusion, and let $p : E \to E_1$ be the projection (so that $pi = 1_{E_1}$).

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "j"] \arrow[d, "f"] & C \arrow[d, dashed, "g_0"] \\ & E_1 \arrow[r, shift left, "i"] & E_1 \oplus E_2 \arrow[l, shift left, "p"] \end{tikzcd}$$

Then, the proof can be completed easily using the diagram as a guide.

□

Corollary 3.1 *Any finite direct sum of injective left R-modules is injective.*

Proof The direct sum of finitely many modules coincides with the direct product. □

Theorem 3.2 ***(Baer Criterion)*** *((Rotman, 2009), Theorem 3.30) A left R-module E is injective if and only if every map $f : I \to E$, where I is an ideal in R, can be extended to R.*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[dl, dashed, "g"] \\ & E & \end{tikzcd}$$

Proof Since any left ideal I is a submodule of R, the existence of an extension g of f is just special case of the definition of injectivity of E.

Suppose we have the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"] & B \\ & E & \end{tikzcd}$$

where A is a submodule of a left R-module B. For notational convenience, let us assume i is the inclusion [this assumption amounts to permitting us to write a instead of $i(a)$ whenever $a \in A$]. We are going to use Zorn‘s lemma. Let X be the set of all ordered pairs (A', g'), where $A \subseteq A' \subseteq B$ and $g' : A' \to E$ extends f; that is $g'|_A = f$. Note that $X \neq \emptyset$, because $(A, f) \in X$. Partially order X by defining

$$(A^{'}, g^{'}) \preceq (A^{''}, g^{''})$$

to mean $A^{'} \subseteq A^{''}$ and $g^{''}$ extends $g^{'}$. The reader may supply the argument that chains in X have upper bounds in X; hence, Zorn's lemma applies, and there exists a maximal element (A_0, g_0) in X. If $A_0 = B$, we are done, and so we may assume that there is some $b \in B$ with $b \notin (A_0)$.
Define

$$I = \{r \in R : rb \in A_0\}.$$

It easy to see that I is a left ideal in R. Define $h : I \to E$ by

$$h(r) = g_0(rb).$$

By hypothesis, there is a map $h^* : R \longrightarrow E$ extending h. Finally, define $A_1 = A_0 + < b >$ and $g_1 : A_1 \longrightarrow E$ by

$$g_1(a_0 + rb) = g_0(a_0) + rh^*(1),$$

where $a_0 \in A_0$ and $r \in R$.
Let us show that g_1 is well defined. If $a_0 + rb = a'_0 + r'b$, then $(r - r')b = a'_0 - a_0 \in A_0$; it follows that $r - r' \in I$. Therefore, $g_0((r - r')b)$ and $h(r - r')$ are defined, and we have

$$g_0(a'_0 - a_0) = g_0((r - r')b) = h(r - r') = h^*(r - r') = (r - r')h^*(1).$$

Thus $g_0(a'_0) - g_0(a_0) = rh^*(1) - r'h^*(1)$ and $g_0(a'_0) + r'h^*(1) = g_0(a_0) + rh^*(1)$, as desired. Clearly $g_1 a_0 = g_0 a_0$ for all $a_0 \in A_0$, so that the map g_1 extends g_0. We conclude that $(A_0, g_0) \preceq (A_1, g_1)$, contradicting the maximality of (A_0, g_0). Therefore $A_0 = B$, the map g_0 is lifting of f, and E is injective. □

Proposition 3.4 *((Rotman, 2009), Proposition 3.31) If R is a left Noetherian ring and $(E_k)_{k\in K}$ is a family of injective left R-modules, then $\bigoplus_{k\in K} E_i$ is an injective left R-module.*

Definition 3.4 *Let M be an R-module over a domain R. If $r \in R$ and $m \in M$, then we say that m is **divisible by** r if there is some $m' \in M$ with $m = rm'$. We say that M is a **divisible module** if each case $m \in M$ is divisible by every nonzero $r \in R$.*

If R is a domain, $r \in R$ and M is an R-module, then the function $\varphi_r : M \to M$, defined by $\varphi_r : m \mapsto rm$, is an R-map. It is clear that M is divisible module if and only if φ_r is surjective for every $r \neq 0$.

Lemma 3.1 *((Rotman, 2009), Lemma 3.33) If R is a domain, then every injective R-module E is a divisible module.*

Corollary 3.2 *Let R be a principal ideal domain.*

(i) *An R-module E is injective if and only if it is divisible.*

(ii) *Every quotient of an injective R-module E is itself injective.*

Corollary 3.3 *((Rotman, 2009), Corollary 3.36) Every abelian group M can be imbedded as a subgroup of some injective abelian group.*

Theorem 3.3 *((Rotman, 2009), Theorem 3.38) For every ring R, every left R-module M can be imbedded as a submodule of an injective left R-module.*

Theorem 3.4 *((Rotman, 2009), Theorem 3.39) If R is a ring for which every direct sum of injective left R-modules is an injective module, then R is left Noetherian.*

Definition 3.5 *Let M and E be left R-modules. Then E is an **essential extension** of M if there is an injective R-map $\alpha : M \to E$ with $S \cap \alpha(M) \neq \{0\}$ for every nonzero submodule $S \subseteq E$. If also $\alpha(M) \subsetneq E$ is called a **proper essential extension** of M.*

Lemma 3.2 *((Rotman, 2009), 3.44) Given a left R-module M, the following conditions are equivalent for a module $E \supseteq M$.*

(i) *E is a maximal essential extension of M; that is, no proper extension of E is an essential extension of M.*

(ii) *E is an injective module and E is an essential extension of M.*

(iii) *E is an injective module and there is no proper injective intermediate submodule E'; that is, there is no injective E' with $M \subseteq E' \subsetneq E$.*

Definition 3.6 *If M is a left R-module, then a left R-module E containing M is an **injective envelope** of M, denoted by $E(M)$ if any of equivalent conditions in Lemma 3.2 hold.*

3.2.1. Injective Cogenerator

Definition 3.7 *Let $\mathcal{U}$ be a non-empty set (class) of objects in a category C. An object A in C is said to be generated by $\mathcal{U}$ or $\mathcal{U}$-generated if, for every pair of distinct morphisms $f, g : A \to B$ in C, there is a morphism $h : U \to A$ with $U \in \mathcal{U}$ and $hf \neq hg$. In this case $\mathcal{U}$ is called a set (class) of generators for A.*

Definition 3.8 *Let M be an R-module. We say that an R-module N is subgenerated by M, or that M is a subgenerator for N, if N is isomorphic to a submodule of an M-generated module.*
A subcategory C of $R-MOD$ is subgenerated by *M, or M is a subgenerator for C, if every object in C is subgenerated by M.*
We denote by $\sigma[M]$ the full subcategory of $R - MOD$ whose objects are all R-modules subgenerated by M.

Definition 3.9 *An injective module Q in $\sigma[M]$ is a cogenerator in $\sigma[M]$ if and only if it cogenerates every simple module in $\sigma[M]$, or equivalently, Q contains every simple module in $\sigma[M]$ as a submodule (up to isomorphism).*

3.3. Flat Modules

Definition 3.10 *If R is a ring, then a right R-module A is **flat** if $A \otimes_R \Box$ is an exact functor; that is, whenever*

$$0 \longrightarrow B' \xrightarrow{i} B \xrightarrow{p} B'' \longrightarrow 0$$

is an exact sequence of left R-modules, then

$$0 \longrightarrow A \otimes_R B' \xrightarrow{1_A \otimes i} A \otimes_R B \xrightarrow{1_A \otimes p} A \otimes_R B'' \longrightarrow 0$$

is an exact sequence of abelian groups. Flatness of left R-modules is defined similarly.

Proposition 3.5 *((Rotman, 2009), Proposition 3.46) Let R be an arbitrary ring.*

(i) *The right R-module R is a flat right R-module.*

(ii) *A direct sum $\bigoplus_j M_j$ of right R-modules is flat if and only if each M_j is flat.*

(iii) *Every projective right R-module P is flat.*

Definition 3.11 *For a right R-module M, the left module $M^+ = Hom(M, \mathbb{Q}/\mathbb{Z})$ is called the character module of M.*

Proposition 3.6 ***(Lambek).*** *A right R-module M is flat if and only if its character module M^+ is an injective left R-module.*

Proof The functors $Hom_R(\square, Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) = Hom_R(\square, M^+)$ and $Hom_{\mathbb{Z}}(\square, \mathbb{Q}/\mathbb{Z}) \circ (M \otimes_R \square)$ are naturally isomorphic, by ((Rotman, 2009), Corollary 2.77). If M is flat, then each of the functors in the composite is exact, for $\mathbb{Q}/\mathbb{Z}$ is $\mathbb{Z}$-injective; hence, $Hom_R(\square, M^+)$ is exact and M^+ is injective.
Conversely, assume that M^+ is an injective left R-module and $A' \to A$ is an injection between left R-modules A' and A. Since $Hom_R(A, M^+) = Hom_R(A, Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}))$, the (second version of the) adjoint isomorphism, ((Rotman, 2009), Theorem 2.76), gives a commutative diagram in which the vertical maps are isomorphisms.

$$\begin{tikzcd}
Hom_R(A, M^+) \arrow[r] \arrow[d] & Hom_R(A', M^+) \arrow[r] \arrow[d] & 0 \\
Hom_{\mathbb{Z}}(M \otimes_R A, \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, "="'] & Hom_{\mathbb{Z}}(M \otimes A', \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, "="] & 0 \\
(M \otimes_R A)^+ \arrow[r] & (M \otimes_R A')^+ \arrow[r] & 0
\end{tikzcd}$$

Exactness of the top row gives exactness of the bottom row. The sequence
$0 \longrightarrow M \otimes_R A' \longrightarrow M \otimes_R A$ is exact, by ((Rotman, 2009), Lemma 3.53), and this gives M is flat. □

Definition 3.12 *Let M be a right R-module and N a submodule of M. N is said to be a pure submodule of M if the induced map $N \otimes L \to M \otimes L$ is monic for each left R-module L.*

Proposition 3.7 *(Rotman, 2009) The following are equivalent for a right module M and its submodule N.*

(1) N is a pure submodule of M.

(2) The induced map $N \otimes F \to M \otimes F$ is monic for each finitely presented left R-module F.

(3) The induced map $Hom(F, M) \to Hom(F, M/N)$ is epic for each finitely presented left R-module F.

Definition 3.13 *A right module M is said to be absolutely pure if it is a pure submodule of $E(M)$.*

CHAPTER 4

POOR MODULES

In this chapter we introduce the notions of relative injectivity and injectivity domains of modules. Poor modules are defined as the modules whose injectivity domains is as small as possible. We review some results from (Alahmadi, Alkan and López-Permouth, 2010), (Alizade and Büyükaşık, 2017) and (Alizade, Büyükaşık, López-Permouth and Yang, 2018) about poor modules.

4.1. Relative Injectivity and Injectivity Domain

Definition 4.1 *A right R-module M is said to be N-injective (or injective relative to N) if for every submodule K of N and every morphism $f : K \to M$ there exists a morphism $\bar{f} : N \to M$ such that $\bar{f}\mid_K = f$.*

$$\begin{tikzcd} K \arrow[r, hook] \arrow[d, "f"'] & N \arrow[dl, dashed, "\bar{f}"] \\ M & \end{tikzcd}$$

Proposition 4.1 *((Mohamed and Müller, 1990), Proposition 1.3) Let N be an A-injective module. If $B \leq A$, then N is B-injective and A/B-injective.*

Proof It is obvious that N is B-injective. Let X/B be a submodule of A/B, and $\varphi : X/B \to N$ be a homomorphism. Let π denote the natural homomorphism of A onto A/B and $\pi' = \pi|_X$. Since N is A-injective, there exists a homomorphism $\theta : A \to N$ that extends $\varphi\pi'$. Now

$$\theta B = \varphi\pi' B = \varphi(0) = 0.$$

Hence $Ker\pi \leq Ker\theta$, and consequently there exists $\psi : A/B \to N$ such that $\psi\pi = \theta$. For every $x \in X$,

$$\psi(x + B) = \psi\pi(x) = \theta(x) = \varphi\pi'(x) = \varphi(x + B).$$

Thus ψ extends φ, and therefore N is A/B injective.

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "\pi'"] & A \arrow[d, "\pi"] \arrow[ddl, dashed, "\theta"'] \\
X/B \arrow[r, hook] \arrow[d, "\varphi"] & A/B \arrow[dl, dashed, "\psi"] \\
N &
\end{tikzcd}$$

□

Definition 4.2 *For a module M, the* ***injectivity domain*** *of M is defined to be the collection of modules N such that M is N-injective, that is, $In^{-1}(M) = \{N \in Mod - R \mid M$ is N-injective $\}$. Clearly, for any right R-module M, semisimple modules in $Mod - R$ are contained in $In^{-1}(M)$, and M is injective if and only if $In^{-1}(M) = Mod - R$.*

4.2. Poor modules

Definition 4.3 *M is called* ***poor*** *if, for every right R-module N, M is N-injective only if N is semisimple.*

Theorem 4.1 *((Er, Lòpez-Permouth and Sökmez, 2011), Proposition 1) Every ring has a poor module.*

Proof Let R be any ring. Let $\{A_\gamma \mid \gamma \in \Gamma\}$ be a complete set of representatives of isomorphism classes of non-semisimple cyclic (right) R-modules. Since, for each $\gamma \in \Gamma$, A_γ is non-semisimple, we can pick a proper essential submodule K_γ of A_γ. Now put $T = \oplus_{\gamma\in\Gamma} K_\gamma$. Let B be a non-semisimple cyclic module such that T is B-injective. Then there is some $\gamma \in \Gamma$ such that $B \cong A_\gamma$. Thus B has a proper essential submodule, say N, isomorphic to K_γ. But then N is B-injective, a contradiction. Therefore, T is poor. □

Definition 4.4 *A module M crumbles if socles split in all factors of M.*

Theorem 4.2 *((Er, Lòpez-Permouth and Sökmez, 2011), Theorem 1) Let R be any ring. The following conditions are equivalent:*

(i) *R has a semisimple poor right module.*

(ii) *Every cyclic right R-module that crumbles is semisimple.*

(iii) *Every right R-module that crumbles is semisimple.*

(iv) *Every Noetherian but not Artinian cyclic R-module has a factor whose Jacobson radical has nonzero socle.*

(v) *Every Noetherian but not Artinian cyclic R-module has a factor with nonzero Jacobson radical.*

The structure of poor abelian groups is as follows

Theorem 4.3 *((Alizade and Büyükaşık, 2017), Theorem 3.1) A group is poor if and only its torsion part has a direct summand isomorphic to $\oplus_{p \in P} \mathbb{Z}_p$.*

Corollary 4.1 *((Alizade and Büyükaşık, 2017), Corollary 3.2) For a group G, the following are equivalent.*

(1) *G is poor.*

(2) *The reduced part of G is poor.*

(3) *$T(G)$ is poor.*

(4) *For each prime p, G has a direct summand isomorphic to $\mathbb{Z}_p$.*

Definition 4.5 *A commutative domain is $h-semilocal$ (or a finite character), if every nonzero ideal is contained in only finitely many maximal ideals.*

Commutative semilocal rings, $h-local$ domains, and in particular Dedekind domains are h-semilocal. It is known that direct sum of nonisomorphic simple $\mathbb{Z}$-modules is poor. We have the following result for h-semilocal domains.

Let S be the direct sum of non-isomorphic non-injective simple R-modules over a ring R. It is known that S is poor over the ring of integers. We have the following generalization.

Proposition 4.2 *Let R be an h-semilocal domain (a domain of finite character). Then S is poor.*

Proof Suppose S is A-injective for some cyclic R-module A. We need to show that A is semisimple. We have two cases: $A \cong R$ or $A \cong R/I$ for some nonzero ideal I of R. In the first case, S is injective by Baer's Criteria. But R is a domain, so R is a field. This means that A is a simple R-module. Now suppose $A \cong R/I$ for some non zero ideal I of R. We claim that $Rad(R/I) = 0$. Suppose there is a nonzero $a + I \in Rad(R/I)$. Then $(aR + I)/I$ is small in R/I. Since $(aR + I)/I$ is cyclic, it has a simple quotient, say $(aR + I)/K$. Note that $I \leq K$. Then $(aR + I)/K$ small in R/K because small modules are closed under homomorphic images. On the other hand, $(aR + I)/K$ is isomorphic to a direct summand of S. So $(aR + I)/K$ is R/K-injective. Then $(aR + I)/K$ is a direct summand of R/K. This contradicts with the smallness of $(aR + I)/K$ in R/K. As a consequence $Rad(R/I) = 0$. This means that $I = \cap_{\lambda\in\Lambda} I_\lambda$, where Λ is an index set and I_λ are maximal ideals of R for each $\lambda \in \Lambda$. Since R is h-semilocal, I is contained in only finitely many maximal ideals. Therefore we may assume that Λ is finite and that the intersection $\cap_{\lambda\in\Lambda} I_\lambda$ is irredundant in the sense that for each $\lambda' \in \Lambda$, $\cap_{\lambda\neq\lambda'} I_\lambda \neq \cap_{\lambda\in\Lambda} I_\lambda$. Then $R/I = R/(\cap_{\lambda\in\Lambda} I_\lambda) \cong \oplus_{\lambda\in\Lambda}(R/I_\lambda)$ is semisimple. Hence in both cases, we have A is semisimple. This proves that S is poor. □

Definition 4.6 *((Alizade, Büyükaşık, López-Permouth and Yang, 2018)) A module is a pauper (or a pauper module) if it is poor and no proper direct summand of it is poor. For a ring R and a class of right modules $\mathcal{C}$,*

- *$\mathcal{C}$ satisfies $(U*)$ for every poor module P in $\mathcal{C}$ there exists a pauper $M \in \mathcal{C}$ such that M is a pure submodule of P.*

Recently, Theorem 4.3 generalized as follows.

Theorem 4.4 *((Alizade, Büyükaşık, López-Permouth and Yang, 2018), Theorem 4.9) A commutative hereditary Noetherian ring R satisfies $(U*)$. In fact, for every right R-module M, the following statements are equivalent.*

(1) *M is poor.*

(2) *$Z(M)$ is poor.*

(3) *For every noninjective simple module V, M has a direct summand isomorphic to V.*

(4) *M has a pure submodule isomorphic to S, where S is the sum of nonisomorphic and noninjective simple R-modules.*

CHAPTER 5

SUBINJECTIVITY AND SUBINJECTIVITY DOMAINS

As an opposite notion of relative injectivity and injectivity domains, the subinjectivity and subinjectivity domains introduced in (Aydoğdu and López-Permouth, 2011). In this section we outline some properties of subinjectivity and subinjectivity domains. In contrast to injectivity domains subinjectivity domains are not closed under factor modules in general. In this chapter we prove that the ring R is right hereditary if and only if the subinjectivity of any right R-module is closed under factor modules.

Definition 5.1 *Given modules M and N, we say that M is N-subinjective if for every module K with $N \leq K$ and every homomorphism $\varphi : N \to M$ there exists a homomorphism $\phi : K \to M$ such that $\phi\mid_N = \varphi$. The subinjectivity domain of a module M, $Jn^{-1}(M)$ is defined to be the collection of all modules N such that M is N-subinjective.*

Lemma 5.1 *((Aydoğdu and López-Permouth, 2011), Lemma 2.2) The following statements are equivalent for any modules M and N:*

(1) *M is N-subinjective.*

(2) *For each $\varphi : N \to M$ and for every module K with $N \trianglelefteq K$, there exists a homomorphism $\phi : K \to M$ such that $\phi\mid_N = \varphi$.*

(3) *For each $\varphi : N \to M$, there exists a homomorphism $\phi : E(N) \to M$ such that $\phi\mid_N = \varphi$.*

(4) *For each $\varphi : N \to M$, there exists an injective extension E of N and a homomorphism $\phi : E \to M$ such that $\phi\mid_N = \varphi$.*

Proof The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. To show (4) $\Rightarrow$ (1), let $N \subseteq N'$ and $\varphi : N \to M$. By assumption, there exists an injective extension E of N and a homomorphism $\phi : E \to M$ such that $\phi\mid_N = \varphi$. Since E is injective, there exists a $\psi : N' \to E$ such that $\psi|_N = i$, where $i : N \to E$ is the inclusion. Then we get that $(\phi\psi)|_N = \varphi$. This gives that $N \in Jn^{-1}(M)$. □

Proposition 5.1 *((Aydoğdu and López-Permouth, 2011), Proposition 2.3) $\bigcap_{M\in Mod-R} Jn^{-1}(M) = \{A \in Mod - R \mid A \text{ is injective}\}$.*

Proof Let $N \in \cap_{M\in Mod-R} Jn^{-1}(M)$. Then $N \in Jn^{-1}(N)$ which means that N is injective.

□

The following two results summarize some properties of subinjectivity and subinjectivity domains.

Proposition 5.2 *((Aydoğdu and López-Permouth, 2011), Proposition 2.4) The following properties hold for a module N:*

(1) *$\prod_{i\in I} M_i$ is N-subinjective if and only if each M_i is N-subinjective.*

(2) *If each M_i is N-subinjective for $i = 1, \ldots, n$, then so is $\bigoplus_{i=1}^{n} M_i$.*

(3) *Every direct summand of an N-subinjective module is an N-subinjective module. Conversely, if N is a finitely generated module and M_i, $i \in I$ is a family of N-subinjective modules indexed in an arbitrary index set I, then $\bigoplus_{i\in I} M_i$ is an N-subinjective module.*

Proof (1) Suppose that M_i is N-subinjective for each $i \in I$. Consider a homomorphism $\varphi : N \to \prod_{i\in I} M_i$. Let $\pi_i : \prod_{i\in I} M_i \to M_i$ be the canonical epimorphism for each $i \in I$. Then there exists a $\phi_i : E(N) \to M_i$ such that $\phi_i|_N = \pi_i\varphi$ for each $i \in I$. Define an R-homomorphism $\psi : E(N) \to \prod_{i\in I} M_i$ via $x \mapsto (\phi_i(x))$. Then $\psi|_N = \varphi$.

For the converse, let $i \in I$ and $\varphi : N \to M$. There exists a $\phi : E(N) \to \prod_{i\in I} M_i$ such that $\phi|_N = e_i\varphi$, where e_i is the inclusion $M_i \to \prod_{i\in I} M_i$. Let $\pi_i : \prod_{i\in I} M_i \to M_i$ be the canonical epimorphism. Then $(\pi_i\phi)|_N = \varphi$. Hence $N \in \bigcap_{i\in I} Jn^{-1}(M_i)$.
The proofs of (2) and (3) are similar to the proof of (1). □

Proposition 5.3 *((Aydoğdu and López-Permouth, 2011), Proposition 2.5) The following properties hold for any ring R and R-modules N and M:*

(1) *If $N = \bigoplus_{i=1}^{n} N_i$, then M is N-subinjective if and only if M is N_i-subinjective for each $i = 1, \ldots, n$.*

(2) *If R is right Noetherian and I is any index set, then M is $\bigoplus_{i\in I} N_i$-subinjective if and only if M is N_i-subinjective for each $i \in I$.*

(3) *If R is right hereditary right Noetherian ring and M is N-subinjective, then M is N/K-subinjective for any submodule K of N.*

(4) *If M is a non-singular N-subinjective module, then M is K-subinjective for any essential extension K of N.*

(5) *If $N \leq M$ and M is N-subinjective, $E(N) \leq M$. In particular, M is M-subinjective if and only if M is injective.*

Proof

(1) Let $\varphi : N_i \to M$, and consider the canonical epimorphism $\pi : N \to N_i$. Since $N \in Jn^{-1}(M)$, there exists a $\phi : E(N_i) \oplus E(\bigoplus_{i \neq j} N_j) \to M$ such that $\phi|_N = \varphi\pi$. Then $\psi = \phi|_{E(N_i)} : E(N_i) \to M$, and hence $\psi|_{N_i} = \varphi$. Now let $\varphi : N \to M$. Then there exists $\psi_i : E(N_i) \to M$ such that $\psi_i|_{N_i} = \varphi\pi_i$ for each $i = 1, \ldots, n$. Define $\psi : \bigoplus_{i=1}^{n} E(N_i) \to M$, $x_1 + \ldots + x_n \mapsto \psi_1(x_1) + \ldots + \psi_n(x_n)$. Hence, we get that $\psi|_N = \varphi$.

(2) Since R is right Noetherian, $E(N) = \bigoplus_{i \in I} E(N_i)$. The rest of the proof is similar to that of (1).

(3) Since R is right Noetherian, we have a decomposition $M = M_1 \oplus M_2$, where M_1 is an injective module and M_2 is a reduced module, i.e., a module which does not have non-zero injective submodules. Then $Jn^{-1}(M) = Jn^{-1}(M_1) \cap Jn^{-1}(M_2)$ by 5.1(1). But since M_1 is injective, its subinjectivity domain consists of all R-modules. Therefore, $Jn^{-1}(M) = Jn^{-1}(M_2)$. On the other hand, R being right hereditary implies that $Jn^{-1}(M_2) = \{N \in Mod - R \mid Hom_R(N, M_2) = 0\}$. It is easy to see that this set is closed under taking homomorphic images.

(4) Let $f : K \to M$ be any homomorphism. Since M is N-subinjective, there exists $g : E(N) \to M$ such that $g|_N = f|_N$. Then $N \subseteq Ker(g - f)$. Because N is an essential submodule of K, $Ker(g-f)$ is essential in K, too. Therefore, $K/Ker(g-f)$ is singular. On the other hand, $K/Ker(g - f)$ is isomorphic to a submodule of the non-singular module M. Hence, $K = Ker(g - f)$ which means that $g|_K = f|_K$.

(5) Since N is essential in $E(N)$, $E(N)$ can be embedded into M because of N-subinjectivity assumption.

□

From Proposition 5.3(3), we see that the subinjectivity domain is closed under factor modules. We generalize this result as follows.

Theorem 5.1 *A ring R is right hereditary if and only if the subinjectivity domain of each right module is closed under homomorphic images.*

Proof Suppose M is N-subinjective for right modules M and N. Let $K \leq N$ and $f : N/K \to M$ be a homomorphism. Consider the following diagram:

$$\begin{tikzcd}
N \arrow[r, "\iota_N"] \arrow[d, "\pi"] & E(N) \arrow[d, "\pi'"] \arrow[ddl, dashed, "\varphi"] \\
\frac{N}{K} \arrow[r] \arrow[d, "f"] & \frac{E(N)}{K} \arrow[dl, dashed, "\varphi'"] \\
M &
\end{tikzcd}$$

where π, π' are canonical epimorhisms. Since M is N-subinjective, there is a $\varphi : E(N) \to M$ such that $\varphi_{|_N} = f\pi$. Clearly $f\pi(K) = 0$, and so $K \subseteq Ker(\varphi)$. Then, by factor theorem, there exists a homomorphism $\varphi' : E(N)/K \to M$, given by $\varphi'(a + K) = \varphi(a)$ for each $a \in E(N)$. Clearly, $\varphi'_{|_{N/K}} = f$. Then M is N/K-subinjective by 5.1(4). This proves the necessity.

Conversely suppose the subinjectivity domain of every right module is closed under homomorphic images. We shall prove that quotients of injective right modules are injective. Let E be an injective module and $K \leq E$. Clearly E/K is E-subinjective. Therefore E/K is E/K-subinjective by the assumption. Hence E/K is injective. Thus R is right hereditary. □

Proposition 5.4 *((Aydoğdu and López-Permouth, 2011), Proposition 2.8)*

(1) *Let $R = R_1 \oplus R_2$ be a ring decomposition. Then M is N-subinjective in Mod-R if and only if MR_i is NR_i-subinjective in Mod-R_i for each $i = 1, 2$.*

(2) *Let I be an ideal of a ring R, and let M and N be R/I-modules. If M is N-subinjective as an R/I-module, then it is N-subinjective as an R-module. The converse holds if N is a pure submodule of $E(N)$.*

Proof

(1) By assumption, we have $K = KR_1 \oplus KR_2$ for any R-module K. Now assume that M is N-subinjective. Let $f_i : NR_i \to MR_i$ be an R_i-homomorphism. We can define an R-homomorphism $f' : N \to M$, $n_1r_1 + n_2r_2 \mapsto f_i(n_ir_i)$, where $n_1, n_2 \in N$, $r_i \in R_i$ for $i = 1, 2$. Then there exists $g : E(NR_1) \oplus E(NR_2) \to M$ such that $g|_N = f'$. Hence, the result follows. For the converse, note that $E(N) \hookrightarrow E(NR_1) \oplus E(NR_2)$ since $E(NR_1) \oplus E(NR_2)$ is an injective R-module.

(2) Let $E_{R/I}(N)$ be the injective hull of $N_{R/I}$. Since $E_{R/I}(N)$ is also an injective R-module, the result follows from 5.1(4). For the reverse implications, let N be pure in $E(N)$. Then N being both pure and essential in $E(N)$ implies that EN has an R/I-module structure.

□

Proposition 5.5 *((Aydoğdu and López-Permouth, 2011), Proposition 2.9)Consider the following statements for a module N:*

(1) *N is projective.*

(2) *Every homomorphic image of N-subinjective module is N-subinjective.*

(3) *Every homomorphic image of an injective module is N-subinjective.*

Then $(1) \Rightarrow (2) \Rightarrow (3)$*, and* $(3) \Rightarrow (1)$ *if the injective hull $E(N)$ of N is projective.*

Proof $(1) \Rightarrow (2)$ Let M be an N-subinjective module. Let $K \leq M$ and let $f : N \to M/K$ be a homomorphism. Since N is projective, there exists a homomorphism $g : N \to M$ such that $\pi g = f$, where $\pi : M \to M/K$ is the canonical epimorphism. But N-subinjectivity of M implies that g can be extended to a homomorphism $h : E(N) \to M$. It follows that the homomorphism $\pi h : E(N) \to M/K$ extends f. $(2) \Rightarrow (3)$ is obvious. For $(3) \Rightarrow (1)$ assume that $E(N)$ is projective. Let M and K be modules such that $K \leq M$, and let $f : N \to M/K$. Then we have $if : N \to E(M)/K$, where $i : M/K \to E(M)/K$ is the inclusion. By hypothesis, $E(M)/K$ is N-subinjective, so there exists $g : E(N) \to E(M)/K$ which extends if. But $E(N)$ is projective. Therefore, there is a homomorphism $h : E(N) \to E(M)$ such that $\pi' h = g$, where $\pi' : E(M) \to E(M)/K$ is the canonical epimorphism. Hence, if we consider $hi : N \to M$, then $\pi hi = f$, where $\pi : M \to M/N$ is the canonical epimorphism. □

CHAPTER 6

INDIGENT MODULES

In this chapter we study some properties of indigent modules and their characterizations over some particular rings. The existence of indigent modules is not known over arbitrary rings. In (Aydoğdu and López-Permouth, 2011), the authors ask whether the direct sum of non-injective uniform right modules is indigent. We give an example to show that this module is not indigent in general. Namely, we show that over a right semiartinian right V-ring, the direct sum of non-injective uniform right modules is not indigent. On the other hand, it is indigent over right PCI-domains. We give a complete characterization of indigent modules over commutative hereditary Noetherian rings. We characterize the commutative rings whose simple modules are injective or indigent. We also prove that every cyclic right module is indigent if and only if the ring is semisimple Artinian.

6.1. Indigent Modules

A right module M is called *indigent* if its subinjectivity domain is exactly the class of injective right modules.

Proposition 6.1 *((Alizade and Durğun, 2017)) Every right Noetherian ring has an indigent right R-module.*

Proof Let Λ be a complete set of representatives of finitely presented left R-modules. Consider the left module $M = \oplus_{F\in\Lambda}F$. Then $M^+ \cong \prod_{F\in\Lambda} F^+$. We claim that M^+ is an indigent right module. To prove this, suppose M^+ is N-subinjective for some right R-module N. Note that $(\bigoplus_{F\in\Lambda} F)^+ \cong \prod_{F\in\Lambda} F^+$. Then the map $0 \to N \oplus M \to E(N) \oplus M$ is monic. This implies that the map $0 \to N \otimes F \to E(N) \otimes F$ is a monomorphism for each finitely presented left R-module F. Therefore N is absolutely pure by Proposition 3.7. Since the ring is right Noetherian, N injective by ((Megibben, C.), Theorem 3) Hence M^+ is indigent. □

Let R be a non-Noetherian ring. Suppose every module is indigent or injective. Then every pure-injective module is injective. Thus R must be von Neumann regular.

Remark 6.1 *Let $\mathfrak{M} = \oplus_{U \in \mathfrak{V}} U$, where $\mathfrak{V}$ is a complete set of non-injective uniform right modules, and let $\mathfrak{N} = \oplus_{N \in \Gamma} N$, where Γ is any complete set of representatives of cyclic modules. The author of (Aydoğdu and López-Permouth, 2011) suspect whether the module $\mathfrak{M}$ is indigent, at least over right Noetherian rings.*

Proposition 6.2 *Let R be a (non-semisimple) right semiartinian right V-ring. Then $\mathfrak{M}$ is not indigent.*

Proof Let U be a nonzero uniform right R-module. Then U has a simple submodule X by the semiartinian condition. Then X is injective, because R is a right V-ring. Then $U = X \oplus Y$, for some $Y \subseteq U$. Since U is uniform and X is nonzero, we must have $Y = 0$. Thus $U = X$ is a simple. Hence every nonzero uniform module is simple over such ring. Then $\mathfrak{M}$ is semisimple. Since R is non-semisimple, there is a non-injective cyclic right module, say N by Osofsky Theorem. Let $f : N \to \mathfrak{M}$ be any homomorphism. Then $f(N)$ is contained in a finitely generated submodule K of $\mathfrak{M}$. Then K is injective, because it is semisimple and finitely generated. Thus there is a $g : E(N) \to K$ such that $g|_N = f$. Hence $\mathfrak{M}$ is N-subinjective. This shows that $\mathfrak{M}$ is not indigent. □

Proposition 6.3 *Let R be a right hereditary right Noetherian ring. Suppose every non-injective right module has a non-injective uniform factor module. Then $\mathfrak{M}$ is indigent.*

Proof Since R is Noetherian without loss of generality, we can assume that $\mathfrak{M}$ has no nonzero injective submodule. Suppose $\mathfrak{M}$ is N-subinjective for some right module N. We claim that N is injective. Suppose the contrary, and let $N = D \oplus N'$, where D is the largest injective submodule of N and $0 \neq N'$ has no nonzero injective submodule. Then $\mathfrak{M}$ is N'-subinjective. Since $N' \neq 0$, N' has a nonzero uniform quotient module which is non-injective by the assumption. Thus there is a nonzero homomorphism $f : N' \to \mathfrak{M}$. Then f extends to a homomorphism $g : E(N') \to M$. Now $0 \neq g(E(N'))$ is an injective submodule of M, by the hereditary assumption. This contradicts with the fact that M has no nonzero injective submodule. Thus N must be injective, and so $\mathfrak{M}$ is indigent. □

Proposition 6.4 *Let R be a right PCI-domain. Then $\mathfrak{M}$ is indigent.*

Proof Since R is a right hereditary right Noetherian, by Proposition 6.3, it is enough to show that every non-injective right module has a non-injective uniform factor module. Let M be a non-injective right R-module. Since singular modules are injective over right PCI-domains, without loos of generality we can assume that M is nonsingular. Let E be the injective hull of M. Then $E = \oplus_{i \in I} E_i$, where E_i are indecomposable by ((Matlis, 1958), Theorem 2.5). Note that E_i are uniform, because they are injective and indecomposable.

Since R is a domain, we have $E_i \cong E(R)$ for each $i \in I$. For each i, let $e : M \to E$ and $\pi_i : E \to E_i$ be the inclusion and projection maps respectively. Since M is non-injective, there is a $j \in I$ such that the map $g = \pi_j e$ is not epic, and so $M/Kerg$ is isomorphic to a proper submodule E_j. Thus $M/Kerg$ is a non-injective uniform factor module of M. Now, $\mathfrak{M}$ is indigent by Proposition 6.3. □

6.2. Indigent modules over Commutative Hereditary Noetherian rings

In this section we shall give a characterization of indigent modules over commutative hereditary Noetherian rings. First we recall a result from (Alizade, Büyükaşık and Er, 2014) which gives a characterization of indigent abelian groups.

Theorem 6.1 *((Alizade, Büyükaşık and Er, 2014), Theorem 27) The following are equivalent for an abelian group G.*

(i) *G is indigent.*

(ii) *$T_p(G) \neq pT_p(G)$ for each prime p.*

(iii) *The reduced part of $T(G)$ contains a submodule isomorphic to $\bigoplus_p \frac{\mathbb{Z}}{p\mathbb{Z}}$, where p ranges over all primes.*

Proof $(ii) \Rightarrow (iii)$ is clear.

$(i) \Rightarrow (ii)$ Suppose $pT_p(G) = T_p(G)$ for some prime p. On the other hand, for a prime $q \neq p$, we always have $qT_p(G) = T_p(G)$. Hence $T_p(G)$ is divisible, and so injective. Now it straightforward to check that G is $\frac{\mathbb{Z}}{p\mathbb{Z}}$-subinjective, obtaining a contradiction.

$(iii) \Rightarrow (i)$ Suppose G is N-subinjective for some abelian group N. We will show that N is injective, equivalently, that $qN = N$ for every prime q. Assume, contrarily, that $pN \neq N$ for some prime p. Since $\frac{N}{pN}$ is nonzero semisimple, N has a factor isomorphic to $\frac{\mathbb{Z}}{p\mathbb{Z}}$.

Now $G = D \oplus B$, where D is divisible and B is reduced. Then $T(G) = T(D) \oplus T(B)$, where $T(D)$ is clearly divisible and $T(B)$ is reduced. So, by assumption, $T(B)$ contains a copy of $\frac{\mathbb{Z}}{p\mathbb{Z}}$. Then there is a nonzero map $f : N \to T(B)$, which, by assumption of N-subinjectivity, extends to some $g : E(N) \to G$. Thus, $Im(g)$ is divisible. Let $\pi : D \oplus B \to B$ be the obvious projection. If $Im(g)$ were not contained in D, $\pi(Im(g))$ would be a nonzero divisible module in B, a contradiction. But then, $Im(f) \subseteq Im(g) \cap B = 0$, again a contradiction. Now the conclusion follows. □

Corollary 6.1 *((Alizade, Büyükaşık and Er, 2014), Corollary 28) An abelian group G is indigent if and only if its torsion part is indigent.*

From Theorem 6.1, we see that the direct sum simple abelian groups is indigent. For right small ring we have the following result.

Proposition 6.5 *Let R be a right small ring and S be the direct sum of non-isomorphic simple right R-modules. The following are equivalent.*

(1) S is indigent.

(2) R is right hereditary, right Noetherian and any module with $Rad(N) = N$ is injective.

Proof (1) $\Rightarrow$ (2) Suppose S is indigent. First note that $Hom(N, S) = 0$ for every module N such that $Rad(N) = N$. This implies S is N-subinjective, and so N is injective because S is indigent. Since R is a small ring, $Rad(E) = E$ for every injective right module E. Now for an injective module E and a submodule K of M, $Rad(E/K) = E/K$. Then $Hom(E/K, S) = 0$, and so E/K is injective by (1). This shows that R is right hereditary. Now let E_i, $i \in I$ be a family of injective right modules. Then $Rad(\oplus E_i) = \oplus Rad(E_i) = \oplus E_i$, and so $Hom(\oplus E_i, S) = 0$. Thus $\oplus E_i$ is injective by (1) again. This proves that R is right Noetherian.

(2) $\Rightarrow$ (1) Suppose S is N-subinjective for some right module N. Assume that N is not injective and lets find a contradiction. By the Noetherian assumption we can assume that N has no nonzero injective submodule. Let $f \in Hom(N, S)$. Since S is N-subinjective, the map f extends to a a map $g : E(N) \to S$. Since R is a small ring, $Rad(E) = E$. Thus $g(E) \leq Rad(S) = 0$ i.e. $g = 0$. Then $f = g|_N = 0$, and so we have $Hom(N, S) = 0$. Hence $Rad(N) = N$, and so N is injective by (2). This proves that S is indigent. □

The following proposition shows that, injective modules, flat module and projective semisimple modules coincide over commutative Noetherian rings.

Proposition 6.6 *Let R be a commutative ring and S a semisimple module. Consider the following statements.*

(1) S is injective;

(2) S is flat;

(3) S is projective.

Then (1) $\Rightarrow$ (2) $\Leftarrow$ (3). *If R is also Noetherian, then* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

Proof (1) $\Rightarrow$ (2) By ((Ware, 1971), Lemma 2.6) and by the fact that direct sum of flat modules is flat.

(3) $\Rightarrow$ (2) Projective modules are flat, so this is clear.

(2) $\Rightarrow$ (1) Over a Noetherian ring, arbitrary direct sum of injective modules is injective. So the proof is clear by ((Ware, 1971), Lemma 2.6).

(2) $\Rightarrow$ (3) If R is Noetherian, then each simple module is finitely presented. Finitely presented flat modules are projective by ((Lam, 1999), Theorem 4.30). Since direct sum of projective modules is projective, semisimple flat modules are projective over Noetherian rings. □

Theorem 6.2 *((McConnell and Robson, 2001), Theorem 4.6) A hereditary Noetherian ring R is a finite direct sum of Artinian hereditary rings and hereditary Noetherian prime rings.*

Proposition 6.7 *Let R be a Dedekind domain and M be an R-module. The following statements are equivalent.*

(1) M is injective.

(2) M is divisible.

(3) M has no maximal submodules i.e. Rad(M) = M.

Proof (1) $\Rightarrow$ (2) Assume that M is injective. Let $m \in M$ and $r_0 \in R$ be nonzero; we must find $x \in M$ with $m = r_0 x$. Define $f : Rr_0 \to M$ by $f(rr_0) = rm$ (note f is well-defined because R is a domain: $rr_0 = r'r_0$ implies that $r = r'$). Since M is injective, there exists $h : R \to M$ extending f. In particular,

$$m = f(r_0) = h(r_0) = r_0 h(1),$$

so that $x = h(1)$ is the element in M required by the definition of divisible.

(2) $\Rightarrow$ (1) Assume that M is divisible R-module. By the Baer Criterion, it is suffices to complete the diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "f"] & R \arrow[dl, dashed] \\ & M & \end{tikzcd}$$

where I is an ideal and i is the inclusion. We may assume that I is nonzero, so that I is invertible: there are elements $a_1, \ldots, a_n \in I$ and $q_1, \ldots, q_n \in FracR$ with $q_i I \subseteq R$ and $I = \Sigma_i q_i a_i$. Since M is divisible, there are elements $m_i \in M$ with $f(a_i) = a_i m_i$. Note for every $b \in I$, that

$$f(b) = f(\Sigma_i q_i a_i b) = \Sigma_i (q_i b) f(a_i) = \Sigma_i (q_i b) a_i m_i = b \Sigma_i (q_i a_i) m_i.$$

Hence, if we define $m = \Sigma_i (q_i a_i) m_i$, then $m \in M$ and $f(b) = bm$ for all $b \in I$. Now we define $g : R \to M$ by $g(r) = rm$; since g extends f, the module M is injective. □

The following result shows that, radical modules are injective over commutative hereditary Noetherian rings.

Proposition 6.8 *Let R be a commutative hereditary Noetherian ring. Then every module with $Rad(N) = N$ is injective.*

Proof By Theorem 6.2 and the commutativity assumption, we have

$$R = e_1 R \oplus \cdots e_t R \oplus f_1 R \oplus \cdots f_k R,$$

where $e_i R$'s are fields and $f_j R$'s are Dedekind domains for each $1 \leq i \leq t$ and $1 \leq j \leq k$. Let $S = \oplus_{i=1}^{t} e_i R$ and $T = \oplus_{i=1}^{k} f_i R$. Let E be a module with $Rad(E) = E$. Then E can not have a simple direct summand. Thus $S \cdot E = 0$, and so E is a T-module. So E has a decomposition as

$$E = f_1 E \oplus f_2 E \oplus \cdots f_k E,$$

where $f_j E$ is an $f_j R$ module and $Rad(f_i R) = f_i R$ for each $j = 1, \cdots, k$. Since $f_j R$ is a Dedekind domain and $Rad(f_j E) = f_j E$, the modules $f_j E$ are injective $f_j R$ for each $j = 1, \cdots, k$. Thus E is injective both as a T-module and as R-module. □

We do not know whether the following result is stated somewhere in the literature, we include it for completeness.

Proposition 6.9 *Let R be a commutative hereditary Noetherian ring. Then $R = S \oplus T$, where S is semisimple, $Soc(T) = 0$ and $T \ll E(T)$.*

Proof By Theorem 6.2, $R = S \oplus T$, where $S = Soc(R)$ and T is a direct sum of Dedekind domains. Now let us prove that $T \ll E(T)$. If not, then $T + K = E(T)$ for some maximal submodule K of $E(T)$. Then

$$T/(T \cap K) \cong E(T)/K$$

is simple, and also injective by the Hereditary condition. Thus $T/(T \cap K)$ is projective by Proposition 6.6, and so $T = T \cap K \oplus U$ for some simple submodule U of R. Then $U \leq Soc(T) = 0$, a contradiction. Therefore $T \ll E(T)$. □

Proposition 6.10 *Let R be a commutative hereditary Noetherian ring and C the direct sum of nonisomorphic singular simple R-modules. Then C is indigent.*

Proof Let $R = S \oplus T$ be as in Proposition 6.9. Since S is projective it is nonsingular. Thus we have $S.C = 0$, and so C is a T-module. Clearly every simple T-module is singular. Therefore C is exactly the direct sum nonisomorphic singular simple T-modules. Now T is a small ring. Thus C is an indigent T-module by Proposition 6.8 and Proposition 6.10. Now let us see C is indigent R-module. Suppose C is N-subinjective for some R-module N. There is a decomposion $N = N.S \oplus N.T$. Since S is projective it is injective by 6.6. Then $N.S$ is injective. Since C is $N.T$-subinjective and C is indigent T-module, $N.T$ is injective T-module. Now it is straightforward to check that both $N.S$ and $N.T$ are injective as R-modules. Hence their direct sum $N = N.S \oplus N.T$ is injective R-module. Therefore C is indigent. □

Proposition 6.11 *Let R be a commutative Noetherian ring and M an indigent R-module. Let C be the direct sum of nonisomorphic singular simple R-modules. Then M contains a submodule isomorphic to C.*

Proof Suppose $Hom(U, M) = 0$ for some singular simple module U. Since U is singular, it is noninjective. Let $E = E(U)$. Then $Soc(E/U) \neq 0$. So there is a nonsemisimple $V \leq E$ such that V/U is simple. As U essential in V, we must have $V/U \cong U$. But then $Hom(V, M) = 0$, and this implies that M is V-subinjective. This contradicts the fact that M is indigent. Thus $Hom(U, M) \neq 0$ for each singular simple module U. □

Proposition 6.12 *Let R be a commutative hereditary noetherian ring and M an R-module. The following are equivalent.*

(1) M is indigent.

(2) The reduced part M' of M is indigent.

(3) $Hom(S, M') \neq 0$ for every singular simple R-module S, where M' is the reduced part of M.

Proof (1) $\Leftrightarrow$ (2) Since R is Noetherian, the module M has a largest injective submodule, say N. Then $M = M' \oplus N$, for some $M' \leq M$. Now it is easy to see that, for a right module K, M is K-subinjective if and only if M' is K-subinjective. Thus M is indigent if and only if M' is indigent.

(2) $\Rightarrow$ (3) By Proposition 6.11.

(3) $\Rightarrow$ (2) Suppose M' is K-subinjective for some R-module K. Suppose K is not injective. Then without loss of generality we can assume that K has no nonzero injective submodule.

Suppose $Hom(S, M') = 0$ for some singular simple module S. Then, as in the proof of Proposition 6.11.

□

6.3. When simple modules are indigent or injective

In this section we give a complete characterization of commutative rings over which each simple is indigent or injective. As a consequence we also characterize the commutative rings whose simple modules are indigent. The rings whose cyclic modules are indigent are shown to be semisimple artinian.

Lemma 6.1 *If R is a commutative ring, then every simple R-module is pure-injective.*

A ring R is called right H-ring if every right module is a direct sum of an injective module and a small module. Every right QF-ring is a right Harada ring by ((Oshiro, 1984), Theorem 4.3).

Theorem 6.3 *Let R be a commutative ring. The following are equivalent.*

(1) Every simple module is indigent or injective.

(2) R is a V-ring, or $R = A \times B$, where B is semisimple, and

(i) A is local, hereditary, Noetherian, small ring, or;

(ii) A is local QF.

Proof (1) $\Rightarrow$ (2) Suppose every simple is indigent or injective. If all simple modules are injective, then R is a V-ring. Now suppose, there is a non-injective simple module U. Then U is indigent by the hypothesis. If U' is any simple module which is not isomorphic to U, then $Hom(U', U) = 0$. That is, U is U'-subinjective, and so U' must be injective because U is indigent. Thus the ring has a unique non-injective simple module, up to isomorphism.

Let $\{E_i\}_{i\in I}$ be an arbitrary family of injective modules. Then $\oplus_{i\in I}E_i$ is a pure submodule of $\prod_{i\in I} E_i$. By Lemma 6.1 the simple module U is pure-injective. Thus U is $\oplus_{i\in I}E_i$-subinjective. As U is indigent, $\oplus_{i\in I}E_i$ must be injective. So that the ring R is Noetherian. Now let B be the sum of the injective simple ideals of R. Then B is injective, because the ring is Noetherian. So $R = A \oplus B$ for some ideal A of R. Then $Hom(B, A) = 0$, and $Hom(A, B) = 0$ by Proposition 6.6. Now, since $Hom(B, U) = 0$, we have

$$0 \neq Hom(R, U) = Hom(A \oplus B, U) \cong Hom(A, U) \oplus Hom(B, U) = Hom(A, U).$$

This implies that A has a simple module isomorphic U. If X is a simple A-module, then we must have $X \cong U$. Otherwise X would be injective by the hypothesis. Then X must be projective by Proposition 6.6, which implies that $A = X' \oplus Y$ for some $X' \cong X$. But then $X' \subseteq A \cap B = 0$, contradiction. Therefore A has a unique simple module, and this simple is isomorphic to U. By the commutativity condition, we get that A is a local ring. We have the following two cases:

Case I: A is a small ring, i.e. $A \ll E(A)$. Then $Rad(E/K) = E/K$ for each injective A-module E and $K \subseteq E$. Thus $Hom(E/K, U) = 0$, and so U is E/K-subinjective. Hence E/K is injective. This proves that R is Hereditary.

Case II: A is not small i.e. $Rad(E(A)) \neq E(A)$. Let us prove that A is QF by showing that A is injective. We know that A is local and non small. Since A is finitely generated, we have $A \nsubseteq Rad(E(A))$. Thus there is a maximal submodule K of $E(A)$ such that $A + K = E(A)$. Then $A/A \cap K \cong E(A)/K$ is simple, and so $A \cap K$ is the unique maximal submodule of A. Moreover $E(A)A \cap K = A/A \cap K \oplus K/A \cap K$. Let $f : A \to U$ be a nonzero homomorphism. Since U is simple $Ker(f) = A \cap K$ is maximal. Let $\pi' : A \to A/A \cap K$ and $\pi : E(A) \to A/A \cap K$ be the natural projections, and $\overline{f} : A/A \cap K \to U$ be the map satisfying $f = \overline{f}\pi'$. It is straightforward to check that the map $g = \overline{f}\pi : E(A) \to U$ extends f i.e. $g|_A = f$. Thus U is A-subinjective, and so A is injective. Being Noetherian and injective, A is QF.

(2) $\Rightarrow$ (1) If R is a V-ring, then each simple is injective. So that (1) s hold. Now assume (i). Note that B is projective R-module, and so injective as an R-module by Proposition 6.6. Let U be a noninjective simple R-module. Then $B.U = 0$, and so U is a simple A-module. Since A is local U is the unique simple A-module up to isomorphism. Hence U is an indigent A-module by Proposition 6.10. Let us prove that U is indigent R-module. Suppose U is M-subinjective for some R-module M. Then $M = M_1 \oplus M_2$, where M_1 is an A-module and M_2 is a B-module. Since B is semisimple and injective R-module. M_2 is an injective R-module. Since U is M_1-subinjective and U is indigent A-module, M_1 is an injective A-module. Since $Hom(B, M_1) = 0$, M_1 is also an injective B-module. Thus M_1 is injective $A \oplus B = R$-module. Therefore $M = M_1 \oplus M_2$ is an injective R-module, and so U is indigent R-module.

Now assume (ii). As in the proof of (i), the ring R has a unique non-injective simple module, say U which is also the unique simple A-module, up to isomorphism. Then since A is local and QF, U is an indigent A-module by ((Alizade, Büyükaşık and Er, 2014), Proposition 32) and ((Oshiro, 1984), Theorem 4.4). Then by similar arguments as in the proof of (i), U is an indigent R-module. This completes the proof. □

The following is a direct consequence of Theorem 6.3.

Corollary 6.2 *Let R be a commutative ring. The following are equivalent.*

(1) Every simple module is indigent.

(2) R is semisimple, or R is a local,

(i) V-ring, or;

(ii) hereditary Noetherian small ring, or;

(iii) QF.

A module is said to be semiartinian in case every homomorphic image of it has an essential socle. A ring R is called right semiartinian if it is semiartinian as a right R-module. In ((Aydoğdu and López-Permouth, 2011),Proposition 4.13), it is shown that if every non-zero cyclic right R-module is indigent, then R is right semiartinian. We have the following result.

Proposition 6.13 *Let R be a ring. Every non-zero cyclic right R-module is indigent if and only if R is semisimple Artinian.*

Proof Sufficiency is clear. To prove the necessity suppose every cyclic right R-module is indigent. Let A and B be two simple right R-modules. Assume that A and B are not isomorphic. Then, A is clearly B-subinjective. As A must be indigent by its choice, then B is injective. But, by our assumption, B is also indigent. Then R is semisimple Artinian since B is both injective and indigent module. W.l.o.g, R has a unique simple non-injective module up to isomorphism. If A is projective, then R is semisimple Artinian. Hence, A has to be a singular module. Note that R is also indigent by our assumption. Hence $Hom(A, R) \neq 0$, otherwise A is injective, a contradiction. Then R has a minimal right ideal, which is isomorphic to A , i.e. R is a right Kasch ring. We have the following two cases:

Case I: $Hom(E(R), A) = 0$. Then R is right small ring by Proposition 2.14. By Proposition 6.5, R is right hereditary, contradicting the singularity of A.

Case II: $Hom(E(R), A) \neq 0$. In this case, R is a right self-injective ring. Consider the following diagram

$$
\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\iota"] \arrow[d, "\pi"] \arrow[dl, dashed, "u"'] & E(R) \arrow[d, "\pi'"] \arrow[dll, dashed, "v"] \\
E(R) \arrow[r, "f"] & A \arrow[r] & 0
\end{tikzcd}
$$

By projectivity of R, there exists a homomorphism $u : R \to E(R)$ such that $fu = \pi$. Now, by injectivity of $E(R)$, there exists a homomorphism $v : E(R)) \to E(R)$ such that $v\iota = u$. Then $fv\iota = fu = \pi$,and so A is R-subinjective. But A is indigent, and so R is a right self-injective. But R is indigent, and so R is semisimple Artinian.

□

CHAPTER 7

TEST MODULES FOR INJECTIVITY BY SUBINJECTIVITY

There is another notion which is defined by subinjectivity. Namely, a module N is said to be a test for injectivity by subinjectivity (t.i.b.s.) if the only N-subinjective modules are injective modules.

In this chapter we give a characterization of t.i.b.s. modules over Dedekind domains.

First we state a result which states that t.i.b.s. modules exist over any ring.

Proposition 7.1 *((Alizade, Büyükaşık and Er, 2014), Proposition 1) Every ring has t.i.b.s. right module.*

Proof Let R be a ring and $N = \bigoplus I$, where I ranges among (proper) essential right ideals of R, and assume that X is an N-subinjective module. Let A be a right ideal of R, and $f : A \to X$ be any homomorphism. We may assume, without loss of generality, that A is essential in R_R. Then, the copy of A in N that is a direct summand of N is essential in an injective submodule, say Q, of $E(N)$. So, there is an embedding $\phi : R_R \to Q$ fixing A. Since X is N-subinjective, $f(\phi^{-1})|_A$ (here, A is the copy in N) extends to some $h : E(N) \to X$. Thus, $h\phi$ is the desired extension of f to $R \to X$. $\square$

Proposition 7.2 *((Alizade, Büyükaşık and Er, 2014), Proposition 2) The following conditions are equivalent for a ring R:*

(i) *Every right R-module is injective or a t.i.b.s.;*

(ii) *Every right R-module is injective or indigent;*

(iii) *If $A, B \in Mod - R$ and A is B-subinjective, then A or B is injective.*

In this case, the class of indigent modules and that of t.i.b.s. modules coincide.

Theorem 7.1 *((Alizade, Büyükaşık and Er, 2014), Theorem 19) The following are equivalent for a ring R:*

(i) *R_R is t.i.b.s.;*

(ii) *R is right hereditary and right Noetherian.*

Corollary 7.1 *((Alizade, Büyükaşık and Er, 2014), Corollary 20) A commutative domain R is Dedekind if and only if it is a t.i.b.s.*

7.1. The structure of t.i.b.s. modules over commutative rings

The structure of t.i.b.s. modules is known over the ring of integers. In this section we shall characterize t.i.b.s. modules over Dedekind domains.

Theorem 7.2 *((Alizade, Büyükaşık and Er, 2014), Theorem 26) An abelian group G is t.i.b.s. if and only if G has a direct summand isomorphic to $\mathbb{Z}$.*

Proof Suppose G is a t.i.b.s. Then $\mathrm{Hom}(G, \mathbb{Z}) \neq 0$. Let $f : G \to \mathbb{Z}$ be a nonzero homomorphism. Then $\frac{G}{Ker(f)} \cong n\mathbb{Z}$ is projective. So that $G = Ker(f) \oplus G'$ with $G' \cong \mathbb{Z}$. Conversely, if $G = A \oplus A'$ with $A' \cong \mathbb{Z}$, then G is a t.i.b.s. since $\mathbb{Z}$ is a t.i.b.s. by 7.1. □

The following lemma can be easily verified by using the properties of subinjectivity (see, (Aydoğdu and López-Permouth, 2011)). The proof is omitted here.

Lemma 7.1 *The following statements are equivalent for a right R-module M.*

(1) M is t.i.b.s.

(2) M^n is t.i.b.s. for some $n \in \mathbb{Z}^+$.

(3) M^n is t.i.b.s. for all $n \in \mathbb{Z}^+$.

(4) $M \oplus N$ is t.i.b.s. for any right module N.

The following theorem is a generalization of ((Alizade, Büyükaşık and Er, 2014), Theorem 26).

Theorem 7.3 *The following are equivalent for a commutative domain R.*

(1) R is Dedekind.

(2) R is t.i.b.s.

(3) Every nonzero ideal of R is t.i.b.s.

(4) A nonzero R-module M is t.i.b.s. exactly when $Hom(M, R) \neq 0$.

Proof (1) $\Leftrightarrow$ (2) By ((Alizade, Büyükaşık and Er, 2014), Corollary 20).

(2) $\Rightarrow$ (3). Let I be nonzero ideal of R. Since R is Dedekind, I is finitely generated and projective. Let P be a maximal ideal of R. We claim that, $P \cdot I \neq I$. Otherwise, we would have $P_P \cdot I_P = I_P$. Now R_P is a DVR with the unique maximal ideal P_P, and I_P is finitely generated. So $I_P = 0$, by Nakayama's Lemma. $I_P = 0$ implies, $t \cdot I = 0$ for some nonzero $t \in R - P$. But R is a domain, and $I \neq 0$, so $t = 0$. Contradiction. Therefore we have $PI \neq I$ for each maximal ideal P of R. Thus I/PI is a nonzero semisimple R-module, and so there is a maximal submodule K of I such that $I/K \cong R/P$ for each maximal ideal P of R. This means that I generates each simple R-module. So that I is a projective generator by ((Anderson and Fuller, 1992), Proposition 17.9). Then there is a positive integer n, such that $I^n \cong R \oplus K$. Hence I is t.i.b.s. by Lemma 7.1.

(3) $\Rightarrow$ (2) is clear.

(3) $\Rightarrow$ (4) Let M be an R-module. Suppose first that $Hom(M, R) \neq 0$. Then there is a nonzero ideal I of R such that $M/K \cong I$ for some $K \leq M$. Since (3) also implies that R is Dedekind, the ideal I is projective. Thus $M = K \oplus I$. By (3), the ideal I is t.i.b.s. Then M is t.i.b.s. by Lemma 7.1. Conversely, suppose M is t.i.b.s. and $Hom(M, R) = 0$. Then R is M-subinjective, and so R is injective by the t.i.b.s. assumption on M. Thus R is a field, and so $Hom(M, R) = 0$ gives $M = 0$, a contradiction. Hence $Hom(M, R) \neq 0$.

(4) $\Rightarrow$ (2) $Hom(R, R) \neq 0$. So R is t.i.b.s. by (4). This proves (2).

$\square$

CHAPTER 8

CONCLUSION

Recently, an opposite notions of poor modules and relative injectivity introduced in (Aydoğdu and López-Permouth, 2011). A right module M is said to be N-subinjective for some right module N, if every homomorphism from N to M can be extended to a homomorphism from the injective hull $E(N)$ of N to M. The subinjectivity domain of M is defined as the collection of all right modules N such that M is N-subinjective. A right module M is called *indigent* if its subinjectivity domain is exactly the class of injective right modules.

In this thesis we consider some problems and also generalize some results related to indigent modules and subinjectivity domains. We prove that subinjectivity domain of any right module is closed under factor modules if and only if the ring is right hereditary. Indigent modules are the modules whose subinjectivity domain is as small as possible, namely the modules whose subinjectivity domain is exactly the class of injective modules. We give a complete characterization of indigent modules over commutative hereditary Noetherian rings. The commutative rings whose simple modules are injective or indigent are fully determined. The rings whose cyclic right modules are indigent are shown to be semisimple Artinian. We also give a characterization of t.i.b.s. modules over Dedekind domains.

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